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A Note on Rectilinearity and Angular Resolution

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Abstract

We connect two aspects of graph drawing, namely angular resolution, and the possibility to draw with all angles an integer multiple of $2\pi/d$. A planar graph with angular resolution at least $\pi/2$ can be drawn with all angles an integer multiple of $\pi/2$ (rectilinear). For $d \neq 4$, $d > 2$, an angular resolution of $2\pi/d$ does *not* imply that the graph can be drawn with all angles an integer multiple of $2\pi/d$. We argue that the exceptional situation for $d = 4$ is due to the absence of triangles in the rectangular grid.

Keywords : Rectilinear drawing, plane graph, angular resolution, integer flow.

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1 Introduction

Angular resolution and rectilinearity are well-studied aspects of plane graphs. The angular resolution of a (plane) graph is the minimum angle made by line segments at a vertex. A graph is rectilinear if it can be drawn with all angles a multiple of $\pi/2$ radians. In this note we give an easy proof of the "folk conjecture" that graphs with an angular resolution at least $\pi/2$ are rectilinear. We generalise rectilinearity and call a graph d-linear if it allows a drawing with all edges a multiple of $2\pi/d$ (thus rectinearity is 4-linearity). Unlike the case $d = 4$, for $d > 4$ it is not the case that an angular resolution of $2\pi/d$ implies d-linearity.

This is the organization of the paper. The remainder of this section introduces some preliminaries, including Tamassia's *flow model* for the angles in a drawing, on which our first result is based. Section 2 proves our positive result (for $d = 4$). Section 3 contains the negative result (for $d > 4$). Section 4 lists conclusions.

1.1 Preliminaries

We assume familiarity of the reader with basic graph notions. A *plane graph* is a planar graph given together with an embedding; this embedding should be respected in a drawing. Given a drawing, its *angular resolution* is the minimum angle made by line segments at any vertex, and the angular resolution of a plane graph is the maximum angular resolution of any drawing. A drawing is called *d-linear* if all angles are an integer multiple of $2\pi/d$ radians.

A vertex of degree δ has δ angles: one between each two successive incident edges. For vertex v and face f, let $d(v, f)$ be the number of angles of v that belong to face f (only for a cutvertex v there is an f for which $d(v, f) \geq 2$). For every face f, we let $a(f)$ denote the number of angles that belong to f. In a biconnected graph, $a(f)$ also equals the number of edges at the border of f and it equals the number of vertices on the border of f.

1.2 The Flow Model for Angles

The embedding contained in a plane graph defines the position of the nodes in a qualitative manner, but to convert the embedding into a drawing, in addition two more things need be specified: the angles between the edges at each node, and the lengths of all edges. Tamassia [3] has shown that the angle values in a drawing satisfy the constraints of a suitably chosen *multi-source multi-sink flow network*. In any drawing, the angles around a node sum up to 2π and if an internal face is drawn as an a-gon its angles sum up to $\pi(a - 2)$ radians. (The angles of the outer face with a edges sum up to $\pi(a + 2)$.)

Because we want rectangular angles to correspond to integers, we shall now express angles in units of $\pi/2$ radians. Thus, with $\alpha_{v,f}$ the angle at node v in face f, the collection of angles in a drawing with angular resolution 1 is a solution for this set of linear equations:

$$\begin{array}{ll} \sum_f \ \alpha_{v,f} = 4 & \text{for all nodes } v \\ \sum_v \ \alpha_{v,f} = 2(a(f) - 2) & \text{for all internal faces } f \\ \sum_v \ \alpha_{v,f} = 2(a(f) + 2) & \text{for outer faces } f \\ \alpha_{v,f} \geq 1 & \text{for all incident } v \text{ and } f \end{array}$$

We refer to these equations as the *network model* for G; observe that all constraints are *integer* numbers. The description of this set of equations as a flow network can be found in [1, 3].

Relations between flows and drawings. The following two results are known.

Theorem 1 *If plane graph G has a drawing with angular resolution at least 1 unit ($\pi/2$ radians), then the associated network model has a solution.*

Theorem 2 (Tamassia [3]) *If the network model associated to plane graph G has an* integer solution*, then G has a rectilinear drawing.*

2 Angular Resolution $\pi/2$ Implies Rectilinear

Our main result is obtained by combining Theorems 1 and 2 with a result from standard flow theory.

Theorem 3 *If a graph has angular resolution at least $\pi/2$, then it is rectilinear.*

Proof. Assume G has angular resolution at least $\pi/2$ radians. By definition, it has a drawing with angular resolution at least 1 unit, hence by Theorem 1 the associated network model has a solution. It is known from flow theory (see, e.g., [2, Chapters 10, 11]) that if a flow network with integer constraints admits a flow, then it admits an integer flow. By Theorem 2, we have that G has a rectilinear drawing. □

3 Angular Resolution and d-Linearity

This section answers the question for what values of d, any plane graph with angular resolution $2\pi/d$ radians is d-linear. The cases $d = 1$ and $d = 2$ are somewhat trivial, as the classes of drawable graphs are collections of isolated edges, or paths, respectively.

The case of *odd* d is somewhat degenerate as, while drawings are supposed to be built up of straight lines, a straight angle at a vertex is not allowed. The cycle with $2d+1$ points can be drawn as a regular $(2d+1)$-gon, witnessing that its angular resolution is at least $2\pi/d$. But it does not have a d-linear drawing, as its angle sum, $(2d-1)\cdot\pi$, is not an integer multiple of $2\pi/d$.

In the remainder of this section $d = 2d'$ is even and larger than 4. Measuring angles in units of π/d' radians, we observe that the angles of any triangle add to exactly d' units. We introduce *rigid triangles* as gadgets with a fixed shape. The graph $T_{d'}$ has a top node t and base nodes b_1 and b_2, forming a cycle. For even d', t has a third neighbor i, inserted between b_2 and b_1 in the planar embedding. Each base node has $\lfloor \frac{d'-3}{2} \rfloor$ neighbors, also located inside the triangle; see Figure 1. The graph $T_{d'}$ has an angular resolution of $2\pi/d$ radians, and in every drawing with that resolution, each segment of each angle measures exactly π/d' radians, that is, the proportions of $T_{d'}$ are fixed in every drawing. The ratio between height and base length of the rigid triangle in such a drawing, $b_{d'}$, can be computed as $b_{d'} = \frac{1}{2}\tan(\lfloor \frac{d'-1}{2} \rfloor \cdot (\frac{\pi}{d'}))$.

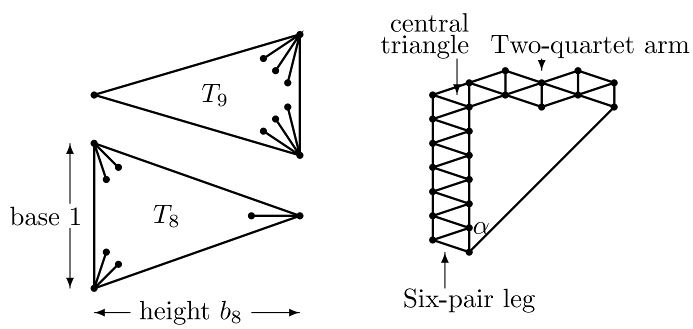

Figure 1: Rigid Triangles and the Crane $C_{d',6,2}$

The crane graph $C_{d',k,l}$ contains $1 + 2k + 4l$ copies of $T_{d'}$ joined together. A central triangle is extended with a leg consisting of k pairs of triangles on one side, and an arm consisting of l quartets of triangles on the other side. In any drawing where all internal angles of the triangles satisfy the constraint for angular resolution π/d', the angle α at the bottom of the drawing satisfies $\tan\alpha = \frac{2l}{k} b_{d'}$. By choosing k and l, any angle between π/d' and $\pi/2$ radians can be approximated arbitrarily closely, contradicting the possibility to draw any crane with all angles a multiple of π/d' radians.

Theorem 4 *For each $d' > 2$, $\beta \geq \pi/d'$, $\epsilon > 0$, there exists a graph $G = C_{d',k,l}$ such that*

1. *G has angular resolution π/d' radians;*
2. *each drawing of G with angular resolution π/d' contains an angle α such that $|\alpha - \beta| < \epsilon$.*

4 Conclusions

Our note compares two types of drawings, namely (1) those where all angles are at least $2\pi/d$ (angular resolution) and (2) those where all angles are an integer multiple of $2\pi/d$. The flow model introduced by Tamassia [3] implies that if angles can be assigned satisfying (1), then it is also possible to assign all angles satisfying (2). However, only in the special case $d = 4$ it is also possible to assign edge lengths in such a way that a drawing results.

When drawing graphs in a rectilinear way, the drawing is built up from rectangular elements and in such drawings it is possible to shift parts of the drawing without disrupting angles in other parts; see Figure 2. Indeed, a rectangle can be stretched in one direction while preserving the orthogonality of its angles. In a drawing containing triangles, this is not possible, because stretching a triangle in one direction changes its angles. We therefore conclude that the special position of $d = 4$ in the studied problem is due to the absence of triangles in a rectilinear grid.

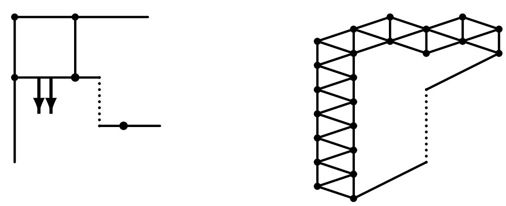

Figure 2: Orthogonal and non-orthogonal movements

The observations in this note can be extended to the situation where the embedding is free; that is, only a (planar) graph is given and the question is, what type of drawings does it admit. For the positive result (for $d = 4$), if the graph has some drawing with angular resolution $\pi/2$, it can be drawn rectilinearly with the same embedding. Our triangles $T_{d'}$ loose their rigidity if the embedding is free, because one can draw the internal nodes on the outside and then modify the triangle shape. By connecting the internal nodes in a cycle this can be prevented; in fact the triangles are modified to enforce the same embedding.

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