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Rectilinear Planarity of Partial 2-Trees

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Abstract. A graph is rectilinear planar if it admits a planar orthogonal drawing without bends. While testing rectilinear planarity is NP-hard in general (Garg and Tamassia, 2001), it is a long-standing open problem to establish a tight upper bound on its complexity for partial 2-trees, i.e., graphs whose biconnected components are series-parallel. We describe a new $O(n^2)$-time algorithm to test rectilinear planarity of partial 2-trees, which improves over the current best bound of $O(n^3 \log n)$ (Di Giacomo et al., 2022). Moreover, for partial 2-trees where no two parallel-components in a biconnected component share a pole, we are able to achieve optimal $O(n)$-time complexity. Our algorithms are based on an extensive study and a deeper understanding of the notion of orthogonal spirality, introduced several years ago (Di Battista et al., 1998) to describe how much an orthogonal drawing of a subgraph is rolled-up in an orthogonal drawing of the graph.

1 Introduction

In an *orthogonal drawing* of a graph each vertex is a distinct point of the plane and each edge is a chain of horizontal and vertical segments. Rectilinear planarity testing asks whether a planar 4-graph (i.e., with vertex-degree at most four) admits a planar orthogonal drawing without edge bends. It is a classical subject of study in graph drawing, partly for its theoretical beauty and partly

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because it is at the heart of the algorithms that compute bend-minimum orthogonal drawings, which find applications in several domains (see, e.g., [4, 12, 14, 22, 23, 24]). Rectilinear planarity testing is NP-hard [18], it belongs to the XP-class when parameterized by treewidth [6], and it is FPT when parameterized by the number of degree-4 vertices [11]. Polynomial-time solutions exist for restricted versions of the problem. Namely, if the algorithm must preserve a given planar embedding, rectilinear planarity testing can be solved in subquadratic time for general graphs [2, 17], and in linear time for planar 3-graphs [26] and for biconnected series-parallel graphs (SP-graphs for short) [8]. When the planar embedding is not fixed, linear-time solutions exist for (families of) planar 3-graphs [13, 20, 25, 28] and for outerplanar graphs [16]. A polynomial-time solution for SP-graphs has been known for a long time [5], but establishing a tight complexity bound for rectilinear planarity testing of SP-graphs remains a long-standing open problem.

In this paper we provide significant advances on this problem. Our main contribution is twofold:

- We present an $O(n^2)$-time algorithm to test rectilinear planarity of partial 2-trees, i.e., graphs whose biconnected components are SP-graphs. This result improves the current best known bound of $O(n^3 \log n)$ [6].

- We give an $O(n)$-time algorithm for those partial 2-trees where no two parallel-components in a block (i.e., a biconnected component) share a pole. We also show a logarithmic lower bound on the possible values of spirality for an orthogonal component of a graph.

Our algorithms are based on an extensive study and a deeper understanding of the notion of orthogonal spirality, introduced in 1998 to describe how much an orthogonal drawing of a subgraph is rolled-up in an orthogonal drawing of the graph [5]. In the concluding remarks we also mention some of the pitfalls behind an $O(n)$-time algorithm for partial 2-trees.

2 Preliminaries

A *planar orthogonal drawing* Γ of a planar graph is a crossing-free drawing that maps each vertex to a distinct point of the plane and each edge to a sequence of horizontal and vertical segments between its end-points [4, 14, 24]. A graph is *rectilinear planar* if it admits a planar orthogonal drawing without bends. A *planar orthogonal representation* H describes the shape of a class of orthogonal drawings in terms of sequences of bends along the edges and angles at the vertices. A drawing Γ of H can be computed in linear time [27]. If H has no bend, it is a planar *rectilinear representation*. Since we only deal with planar drawings, we just use the term "rectilinear representation" in place of "planar rectilinear representation".

SP-graphs and SPQ*-trees. A *two-terminal SP-graph* is a graph inductively defined as follows:

- **Base case**. A single edge (s, t) is a two-terminal SP-graph with terminals s and t.

- **Inductive case**. Let $G_1, G_2, \ldots, G_p$, with $p \geq 2$, be a set of two-terminal SP-graphs, where each G_i has terminals s_i and t_i; two inductive operations are possible:

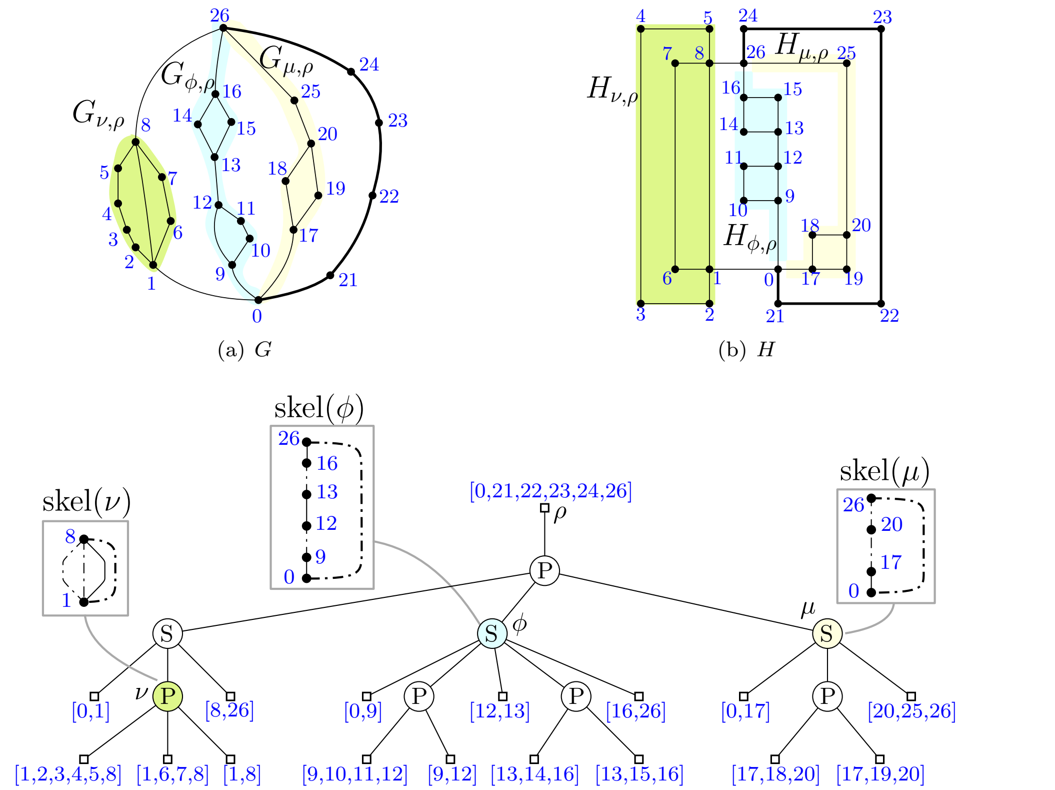
  – **Series-composition**. The graph G obtained by the union of all G_i in which t_i is identified with s_{i+1}, for $i = 1, \ldots, p-1$ is a two-terminal SP-graph with terminals $s = s_1$ and $t = t_p$, called a *series-component*.

– **Parallel-composition**. The graph G obtained by the union of all G_i where all terminals s_i are identified in a unique vertex s, and all terminals t_i are identified in a unique vertex t, is a two-terminal SP-graph with terminals $s = s_1 = \cdots = s_p$ and $t = t_1 = \cdots = t_p$, called a *parallel-component*.

An *SP-graph* is any biconnected two-terminal SP-graph. Such a graph can be described by a decomposition-tree called *SPQ-tree*, which contains three types of nodes: *S-*, *P-*, and *Q-nodes*. The degree-1 nodes of T are Q-nodes, each corresponding to a distinct edge of G. If ν is an S-node (resp. a P-node) it represents a series-component (resp. a parallel-component), denoted as skel(ν) and called the *skeleton* of ν. If ν is an S-node, skel(ν) is a simple cycle of length at least three; if ν is a P-node, skel(ν) is a bundle of at least three multiple edges. A property of T is that any two S-nodes, as well as any two P-nodes, are never adjacent in the tree. A *real edge* (resp. *virtual edge*) in skel(ν) corresponds to a Q-node (resp. an S- or a P-node) adjacent to ν in T.

Testing whether a simple cycle is rectilinear planar is trivial (if and only if it has at least four vertices). Hence, we shall assume that G is a biconnected SP-graph different from a simple cycle and we use a variant of the SPQ-tree called *SPQ*-tree* (refer to Fig. 1). In an SPQ*-tree, each degree-1 node of T is a *Q*-node*, and represents a maximal chain of edges of G (possibly a single edge) starting and ending at vertices of degree larger than two and passing through a sequence of degree-2 vertices only (possibly none). If ν is an S- or a P-node, an edge of skel(ν) corresponding to a Q*-node μ is virtual if μ is a chain of at least two edges, else it is a real edge.

For any given Q*-node ρ of T, denote by T_ρ the tree T rooted at ρ. Also, for any node ν of T_ρ, denote by $T_\rho(\nu)$ the subtree of T_ρ rooted at ν. The chain of edges represented by ρ is the *reference chain* of G with respect to T_ρ. If ν is an S- or a P-node distinct from the root child of T_ρ, then skel(ν) contains a virtual edge that has a counterpart in the skeleton of its parent; this edge is the *reference edge* of skel(ν). If ν is the root child, the *reference edge* of skel(ν) is the edge corresponding to ρ. For any S- or P-node ν of T_ρ, the end-vertices of the reference edge of skel(ν) are the *poles* of ν and of skel(ν). We remark that skel(ν) does not change if we change ρ. However, if ν is an S-node, its poles depend on ρ; namely, if ρ' is a Q*-node in the subtree $T_\rho(\nu)$, the poles of ν in $T_{\rho'}$ are different from those in T_ρ. Conversely, the poles of a P-node stay the same independent of the root of T. For a Q*-node ν of T_ρ (including ρ), the *poles* of ν are the end-vertices of the corresponding chain, and do not change when the root of T changes. For any S- or P-node ν of T_ρ, the *pertinent graph* $G_{\nu,\rho}$ of ν is the subgraph of G formed by the union of the chains represented by the leaves in the subtree $T_\rho(\nu)$. The *poles* of $G_{\nu,\rho}$ are the poles of ν. The *pertinent graph* of a Q*-node ν (including the root) is the chain represented by ν, and its *poles* are the poles of ν. Any graph $G_{\nu,\rho}$ is also called a *component* of G (with respect to ρ). If μ is a child of ν, we call $G_{\mu,\rho}$ a *child component* of ν. If H is a rectilinear representation of G, for any node ν of T_ρ, the restriction $H_{\nu,\rho}$ of H to $G_{\nu,\rho}$ is a *component* of H (with respect to ρ). Tree T_ρ is used to describe all planar embeddings of G having the reference chain on the external face. These embeddings are obtained by permuting in all possible ways the edges of the skeletons of the P-nodes distinct from the reference edges, around the poles. For each P-node ν, each permutation of the edges in skel(ν) corresponds to a different left-to-right order of the children of ν in T_ρ and of their associated components. Namely, assume given an *st-numbering* of G such that s and t coincide with the poles of ρ. We recall that an *st*-numbering is a labeling of the n vertices of G, with numbers in the set $\{1, \ldots, n\}$, such that each vertex gets a different number, s gets number 1, t gets number n, and each other vertex $v \notin \{s, t\}$ is adjacent to both a vertex with smaller number and a vertex with larger number. It is well-known that an n-vertex graph G admits an *st*-numbering if and only if $G \cup (s, t)$ is biconnected, and such a numbering can be computed in $O(n)$ time [15].



(a) G (b) H (c) T_ρ

Figure 1: (a) An SP-graph G. (b) A rectilinear representation H of G. (c) The SPQ*-tree T_ρ of G, where ρ represents the thick chain; Q*-nodes are small squares; the left-to-right order of the children of each P-node reflects the embedding of H. The components and the skeletons of the nodes ν, μ, ϕ are shown: virtual edges are dashed and the reference edge is thicker.

For each P-node ν of T_ρ, let u and v be its poles where u precedes v in the st-numbering. Denote by e_ν the reference edge of skel(ν), by $e_1, \dots, e_h$ the edges of skel(ν) distinct from e_ν, and by $\mu_1, \dots, \mu_h$ the children of ν corresponding to $e_1, \dots, e_h$. Each permutation of $e_1, \dots, e_h$ defines a class of planar embeddings of $G_{\nu,\rho}$ with u and v on the external face, where the components $G_{\mu_1,\rho}, \dots, G_{\mu_h,\rho}$ are incident to u and v in the order of the permutation. More precisely, if $e_{i_1}, \dots, e_{i_h}$ is one of these permutations ($i_j \in \{1, \dots, h\}$), the clockwise (resp. counterclockwise) sequence of edges incident to u (resp. v) in skel(ν) is $e_\nu, e_{i_1}, \dots, e_{i_h}$; we say that, according to this permutation, $\mu_{i_1}, \dots, \mu_{i_h}$ and their corresponding components appear in this left-to-right order.

We finally recall that the SPQ*-tree T of an n-vertex graph G can be computed in $O(n)$ time [4, 19, 21].

Partial 2-trees and BC-trees. A 1-connected graph G is a *partial 2-tree* if every biconnected component of G is an SP-graph. A biconnected component of G is also called a *block*. A block is *trivial* if it consists of a single edge. The *block-cutvertex tree* $\mathcal{T}$ of G, also called *BC-tree* of

G, describes the decomposition of G in terms of its blocks (see, e.g., [4]). Each node of $\mathcal{T}$ either represents a block of G or it represents a cutvertex of G. A *block-node* (resp. a *cutvertex-node*) of $\mathcal{T}$ is a node that represents a block (resp. a cutvertex) of G. There is an edge between two nodes of $\mathcal{T}$ if and only if one node represents a cutvertex of G and the other node represents a block that contains the cutvertex.

3 Rectilinear Planarity Testing of Partial 2-Trees

Let G be a partial 2-tree. We describe a rectilinear planarity testing algorithm that visits the block-cutvertex tree (BC-tree) of G and the SPQ*-tree of each block of G, for each possible choice of the roots of both decomposition trees. Our algorithm revisits the notion of "spirality values" for the blocks of G, and introduces new concepts to efficiently compute these values (Section 3.1). It is based on a combination of dynamic programming techniques (Section 3.2).

3.1 Spirality of SP-graphs

Let G be a degree-4 SP-graph and let H be an orthogonal representation of G. Let T_ρ be a rooted SPQ*-tree of G, let $H_{\nu,\rho}$ be a component of H (i.e., the restriction of H to $G_{\nu,\rho}$), and let $\{u,v\}$ be the poles of ν, conventionally ordered according to an *st*-numbering of G, where s and t are the poles of ρ. For each pole $w \in \{u,v\}$, let $\text{indeg}_\nu(w)$ and $\text{outdeg}_\nu(w)$ be the degree of w inside and outside $H_{\nu,\rho}$, respectively. Define two (possibly coincident) *alias vertices* of w, denoted by w' and w'', as follows: (*i*) if $\text{indeg}_\nu(w) = 1$, then $w' = w'' = w$; (*ii*) if $\text{indeg}_\nu(w) = \text{outdeg}_\nu(w) = 2$, then w' and w'' are dummy vertices, each splitting one of the two distinct edge segments incident to w outside $H_{\nu,\rho}$; (*iii*) if $\text{indeg}_\nu(w) > 1$ and $\text{outdeg}_\nu(w) = 1$, then $w' = w''$ is a dummy vertex that splits the edge segment incident to w outside $H_{\nu,\rho}$.

Let A^w be the set of distinct alias vertices of a pole w. Let P^{uv} be any simple path from u to v inside $H_{\nu,\rho}$ and let u' and v' be the alias vertices of u and of v, respectively. The path $S^{u'v'}$ obtained concatenating (u',u), P^{uv}, and (v,v') is called a *spine* of $H_{\nu,\rho}$. Denote by $n(S^{u'v'})$ the number of right turns minus the number of left turns encountered along $S^{u'v'}$ while moving from u' to v'. The *spirality* $\sigma(H_{\nu,\rho})$ of $H_{\nu,\rho}$, introduced in [5], is either an integer or a semi-integer number, defined based on the following cases (see Fig. 2 for an example): (*i*) If $A^u = \{u'\}$ and $A^v = \{v'\}$ then $\sigma(H_\nu) = n(S^{u'v'})$. (*ii*) If $A^u = \{u'\}$ and $A^v = \{v',v''\}$ then $\sigma(H_\nu) = \frac{n(S^{u'v'})+n(S^{u'v''})}{2}$. (*iii*) If $A^u = \{u',u''\}$ and $A^v = \{v'\}$ then $\sigma(H_\nu) = \frac{n(S^{u'v'})+n(S^{u''v'})}{2}$. (*iv*) If $A^u = \{u',u''\}$ and $A^v = \{v',v''\}$ assume, without loss of generality, that (u,u') precedes (u,u'') counterclockwise around u and that (v,v') precedes (v,v'') clockwise around v; then $\sigma(H_\nu) = \frac{n(S^{u'v'})+n(S^{u''v''})}{2}$.

It is proved that the spirality of $H_{\nu,\rho}$ does not depend on the choice of P^{uv} [5]. Also, a component $H_{\nu,\rho}$ of H can always be substituted by any other component $H'_{\nu,\rho}$ with the same spirality, getting a new valid orthogonal representation with the same set of bends on the edges of H that are not in $H_{\nu,\rho}$ (see [5] and also Theorem 1 in [10]). For brevity, we shall denote by σ_ν the spirality of an orthogonal representation of $G_{\nu,\rho}$. Lemmas 1 to 3 relate, for any S- or P-node ν, the values of spirality for a rectilinear representation of $G_{\nu,\rho}$ to the values of spirality of the rectilinear representations of the child components of $G_{\nu,\rho}$ (i.e., the components corresponding to the children of ν). They rephrase known results proved in [5], specialized to rectilinear representations. See Fig. 3 for a schematic illustration.

Lemma 1 ([5], Lemma 4.2) *Let ν be an S-node of T_ρ with children $\mu_1,\dots,\mu_h$ $(h \geq 2)$. $G_{\nu,\rho}$ has*

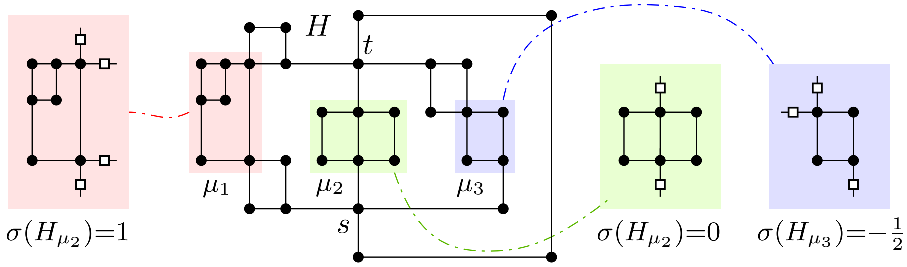


Figure 2: An orthogonal representation H and three of its components with respect to the reference chain with poles s and t. For each component, its alias vertices (white squares) and its spirality are reported.

a rectilinear representation with spirality σ_ν if and only each G_{μ_i} $(1 \leq i \leq h)$ has a rectilinear representation with spirality σ_{μ_i}, such that $\sigma_\nu = \sum_{i=1}^{h} \sigma_{\mu_i}$.

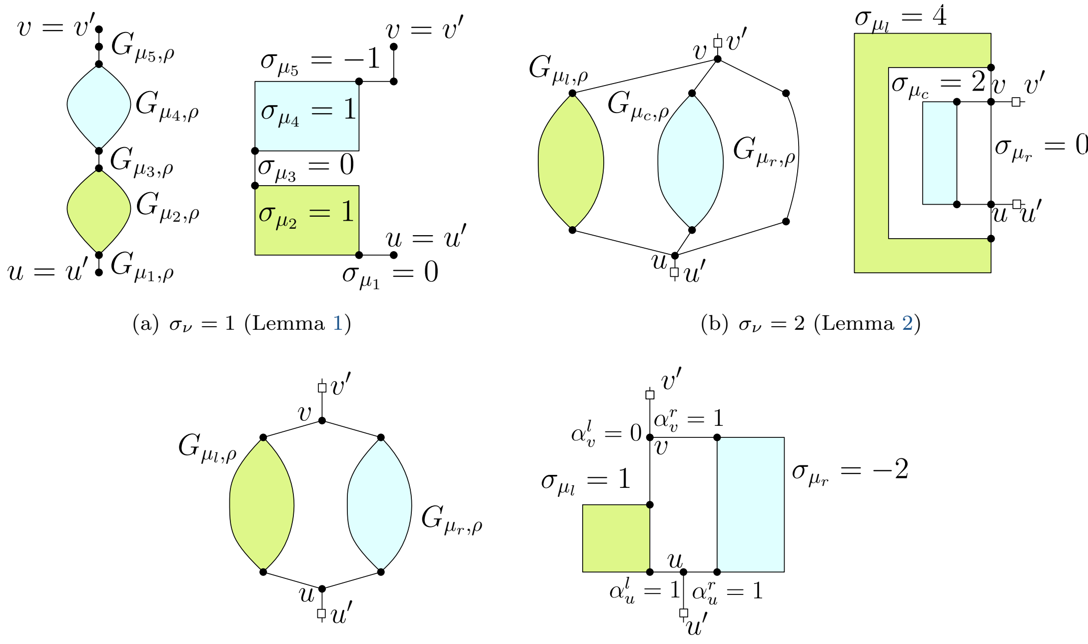
Lemma 2 ([5], Lemma 4.3) *Let ν be a P-node of T_ρ with three children μ_l, μ_c, and μ_r. $G_{\nu,\rho}$ has a rectilinear representation with spirality σ_ν, where $G_{\mu_l,\rho}$, $G_{\mu_c,\rho}$, $G_{\mu_r,\rho}$ are in this left-to-right order, if and only if there exist values σ_{μ_l}, σ_{μ_c}, σ_{μ_r} such that: (i) $G_{\mu_l,\rho}$, $G_{\mu_c,\rho}$, $G_{\mu_r,\rho}$ have rectilinear representations with spirality σ_{μ_l}, σ_{μ_c}, σ_{μ_r}, respectively; and (ii) $\sigma_\nu = \sigma_{\mu_l} - 2 = \sigma_{\mu_c} = \sigma_{\mu_r} + 2$.*

For a P-node ν of T_ρ with two children we need some more notation. Let H be an orthogonal representation of G with ρ on the external face and let $H_{\nu,\rho}$ be the restriction of H to $G_{\nu,\rho}$. For each pole $w \in \{u, v\}$ of ν, the *leftmost angle* (resp. *rightmost angle*) at w in $H_{\nu,\rho}$ is the angle formed by the leftmost (resp. rightmost) external edge and the leftmost (resp. rightmost) internal edge of $H_{\nu,\rho}$ incident to w. Define two binary variables α_w^l and α_w^r as follows: $\alpha_w^l = 0$ ($\alpha_w^r = 0$) if the leftmost (rightmost) angle at w in H is 180°, while $\alpha_w^l = 1$ ($\alpha_w^r = 1$) if this angle is 90°. Also define two variables k_w^l and k_w^r as follows: $k_w^d = 1$ if $\mathrm{indeg}_{\mu_d}(w) = \mathrm{outdeg}_\nu(w) = 1$, while $k_w^d = 1/2$ otherwise, for $d \in \{l, r\}$.

Lemma 3 ([5], Lemma 4.4) *Let ν be a P-node of T_ρ with two children μ_l and μ_r, and poles u and v. $G_{\nu,\rho}$ has a rectilinear representation with spirality σ_ν, where $G_{\mu_l,\rho}$ and $G_{\mu_r,\rho}$ are in this left-to-right order, if and only if there exist values σ_{μ_l}, σ_{μ_r}, α_u^l, α_u^r, α_v^l, α_v^r such that: (i) $G_{\mu_l,\rho}$ and $G_{\mu_r,\rho}$ have rectilinear representations with spirality σ_{μ_l} and σ_{μ_r}, respectively; (ii) $\alpha_w^l, \alpha_w^r \in \{0, 1\}$, $1 \leq \alpha_w^l + \alpha_w^r \leq 2$ with $w \in \{u, v\}$; and (iii) $\sigma_\nu = \sigma_{\mu_l} - k_u^l \alpha_u^l - k_v^l \alpha_v^l = \sigma_{\mu_r} + k_u^r \alpha_u^r + k_v^r \alpha_v^r$.*

Spirality sets. Let G be an n-vertex SP-graph (distinct from a simple cycle), T_ρ be a rooted SPQ*-tree of G, and ν be a node of T_ρ. We say that $G_{\nu,\rho}$, or directly ν, *admits spirality σ_ν* in T_ρ if there exists a rectilinear representation $H_{\nu,\rho}$ with spirality σ_ν in some orthogonal representation H of G. The *rectilinear spirality set* $\Sigma_{\nu,\rho}$ of ν in T_ρ (and of $G_{\nu,\rho}$) is the set of spirality values for which $G_{\nu,\rho}$ admits a rectilinear representation. $\Sigma_{\nu,\rho}$ is representative of all "shapes" that $G_{\nu,\rho}$ can take in a rectilinear representation of G with the reference chain on the external face, if one exists. If $G_{\nu,\rho}$ is not rectilinear planar, $\Sigma_{\nu,\rho}$ is empty. Let n_ν be the number of vertices of $G_{\nu,\rho}$. The following holds.

Property 1 $|\Sigma_{\nu,\rho}| \leq 2n_\nu$. *Also, for each $\sigma_\nu \in \Sigma_{\nu,\rho}$ we have $|\sigma_\nu| \leq n_\nu$.*



(a) $\sigma_\nu = 1$ (Lemma 1) (b) $\sigma_\nu = 2$ (Lemma 2)

(c) $\sigma_\nu = 0$ (Lemma 3)

Figure 3: Illustrations for Lemmas 1 to 3 (alias vertices are small squares).

Proof: The spirality value of any rectilinear representation of $G_{\nu,\rho}$ is either an integer or a semi-integer value that cannot exceed the length of the shortest path between the poles of $G_{\nu,\rho}$. Since any simple path in $G_{\nu,\rho}$ has at most n_ν vertices and since for each spirality value σ_ν admitted by ν, the spirality value $-\sigma_\nu$ is also admitted by ν, the statement follows. □

3.2 Testing Algorithm

We first consider SP-graphs (which are biconnected according to our definition) and then partial 2-trees that are not biconnected.

3.2.1 SP-Graphs.

Let G be an SP-graph. Our rectilinear planarity testing algorithm for G elaborates and refines ideas of [5]. It is based on a dynamic programming technique that visits the SPQ*-tree of G for each possible choice of the root; for each tree, either the root is reached and a rectilinear representation is found (in which case the test stops and returns the solution), or a node with empty rectilinear spirality set is encountered (in which case the visit is interrupted and the tree is discarded). With respect to [5], our algorithm exploits two fundamental ingredients: (a) a more careful analysis that leads to an $O(n^2)$-time procedure to compute the spirality sets of all nodes for a given rooted SPQ*-tree; (b) a re-usability principle that makes it possible to process all rooted SPQ*-trees in the same asymptotic time needed to process a single SPQ*-tree.

Similar to [5] and [7], in the reminder of this section we shall assume to work with a variant of the SPQ*-tree having the property that each S-node has exactly two children. We call this tree a *normalized* SPQ*-tree. Observe that every SPQ*-tree can be easily transformed into a normalized SPQ*-tree by recursively splitting a series with more than two children into multiple series with two children.[1] In contrast to the original definition of SPQ*-tree, in a normalized tree two S-nodes can be adjacent. We remark that a normalized tree still has $O(n)$ nodes and that it can be easily computed in $O(n)$ time from the original SPQ*-tree.

In the following we first describe our rectilinear planarity testing algorithm and then we prove, through a sequence of technical lemmas, that it can be executed in quadratic time.

Description and correctness of the testing algorithm. Assume that G is not a simple cycle, otherwise the test is trivial. Let T be a normalized SPQ*-tree of G and let $\{\rho_1, \dots, \rho_h\}$ be a sequence of all Q*-nodes of T. Denote by ℓ_i the length of the chain corresponding to ρ_i; the spirality set of ρ_i consists of all integer values in the interval $[-(\ell_i - 1), (\ell_i - 1)]$. Namely, the spirality value $-(\ell_i - 1)$ (resp. $(\ell_i - 1)$) is taken when there is a left (resp. right) turn at every vertex of the chain. For each $i = 1, \dots, h$, the testing algorithm performs a post-order visit of T_{ρ_i}. During this visit of T_{ρ_i}, for every non-root node ν of T_{ρ_i} the algorithm computes the set Σ_{ν,ρ_i} by combining the spirality sets of the children of ν, according to the relations given in Lemmas 1–3. If $\Sigma_{\nu,\rho_i} = \emptyset$, the algorithm stops the visit, discards T_{ρ_i}, and starts visiting $T_{\rho_{i+1}}$ (if $i < h$). If the algorithm reaches the root child ν and if $\Sigma_{\nu,\rho_i} \neq \emptyset$, it checks whether G is rectilinear planar by verifying if there exists a value $\sigma_\nu \in \Sigma_{\nu,\rho_i}$ and a value $\sigma_{\rho_i} \in \Sigma_{\rho_i,\rho_i} = [-(\ell_i - 1), (\ell_i - 1)]$ such that $\sigma_\nu - \sigma_{\rho_i} = 4$. We call this property the *root condition*. If the root condition holds, the test is positive and the algorithm does not visit the remaining trees; otherwise it discards T_{ρ_i} and starts visiting $T_{\rho_{i+1}}$ (if $i < h$).

The correctness of the dynamic programming approach followed by the algorithm is an immediate consequence of the spirality properties described in the previous section. Also, denoted by s and t the poles of ν (which coincide with those of ρ_i), the final condition $\sigma_\nu - \sigma_{\rho_i} = 4$ is necessary and sufficient for the existence of a rectilinear representation due to the following observations: (i) In any orthogonal representation of G, the difference k between the number of right and left turns encountered walking clockwise along the boundary of any simple cycle that contains the reference chain is $k = 4$; (ii) since the alias vertices of the poles of ν are vertices that subdivide the two edges of the reference chain incident to s and t, the value k equals the spirality of σ_ν plus the difference $\overline{\sigma}_{\rho_i}$ between the number of right and the number of left turns along the reference chain, going from t to s; (iii) $\overline{\sigma}_{\rho_i} = -\sigma_{\rho_i}$, where $\sigma_{\rho_i} \in [-(\ell_i - 1), (\ell_i - 1)]$ is the spirality of the chain corresponding to ρ_i.

From now on we refer to the algorithm described above as RectPlanTest-SP(G), where G is the input graph.

Complexity of the testing algorithm. We prove that, for each type of node (i.e., Q*, P, or S), computing the spirality sets of all nodes of that type, over all T_{ρ_i} ($i \in \{1, \dots, h\}$), takes $O(n^2)$ time. Thanks to Property 1, every time the algorithm visits a node ν of T_{ρ_i}, it stores at ν a list of integers or semi-integers values of length at most $2n_\nu$ that represents Σ_{ν,ρ_i}. Also, it stores at ν a Boolean array of size $2n_\nu$ that reports which of the $2n_\nu$ candidate spirality values is actually in Σ_{ν,ρ_i}. This array allows us to know in $O(1)$ time whether a specific value of spirality belongs to Σ_{ν,ρ_i} or not.

[1] Note that [5] and [7] adopt the term "canonical" instead of "normalized". However, since there are in general several ways of splitting a series into multiple series (i.e., the normalized tree is not uniquely defined), we prefer to avoid the term "canonical".

Lemma 4 RectPlanTest-SP(G) *computes the spirality sets of all Q*-nodes over all* T_{ρ_i} *(*$i \in \{1,\dots,h\}$*) in* $O(n)$ *time.*

Proof: For each T_{ρ_i}, a Q*-node ν admits all integer spirality values in the interval $[-(\ell-1),(\ell-1)]$, where ℓ is the length of the chain corresponding to ν. The value ℓ can be stored at ν when T is computed. Since T is computed in $O(n)$ time and the sum of the lengths of all chains represented by Q*-nodes is $O(n)$, the statement follows. □

Lemma 5 RectPlanTest-SP(G) *computes the spirality sets of all P-nodes over all* T_{ρ_i} *(*$i \in \{1,\dots,h\}$*) in* $O(n^2)$ *time.*

Proof: Let T_{ρ_i} be the currently visited tree in the algorithm RectPlanTest-SP(G), and let ν be a P-node of T_{ρ_i}. Denote by δ_ν the degree of ν. Notice that $\delta_\nu \leq 4$, as ν has either two or three children. If the parent of ν in T_{ρ_i} coincides with the parent of T_{ρ_j} for some $j \in \{1,\dots,i-1\}$, and if Σ_{ν,ρ_j} was previously computed, then the algorithm does not need to compute Σ_{ν,ρ_i}, because $\Sigma_{\nu,\rho_i} = \Sigma_{\nu,\rho_j}$. Hence, for each P-node ν, the number of computations of its rectilinear spirality sets that are performed over all possible trees T_{ρ_i} is at most $\delta_\nu = 4$ (one for each different way of choosing the parent of ν).

Consider a P-node ν whose spirality set needs to be computed for the first time in T_{ρ_i}. If ν has three children, Σ_{ν,ρ_i} is computed in $O(n)$ time. Namely, it is sufficient to check, for each of the six permutations of the children of ν and for each value in the rectilinear spirality set of one of the three children, whether the sets of the other two children contain the values that satisfy condition *(ii)* of Lemma 2. If ν has two children, Σ_{ν,ρ_i} is computed in $O(n)$ with a similar approach: For each of the two permutations of the children of ν, for each value in the rectilinear spirality set of one of the two children, and for each combination of the values α_w^d ($w \in \{u,v\}, d \in \{l,r\}$) defined in Lemma 3, check whether the set of the other children contains the value that satisfies condition *(iii)* of Lemma 3. Note that, by Property 1, there are $O(n)$ possible spirality values that must be checked for each P-node ν; also, checking whether a specific value of spirality exists in the set of a child of ν takes $O(1)$ time, thanks to the Boolean array stored at each child of ν, which informs about the spirality values admitted by that child.

Therefore, since the SPQ*-tree contains $O(n)$ P-nodes in total, since the spirality set of each P-node in a rooted tree is computed in $O(n)$ time, and since the spirality set of each P-node needs to be computed at most four times over all T_{ρ_i} ($i \in \{1,\dots,h\}$), the time needed to compute the spirality sets of all P-nodes over all sequence of rooted SPQ*-trees is $O(n^2)$. □

For the S-nodes we need a more careful analysis. Our ingredients are similar to those used by Chaplick et al. [1] to efficiently test upward planarity testing of digraphs whose underlying undirected graphs are series-parallel. Recall that, since T_{ρ_i} is a normalized SPQ*-tree, each S-node ν has exactly two children, which we denote by $\mu_1(\nu)$ and $\mu_2(\nu)$. Also, we denote by n_1^ν and n_2^ν the number of vertices of the pertinent graphs $G_{\mu_1(\nu),\rho_i}$ and $G_{\mu_2(\nu),\rho_i}$, respectively.

We start by proving an upper bound to the sum of the products of the sizes of the pertinent graphs for the children of the S-nodes in a tree T_{ρ_i}. For our purposes, it is enough to restrict the attention to T_{ρ_1}, although the result holds for any T_{ρ_i}.

Lemma 6 *Let* $\mathcal{S}$ *be the set of all S-nodes in* T_{ρ_1}*. We have* $\sum_{\nu \in \mathcal{S}} n_1^\nu \cdot n_2^\nu = O(n^2)$*.*

Proof: Let ξ be any node of T_{ρ_1} distinct from ρ_1. Let $T_{\rho_1}(\xi)$ be the subtree of T_{ρ_1} rooted at ξ, and let $\mathcal{S}(\xi) \subseteq \mathcal{S}$ be the set of S-nodes in $T_{\rho_1}(\xi)$. Denote by $s(\xi) = \sum_{\nu \in \mathcal{S}(\xi)} n_1^\nu \cdot n_2^\nu$. Also, let n_ξ and m_ξ be the number of vertices and the number of edges of G_{ξ,ρ_1}, respectively. We will prove that $s(\xi) \leq 4m_\xi^2$. When ξ is the child of ρ_1, the statement follows by observing that $m_\xi = O(n_\xi)$ and that $n_\xi = n - \ell_1 + 1$, where ℓ_1 is the length of the reference chain. To prove that $s(\xi) \leq 4m_\xi^2$ we proceed by induction on the depth d of $T_{\rho_1}(\xi)$. In the base case $d = 0$ and ξ is a Q*-node (i.e., it is a leaf); we have $s(\xi) = 0 < m_\xi$. In the inductive case, $d \geq 1$ and we assume (by the inductive hypothesis) that the property holds for every node in the subtree $T_{\rho_1}(\xi)$. There are two cases:

Case 1: ξ is an S-node. Let μ_1 and μ_2 be the children of ξ. We have $s(\xi) = n_{\mu_1} n_{\mu_2} + s(\mu_1) + s(\mu_2)$. By using the inductive hypothesis and since $n_{\mu_i} \leq m_{\mu_i} + 1$ ($i \in \{1, 2\}$), we get $s(\xi) \leq m_{\mu_1} m_{\mu_2} + m_{\mu_1} + m_{\mu_2} + 1 + 4m_{\mu_1}^2 + 4m_{\mu_2}^2 \leq 4(m_{\mu_1} + m_{\mu_2})^2$. Since $m_{\mu_1} + m_{\mu_2} = m_\xi$, we have $s(\xi) \leq 4m_\xi^2$.

Case 2: ξ is a P-node. Let $\mu_1, \dots, \mu_k$ be the children of ξ, with $k \in \{2, 3\}$. We have $s(\xi) = s(\mu_1) + \dots + s(\mu_k)$. By inductive hypothesis and since $m_{\mu_1} + \dots + m_{\mu_k} = m_\xi$, we get $s(\xi) \leq 4m_{\mu_1}^2 + \dots + 4m_{\mu_k}^2 \leq 4(m_{\mu_1} + \dots + m_{\mu_k})^2 = 4m_\xi^2$. □

The next lemma provides an upper bound to the time required to compute the spirality set of an S-node, looking at the size of the pertinent graphs of its two children and at the size of the remaining part of the graph. For an S-node ν of a normalized tree T_{ρ_i}, denote by n_0^ν the number of vertices of the graph $(G \setminus G_{\nu,\rho_i}) \cup \{u, v\}$, where u and v are the poles of ν. In other words, n_0^ν is the number of vertices incident to the edges of G that are not in the pertinent graph of ν. Also, as in the previous lemma, let $\mu_1(\nu)$ and $\mu_2(\nu)$ be the two children of ν in T_{ρ_i} and let n_1^ν and n_2^ν denote the number of vertices of their pertinent graphs. We prove the following.

Lemma 7 *Let ν be an S-node of T_{ρ_i} for which the spirality sets $\Sigma_{\mu_1(\nu),\rho_i}$ and $\Sigma_{\mu_2(\nu),\rho_i}$ are given and non-empty. The spirality set Σ_{ν,ρ_i} can be computed in $O(\min\{n_1^\nu \cdot n_2^\nu, n_1^\nu \cdot n_0^\nu, n_2^\nu \cdot n_0^\nu\})$ time.*

Proof: Suppose first that $n_0^\nu = \max\{n_0^\nu, n_1^\nu, n_2^\nu\}$. In this case the spirality set Σ_{ν,ρ_i} is computed as in [5], by looking at all distinct values (all integers or all semi-integers) that result from the sum of a value in $\Sigma_{\mu_1(\nu),\rho_i}$ with a value in $\Sigma_{\mu_2(\nu),\rho_i}$. That is, Σ_{ν,ρ_i} is the Cartesian sum of $\Sigma_{\mu_1(\nu),\rho_i}$ and $\Sigma_{\mu_2(\nu),\rho_i}$, which can be computed in $O(n_1^\nu \cdot n_2^\nu) = O(\min\{n_1^\nu \cdot n_2^\nu, n_1^\nu \cdot n_0^\nu, n_2^\nu \cdot n_0^\nu\})$.

Suppose vice versa that $\max\{n_0^\nu, n_1^\nu, n_2^\nu\}$ is one among n_1^ν and n_2^ν, say for example $n_2^\nu = \max\{n_0^\nu, n_1^\nu, n_2^\nu\}$ (if the maximum is n_1^ν, the argument is analogous). The spirality values admitted by ν must be in the interval $[-(n_0^\nu + 4), +(n_0^\nu + 4)]$, because the number of right turns minus the number of left turns walking counterclockwise on the boundary of any cycle of a rectilinear representation of G equals 4, and because any rectilinear representation of G restricted to $G \setminus G_{\nu,\rho_i}$ cannot have more than n_0 turns in the same direction (either left or right). Also, recall that the spirality values admitted by ν are either all integer or all semi-integer numbers, depending on the in-degree and out-degree of the poles of ν. Hence, to construct the spirality set Σ_{ν,ρ_i}, we can consider every pair $\{\sigma_\nu, \sigma_1\}$, with σ_ν being either an integer or a semi-integer in $[-(n_0^\nu + 4), +(n_0^\nu + 4)]$ and $\sigma_1 \in \Sigma_{\mu_1(\nu),\rho_i}$, and for each such pair we check whether there exists a value $\sigma_2 \in \Sigma_{\mu_2(\nu),\rho_i}$ such that $\sigma_1 + \sigma_2 = \sigma_\nu$. In the positive case, the value σ_ν is inserted in Σ_{ν,ρ_i}, otherwise this value is discarded. Since there are $O(n_0^\nu \cdot n_1^\nu)$ distinct pairs $\{\sigma_\nu, \sigma_1\}$ and since for each pair we can check in $O(1)$ time whether there exists a value σ_2 that satisfies $\sigma_1 + \sigma_2 = \sigma_\nu$ (thanks to the Boolean array stored at $\mu_2(\nu)$), this procedure takes $O(n_0^\nu \cdot n_1^\nu) = O(\min\{n_1^\nu \cdot n_2^\nu, n_1^\nu \cdot n_0^\nu, n_2^\nu \cdot n_0^\nu\})$ time. □

We finally establish the time complexity of RectPlanTest-SP(G) to compute the spirality sets of all S-nodes over all sequence of normalized rooted SPQ*-trees of G.

Lemma 8 RectPlanTest-SP(G) *computes the spirality sets of all S-nodes over all T_{ρ_i} ($i \in \{1, \dots, h\}$) in $O(n^2)$ time.*

Proof: Let T_{ρ_i} be the currently visited tree in the algorithm RectPlanTest-SP(G). As for the P-nodes, if the parent of ν in T_{ρ_i} coincides with the parent of T_{ρ_j} for some $j \in \{1, \dots, i-1\}$, and if Σ_{ν,ρ_j} was previously computed, then the algorithm does not need to compute Σ_{ν,ρ_i}, because $\Sigma_{\nu,\rho_i} = \Sigma_{\nu,\rho_j}$. Hence, for each S-node ν, the number of computations of its rectilinear spirality sets that are performed over all possible trees T_{ρ_i} is at most 3 (one for each different way of choosing the parent of ν).

Suppose that, for an S-node ν, $\mu_1(\nu)$ and $\mu_2(\nu)$ are the children of ν in the first rooted tree T_{ρ_1}, and that n_1^ν and n_2^ν are the number of vertices of $G_{\mu_1(\nu),\rho_1}$ and $G_{\mu_2(\nu),\rho_1}$, respectively. By Lemma 7, every time RectPlanTest-SP(G) needs to compute the spirality set of an S-node ν in a tree T_{ρ_i}, it spends $O(n_1^\nu \cdot n_2^\nu)$ time. Denote by $\mathcal{S}$ the set of all S-nodes in T_{ρ_1}. Since the spirality set of each S-node has to be computed at most three times over all T_{ρ_i} ($i = 1, \dots, h$), the time required to compute the spirality sets of all S-nodes over all T_{ρ_i} is $O(\sum_{\nu \in \mathcal{S}} n_1^\nu \cdot n_2^\nu)$, which, by Lemma 6, is $O(n^2)$. □

We are now ready to prove the main result of this subsection.

Lemma 9 *Let G be an n-vertex SP-graph. There exists an $O(n^2)$-time algorithm that tests whether G is rectilinear planar and that computes a rectilinear representation of G in the positive case.*

Proof: Consider the algorithm RectPlanTest-SP(G) described above. By Lemmas 4 and 5, and 8, this algorithm spends $O(n^2)$ time to compute the spirality sets of all nodes, over all sequence $T_{\rho_1}, \dots, T_{\rho_h}$ of normalized trees. Also, for each visited tree T_{ρ_i} ($i \in \{1, \dots, h\}$), if the spirality set of the root child ν is not empty, the algorithm takes $O(n)$ time to check the root condition, i.e., whether there exist two values $\sigma_\nu \in \Sigma_{\nu,\rho_i}$ and $\sigma_{\rho_i} \in \Sigma_{\rho_i,\rho_i}$ such that $\sigma_\nu - \sigma_{\rho_i} = 4$. Therefore, RectPlanTest-SP(G) can be executed in $O(n^2)$ time.

Construction algorithm. Suppose now that the test is positive for some rooted tree T_{ρ_i}, with $1 \leq i \leq h$. This implies that the final condition $\sigma_\nu - \sigma_{\rho_i} = 4$ holds when ν is the root child, for some suitable values $\sigma_\nu \in \Sigma_{\nu,\rho_i}$ and $\sigma_{\rho_i} \in [-(\ell_i - 1), (\ell_i - 1)]$. In order to construct a rectilinear planar representation of G with the reference edge corresponding to ρ_i on the external face, we proceed as follows: First we assign spirality σ_ν to the root child ν; then we visit T_{ρ_i} top-down and assign a suitable value of spirality to each visited node, according to the spirality value already assigned to its parent; for each P-node, we also determine the permutation of its children that yields the desired value of spirality. Once the spirality values of all nodes have been assigned and the permutation of the children of each P-node has been fixed, we apply the algorithm in [8] (which works for plane SP-graphs) to construct a rectilinear representation of G in linear time. More in detail, suppose that during the top-down visit we have assigned a spirality value σ_ν to a node ν. If ν is not a Q*-node, we determine the spirality values that can be assigned to its children based on whether ν is a P-node or an S-node, namely:

- *ν is a P-node with three children.* By Lemma 2, we check, for each of the six left-to-right orders (permutations) μ_l, μ_c, μ_r of the three children of ν, whether $\Sigma_{\mu_l,\rho_i}, \Sigma_{\mu_c,\rho_i}$, and Σ_{μ_r,ρ_i} contain the values $\sigma_{\mu_l} = \sigma_\nu + 2$, $\sigma_{\mu_c} = \sigma_\nu$, and $\sigma_{\mu_r} = \sigma_\nu - 2$, respectively. If so, assign these values of spiralities to three children of ν and fix this order of the children for ν. This test takes $O(1)$ time.

- *ν is a P-node with two children.* Let u and v be the poles of ν in T_{ρ_i}. By Lemma 3, we check, for each left-to-right order (permutation) μ_l, μ_r of the two children of ν, whether there exists a combination of values α_u^l, α_u^r, α_v^l, α_v^r and two values $\sigma_{\mu_l} \in \Sigma_{\mu_l,\rho_i}$ and $\sigma_{\mu_r} \in \Sigma_{\mu_r,\rho_i}$ such that:

$\sigma_{\mu_l} = \sigma_\nu + k^l_u\alpha^l_u + k^l_v\alpha^l_v$ and $\sigma_{\mu_r} = \sigma_\nu - k^r_u\alpha^r_u - k^r_v\alpha^r_v$. Since each α^d_w ($w \in \{u,v\}, d \in \{l,r\}$) is a binary variable, this test takes $O(1)$ time.

- *ν is an S-node.* Let μ_1 and μ_2 be the two children of ν. By Lemma 1, we check the existence of two values $\sigma_{\mu_1} \in \Sigma_{\mu_1,\rho_i}$ and $\sigma_{\mu_2} \in \Sigma_{\mu_2,\rho_i}$, such that $\sigma_{\mu_1} + \sigma_{\mu_2} = \sigma_\nu$. This takes $O(n)$ time.

By the analysis above, the time complexity of the construction is dominated by the assignment of spirality values to the children of the S-nodes, which takes in total $O(n^2)$ time. □

3.2.2 1-connected partial 2-trees

We now extend the result of Lemma 9 to partial 2-trees that consist of multiple blocks. The main difficulty in this case is to handle the angle constraints that may be required at the cutvertices of the input graph G. Indeed, one cannot simply test the rectilinear planarity of each single block independently, as it might be impossible to merge the rectilinear representations of the different blocks into a rectilinear representation for G without additional angle constraints at the cutvertices. For example, suppose that c is a cutvertex shared by two blocks B_1 and B_2, each having two edges incident to c; we cannot accept any rectilinear representation of B_1 in which the two edges incident to c form angles of 180°, as such a representation does not leave enough space to attach the two edges of B_2 incident to c.

We prove the following result.

Theorem 1 *Let G be an n-vertex partial 2-tree. There exists an $O(n^2)$-time algorithm that tests whether G is rectilinear planar and that computes a rectilinear representation of G in the positive case.*

Proof: Let $\mathcal{T}$ be the BC-tree of G, and let $B_1, \dots, B_q$ be the blocks of G ($q \geq 2$). We denote by $\beta(B_i)$ the block-node of $\mathcal{T}$ corresponding to B_i ($1 \leq i \leq q$) and by $\mathcal{T}_{B_i}$ the tree $\mathcal{T}$ rooted at $\beta(B_i)$. For a cutvertex c of G, we denote by $\chi(c)$ the node of $\mathcal{T}$ that corresponds to c. Each $\mathcal{T}_{B_i}$ describes a class of planar embeddings of G such that, for each non-root node $\beta(B_j)$ ($1 \leq j \leq q$) with parent node $\chi(c)$ and grandparent node $\beta(B_k)$, the cutvertex c and B_k lie on the external face of B_j. We say that G is *rectilinear planar with respect to $\mathcal{T}_{B_i}$* if it is rectilinear planar for some planar embedding in the class described by $\mathcal{T}_{B_i}$. To check whether G is rectilinear planar with respect to $\mathcal{T}_{B_i}$, we have to perform a constrained rectilinear planarity testing for every block $B_1, \dots, B_q$ to guarantee that the rectilinear representations of the different blocks can be merged together at the shared cutvertices. We first define the types of constraints that we need to impose on the angles at the cutvertices of B_j in each $\mathcal{T}_{B_i}$. Then we explain how to perform the rectilinear planarity testing algorithm with respect to $\mathcal{T}_{B_i}$, over all $i = 1, \dots, q$, while considering these constraints.

Types of constraints for a block $\mathbf{B_j}$ in a rooted BC-tree $\mathcal{T}_{\mathbf{B_i}}$. The constraints for each block B_j in tree $\mathcal{T}_{B_i}$ depend on whether $j = i$ or not and on the angles that we may have to impose on each cutvertex c of B_j. We denote by $\deg(c)$ the degree of c in G and by $\deg(c|B_j)$ the degree of c in B_j.

Case $j = i$ ($\beta(B_j)$ is the root). Let c' be a cutvertex of B_j and let B_k be one of the blocks that share c' with B_j. Note that a rectilinear representation of B_k (if any) must have c' on its external face, as $\chi(c')$ is the parent of $\beta(B_k)$ in $\mathcal{T}_{B_j}$. We distinguish two subcases: (*i*) If $\deg(c'|B_k) = \deg(c'|B_j) = 2$, there is not a third block that contains c'. We constraint c' to have a reflex angle (i.e., an angle of 270°) in any rectilinear representation of B_j (if any). We call this type of constraint a *reflex-angle constraint* on c'. This constraint is necessary and sufficient to merge a rectilinear representation of B_k having a reflex angle at c' on the external face (if any) to the one of B_j. Indeed, if both the angles at c' in the representation of B_j were smaller than 270°, then there would not be enough space to embed the representation of B_k on one of the two faces of B_j incident to c', because $\deg(c'|B_k) = 2$; this proves the necessity of the constraint. On the other hand, if a face f incident to c' in the representation of B_j has an angle of 270° at c', then we can easily merge the representation of B_j with a representation of B_k having an external reflex angle at c' by embedding the representation of B_k on face f (there will be four angles of 90° at c' in the final representation); this proves the sufficiency of the constraint. Note that, the constraint that forces a representation of B_k to have an external reflex angle at c' is treated when we consider the case $j \neq i$. (*ii*) In all other cases, we do not need to impose any constraints on c'; indeed, either $\deg(c'|B_j) = 1$ or $\deg(c'|B_k) = 1$, and any rectilinear representation of B_k with c' on the external face is embeddable in one of the faces incident to c' in a rectilinear representation of B_j.

Case $j \neq i$ ($\beta(B_j)$ is not the root). Let $\chi(c)$ be the parent node of $\beta(B_j)$; we must restrict to those rectilinear representations of B_j with c on the external face. If $\deg(c|B_j) = 1$ then B_j is a trivial block and we do not need to impose any constraint for B_j. Hence, assume that $\deg(c|B_j) \geq 2$ and let $\beta(B_k)$ be the parent node of $\chi(c)$ in $\mathcal{T}_{B_i}$. We distinguish different types of *external constraints* on c, based on the following subcases: (*i*) If $\deg(c) = 4$ and $\deg(c|B_k) = \deg(c|B_j) = 2$, then we impose an *external reflex-angle constraint* on c, which forces c to have a reflex angle on the external face f of any rectilinear representation of B_j. A rectilinear representation of B_k (if any) will be embedded in f. (*ii*) If $\deg(c) = 4$ with $\deg(c|B_k) = 1$ (i.e., B_k is a trivial block) and $\deg(c|B_j) = 2$, then B_j has a sibling B_h, which is a trivial block. In this case, we impose an *external non-right-angle constraint* on c, which forces c to have an angle larger than 90° (i.e., either a flat or a reflex angle) on the external face f; a rectilinear representation of B_k (if any) will be embedded in f, while a rectilinear representation of B_h (if any) will be embedded either in f (if c has a reflex angle in f) or in the other face of B_j incident to c (if c has a flat angle in f). (*iii*) If $\deg(c) = 4$ with $\deg(c|B_k) = 1$ and $\deg(c|B_j) = 3$, we impose an *external flat-angle constraint* on c, which forces c to have its unique flat angle on the external face f; again, a rectilinear representation of B_k (if any) will be embedded in f. (*iv*) If $\deg(c) = 3$ and $\deg(c|B_j) = 2$ (which implies $\deg(c|B_k) = 1$), we impose an external non-right-angle constraint on c, as in case (*ii*). Observe that, by definition, there is at most one external constraint on c in B_j. Additionally, for any cutvertex $c' \neq c$ of B_j, we impose a reflex-angle constraint on c' when there is exactly one block B_h that shares c' with B_j and $\deg(c'|B_j) = \deg(c'|B_h) = 2$.

Testing algorithm. We describe how to test in $O(n^2)$ time whether G admits a rectilinear representation with respect to $\mathcal{T}_{B_i}$, over all $i = 1, \ldots, q$. The test consists of two main phases.

Phase 1 (pre-processing). In this phase, for each block B_j, we consider all possible *configurations* of the cutvertex-nodes incident to $\beta(B_j)$ in which either all these cutvertex-nodes are children of $\beta(B_j)$ in a rooted BC-tree of G (i.e., $\beta(B_j)$ is the root) or one of them is chosen as the parent of $\beta(B_j)$ and the remaining ones are the children of $\beta(B_j)$. For each configuration, we store at $\beta(B_j)$ a Boolean *local label* that is true if and only if B_j is rectilinear planar with respect to a rooted BC-tree that has the given configuration for the cutvertex nodes incident to $\beta(B_j)$ (see the initial part of the proof for the definition of rectilinear planarity with respect to a given rooted BC-tree). Note that, in this way, for each block-node we store a number of local labels equal to its degree plus one. Thus, in total we store $O(n)$ local labels at the nodes of the BC-tree. These local labels allow us to (quickly) check the realizability of a block in the second phase of the algorithm (see

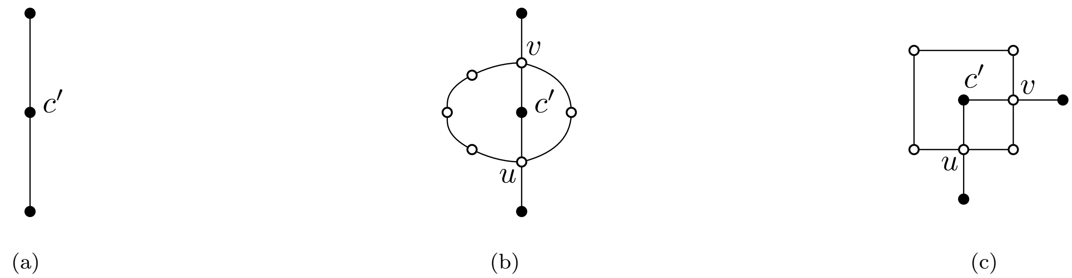


Figure 4: (*a*) A degree-2 cutvertex c'; (*b*) the reflex-angle gadget; (*c*) a rectilinear representation of the reflex-angle gadget, which forces c' to form a reflex angle.

later), over all possible choices of the root of the BC-tree. To compute the local label associated with each configuration of the cutvertex-nodes incident to $\beta(B_j)$, we execute the following steps:

- Step 1. For each cutvertex c' of B_j such that we need to impose on c' either a reflex-angle constraint or an external reflex-angle constraint in some configuration, we enhance B_j with a gadget, called a *reflex-angle gadget* for c', depicted in Figures 4(a) and 4(b). It consists of two vertices u and v, each subdividing one of the two edges incident to c' in B_j, and of two edge-disjoint paths connecting u and v, one having length two and the other having length four. Call B'_j the block resulting from B_j after the addition of all these reflex-angle gadgets. B'_j is still an SP-graph and each cutvertex c' with a reflex-angle constraint gadget will be forced to have a reflex angle in any rectilinear representation of the block. Indeed, since a rectilinear representation has no edge bends and since u and v have degree four, the shape of the reflex-angle gadget is necessarily a rectangle whose corners are its four degree-2 vertices, and c' is necessarily inside this rectangle and has an angle of 270° (see Figure 4(c)). Also, since each reflex-angle gadget consists of a constant number of nodes and edges, the size of B'_j is linear in the size of B_j. From a rectilinear representation of B'_j we will obtain a constrained rectilinear representation of B_j by simply ignoring the reflex-angle gadgets (once we have possibly exchanged the identity of c' with the degree-2 vertex of the path of the gadget having length two).

- Step 2. Execute on B'_j the non-constrained planarity testing algorithm of Lemma 9, over all possible roots of the SPQ*-tree of B'_j. However, during the test on each rooted SPQ*-tree, and similarly to what is done in [5], for each node ν and for each value σ_ν in the spirality set of ν, we also store at ν a different 4-tuple for each possible combination of the leftmost and rightmost external angles at the poles u and v of ν that are compatible with σ_ν. We recall that the leftmost (resp. rightmost) external angle at a pole $w \in \{u, v\}$ is the angle formed by the leftmost (resp. rightmost) external edge and the leftmost (resp. rightmost) internal edge incident to w. Note that, there are at most four tuples for each spirality value admitted by ν, because each pole of ν has either degree three or degree four in the block, and its leftmost and rightmost external angles are either of 90° or of 180°.

- Step 3. For each distinct configuration of the cutvertex-nodes incident to $\beta(B_j)$ we decide its corresponding Boolean local label, based on the output of the previous step and on whether the configuration requires an external angle constraint at a cutvertex of B_j or not. Namely, if the configuration is such that all cutvertex-nodes incident to $\beta(B_j)$ are children of $\beta(B_j)$ (which models the case when $\beta(B_j)$ is the root of the BC-tree), there is no external angle constraints on the cutvertices of B_j, hence the local label is true if and only if B'_j was rectilinear planar in Step 2. Consider vice versa a configuration such that $\chi(c)$ is the parent of $\beta(B_j)$, for a cutvertex c in B_j. Clearly, if B'_j was not rectilinear planar in Step 2, the local label for the configuration is false. However, if B'_j was rectilinear planar in Step 2, we must check whether it remains rectilinear planar with the additional external angle-constraint on c. We distinguish the following cases:

(i) If there is an external reflex-angle-constraint on c, consider the output of the testing algorithm of Step 2 restricted to the SPQ*-tree of B'_j whose reference chain is the path of length four of the reflex-angle gadget for c. The local label is set to true if and only if the test for this rooted tree was positive, as it equals to saying that B_j is rectilinear planar with c on the external face and with a reflex angle on the external face.

(ii) If there is an external non-right-angle constraint on c, we know that $\deg(c|B_j) = 2$. We restrict the output of the testing algorithm of Step 2 to the only root ρ of the SPQ*-tree whose reference chain π contains c. Denote by ℓ the length of π and let s and t be the two poles of π. Since c is not allowed to have a 90° angle on the external face, the spirality σ_ρ is restricted to take values in the range $[-(\ell-1), (\ell-2)]$, instead of $[-(\ell-1), (\ell-1)]$ ($\sigma_\rho = (\ell-1)$ corresponds to having a 90° angle on the external face at all degree-2 vertices of π). Hence, we just repeat the checking of the root condition under this restriction, and we set the local label for the configuration to true if and only if the checking remains positive.

(iii) Finally, if there is an external flat-angle constraint on c, we know that $\deg(c|B_j) = 3$. Denote by π_1, π_2, and π_3 the three chains incident to c in B_j. We restrict the output of the testing algorithm of Step 2 to the roots of SPQ*-tree of B_j corresponding to π_1, π_2, and π_3. For each of these roots, we remove from the spirality set of the root child those values whose associated 4-tuples require a 90° angle at c on the external face. After this removal, the local label for the configuration is set to true if and only if we can still satisfy the root condition, as described in the proof of Lemma 9.

Concerning the time complexity of Phase 1, for each block B_j, denote by n_{B_j} the number of vertices of B_j. We have the following: Step 1 is easily executed in $O(n_{B_j})$ time; Step 2 is executed in $O(n^2_{B_j})$ time by Lemma 9; Step 3 is executed in $O(n_{B_j})$ time for each distinct configuration, and hence in $O(n^2_{B_j})$ over all $O(n_{B_j})$ configurations. Summing up over all B_j ($i = 1, \dots, q$), we have that Phase 1 takes $O(n^2)$ time.

Phase 2. After the pre-processing phase, we first consider the rooted BC-tree $\mathcal{T}_{B_1}$. We visit $\mathcal{T}_{B_1}$ bottom-up and for each node γ of $\mathcal{T}_{B_1}$ (either a block-node or a cutvertex-node) we compute a Boolean *cumulative label* that is either true or false depending on whether all blocks in the subtree of $\mathcal{T}_{B_1}$ rooted at γ (included γ) have a cumulative label true or not. Namely, for a leaf $\gamma = \beta(B_j)$, its cumulative label coincides with the local label of $\beta(B_j)$ for its current configuration of cutvertices. For each cutvertex-node, its cumulative label is the Boolean logic AND of the cumulative labels of its children. For each internal block-node, its cumulative label is the Boolean logic AND of its children and of its local label. Computing the cumulative labels of each node of $\mathcal{T}_{B_1}$ takes $O(n)$ time. At this point, one of the following three cases holds (each case leads to an answer of the testing algorithm, and we explain why this answer is correct):

Case 1. The cumulative label of the root is true. In this case the test is positive, because G is rectilinear planar with respect to $\mathcal{T}_{B_1}$.

Case 2. There are two block-nodes γ_1 and γ_2 in $\mathcal{T}_{B_1}$ with cumulative label false and that are along two distinct paths from a leaf to the root (which implies that there is a node with two children whose cumulative labels are false). In this case the test is negative, because for any other $\mathcal{T}_{B_i}$ ($i = 2, \dots, q$), at least one of the subtrees rooted at γ_1 and γ_2 remains unchanged.

Case 3. All block-nodes with cumulative label false (possibly one block-node) are on the same path from a leaf to the root. In this case, let $\beta(B_j)$ be the deepest node along this path (note that $\beta(B_j)$ could also be the root). The rest of the test can be restricted to considering all rooted BC-trees whose root $\beta(B_i)$ is a leaf of the subtree rooted at $\beta(B_j)$. For each of these roots we repeat the procedure above, by visiting $\mathcal{T}_{B_i}$ bottom-up and by computing the cumulative label of each node γ of $\mathcal{T}_{B_i}$ only if the subtree of γ has changed with respect to any previous visit (otherwise we just reuse the cumulative label of γ computed in a previous tree without visiting its subtree again). Also, for a node γ whose parent has changed, the cumulative label of γ can be easily computed in $O(1)$ time by looking at the cumulative label of γ in $\mathcal{T}_{B_1}$, at the cumulative label of the child of γ in $\mathcal{T}_{B_1}$ that becomes its parent in $\mathcal{T}_{B_i}$, and at the cumulative label of the parent of γ in $\mathcal{T}_{B_1}$ (if $\gamma \neq \beta(B_1)$) that becomes its child in $\mathcal{T}_{B_i}$ (γ has at most one child whose cumulative label is false).

Concerning the time complexity of Phase 2, for each node γ of $\mathcal{T}$ of degree δ_γ, the cumulative label of γ is computed in $O(\delta_\gamma)$ time for $\mathcal{T}_{B_1}$ and in $O(1)$ in each subsequent rooted BC-tree in which γ changes the parent. Since each node γ changes its parent $O(\delta_\gamma)$ times, summing up over all γ, we have that Phase 2 takes in total $O(n)$ time.

Thanks to the analysis discussed above, Phase 1 and Phase 2 together allow us to verify whether G is rectilinear planar over all possible planar embeddings of G. Also, since Phase 1 takes $O(n^2)$ time and Phase 2 takes $O(n)$ time, the overall test is executed in $O(n^2)$ time. Also, if the test is positive, with the same strategy as in Lemma 9, we construct a rectilinear representation of each block and, thanks to the given angle constraints at the cutvertices, we just merge all the representations together in order to compute a rectilinear representation of G. Since constructing a representation for each block B_j takes $O(n^2_{B_j})$ (see Lemma 9), the overall time of the construction algorithm is $O(n^2)$. □

4 Independent-Parallel Partial 2-Trees

In this section we show that the rectilinear planarity testing problem can be solved in linear-time for a meaningful subclass of partial 2-trees, which we call "independent-parallel". In the final remark we also discuss the difficulties of extending this result to a larger subclass of partial 2-trees.

An *independent-parallel SP-graph* is a (biconnected) SP-graph in which no two P-components share a pole (the graph in Fig. 1(a) is an independent-parallel SP-graph). An *independent-parallel partial 2-tree* is a partial 2-tree such that every block is an independent-parallel SP-graph.

The first step towards a linear-time testing algorithm for independent-parallel partial 2-trees is to design a linear-time testing algorithm for independent-parallel SP-graphs, i.e., to improve the complexity stated in Lemma 9 when we restrict to this subclass of SP-graphs. To this aim, we ask whether the components of an independent-parallel SP-graph have spirality sets of constant size, as for the case of planar 3-graphs [13, 28]. Unfortunately, this is not the case for SP-graphs with degree-4 vertices, even when they are independent-parallel. Namely, in Section 4.1 we describe an infinite family of independent-parallel SP-graphs whose rectilinear representations require that some components have spirality $\Omega(\log n)$.

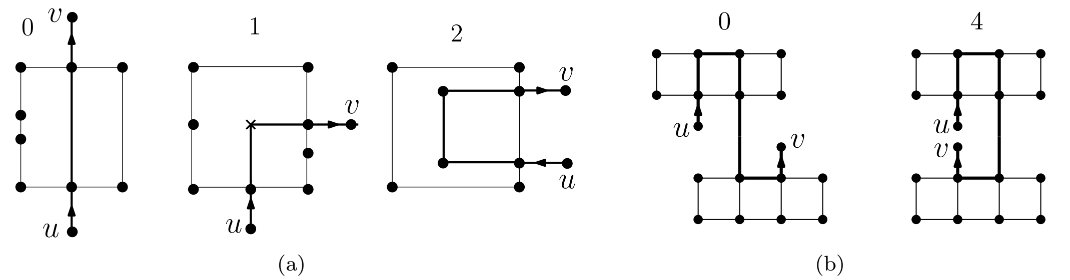


Figure 5: Two components that are: (a) rectilinear planar for spiralities 0 and 2, but not 1 (which requires a bend, shown as a cross); (b) rectilinear planar only for spiralities 0 and 4. In bold, an arbitrary path from the pole u to the pole v.

Moreover, it is not obvious how to describe the spirality sets for independent-parallel SP-graphs with degree-4 vertices in $O(1)$ space. See for example the irregular behavior of the spirality sets of the components in Fig. 5(a) and Fig. 5(b). In particular, the component in Fig. 5(a) corresponds to a parallel-node with three children; by Lemma 2 this component cannot have a rectilinear representation with spirality 1, as it would imply that the spirality values of the representations of the three children are all different from 0, which is however the only value of spirality admitted by the child corresponding to a single edge. A similar analysis can be done for the component in Fig. 5(b). The absence of regularity is an obstacle to the design of a succinct description based on whether a component is rectilinear planar for consecutive spirality values. By carefully analyzing the spirality properties of independent-parallel SP-graphs, in Sections 4.2 and 4.3 we show how to overcome these difficulties and design a linear-time rectilinear planarity testing algorithm for this graph family.

In the remainder of the paper we assume to work with the basic definition of SPQ*-tree given in Section 2, i.e., unlike Section 3, we will no longer work with normalized SPQ*-trees. This implies in particular that an S-node can have many children and that there cannot be two adjacent S-nodes in an SPQ*-tree.

4.1 Spirality Lower Bound

Theorem 2 *For infinitely many integer values of n, there exists an n-vertex independent-parallel SP-graph for which every rectilinear representation has a component with spirality $\Omega(\log n)$.*

Proof: For any arbitrarily large even integer $N \geq 2$, we construct an independent-parallel SP-graph G with $n = O(3^N)$ vertices such that every rectilinear representation of G has a component with spirality larger than N. Let $L = \frac{N}{2} + 1$. For any $k \in \{0, \ldots, L\}$, let G_k be the SP-graph inductively defined as follows: (i) G_0 is a chain of $N+4$ vertices; (ii) G_1 is a parallel composition of three copies of G_0, with coincident poles (Fig. 6(a)); (iii) for $k \geq 2$, G_k is a parallel composition of three series, each starting and ending with an edge, and having G_{k-1} in the middle (Fig. 6(b)). The graph G is obtained by composing in a cycle two chains p_1 and p_2, of three edges each, with two copies of G_L (Fig. 6(c)). The graph G_L for $N = 4$ is in Fig. 6(d). About the number n of vertices of G, let n_k be the number of vertices of G_k. We have $n_0 = N + 4$ and $n_k = O(3^k N)$ for $k \leq N$. Hence, $n_L = O(3^{\frac{N}{2}} N)$ and, since $N \leq 3^{\frac{N}{2}}$, $n_L = O(3^N)$. It follows that $n = O(3^N)$.

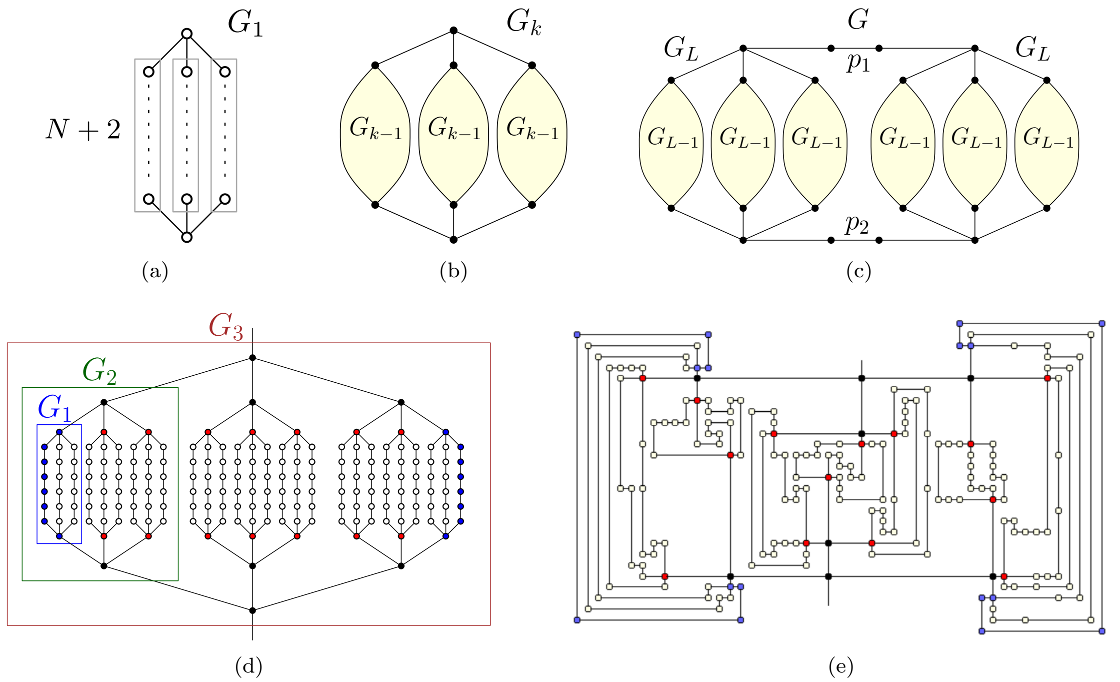


Figure 6: (a)–(c) The graph family of Theorem 2, with $L = \frac{N}{2}+1$. (d)–(e) Graph G_L for $N = 4$ and a rectilinear representation of G_L (computed by the GDToolkit library [3]); the two components G_0 with blue vertices have spirality $N + 2 = 6$ (left) and $-(N + 2) = -6$ (right), respectively.

Consider first the rooted SPQ*-tree T_ρ of G, where ρ represents p_1. All the planar embeddings of G encoded by T_ρ have p_1 (and p_2) on the external face of G, and by symmetry of the construction they are all equivalent. Any rectilinear representation H of G with an embedding encoded by T_ρ requires that the restriction of H to each copy of G_L has spirality zero and, at the same time, the restriction of H to one of the copies of G_0 in G_L has spirality $N+2$. Indeed, due to Lemma 2, for each rectilinear representation H_k of G_k, the leftmost (resp. rightmost) child component of H_k has spirality that is two units larger (resp. smaller) than the spirality of H_k. Hence, if there existed a rectilinear representation of G_L with spirality greater (resp. smaller) than zero, it would contain a representation of a copy of G_0 with spirality greater than $N+2$ (resp. less than $-(N+2)$), which is impossible, as the absolute value of spirality of any copy of G_0 is at most $N+2$. See Fig. 6(e), where $N = 4$.

On the other hand, if we consider the planar embeddings encoded by T when rooted at a Q*-node whose chain p belongs to a copy of G_L, the same argument as above applies to the copy of G_L that does not contain p; namely, any rectilinear representation of this copy must contain a component with spirality $N + 2$. □

4.2 Rectilinear Spirality Sets

Let G be an independent-parallel SP-graph, T be the SPQ*-tree of G, and ρ be a Q*-node of T. Each pole w of a P-node ν of T_ρ is such that $\text{outdeg}_\nu(w) = 1$; if ν is an S-node, either $\text{indeg}_\nu(w) = 1$

or $\text{outdeg}_\nu(w) = 1$. In all cases, $\text{outdeg}_\nu(w) = 1$ when $\text{indeg}_\nu(w) > 1$. For any node ν of T_ρ, denote by $\Sigma^+_{\nu,\rho}$ (resp. $\Sigma^-_{\nu,\rho}$) the subset of non-negative (resp. non-positive) values of $\Sigma_{\nu,\rho}$. Clearly, $\Sigma_{\nu,\rho} = \Sigma^+_{\nu,\rho} \cup \Sigma^-_{\nu,\rho}$. Note that, for any value $\sigma_\nu \in \Sigma_{\nu,\rho}$, we also have that $-\sigma_\nu \in \Sigma_{\nu,\rho}$. Indeed, if $G_{\nu,\rho}$ admits a rectilinear representation with spirality σ_ν for some embedding, by flipping this embedding around the poles of $G_{\nu,\rho}$, we can obtain a rectilinear representation of $G_{\nu,\rho}$ with spirality $-\sigma_\nu$. Hence, $\sigma_\nu \in \Sigma^+_{\nu,\rho}$ if and only if $-\sigma_\nu \in \Sigma^-_{\nu,\rho}$, and we can restrict the study of the properties of $\Sigma_{\nu,\rho}$ to $\Sigma^+_{\nu,\rho}$, which we call the *non-negative rectilinear spirality set* of ν in T_ρ (or of $G_{\nu,\rho}$).

The main result of this subsection is Theorem 3, which proves that there is a limited number of possible structures for the sets $\Sigma^+_{\nu,\rho}$ of independent-parallel SP-graphs, which can be succinctly described (see also Fig. 7). Let m and M be two non-negative integers with $m < M$: $[M]$ is a *trivial interval* and denotes the singleton $\{M\}$; $[m, M]^1$ is a *jump-1 interval* and denotes the set of all integers in the interval $[m, M]$, i.e., $\{m, m+1, \dots, M-1, M\}$; If m and M have the same parity, $[m, M]^2$ is a *jump-2 interval* and denotes the set of values $\{m, m+2, \dots, M-2, M\}$.

Theorem 3 *Let G be a rectilinear planar independent-parallel SP-graph and let $G_{\nu,\rho}$ be a component of G. The non-negative rectilinear spirality set $\Sigma^+_{\nu,\rho}$ of $G_{\nu,\rho}$ has one the following six structures:* $[0]$, $[1]$, $[1,2]^1$, $[0,M]^1$, $[0,M]^2$, $[1,M]^2$.

To prove Theorem 3 we give a series of key lemmas that state important properties of the spirality values admitted by the components of an independent-parallel SP-graph. For brevity, if the non-negative spirality set of ν is trivial, jump-1, or jump-2, we also say that ν is trivial, jump-1, or jump-2, respectively.

Lemma 10 *Let $G_{\nu,\rho}$ be a component that admits spirality $\sigma_\nu \geq 2$. The following properties hold: (a) if $\sigma_\nu = 2$, $G_{\nu,\rho}$ admits spirality $\sigma'_\nu = 0$ or $\sigma'_\nu = 1$; (b) if $\sigma_\nu > 2$, $G_{\nu,\rho}$ admits spirality $\sigma'_\nu = \sigma_\nu - 2$; (c) if $\sigma_\nu = 4$, $G_{\nu,\rho}$ admits spirality $\sigma'_\nu = 0$.*

Proof: The proof is by induction on the depth of the subtree $T_\rho(\nu)$. In the base case ν is a Q*-node and the three properties trivially hold for $G_{\nu,\rho}$. In the inductive case, ν is either an S-node, or a P-node with three children, or a P-node with two children. We analyze the three cases separately. The most involved case is when ν is a P-node with two children.

- ν is an S-node. We inductively prove the three properties.

  Proof of Property (a). If ν admits spirality $\sigma_\nu = 2$, by Lemma 1, ν has a child μ that admits spirality $\sigma_\mu > 0$. If $\sigma_\mu = 1$, μ also admits spirality -1, and ν admits spirality 0. If $\sigma_\mu = 2$, by inductively using Property (a), μ also admits 0 or 1, and so does ν. If $\sigma_\mu > 2$, by inductively using Property (b), μ admits spirality $\sigma_\mu - 2$, and ν admits spirality 0.

  Proof of Property (b). If ν admits spirality $\sigma_\nu > 2$, by Lemma 1 one of the following subcases holds: (i) ν has child μ that admits spirality $\sigma_\mu > 2$; by inductively using Property (b), μ admits spirality $\sigma_\mu - 2$ and, by Lemma 1, ν admits spirality $\sigma_\nu - 2$. (ii) ν has child μ that admits spirality 1; in this case μ admits spirality -1, and ν admits spirality $\sigma_\nu - 2$. (iii) ν has two children μ_1 and μ_2 such that μ_1 and μ_2 both admit spirality 2; by inductively using Property (a), either one of them also admits spirality 0 or they both admit spirality 1. In any case, ν admits spirality $\sigma_\nu - 2$.

  Proof of Property (c). If ν admits spirality $\sigma_\nu = 4$, by Lemma 1, one of the following cases holds: (i) ν has a child μ that admits spirality 4; if so, by inductively using Property (c), ν admits spirality 0. (ii) ν has a child μ that admits spirality $\sigma_\mu > 4$; if so, by inductively

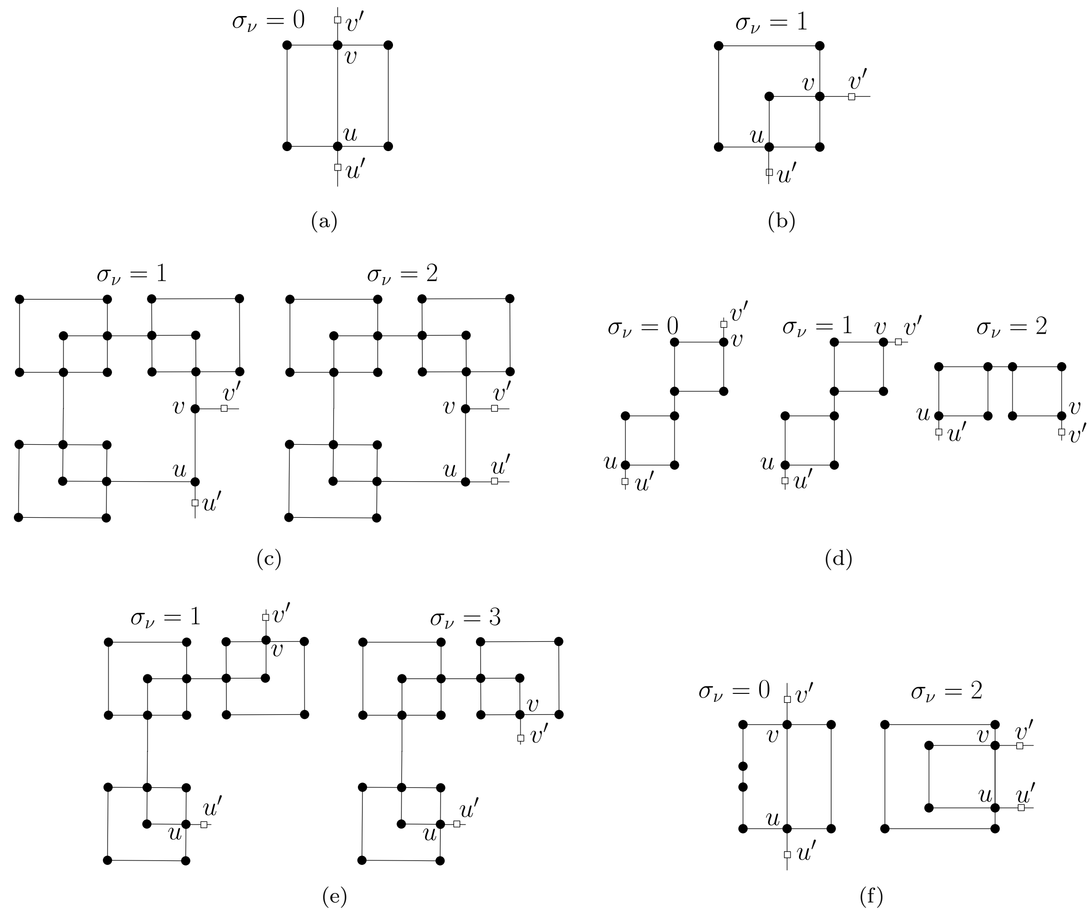


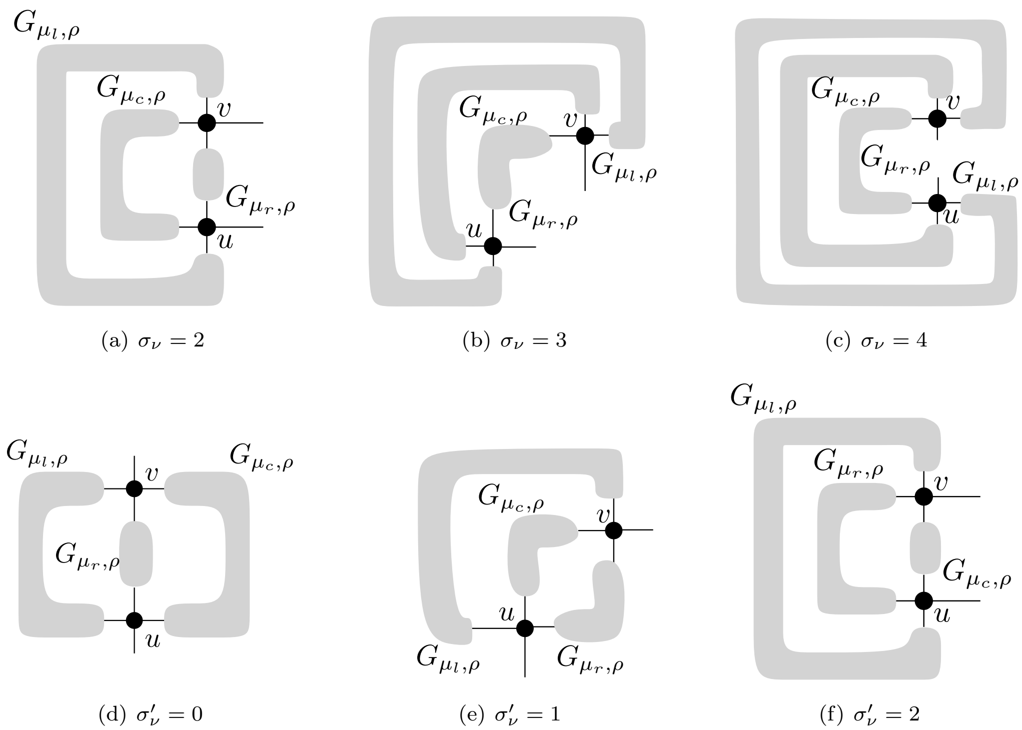
Figure 7: Examples of non-negative spirality sets for each of the six structures in Theorem 3: (a) $[0]$; (b) $[1]$; (c) $[1,2]^1$; (d) $[0,2]^1$; (e) $[1,3]^2$; (f) $[0,2]^2$.

applying Property (b) twice, μ admits spirality $\sigma_\mu - 4$, and hence ν admits spirality 0. (iii) ν has two children μ_1 and μ_2, each admitting spirality either 1 or 3; note that if μ_i $(i \in \{1,2\})$ admits spirality 1, it also admits spirality -1 and if μ_i admits spirality 3, it also admits spirality 1 by inductively using Property (b); this implies that ν admits spirality $\sigma_\nu - 4$.

- ν is a P-node with three children. Let $H_{\nu,\rho}$ be a rectilinear representation of $G_{\nu,\rho}$ with spirality σ_ν. Let μ_l, μ_c, and μ_r be the children of ν such that $G_{\mu_l,\rho}$, $G_{\mu_c,\rho}$, and $G_{\mu_r,\rho}$ appear in this left-to-right order in $H_{\nu,\rho}$. By Lemma 2 we have $\sigma_{\mu_l} = \sigma_\nu + 2$, $\sigma_{\mu_c} = \sigma_\nu$, and $\sigma_{\mu_r} = \sigma_\nu - 2$.

  Proof of Property (a). If $\sigma_\nu = 2$, we have $\sigma_{\mu_l} = 4$, $\sigma_{\mu_c} = 2$, and $\sigma_{\mu_r} = 0$; see Fig. 8(a). By inductively applying Property (b), μ_l admits spirality 2. Also μ_c admits spirality -2. Hence, exchanging $G_{\mu_c,\rho}$ with $G_{\mu_r,\rho}$ in the left-to-right order, by Lemma 2 we have that ν admits spirality 0; see Fig. 8(b).

  Proof of Property (b). If $\sigma_\nu > 2$, we distinguish three cases: (i) $\sigma_\nu = 3$, which implies $\sigma_{\mu_l} = 5$,



(a) $\sigma_\nu = 2$ (b) $\sigma_\nu = 3$ (c) $\sigma_\nu = 4$

(d) $\sigma'_\nu = 0$ (e) $\sigma'_\nu = 1$ (f) $\sigma'_\nu = 2$

Figure 8: Illustration of Lemma 10 for a P-node with three children.

$\sigma_{\mu_c} = 3$, and $\sigma_{\mu_r} = 1$. By inductively applying Property (b), μ_l and μ_c admit spirality 3 and 1, respectively. Also, μ_r admits spirality -1. By Lemma 2, ν admits spirality $\sigma_\nu - 2$. (ii) $\sigma_\nu = 4$, which implies $\sigma_{\mu_l} = 6$, $\sigma_{\mu_c} = 4$, and $\sigma_{\mu_r} = 2$. By inductively applying Property (b), μ_l admits spirality 4; also, by inductively applying Property (c), μ_c admits spirality 0. Hence, exchanging $G_{\mu_c,\rho}$ with $G_{\mu_r,\rho}$ in the left-to-right order, by Lemma 2 we have that ν admits spirality $\sigma_\nu - 2 = 2$. (iii) $\sigma_\nu > 4$, which implies $\sigma_{\mu_l} > 2$, $\sigma_{\mu_c} > 2$, and $\sigma_{\mu_r} > 2$. By inductively applying Property (b), μ_l, μ_c, and μ_r admit spirality $\sigma_{\mu_l} - 2$, $\sigma_{\mu_c} - 2$, and $\sigma_{\mu_r} - 2$, respectively. Hence ν admits spirality $\sigma_\nu - 2$.

Proof of Property (c). If $\sigma_\nu = 4$, we have $\sigma_{\mu_l} = 6$, $\sigma_{\mu_c} = 4$, and $\sigma_{\mu_r} = 2$. By inductively applying Property (b) twice, we have that μ_l admits spirality 2. By inductively applying Property (c), we have that μ_c admits spirality 0. Finally, μ_r admits spirality -2. Hence, by Lemma 2, ν admits spirality $\sigma_\nu - 4 = 0$.

- ν is a P-node with two children. Let $H_{\nu,\rho}$ be a rectilinear representation of $G_{\nu,\rho}$ with spirality σ_ν. Let $G_{\mu_l,\rho}$ and $G_{\mu_r,\rho}$ be the left child and the right child of $G_{\nu,\rho}$ in $H_{\nu,\rho}$, respectively. By Lemma 3, we have $\sigma_\nu = \sigma_{\mu_l} - \alpha_u^l - \alpha_v^l = \sigma_{\mu_r} + \alpha_u^r + \alpha_v^r$. Lemma 3 implies $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 4]$. Without loss of generality, we assume that $\alpha_v^l \geq \alpha_u^l$.

Proof of Property (a). If $\sigma_\nu = 2$, we analyze separately the cases when $\sigma_{\mu_l} - \sigma_{\mu_r}$ equals 2, 3 or 4 (see Lemma 3).

Case $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$. There are three subcases: (i) $\sigma_{\mu_l} = 2$, $\sigma_{\mu_r} = 0$, and $\alpha_u^l = \alpha_v^l = 0$;

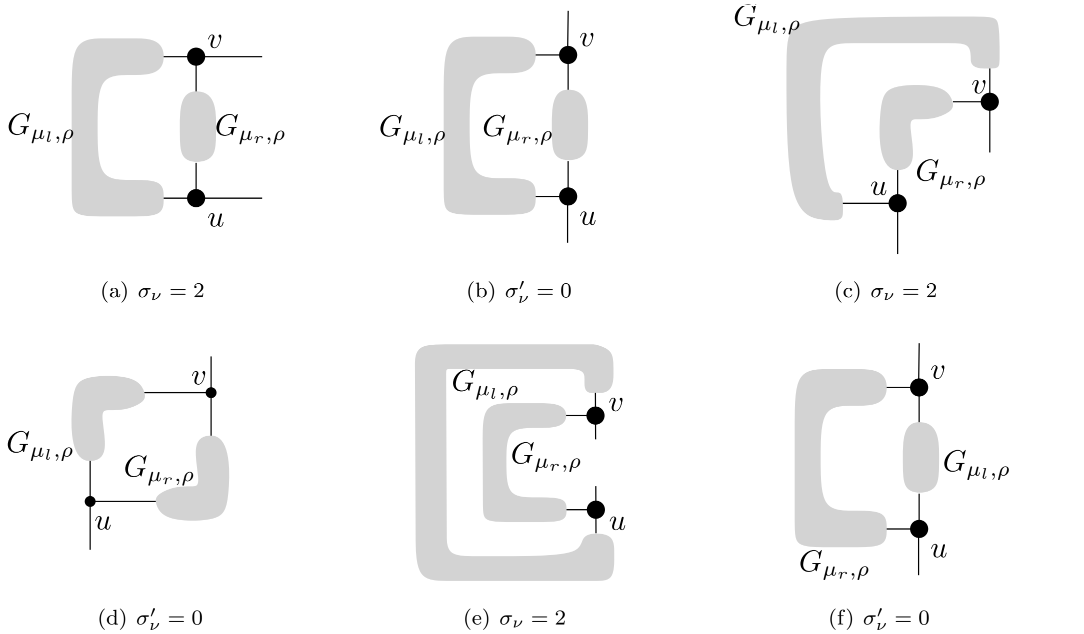


Figure 9: Illustration for the proof of Property (a) of Lemma 10 for a P-component with two children for the case $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$.

see Fig. 9(a). For $\alpha^l_u = \alpha^l_v = 1$ and $\alpha^r_u = \alpha^r_v = 0$, by Lemma 3, $G_{\nu,\rho}$ admits spirality $\sigma_\nu - 2 = 0$; see Fig. 9(b). (ii) $\sigma_{\mu_l} = 3$, $\sigma_{\mu_r} = 1$, $\alpha^r_v = 0$, and $\alpha^l_u = 0$; see Fig. 9(c). By inductively using Property (b), $G_{\mu_l,\rho}$ admits spirality 1. Also, $G_{\mu_r,\rho}$ admit spirality $\sigma_{\mu_r} = -1$. For $\alpha^l_u = \alpha^r_v = 0$ (which implies $\alpha^r_u = \alpha^l_v = 1$), by Lemma 3, $G_{\nu,\rho}$ admits spirality 0; see Fig. 9(d). (iii) $\sigma_{\mu_l} = 4$, $\sigma_{\mu_r} = 2$, and $\alpha^r_u = \alpha^r_v = 1$; see Fig. 9(e). By inductively using Property (c), $G_{\mu_l,\rho}$ admits spirality 0. Hence, exchanging $G_{\mu_l,\rho}$ and $G_{\mu_r,\rho}$ in the left-to-right order, and for $\alpha^r_u = \alpha^r_v = 0$, by Lemma 3, $G_{\nu,\rho}$ admits spirality 0; see Fig. 9(f).

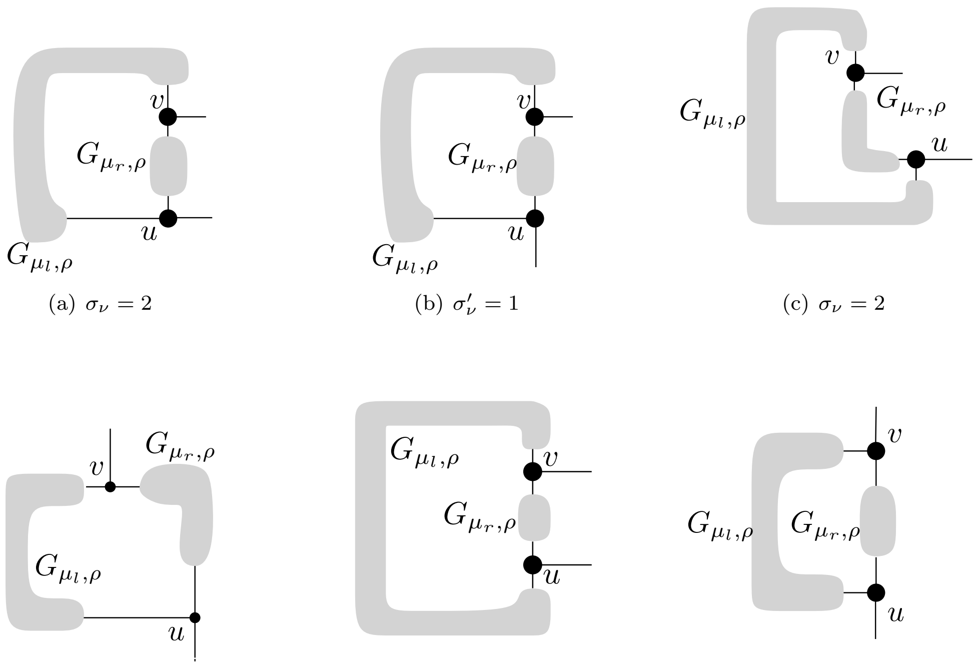
Case $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. In this case, for one of the two poles $\{u, v\}$ of ν, say v, we have $\alpha^l_v = \alpha^r_v = 1$. There are two subcases: (iv) $\sigma_{\mu_l} = 3$ and $\sigma_{\mu_r} = 0$; see Fig. 10(a). In this case $\alpha^r_u = 1$. For $\alpha^r_u = 0$, by Lemma 3, $G_{\nu,\rho}$ admits spirality 1; see Fig. 10(b). (v) $\sigma_{\mu_l} = 4$ and $\sigma_{\mu_r} = 1$; see Fig. 10(c). By inductively using Property (b), $G_{\mu_l,\rho}$ admits spirality 2. Also, $G_{\mu_r,\rho}$ admits spirality -1. For $\alpha^r_u = 0$, by Lemma 3, $G_{\nu,\rho}$ admits spirality 0; see Fig. 10(d).

Case $\sigma_{\mu_l} - \sigma_{\mu_r} = 4$; see Fig. 10(e). We have $\sigma_{\mu_l} = 4$ and $\sigma_{\mu_r} = 0$. By inductively using Property (b), $G_{\mu_l,\rho}$ admits spirality 2. For $\alpha^l_u = \alpha^l_v = 0$, by Lemma 3, $G_{\nu,\rho}$ admits spirality 0; see Fig. 10(f).

Proof of Property (b). If $\sigma_\nu > 2$, we have three cases: $\sigma_\nu = 3$; $\sigma_\nu = 4$; $\sigma_\nu > 4$.

- If $\sigma_\nu = 3$, we perform a subcase analysis based on the value of $\sigma_{\mu_l} - \sigma_{\mu_r}$.

  Case $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$. There are three subcases: (i) If $\sigma_{\mu_l} = 3$ and $\sigma_{\mu_r} = 1$, we have $\alpha^r_u = \alpha^r_v = 0$. For $\alpha^r_u = \alpha^r_v = 1$ and $\alpha^l_u = \alpha^l_v = 0$, by Lemma 3, $G_{\nu,\rho}$ admits



(a) $\sigma_\nu = 2$ (b) $\sigma'_\nu = 1$ (c) $\sigma_\nu = 2$

(d) $\sigma'_\nu = 0$ (e) $\sigma_\nu = 2$ (f) $\sigma'_\nu = 0$

Figure 10: Illustration for the proof of Property (a) of Lemma 10 for a P-component with two children for the cases $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$ and $\sigma_{\mu_l} - \sigma_{\mu_r} = 4$.

spirality $\sigma_\nu - 2 = 1$. (ii) If $\sigma_{\mu_l} = 4$ and $\sigma_{\mu_r} = 2$, by inductively using Property (c), $G_{\mu_l,\rho}$ admits spirality 0. Exchanging $G_{\mu_l,\rho}$ and $G_{\mu_r,\rho}$ in the left-to-right order and for $\alpha^l_u = \alpha^r_v = 0$, by Lemma 3, $G_{\nu,\rho}$ admits spirality $\sigma_\nu - 2 = 1$. (iii) If $\sigma_{\mu_l} = 5$ and $\sigma_{\mu_r} = 3$, by inductively using Property (b), $G_{\mu_l,\rho}$ and $G_{\mu_r,\rho}$ admit spirality values $\sigma_{\mu_l} - 2$ and $\sigma_{\mu_r} - 2$, respectively. Hence $G_{\nu,\rho}$ admits spirality $\sigma_\nu - 2 = 1$.

Case $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. As in proof for Property (a), assume, without loss of generality, that $\alpha^l_v = \alpha^r_v = 1$. The following subcases hold: (iv) If $\sigma_{\mu_l} = 4$ and $\sigma_{\mu_r} = 1$, by inductively using Property (b), $G_{\mu_l,\rho}$ admits spirality 2. Also, G_{μ_r} admits spirality -1. For $\alpha^l_u = 0$, by Lemma 3, $G_{\nu,\rho}$ admits spirality $\sigma_\nu - 2 = 1$. (v) If $\sigma_{\mu_l} = 5$ and $\sigma_{\mu_r} = 2$, by inductively using Property (a), G_{μ_r} admits spirality either 1 or 0. Suppose first that G_{μ_r} admits spirality 1. By inductively using Property (b), $G_{\mu_l,\rho}$ admits spirality 3. For $\alpha^r_v = \alpha^r_u = 0$, by Lemma 3, $G_{\nu,\rho}$ admits spirality $\sigma_\nu - 2 = 1$. Suppose now that $G_{\mu_r,\rho}$ admits spirality 0. As before, $G_{\mu_l,\rho}$ admits spirality 3. For $\alpha^r_u = 0$, we have again that $G_{\nu,\rho}$ admits spirality $\sigma_\nu - 2 = 1$; see Fig. 10(b).

Case $\sigma_{\mu_l} - \sigma_{\mu_r} = 4$. We have $\sigma_{\mu_l} = 5$ and $\sigma_{\mu_r} = 1$. By inductively using Property (b), $G_{\mu_l,\rho}$ admits spirality 3, and then for $\alpha^r_v = 0$ and $\alpha^r_u = 0$, we have that $G_{\nu,\rho}$ admits spirality $\sigma_\nu - 2 = 1$.

- If $\sigma_\nu = 4$, as before, the subcase analysis is based on the value of $\sigma_{\mu_l} - \sigma_{\mu_r}$.

Case $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$. (vi) If $\sigma_{\mu_l} = 4$ and $\sigma_{\mu_r} = 2$, we have $\alpha^l_u = \alpha^l_v = 0$. For $\alpha^l_u = \alpha^l_v = 1$ and $\alpha^r_u = \alpha^r_v = 0$, by Lemma 3, $G_{\nu,\rho}$ admits spirality $\sigma_\nu - 2 = 2$. (vii) If $\sigma_{\mu_l} = 5$ and $\sigma_{\mu_r} = 3$ or $\sigma_{\mu_l} = 6$ and $\sigma_{\mu_r} = 4$, by inductively using Property (b), $G_{\mu_l,\rho}$ and $G_{\mu_r,\rho}$ admit spirality values $\sigma_{\mu_l} - 2$ and $\sigma_{\mu_r} - 2$, respectively. Hence, $G_{\nu,\rho}$ admits spirality $\sigma_\nu - 2 = 2$.

Case $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. $(viii)$ If $\sigma_{\mu_l} = 5$ and $\sigma_{\mu_r} = 2$, by inductively using Property (b), G_{μ_l} admits spirality 3. Also, by inductively using Property (a), G_{μ_r} admits spirality either 0 or 1. In the first case, for $\alpha^l_u = 0$, we have that $G_{\nu,\rho}$ admits spirality $\sigma_\nu - 2 = 2$; see Fig. 10(a) the property holds for $\alpha^r_u = 0$; see Fig. 10(c).

(ix) If $\sigma_{\mu_l} = 6$ and $\sigma_{\mu_r} = 3$, by inductively using Property (b), $G_{\mu_l,\rho}$ and $G_{\mu_r,\rho}$ admit spirality values $\sigma_{\mu_l} - 2$ and $\sigma_{\mu_r} - 2$, respectively. Hence, $G_{\nu,\rho}$ admits spirality $\sigma_\nu - 2 = 2$.

Case $\sigma_{\mu_l} - \sigma_{\mu_r} = 4$. We have $\sigma_{\mu_l} = 6$ and $\sigma_{\mu_r} = 2$. By Property (b), G_{μ_l} admits spirality 4, and by for $\alpha^l_u = \alpha^l_v = 1$, $G_{\nu,\rho}$ admits spirality $\sigma_\nu - 2 = 2$; see Fig. 9(e).

– If $\sigma_\nu > 4$, we always have $\sigma_{\mu_r} > 2$ (and $\sigma_{\mu_l} > 2$). By inductively using Property (b), $G_{\mu_l,\rho}$ and $G_{\mu_r,\rho}$ admit spirality values $\sigma_{\mu_l} - 2$ and $\sigma_{\mu_r} - 2$, respectively. Hence, $G_{\nu,\rho}$ admits spirality $\sigma_\nu - 2 = 2$.

Proof of Property (c). If $\sigma_\nu = 4$, we still perform a case analysis based on $\sigma_{\mu_l} - \sigma_{\mu_r}$.

Case $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$. There are three subcases. (i) Suppose $\sigma_{\mu_l} = 4$ and $\sigma_{\mu_r} = 2$. By inductively using Property (c), G_{μ_l} admits spirality 0. Exchanging $G_{\mu_l,\rho}$ and $G_{\mu_r,\rho}$, and for $\alpha^l_u = \alpha^l_v = 0$, we have that $G_{\nu,\rho}$ admits spirality 0. (ii) Suppose $\sigma_{\mu_l} = 5$ and $\sigma_{\mu_r} = 3$. By inductively using Property (b) (applied twice), $G_{\mu_l,\rho}$ and $G_{\mu_r,\rho}$ admit spirality 1; hence, G_{μ_r} also admits -1. For $\alpha^r_v = \alpha^l_u = 0$, we have that $G_{\nu,\rho}$ admits spirality 0; see Fig. 9(d). (iii) Suppose $\sigma_{\mu_l} = 6$ and $\sigma_{\mu_r} = 4$. By inductively using Property (b) (applied twice), $G_{\mu_l,\rho}$ admits spirality 2, and hence, by inductively using Property (c), it also admits spirality 0. For $\alpha^l_u = \alpha^l_v = 0$, $G_{\nu,\rho}$ admits spirality 0; see Fig. 9(b).

Case $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. We have the following subcases. (iv) Suppose $\sigma_{\mu_l} = 5$ and $\sigma_{\mu_r} = 2$. By inductively using Property (b) (applied twice), $G_{\mu_l,\rho}$ admits spirality 1. Also, $G_{\mu_l,\rho}$ admits spirality -2. For $\alpha^l_v = 0$, $G_{\nu,\rho}$ admits spirality 0. (v) Suppose $\sigma_{\mu_l} = 6$ and $\sigma_{\mu_r} = 3$. By inductively using Property (b) (applied twice), $G_{\mu_l,\rho}$ admits spirality 2 and $G_{\mu_r,\rho}$ admits spirality 1, and hence also spirality -1. For $\alpha^l_u = 0$, we have that $G_{\nu,\rho}$ admits spirality 0.

Case $\sigma_{\mu_l} - \sigma_{\mu_r} = 4$. We have $\sigma_{\mu_l} = 6$ and $\sigma_{\mu_r} = 4$. By inductively using Property (b) (applied twice), $G_{\mu_l,\rho}$ admits spirality 2, and by inductively using Property (c), $G_{\mu_r,\rho}$ admits spirality 0. For $\alpha^l_u = \alpha^l_v = 0$, $G_{\nu,\rho}$ admits spirality 0; see Fig. 9(b).

This concludes the analysis for the different types of nodes in the SPQ*-tree of G. □

Lemma 10 immediately implies the following.

Corollary 1 *If $G_{\nu,\rho}$ admits spirality $\sigma_\nu > 2$, then $G_{\nu,\rho}$ admits spirality for every value in $[1, \sigma_\nu]^2$ when σ_ν is odd, or for every value in $[0, \sigma_\nu]^2$ when σ_ν is even.*

The next lemma states an interesting property that is used to prove Lemma 12.

Lemma 11 *Let ν be a P-node with two children and suppose that $G_{\nu,\rho}$ admits spirality $\sigma_\nu \geq 0$. There exists a rectilinear representation of $G_{\nu,\rho}$ with spirality σ_ν such that the difference of spirality between the left child component and the right child component of $G_{\nu,\rho}$ is either 2 or 3.*

Proof: Let $H_{\nu,\rho}$ be any rectilinear representation of $G_{\nu,\rho}$ with spirality σ_ν. Also, let σ_{μ_l} and σ_{μ_r} be the spiralities of the left child component $H_{\mu_l,\rho}$ and of the right child component $H_{\mu_r,\rho}$ of $H_{\nu,\rho}$, respectively. Let $G_{\mu_l,\rho}$ and $G_{\mu_r,\rho}$ be the underlying graphs of $H_{\mu_l,\rho}$ and $H_{\mu_r,\rho}$. By Lemma 3, we have $2 \leq \sigma_{\mu_l} - \sigma_{\mu_r} \leq 4$. We show that if $\sigma_{\mu_l} - \sigma_{\mu_r} = 4$, one can construct a representation $H'_{\nu,\rho}$ of $G_{\nu,\rho}$ with spirality $\sigma'_\nu = \sigma_\nu$ such $\sigma'_{\mu_l} - \sigma'_{\mu_r} \in [2,3]$. Since $\sigma_{\mu_l} - \sigma_{\mu_r} = 4$, we have $\alpha_u^l = \alpha_v^l = \alpha_u^r = \alpha_v^r = 1$, where u and v are the poles of ν. We distinguish between two cases:

- Case $\sigma_\nu = 0$. We have $\sigma_{\mu_l} = 2$ and $\sigma_{\mu_r} = -2$; see Fig. 11(a). By Property (a) of Lemma 10, both $G_{\mu_l,\rho}$ and $G_{\mu_r,\rho}$ admit spirality 0 or 1. Assume first that $G_{\mu_l,\rho}$ admits spirality 1. We can construct $H'_{\nu,\rho}$ by merging in parallel two representations $H'_{\mu_l,\rho}$ of $G_{\mu_l,\rho}$ and $H'_{\mu_r,\rho}$ of $G_{\mu_r,\rho}$ (in the same left-to-right order they have in $H_{\nu,\rho}$) in such a way that: $H'_{\mu_l,\rho}$ has spirality $\sigma'_{\mu_l} = 1$, $\sigma'_{\mu_r} = \sigma_{\mu_r} = -2$, ${\alpha'}_u^l = 0$, and ${\alpha'}_v^l = {\alpha'}_u^r = {\alpha'}_v^r = 1$; see Fig. 11(b). Assume now that $G_{\mu_l,\rho}$ does not admit spirality 1 but admits spirality 0. We can construct $H'_{\nu,\rho}$ by merging in parallel two representations $H'_{\mu_l,\rho}$ of $G_{\mu_l,\rho}$ and $H'_{\mu_r,\rho}$ of $G_{\mu_r,\rho}$ (in the same left-to-right order they have in $H_{\nu,\rho}$) in such a way that: $H'_{\mu_l,\rho}$ has spirality $\sigma'_{\mu_l} = 0$, $\sigma'_{\mu_r} = \sigma_{\mu_r} = -2$, ${\alpha'}_u^l = {\alpha'}_v^l = 0$, and ${\alpha'}_u^r = {\alpha'}_v^r = 1$; see Fig. 11(c). In both cases $H'_{\nu,\rho}$ has spirality $\sigma'_\nu = \sigma_\nu$ and $\sigma'_{\mu_l} - \sigma'_{\mu_r} \in [2,3]$.

- Case $\sigma_\nu > 0$. We have $\sigma_{\mu_l} \geq 3$ (because $\sigma_\nu = \sigma_{\mu_l} - \alpha_u^l - \alpha_v^l$ by Lemma 3, and $\alpha_u^l + \alpha_v^l = 2$ by hypothesis); see Fig. 11(d), where $\sigma_\nu = 2$. Hence, by Property (b) of Lemma 10, $G_{\mu_l,\rho}$ admits spirality $\sigma'_{\mu_l} = \sigma_{\mu_l} - 2$. We can construct $H'_{\nu,\rho}$ by merging in parallel two representations $H'_{\mu_l,\rho}$ of $G_{\mu_l,\rho}$ and $H'_{\mu_r,\rho}$ of $G_{\mu_r,\rho}$ (in the same left-to-right order they have in $H_{\nu,\rho}$) in such a way that: $H'_{\mu_l,\rho}$ has spirality $\sigma'_{\mu_l} = \sigma_{\mu_l} - 2$, $\sigma'_{\mu_r} = \sigma_{\mu_r}$, ${\alpha'}_u^l = {\alpha'}_v^l = 0$, and ${\alpha'}_u^r = {\alpha'}_v^r = 1$. This way, $H_{\nu,\rho}$ has spirality $\sigma'_\nu = \sigma_\nu$ and $\sigma'_{\mu_l} - \sigma'_{\mu_r} = 2$; see Fig. 11(e), where $\sigma_\nu = 2$.

This concludes the analysis for different values of σ_ν. □

Lemma 12 *Let $\Sigma^+_{\nu,\rho}$ be a non-trivial interval with maximum value $M > 2$. If $\Sigma^+_{\nu,\rho}$ contains an integer with parity different from that of M, then $\Sigma^+_{\nu,\rho} = [0,M]^1$.*

Proof: Assume that M is odd (if M is even the proof is similar). By hypothesis $M \geq 3$. We prove that, if Σ^+_ν contains a value σ_ν whose parity is different from the one of M, then $\Sigma^+_{\nu,\rho} = [0,M]^1$. The proof is by induction on the depth of the subtree $T_\rho(\nu)$. If ν is a Q*-node, then $\Sigma^+_{\nu,\rho} = [0,M]^1$ and the statement trivially holds. In the inductive case, ν is either an S-node or a P-node. By Corollary 1, $G_{\nu,\rho}$ admits spirality σ'_ν for every $\sigma'_\nu \in [1,M]^2$. We analyze separately the case when ν is an S-node, a P-node with three children, or a P-node with two children.

- ν is an S-node. We prove that for any value $\sigma'_\nu \in [1,M]^2$, $G_{\nu,\rho}$ also admits spirality $\sigma'_\nu - 1$. This immediately implies that $\Sigma^+_{\nu,\rho} = [0,M]^1$. We first prove the following claim:

  Claim 1 *There exists a child μ of ν in T_ρ that is jump-1.*

  Proof of the claim: Let $H_{\nu,\rho}$ be a representation of $G_{\nu,\rho}$ with spirality σ_ν and let $H'_{\nu,\rho}$ be a representation of $G_{\nu,\rho}$ with spirality $\sigma'_\nu = \sigma_\nu + 1$. Note that $\sigma'_\nu \in [1,M]^2$, thus $H'_{\nu,\rho}$ exists.

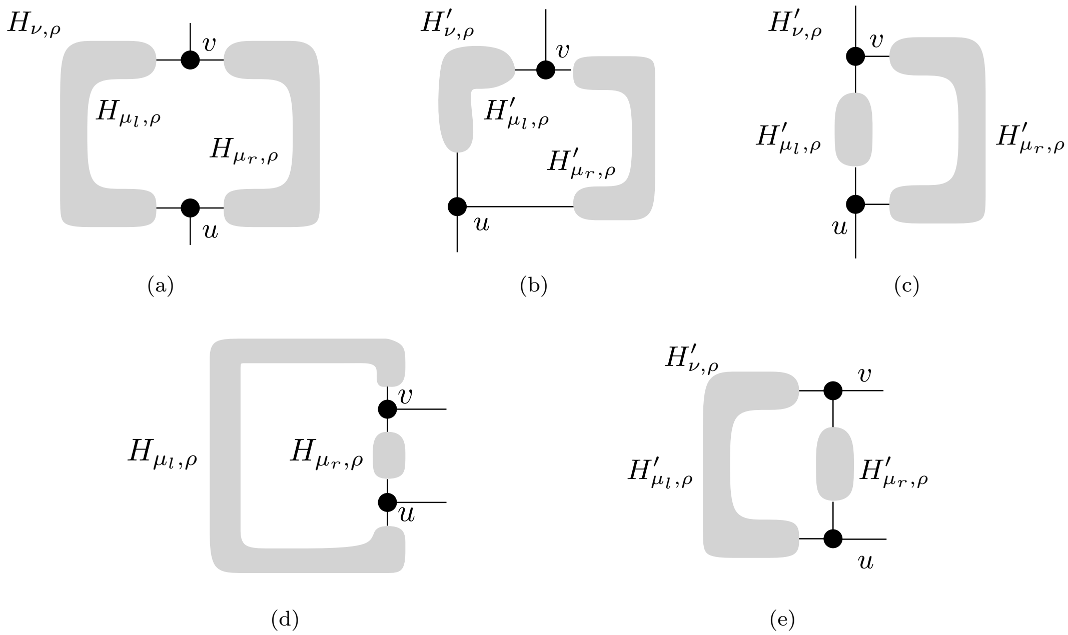


Figure 11: Illustration for the proof of Lemma 11.

By Lemma 1, since the spiralities of $H_{\nu,\rho}$ and of $H'_{\nu,\rho}$ have different parities, ν must have a child μ such that $H_{\mu,\rho}$ has odd spirality in $H_{\nu,\rho}$ and even spirality in $H'_{\nu,\rho}$, or vice versa. Let M_μ be the maximum spirality admitted by μ. Since μ admits both an even and an odd value of spirality, we have: If $M_\mu = 1$, μ admits 0 and $\Sigma^+_{\mu,\rho} = [0,1]^1$; if $M_\mu = 2$, by Property (a) of Lemma 10 and since μ admits spirality 1, either $\Sigma^+_{\mu,\rho} = [0,2]^1$ or $\Sigma^+_{\mu,\rho} = [1,2]^1$; if $M_\mu > 2$, by inductive hypotesis $\Sigma^+_{\mu,\rho} = [0, M_\mu]^1$. Hence, μ is always jump-1. ■

Let μ be a child of ν having a jump-1 interval, which always exists by the previous claim. For any value $\sigma'_\nu \in [1, M]^2$, let $H'_{\nu,\rho}$ be a rectilinear representation of $G_{\nu,\rho}$ with spirality σ'_ν. Let σ_μ be the spirality of the restriction of $H'_{\nu,\rho}$ to $G_{\mu,\rho}$. Suppose first that $\sigma_\mu > -M_\mu$. Since by inductive hypothesis μ admits spirality $\sigma_\mu - 1$ then, by Lemma 1, ν admits $\sigma'_\nu - 1$. Suppose now that $\sigma_\mu = -M_\mu$. Since $\sigma'_\nu > 0$, by Lemma 1, there exists a child $\phi \neq \mu$ of ν such that the restriction of $H'_{\nu,\rho}$ to $G_{\phi,\rho}$ has spirality $\sigma_\phi > 0$. Observe that ϕ also admits either spirality $\sigma_\phi - 1$ or spirality $\sigma_\phi - 2$. Indeed, if $\sigma_\phi > 2$, then ϕ admits spirality $\sigma_\phi - 2$ by Property (b) of Lemma 10; if $\sigma_\phi = 2$ it also admits spirality 0 or 1 by Property (a) of Lemma 10; if $\sigma_\phi = 1$ then it also admits spirality -1. In the case that ϕ admits spirality $\sigma_\phi - 1$, by Lemma 1, ν admits spirality $\sigma'_\nu - 1$. In the case that ϕ admits spirality $\sigma_\phi - 2$, then μ admits spirality $\sigma_\mu + 1$ (because μ is jump-1 and we are assuming $\sigma_\mu = -M_\mu < M_\mu$), and hence ν admits spirality $\sigma_\phi - 2 + 1 = \sigma'_\nu - 1$.

- ν is a P-node with three children. In this case every child μ of ν is jump-1. Indeed, since ν admits an even and an odd value of spirality, by Lemma 2, the same holds for μ. As for the case of an S-node, if M_μ is the maximum value of spirality admitted by ν, we have the following: If $M_\mu = 1$, $\Sigma^+_\mu = [0,1]^1$; if $M_\mu = 2$, either $\Sigma^+_{\mu,\rho} = [0,2]^1$ or $\Sigma^+_{\mu,\rho} = [1,2]^1$; if

$M_\mu > 2$, by inductive hypotesis $\Sigma^+_{\mu,\rho} = [0, M_\mu]^1$. Hence, μ is jump-1.

Assume first that $M > 3$. Let $H_{\nu,\rho}$ be a representation of $G_{\nu,\rho}$ with spirality M. By Lemma 2, every child μ of ν, is such that the restriction of $H_{\nu,\rho}$ to $G_{\mu,\rho}$ has spirality $\sigma_\mu \geq 2$. Since μ is jump-1, then μ also admits spirality $\sigma_\mu - 1$. This implies that, ν admits a representation with spirality $M - 1$. Since $M - 1 > 2$, by Corollary 1, ν admits all values of spirality in the set $[0, M-1]^2$, and hence $\Sigma^+_{\nu,\rho} = [0, M-1]^2 \cup [1, M]^2 = [0, M]^1$.

Assume now that $M = 3$. Let $H_{\nu,\rho}$ be a representation of $G_{\nu,\rho}$ with spirality M. The restrictions of $H_{\nu,\rho}$ to the three child components $G_{\mu_l,\rho}$, $G_{\mu_c,\rho}$, and $G_{\mu_r,\rho}$ of $G_{\nu,\rho}$, have spiraly values 5, 3, and 1, respectively. Since μ_l is jump-1, by the inductive hypothesis it admits spirality for all values in the set $[0, 5]^1$. Similarly, since μ_c is jump-1, by the inductive hypothesis it admits spirality for all values in the set $[0, 3]^1$. Also, since μ_r is jump-1, it admits spirality 0 or 2. If μ_r admits spirality 0, then ν admits spirality $M - 1 = 2$ for a representation in which $G_{\mu_l,\rho}$, $G_{\mu_c,\rho}$, and $G_{\mu_r,\rho}$ appear in this left-to-right order (and have spirality values 4, 2, and 0, respectively). If μ_r admits spirality 2 but not spirality 0, then ν admits spirality $M - 1 = 2$ for a representation in which $G_{\mu_l,\rho}$, $G_{\mu_r,\rho}$, and $G_{\mu_c,\rho}$ appear in this order (and again have spirality values 4, 2, and 0, respectively). Hence, so far we have proved that ν admits spirality for all values in the set $[1, 3]^1$. Finally, as showed in the proof of Property (a) of Lemma 10 for a P-node with three children, the fact that ν admits spirality 2 implies that it also admits spirality 0 (see Figs. 8(a) and 8(d)).

- ν is a P-node with two children. Let $H_{\nu,\rho}$ be a rectilinear representation of $G_{\nu,\rho}$ with spirality M. Let σ_{μ_l} and σ_{μ_r} be the spirality values of the restrictions of $H_{\nu,\rho}$ to the left and right child components $G_{\mu_l,\rho}$ and $G_{\mu_r,\rho}$ of $G_{\nu,\rho}$, respectively. Also, let $\{u, v\}$ be the poles of ν. By Lemma 11, we can assume $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 3]$, which implies that there exists $w \in \{u, v\}$ such that $\alpha^l_w = 0$, as $H_{\nu,\rho}$ has the maximum value of spirality admitted by ν. By Lemma 3, for $\alpha^l_w = 1$ and $\alpha^r_w = 0$ we can obtain a rectilinear representation of $G_{\nu,\rho}$ with spirality $M - 1$. If $M > 3$ then $M - 1 > 2$ and, by Corollary 1, ν admits spirality for all values in the set $[0, M-1]^2$, and hence $\Sigma^+_{\mu,\rho} = [0, M-1]^2 \cup [1, M]^2 = [0, M]^1$. If $M = 3$, by Property (a) of Lemma 10, we have either $[0, 3]^1 \in \Sigma^+_{\mu,\rho}$ or $[1, 3]^1 \in \Sigma^+_{\mu,\rho}$. In the former case, $\Sigma^+_\mu = [0, M]^1$. In the latter case, using a case analysis similar to the proof of Property (a) of Lemma 10 for the P-nodes with two children, it can be proved that 0 is also admitted by ν, and again $\Sigma^+_\mu = [0, M]^1$.

This concludes the analysis for different types of nodes in the SPQ*-tree of G. □

We are now ready to prove the main result of this subsection.

Proof of Theorem 3. Let M be the maximum value in $\Sigma^+_{\nu,\rho}$. If $M = 0$ then $\Sigma^+_{\nu,\rho} = [0]$. If $M = 1$ then either $\Sigma^+_{\nu,\rho} = [1]$ or $\Sigma^+_{\nu,\rho} = [0, 1]^1 = [0, M]^1$. Suppose $M = 2$; by Property (a) of Lemma 10, $G_{\nu,\rho}$ admits spirality 0, or 1, or both, i.e., $\Sigma^+_{\nu,\rho} = [0, 2]^2 = [0, M]^2$, or $\Sigma^+_{\nu,\rho} = [1, 2]^1$, or $\Sigma^+_{\nu,\rho} = [0, 2]^1 = [0, M]^1$. Finally, suppose that $M > 2$. If $G_{\nu,\rho}$ admits a value of spirality whose parity is different from M, by Lemma 12 $\Sigma^+_{\nu,\rho} = [0, M]^1$; else, by Corollary 1, either $\Sigma^+_{\nu,\rho} = [1, M]^2$ (if M is odd) or $\Sigma^+_{\nu,\rho} = [0, M]^2$ (if M is even).

4.3 Rectilinear Planarity Testing

Let G be an independent-parallel SP-graph that is not a simple cycle, T be its SPQ*-tree, and $\{\rho_1, \ldots, \rho_h\}$ be the Q*-nodes of T. The rectilinear planarity testing for G follows a strategy similar to the one described in Section 3 for testing general SP-graphs. For each possible choice of the root $\rho \in \{\rho_1, \ldots, \rho_h\}$, the algorithm visits T_ρ bottom-up in post-order and computes, for each visited node ν, the non-negative spirality set $\Sigma^+_{\nu,\rho}$, based on the sets of the children of ν. $\Sigma^+_{\nu,\rho}$ is representative of all "shapes" that $G_{\nu,\rho}$ can take in a rectilinear representation of G with the reference chain on the external face. The key lemmas used to show that we can efficiently execute this procedure over all SPQ*-tree T_ρ of G are Lemmas 13, 15, 16, 17, and 18. From now on, we say that a node ν in T_ρ is *trivial*, or *jump-1*, or *jump-2*, if $\Sigma^+_{\nu,\rho}$ is a trivial interval, or a jump-1 interval, or a jump-2 interval, respectively.

Q*-nodes. Each chain of length ℓ can turn at most $\ell - 1$ times (one turn for each vertex). Therefore, for a Q*-node ν of T_ρ, we have $\Sigma^+_{\nu,\rho} = [0, \ell - 1]^1$, and the following lemma holds, assuming that each Q*-node is equipped with the length of its corresponding chain when we compute the SPQ*-tree T of G.

Lemma 13 *Let G be an independent-parallel SP-graph, T_ρ be a rooted SPQ*-tree of G, and ν be a Q*-node of T_ρ. The set $\Sigma^+_{\nu,\rho}$ can be computed in $O(1)$ time.*

S-nodes. Lemma 15 establishes the complexity of computing the spirality sets of the S-nodes. To prove it, we first state the following key property.

Lemma 14 *Let ν be an S-node of T_ρ. Node ν is jump-1 if and only if at least one of its children is jump-1. Also, $\Sigma^+_{\nu,\rho} = [1, 2]^1$ if and only if ν has exactly one child with non-negative rectilinear spirality set $[1, 2]^1$ and all the other children with non-negative rectilinear spirality set $[0]$.*

Proof: We prove that ν is jump-1 if and only if at least one of its children is jump-1. Suppose first that ν is jump-1 and suppose by contradiction that all its children are trivial or jump-2. This implies that for each child μ of ν, $\Sigma^+_{\mu,\rho}$ contains only even values or only odd values. Denote by j the number of children of ν whose non-negative rectilinear spirality set contain only odd values. By Lemma 1, the spirality of any rectilinear representation of $G_{\nu,\rho}$ is the sum of the spirality values of all child components. It follows that $G_{\nu,\rho}$ admits only even values of spirality values if j is even and only odd values of spirality values if j is odd, which contradicts the hypothesis that ν is jump-1. Suppose vice versa that ν has at least a child μ that is jump-1. Denote by M the maximum value in $\Sigma^+_{\nu,\rho}$ and by M_μ the maximum value in $\Sigma^+_{\mu,\rho}$. Let $H_{\nu,\rho}$ be any rectilinear representation of $G_{\nu,\rho}$ having spirality M, and let $H_{\mu,\rho}$ be its restriction to $G_{\mu,\rho}$. By Lemma 1, $H_{\mu,\rho}$ has spirality M_μ. Also, since μ is jump-1, by Lemma 1 we can obtain a rectilinear representation $H'_{\nu,\rho}$ of $G_{\nu,\rho}$ with spirality $M - 1$ by simply replacing $H_{\mu,\rho}$ in $H_{\nu,\rho}$ with a rectilinear representation of $G_{\mu,\rho}$ having spirality $M_\mu - 1$. Therefore, by Theorem 3, ν is jump-1.

We now show the second part of the lemma. Suppose first that ν has exactly one child μ with non-negative rectilinear spirality set $[1, 2]^1$ and all the other children with non-negative rectilinear spirality set $[0]$. Clearly, by Lemma 1, $\Sigma^+_{\nu,\rho} = \Sigma^+_{\mu,\rho}$, i.e., $\Sigma^+_{\nu,\rho} = [1, 2]^1$. Suppose vice versa that $\Sigma^+_{\nu,\rho} = [1, 2]^1$. By Lemma 1, the sum of the spiralities admitted by the child components of ν cannot be larger than two. If exactly one child of ν has non-negative rectilinear spirality set $[1, 2]^1$ and all the other children have non-negative rectilinear spirality set $[0]$, we are done. Otherwise, one of the following two cases must be considered: (i) There are two children μ and μ' of ν such that the maximum value of spirality admitted by $G_{\mu,\rho}$ and $G_{\mu',\rho}$ is 1 and any other child ν has non-negative rectilinear spirality set $[0]$; this case is ruled out by observing that $G_{\mu,\rho}$ (and $G_{\mu',\rho}$) would also admit spirality -1 and thus, by Lemma 1, $G_{\nu,\rho}$ would also admit spirality 0. (ii) ν has a child μ for which either $\Sigma^+_{\mu,\rho} = [0,2]^1$ or $\Sigma^+_{\mu,\rho} = [0,2]^2$ and any other child of ν has non-negative rectilinear spirality set $[0]$; again, this case is ruled out because it would imply that also $G_{\nu,\rho}$ admits spirality 0. □

Lemma 15 *Let G be an independent-parallel SP-graph, T be the SPQ*-tree of G, ν be an S-node of T with δ_ν children, and $\rho_1, \rho_2, \dots, \rho_h$ be a sequence of Q*-nodes of T such that, for each child μ of ν in T_{ρ_i}, the set Σ^+_{μ,ρ_i} is given. Σ^+_{ν,ρ_i} can be computed in $O(\delta_\nu)$ time for $i = 1$ and in $O(1)$ time for $2 \leq i \leq h$.*

Proof: For any $i = 1, \dots, h$, let x_{ν,ρ_i} and y_{ν,ρ_i} be the number of children of ν in T_{ρ_i} with non-negative spirality set $[0]$ and $[1,2]^1$, respectively. Also, let z_{ν,ρ_i} be the number of children that are jump-1 (clearly, $z_{\nu,\rho_i} \geq y_{\nu,\rho_i}$). Let M_{ν,ρ_i} be the maximum value in Σ^+_{ν,ρ_i}. First, we show how to compute Σ^+_{ν,ρ_i} in $O(1)$ time given x_{ν,ρ_i}, y_{ν,ρ_i}, z_{ν,ρ_i}, and M_{ν,ρ_i}. By Lemma 14, Σ^+_{ν,ρ_i} is jump-1 if and only if $z_{\nu,\rho_i} > 0$. Suppose that Σ^+_{ν,ρ_i} is jump-1. If $M_{\nu,\rho_i} \neq 2$, by Theorem 3, $\Sigma^+_{\nu,\rho_i} = [0, M_{\nu,\rho_i}]^1$. If $M_{\nu,\rho_i} = 2$, Lemma 14 implies $\Sigma^+_{\nu,\rho_i} = [1,2]^1$ if $x_{\nu,\rho_i} + y_{\nu,\rho_i} = \delta_\nu$ and $y_{\nu,\rho_i} = 1$; otherwise $\Sigma^+_{\nu,\rho_i} = [0,2]^1$. Suppose now that Σ^+_{ν,ρ_i} is not jump-1. By Theorem 3, we have: $\Sigma^+_{\nu,\rho_i} = [0]$ if $M_{\nu,\rho_i} = 0$ and $\Sigma^+_{\nu,\rho_i} = [1]$ if $M_{\nu,\rho_i} = 1$; $\Sigma^+_{\nu,\rho_i} = [1, M_{\nu,\rho_i}]^2$ if $M_{\nu,\rho_i} > 1$ and M_{ν,ρ_i} is odd; $\Sigma^+_{\nu,\rho_i} = [0, M_{\nu,\rho_i}]^2$ if $M_{\nu,\rho_i} > 1$ and M_{ν,ρ_i} is even.

We now show how to compute x_{ν,ρ_i}, y_{ν,ρ_i}, z_{ν,ρ_i}, and M_{ν,ρ_i} for $i = 1, \dots, h$. If $i = 1$, given Σ^+_{μ,ρ_1} for every child μ of ν in T_{ρ_1}, then x_{ν,ρ_1}, y_{ν,ρ_1}, and z_{ν,ρ_1} are computed in $O(\delta_\nu)$ time by just visiting each child of ν. Also, since by Lemma 1 the maximum spirality admitted by $G_{\nu,\rho}$ is the sum of the maximum spirality values admitted by the children of ν in T_{ρ_1}, we also compute M_{ν,ρ_1} and Σ^+_{ν,ρ_1} in $O(\delta_\nu)$ time. We store at ν the values x_{ν,ρ_1}, y_{ν,ρ_1}, z_{ν,ρ_1}, and M_{ν,ρ_1}.

Let $i \in \{2, \dots, h\}$. Let μ_1 be the parent of ν in T_{ρ_1} and let μ_i be the parent of ν in T_{ρ_i}. Note that, μ_1 is a child of ν in T_{ρ_i} and μ_i is a child of ν in T_{ρ_1}. Any other child of ν in T_{ρ_1} is also a child of ν in T_{ρ_i} and vice versa. To compute Σ^+_{ν,ρ_i} in $O(1)$ time, we compute x_{ν,ρ_i}, y_{ν,ρ_i}, z_{ν,ρ_i}, M_{ν,ρ_i} as follows:

(*i*) Let $g_{\mu_i} = 1$ if $\Sigma^+_{\mu_i,\rho_1} = [0]$ and $g_{\mu_i} = 0$ otherwise. Also, let $g_{\mu_1} = 1$ if $\Sigma_{\mu_1,\rho_i} = [0]$ and $g_{\mu_1} = 0$ otherwise. We have $x_{\nu,\rho_i} = x_{\nu,\rho_1} - g_{\mu_i} + g_{\mu_1}$.

(*ii*) Let $g'_{\mu_i} = 1$ if $\Sigma^+_{\mu_i,\rho_1} = [1,2]^1$ and $g'_{\mu_i} = 0$ otherwise. Also, let $g'_{\mu_1} = 1$ if $\Sigma_{\mu_1,\rho_i} = [1,2]^1$ and $g_{\mu_1} = 0$ otherwise. We have $y_{\nu,\rho_i} = y_{\nu,\rho_1} - g'_{\mu_i} + g'_{\mu_1}$.

(*iii*) Let $g''_{\mu_i} = 1$ if Σ_{μ_i,ρ_1} is jump-1 and $g''_{\mu_i} = 0$ otherwise. Also, let $g''_{\mu_1} = 1$ if $\Sigma^+_{\mu_1,\rho_i}$ is jump-1 and $g''_{\mu_1} = 0$ otherwise. We have $z_{\nu,\rho_i} = z_{\nu,\rho_1} - g''_{\mu_i} + g''_{\mu_1}$.

(*iv*) $M_{\nu,\rho_i} = M_{\nu,\rho_1} - M_{\mu_i,\rho_1} + M_{\mu_1,\rho_i}$. □

P-nodes. For a P-node ν, $\Sigma^+_{\nu,\rho}$ can be computed in $O(1)$ time, independent of ρ. We treat separately the case of a P-node with three children (Lemma 16) and the case of a P-node with two children (Lemma 17).

Lemma 16 *Let G be an independent-parallel SP-graph, T_ρ be a rooted SPQ*-tree of G, and ν be a P-node of T_ρ with three children. If for each child μ of ν in T_ρ the set $\Sigma^+_{\mu,\rho}$ is given then $\Sigma^+_{\nu,\rho}$ can be computed in $O(1)$ time.*

Proof: Observe that, by Lemma 2, for any given integer value $\sigma_\nu \geq 0$, one can test in $O(1)$ time whether $G_{\nu,\rho}$ admits spirality σ_ν. It suffices to test if there exists a child of ν that admits spirality σ_ν, another child that admits spirality $\sigma_\nu + 2$, and the remaining child that admits spirality $\sigma_\nu - 2$. Testing this condition requires a constant number of checks.

By Theorem 3, $G_{\nu,\rho}$ is rectilinear planar if and only if it admits spirality either 0 or 1. Based on the previous observation, we can check this property in $O(1)$ time; if it does not hold, then $\Sigma^+_{\nu,\rho} = \emptyset$. Otherwise, we determine the maximum value M in $\Sigma^+_{\nu,\rho}$. By Theorem 3, it suffices to find a value σ_ν such that ν admits spirality σ_ν but not spirality values $\sigma_\nu + 1$ and $\sigma_\nu + 2$; if we find such a value, then $M = \sigma_\nu$. Using this observation, we prove that we can find M in $O(1)$ time.

For each $i = 0, ..., 4$, we can first check in $O(1)$ time whether $M = i$. If this is not the case, then $M > 4$. To find M in this case, we first give an interesting property. Consider the maximum values in the non-negative rectilinear spirality sets of the children of ν. Denote by $\mu_{\max}$ (resp. $\mu_{\min}$) any child of ν whose maximum value is not smaller than (resp. larger than) any other maximum values. Also denote by μ_{mid} the remaining child. We prove the following claim.

Claim 2 *Let M be the maximum value in $\Sigma^+_{\nu,\rho}$. If $M > 4$ then $G_{\nu,\rho}$ admits spirality M for an embedding where $G_{\mu_{\max},\rho}$, $G_{\mu_{\text{mid}},\rho}$, and $G_{\mu_{\min},\rho}$ appear in this left-to-right order.*

Proof of the claim: Let $H_{\nu,\rho}$ be a rectilinear representation of $G_{\nu,\rho}$ with spirality $M > 4$. If $G_{\mu_{\max},\rho}$, $G_{\mu_{\text{mid}},\rho}$, and $G_{\mu_{\min},\rho}$ appear in this order in $H_{\nu,\rho}$ we are done. Hence, suppose this is not the case; we prove that there exists another rectilinear representation $H'_{\nu,\rho}$ of $G_{\nu,\rho}$ with spirality M and such that $G_{\mu_{\max},\rho}$, $G_{\mu_{\text{mid}},\rho}$, and $G_{\mu_{\min},\rho}$ appear in this left-to-right order in the planar embedding of $H'_{\nu,\rho}$.

Let μ_l, μ_c, and μ_r be the children of ν that correspond to the left, the central, and the right component of $H_{\nu,\rho}$, respectively. Denote by σ_{μ_d} the spirality of the restriction of $H_{\nu,\rho}$ to $G_{\mu_d,\rho}$, with $d \in \{l, c, r\}$. By Lemma 2, it suffices to show that $G_{\mu_{\max},\rho}$, $G_{\mu_{\text{mid}},\rho}$, and $G_{\mu_{\min},\rho}$ admit spirality values σ_{μ_l}, σ_{μ_c}, and σ_{μ_r}, respectively. Observe that, since $M > 4$, by Lemma 2 we have $\sigma_{\mu_d} > 0$.

Let $d, d' \in \{l, c, r\}$ with $d \neq d'$ and let M_{μ_d} be the maximum value of spirality in $\Sigma^+_{\mu_d,\rho}$. The following property holds.

Property 2 *If $\sigma_{\mu_{d'}} \leq M_{\mu_d}$ then $G_{\mu_d,\rho}$ admits spirality $\sigma_{\mu_{d'}}$.*

Indeed, since $M > 4$, by Lemma 2 we have $M_{\mu_d} \geq 3$: If μ_d is jump-1, $G_{\mu_d,\rho}$ admits spirality $\sigma_{\mu_{d'}}$ by Theorem 3; if μ_d is not jump-1, M, M_{μ_d}, and $\sigma_{\mu_{d'}}$ have the same parity by Lemma 2, and $G_{\mu_d,\rho}$ admits spirality $\sigma_{\mu_{d'}}$ by Theorem 3.

We now show separately that: (a) $G_{\mu_{\max},\rho}$ admits spirality σ_{μ_l}, (b) $G_{\mu_{\text{mid}},\rho}$ admits spirality σ_{μ_c}, and (c) $G_{\mu_{\min},\rho}$ admits spirality σ_{μ_r}.

Proof of (a): Since by definition $M_{\mu_{\max}} \geq M_{\mu_l} \geq \sigma_{\mu_l}$, by Property 2 we have that $G_{\mu_{\max},\rho}$ admits spirality σ_{μ_l}.

Proof of (b): If $\mu_{\text{mid}} = \mu_c$ we are done. Else, suppose that $\mu_{\text{mid}} = \mu_l$. Since $\sigma_{\mu_l} \geq \sigma_{\mu_c}$, we have $M_{\text{mid}} = M_l \geq \sigma_{\mu_l} > \sigma_{\mu_c}$ and, consequently, by Property 2 $G_{\mu_{\text{mid}},\rho}$ admits spirality σ_{μ_c}. Finally, suppose that $\mu_{\text{mid}} = \mu_r$. If $\mu_{\min} = \mu_c$ then $M_{\mu_{\text{mid}}} \geq M_{\mu_{\min}} \geq \sigma_{\mu_c}$; if $\mu_{\min} = \mu_l$ then $M_{\mu_{\text{mid}}} \geq M_{\mu_{\min}} \geq \sigma_{\mu_l} > \sigma_{\mu_c}$. Hence, by by Property 2, $G_{\mu_{\text{mid}},\rho}$ admits spirality σ_{μ_c}.

Proof of (c): Since $\sigma_{\mu_r} < \sigma_{\mu_c} < \sigma_{\mu_l}$, for any $\mu \in \{\mu_{\min}, \mu_{\text{mid}}, \mu_{\max}\}$, $\sigma_{\mu_r} \leq M_\mu$. Hence, by the claim, $G_{\mu_{\min},\rho}$ admits spirality σ_{μ_r}. ■

By Claim 2, to compute M when $M > 4$, we can restrict to consider only rectilinear representations of $G_{\nu,\rho}$ where $G_{\mu_{\max},\rho}$, $G_{\mu_{\text{mid}},\rho}$, and $G_{\mu_{\min},\rho}$ occur in this left-to-right order. Let $\overline{M} = \min\{M_{\mu_{\max}} - 2, M_{\mu_{\text{mid}}}, M_{\mu_{\min}} + 2\}$. By Lemma 2, we have $M \leq \overline{M}$. We test in $O(1)$ time whether ν is jump-1; by Theorem 3, it is sufficient to check whether $G_{\nu,\rho}$ either admits both spirality values 0 and 1 or both spirality values 1 and 2. If ν is jump-1, by Lemma 2, all the children of ν are jump-1. Hence, $\mu_{\max}$, μ_{mid}, and $\mu_{\min}$ admit spiralities $\overline{M} - 2$, $\overline{M}$, and $\overline{M} + 2$, respectively, which implies that $M = \overline{M}$. Suppose vice versa that ν is not jump-1. In this case, we check in $O(1)$ if $G_{\nu,\rho}$ admits spirality $\overline{M}$. If so, $M = \overline{M}$. Otherwise, M and $\overline{M}$ have opposite parity, which implies that M and $\overline{M} - 1$ have the same parity, and $M \leq \overline{M} - 1$. Since $\overline{M} - 1 = \min\{M_{\mu_{\max}} - 2, M_{\mu_{\text{mid}}}, M_{\mu_{\min}} + 2\} - 1$, we have that $\mu_{\max}$, μ_{mid}, and $\mu_{\min}$ admit spirality values $(\overline{M} - 2) - 1$, $\overline{M} - 1$, and $(\overline{M} + 2) - 1$, respectively, i.e., $M = \overline{M} - 1$.

Based on M, we finally determine the structure of $\Sigma^+_{\nu,\rho}$ in $O(1)$ time. Namely, we check in $O(1)$ time if ν is jump-1; thanks to Theorem 3 it suffices to check whether $G_{\nu,\rho}$ admits spirality values 0 and 1 or spirality values 1 and 2. Suppose that ν is jump-1; if it contains 0, then $\Sigma^+_{\nu,\rho} = [0, M]^1$; else $\Sigma^+_{\nu,\rho} = [1, 2]^1$. Suppose vice versa that ν is not jump-1. If $M \leq 1$ then $\Sigma^+_{\nu,\rho} = [M]$. Otherwise, if M is odd $\Sigma^+_{\nu,\rho} = [1, M]^2$ and if M is even $\Sigma^+_{\nu,\rho} = [0, M]^2$. □

Lemma 17 *Let G be an independent-parallel SP-graph, T_ρ be a rooted SPQ*-tree of G, and ν be a P-node of T_ρ with two children. If for each child μ of ν in T_{ρ_i}, the set Σ^+_{μ,ρ_i} is given, then $\Sigma^+_{\nu,\rho}$ can be computed in $O(1)$ time.*

Proof: We follow the same proof strategy as for Lemma 16. By Lemma 3, for any given integer σ_ν, one can test in $O(1)$ time whether $G_{\nu,\rho}$ admits spirality σ_ν. Indeed, it suffices to test whether there are four binary numbers α^l_u, α^l_v, α^r_u, and α^r_v such that $1 \leq \alpha^l_u + \alpha^r_u \leq 2$, $1 \leq \alpha^l_v + \alpha^r_v \leq 2$, and for which one child of ν admits spirality $\sigma_\nu + \alpha^l_u + \alpha^l_v$ and the other child of ν admits spirality $\sigma_\nu - \alpha^r_u + \alpha^r_v$. Testing this condition requires a constant number of checks.

By Theorem 3, $G_{\nu,\rho}$ is rectilinear planar if and only if it admits spirality either 0 or 1. Based on the reasoning above, we can check this property in $O(1)$ time; if it does not hold, then $\Sigma^+_{\nu,\rho} = \emptyset$. Otherwise, we determine the maximum value M in $\Sigma^+_{\nu,\rho}$. By Theorem 3, it suffices to find a value σ_ν such that ν admits spirality σ_ν but it does not admit spirality $\sigma_\nu + 1$ and $\sigma_\nu + 2$; if we find such a value, then $M = \sigma_\nu$. We prove how to find M in $O(1)$ time.

For each $i = 0, ..., 4$, we first check in $O(1)$ time whether $M = i$. If this is not the case, then $M > 4$. To find M in this case, we claim a property similar to the case of a P-node with three children. Denote by $\mu_{\max}$ a child of ν whose maximum value is not smaller than the other. Let $\mu_{\min}$ be the remaining child.

Claim 3 *Let M be the maximum value in $\Sigma^+_{\nu,\rho}$. If $M > 4$, there exists a rectilinear representation of $G_{\nu,\rho}$ with spirality M where $G_{\mu_{\max},\rho}$ and $G_{\mu_{\min},\rho}$ appear in this left-to-right order.*

Proof of the claim: Let $H_{\nu,\rho}$ be a rectilinear representation of $G_{\nu,\rho}$ with spirality M. If $G_{\mu_{\max},\rho}$ is the left child in $H_{\nu,\rho}$, we are done. Otherwise we show that there exists a rectilinear representation $H'_{\nu,\rho}$ of $G_{\nu,\rho}$ with spirality M such that $G_{\mu_{\max},\rho}$ is the left child. Since $H_{\nu,\rho}$ has the maximum possible value of spirality, we have $\alpha^r_u = \alpha^r_v = 1$. Since $\mu_{\min}$ is the left child in $H_{\nu,\rho}$ and $M > 4$, by Lemma 3, $\sigma_{\mu_{\min}} > 2$. By Property (b) of Lemma 10, there exists a rectilinear representation $H'_{\mu_{\min},\rho}$ of $G_{\mu_{\min},\rho}$ with spirality $\sigma'_{\mu_{\min},\rho} = \sigma_{\mu_{\min}} - 2$. Also, by Lemma 11 we can assume that $\sigma_{\mu_{\min}} - \sigma_{\mu_{\max}} = g$, with $2 \leq g \leq 3$. We have $M_{\mu_{\max}} \geq M_{\mu_{\min}} \geq \sigma_{\mu_{\min}} = g + \sigma_{\mu_{\max}}$. Hence, by Theorem 3, if $\sigma_{\mu_{\max}}$ and $M_{\mu_{\max}}$ have different parities, then $\mu_{\max}$ is jump-1, otherwise it is jump-2. In both cases, $\mu_{\max}$ admits spirality $\sigma'_{\mu_{\max}} = \sigma_{\mu_{\max}} + 2$. We have $\sigma'_{\mu_{\max}} - \sigma'_{\mu_{\min}} = \sigma_{\mu_{\max}} + 2 - \sigma_{\mu_{\max}} - 2 = \sigma_{\mu_{\max}} - \sigma_{\mu_{\max}} = g$. Hence, by Lemma 3, there exists a rectilinear representation $H'_{\nu,\rho}$ that contains $H'_{\mu_{\min},\rho}$ and $H'_{\mu_{\max},\rho}$ in this left-to-right order and such that the spirality σ'_ν of $H'_{\nu,\rho}$ is $\sigma'_\nu = \sigma'_{\mu_{\min}} + 2 = \sigma_{\mu_{\min}} \geq \sigma_{\mu_{\max}} + 2 = M$. ■

When $M > 4$, by Lemma 3, we have $M_{\mu_{\min}} > 2$ and $M_{\mu_{\max}} > 2$. By the claim above we can restrict to consider only rectilinear representations of $G_{\nu,\rho}$ where $G_{\mu_{\max},\rho}$ and $G_{\mu_{\min},\rho}$ are the left and right child, respectively. Also, we can restrict to consider $\alpha_u^r = \alpha_v^r = 1$. By Lemma 3, $M \leq \min\{M_{\mu_{\max}}, M_{\mu_{\min}} + 2\}$.

Suppose first that $M_{\mu_{\max}} \geq M_{\mu_{\min}} + 2$, which implies $M \leq M_{\mu_{\min}} + 2$. We show that in fact $M = M_{\mu_{\min}} + 2$, i.e., ν admits spirality $M_{\mu_{\min}} + 2$. Since $M_{\mu_{\max}} \geq M_{\mu_{\min}} + 2$, we have that $\mu_{\max}$ admits spirality $M_{\mu_{\min}} + 2$ or $M_{\mu_{\min}} + 3$; this implies that we can realize a rectilinear representation of $G_{\nu,\rho}$ whose restrictions to $G_{\mu_{\min},\rho}$ and to $G_{\mu_{\max},\rho}$ have spiralities $\sigma_{\mu_{\min}} = M_{\mu_{\min}}$ and $\sigma_{\mu_{\max}} \in [M_{\mu_{\min}} + 2, M_{\mu_{\min}} + 3]$, respectively. By Lemma 3, if $\sigma_{\mu_{\max}} = M_{\mu_{\min}} + 2$ then $G_{\nu,\rho}$ admits spirality $M_{\mu_{\min}} + 2$ for $\alpha_u^l = \alpha_v^l = 0$. If $\sigma_{\mu_{\max}} = M_{\mu_{\min}} + 3$ then $G_{\nu,\rho}$ admits spirality $M_{\mu_{\min}} + 2$ for $\alpha_u^l = 1$ and $\alpha_v^l = 0$.

Suppose vice versa that $M_{\mu_{\max}} < M_{\mu_{\min}} + 2$, which implies $M \leq M_{\mu_{\max}}$. In this case we show that either $M = M_{\mu_{\max}}$ or $M = M_{\mu_{\max}} - 1$. Since $M_{\mu_{\min}} > M_{\mu_{\max}} - 2$, we have that $\mu_{\min}$ admits spirality $M_{\mu_{\max}} - 2$ or $M_{\mu_{\max}} - 3$. If $\mu_{\min}$ admits spirality $M_{\mu_{\max}} - 2$, then we can realize a rectilinear representation of $G_{\nu,\rho}$ whose restrictions to $G_{\mu_{\min},\rho}$ and to $G_{\mu_{\max},\rho}$ have spiralities $\sigma_{\mu_{\max}} = M_{\mu_{\max}}$ and $\sigma_{\mu_{\min}} = M_{\mu_{\max}} - 2$, respectively. By Lemma 3, this representation has spirality $M_{\mu_{\max}}$, which implies $M = M_{\mu_{\max}}$. If $\mu_{\min}$ does not admit spirality $M_{\mu_{\max}} - 2$, it admits spirality $M_{\mu_{\max}} - 3$ and $M \leq M_{\mu_{\max}} - 1$. In this case we realize a rectilinear representation of $G_{\nu,\rho}$ whose restrictions to $G_{\mu_{\min},\rho}$ and to $G_{\mu_{\max},\rho}$ have spiralities $\sigma_{\mu_{\max}} = M_{\mu_{\max}}$ and $\sigma_{\mu_{\min}} = M_{\mu_{\max}} - 3$. By Lemma 3, this representation has spirality $M_{\mu_{\max}} - 1$, which implies that $M = M_{\mu_{\max}} - 1$.

Based on M, we finally determine the structure of $\Sigma^+_{\mu,\rho}$ in $O(1)$ time. We have that G_ν admits a rectilinear representation with spirality M and $M-1$. Indeed, if ν has spirality M, then $\alpha_u^r = \alpha_v^r = 1$ and, by Lemma 11, at least one of α_u^l and α_v^l equals 0, say for example $\alpha_u^l = 0$. For $\alpha_u^l = 1$ we get spirality $M - 1$. Hence, by Theorem 3, ν is jump-1: If $M = 2$ and G_ν does not admit a representation with spirality 0, $\Sigma^+_{\mu,\rho} = [1, 2]^1$. Otherwise, $\Sigma^+_{\mu,\rho} = [0, M]^1$. □

Let ν be the root child of T_ρ and suppose that $\Sigma^+_{\nu,\rho}$ has been computed. We prove the following.

Lemma 18 *Let G be an independent-parallel SP-graph, T_ρ be a rooted SPQ*-tree of G, and ν be the child of ρ in T_ρ. If $G_{\nu,\rho}$ is rectilinear planar, one can test whether G is rectilinear planar in $O(1)$ time.*

Proof: As we already observed in the proof of Lemma 9, G is rectilinear planar if and only if there exist two values $\sigma_\nu \in \Sigma_{\nu,\rho}$ and $\sigma_\rho \in \Sigma_{\rho,\rho}$, such that $\sigma_\nu - \sigma_\rho = 4$. We show that, since G is independent-parallel, this condition is true if and only if there exist two values $\sigma'_\nu \in \Sigma^+_{\nu,\rho}$ and $\sigma'_\rho \in \Sigma^+_{\rho,\rho}$ such that $\sigma'_\nu + \sigma'_\rho = 4$.

Suppose first that there exist two values $\sigma'_\nu \in \Sigma^+_{\nu,\rho}$ and $\sigma'_\rho \in \Sigma^+_{\rho,\rho}$ such that $\sigma'_\nu + \sigma'_\rho = 4$. In this case we know that $-\sigma'_\rho \in \Sigma_{\rho,\rho}$. Hence, for $\sigma_\nu = \sigma'_\nu$ and $\sigma_\rho = -\sigma'_\rho$ we have $\sigma_\nu - \sigma_\rho = 4$.

Suppose now that $\sigma_\nu - \sigma_\rho = 4$ with $\sigma_\nu \in \Sigma_{\nu,\rho}$ and $\sigma_\rho \in \Sigma_{\rho,\rho}$. Three cases are possible:

Case 1: $\sigma_\nu \geq 0$ and $\sigma_\rho < 0$. In this case $\sigma'_\rho = -\sigma_\rho \in \Sigma^+_{\rho,\rho}$ and setting $\sigma'_\nu = \sigma_\nu$, we have $\sigma'_\nu + \sigma'_\rho = 4$.

Case 2: $\sigma_\nu < 0$ and $\sigma_\rho < 0$. We have $-\sigma_\nu + \sigma_\rho + 8 = 4$. Also, we have $\sigma_\rho < -4 \Rightarrow -2\sigma_\rho > 8 \Rightarrow -\sigma_\rho > \sigma_\rho + 8$. This implies that $\sigma_\rho + 8 \in \Sigma_{\rho,\rho}$. Therefore, for $\sigma''_\nu = -\sigma_\nu \in \Sigma^+_{\nu,\rho}$ and $\sigma''_\rho = \sigma_\rho + 8 \in \Sigma_{\rho,\rho}$, we have $\sigma''_\nu + \sigma''_\rho = 4$. If $\sigma''_\nu \leq 4$ then $\sigma''_\rho \in [0, 4]$, and we are done. Assume vice versa that $\sigma''_\nu > 4$. Observe that σ''_ν and σ''_ρ have the same parity. If they are both even numbers, by Theorem 3, we have $\sigma'_\nu = 4 \in \Sigma^+_{\nu,\rho}$, and for $\sigma'_\rho = 0 \in \Sigma^+_{\rho,\rho}$ we have $\sigma'_\nu + \sigma'_\rho = 4$. If they are both odd numbers, by Theorem 3, we have $\sigma'_\nu = 3 \in \Sigma^+_{\nu,\rho}$, and for $\sigma'_\rho = 1 \in \Sigma^+_{\rho,\rho}$ we have $\sigma'_\nu + \sigma'_\rho = 4$.

Case 3: $\sigma_\nu \geq 0$ and $\sigma_\rho \geq 0$. In this case we have $\sigma_\nu \geq 4$. If $\sigma_\nu = 4$ then $\sigma_\rho = 0$ and we are done. Assume vice versa that $\sigma_\nu > 4$ and $\sigma_\rho \geq 1$. Observe that σ_ν and σ_ρ have the same parity. If they are both even numbers, by Theorem 3 we have $\sigma'_\nu = 4 \in \Sigma^+_{\nu,\rho}$, and for $\sigma'_\rho = 0 \in \Sigma^+_{\rho,\rho}$ we have $\sigma'_\nu + \sigma'_\rho = 4$. If they are both odd numbers, by Theorem 3, we have $\sigma'_\nu = 3 \in \Sigma^+_{\nu,\rho}$, and for $\sigma'_\rho = 1 \in \Sigma^+_{\rho,\rho}$ we have $\sigma'_\nu + \sigma'_\rho = 4$.

Based on the characterization above, if ℓ is the length of the chain corresponding to ρ, for any pair $\sigma'_\nu \in \{0, 1, 2, 3, 4\}$ and $\sigma'_\rho \in \{0, 1, 2, 3, 4\}$ such that $\sigma'_\nu + \sigma'_\rho = 4$ we just test whether $\sigma'_\nu \in \Sigma^+_{\nu,\rho}$ and $\sigma'_\rho \in \{0, \dots, \ell - 1\}$. This requires a constant number of checks. If the test is positive, then there exists a rectilinear representation of G such that its restriction to ν has spirality σ'_ν and its restriction to ρ has spirality $-\sigma'_\rho$. □

We now prove that rectilinear planarity testing of independent-parallel SP-graphs can be solved in linear time.

Lemma 19 *Let G be an n-vertex independent-parallel SP-graph. There exists an $O(n)$-time algorithm that tests whether G is rectilinear planar and that computes a rectilinear representation of G in the positive case.*

Proof: If G is a simple cycle, the test is trivial, as G is rectilinear planar if and only if it contains at least four vertices. Assume that G is not a simple cycle. Let T be the SPQ*-tree of G, and let $\rho_1, \dots, \rho_h$ be the Q*-nodes of T. For each $i = 1, \dots, h$, the testing algorithm performs a post-order visit of T_{ρ_i}. During this visit, for every non-root node ν of T_{ρ_i} the algorithm computes Σ^+_{ν,ρ_i} by using Lemmas 13, 15, 16, and 17. If $\Sigma^+_{\nu,\rho_i} = \emptyset$, the algorithm stops the visit, discards T_{ρ_i}, and starts visiting $T_{\rho_{i+1}}$ (if $i < h$). If the algorithm achieves the root child ν and $\Sigma^+_{\nu,\rho_i} \neq \emptyset$, it checks whether G is rectilinear planar by using Lemma 18: if so, the test is positive and the algorithm does not visit the remaining trees; otherwise it discards T_{ρ_i} and starts visiting $T_{\rho_{i+1}}$ (if $i < h$).

We now analyze the time complexity of the testing algorithm. Suppose that one of the trees T_{ρ_i} is considered, and let ν be a node of T_{ρ_i}. Denote by δ_ν the number of children of ν. If the parent of ν in T_{ρ_i} coincides with the parent of T_{ρ_j} for some $j \in \{1, \dots, i-1\}$, and if Σ^+_{ν,ρ_j} was previously computed, then the algorithm does not need to compute Σ^+_{ν,ρ_i}, because $\Sigma^+_{\nu,\rho_i} = \Sigma^+_{\nu,\rho_j}$. Hence, for each node ν, the number of computations of its non-negative rectilinear spirality set that must be performed over all possible trees T_{ρ_i} is $\delta_\nu + 1 = O(\delta_\nu)$ (one for each different way of choosing the parent of ν).

If ν is a Q*-node or a P-node, by Lemmas 13, 16, and 17, computing Σ^+_{ν,ρ_i} (if not already available) takes $O(1)$ time. Hence, since the sum of the degrees of the nodes in the tree is $O(n)$, the computation of the non-negative rectilinear spirality sets of all Q*-nodes and P-nodes takes $O(n)$ time, over all visits of T_{ρ_i} $(i = 1, \dots, h)$.

If ν is an S-node, by Lemma 15 the algorithm spends $O(\delta_\nu)$ time to compute the non-negative rectilinear spirality set of ν the first time it visits a tree T_{ρ_i} for which Σ^+_{ν,ρ_i} is non-empty (i.e., the first time the pertinent graph of each child of ν is rectilinear planar), and $O(1)$ to compute the non-negative rectilinear spirality set of ν in the remaining trees for which this set is not already available. Hence, also in this case, the computation of the non-negative rectilinear spirality sets of all S-nodes takes $O(n)$ time, over all visits of T_{ρ_i} $(i = 1, \dots, h)$.

It follows that the testing algorithm takes $O(n)$ time.

Construction algorithm. If the testing is positive, a rectilinear representation of G can be constructed in linear time by the same top-down strategy described in Lemma 9. However, to achieve overall linear-time complexity, we have to show how to efficiently assign target spirality values to the children of an S-node for which it is known its target spirality value. Let ν be an S-node of T_{ρ_i} with children $\mu_1, \dots, \mu_s$ $(j \in \{1, \dots, s\})$ and suppose that $\sigma_\nu \in \Sigma_{\nu,\rho_i}$ is the target value of spirality of ν. We must find a value $\sigma_{\mu_j} \in \Sigma_{\mu_j,\rho_i}$ for each $j = 1, \dots, s$ such that $\sum_{i=1}^{s} \sigma_{\mu_j} = \sigma_\nu$. Let M_j

be the maximum value of $\Sigma^+_{\mu_j,\rho_i}$ for any $j \in \{1,\dots,s\}$. Without loss of generality, we assume that $\sigma_\nu \geq 0$. Indeed, if $\sigma_\nu < 0$ we can find spirality values for the children of ν such that their sum equals $-\sigma_\nu$ and then we can change the sign of each of them.

We initially set $\sigma_{\mu_j} = M_j$ for each $j = 1,\dots,s$ and we consider $\Delta = \sum_{i=1}^{s} \sigma_{\mu_j} - \sigma_\nu$. Clearly, $\Delta \geq 0$. If $\Delta = 0$ we are done; note that, this is always the case when Σ^+_{ν,ρ_i} is a trivial interval. Suppose $\Delta > 0$. If $\Sigma^+_{\nu,\rho_i} = [1,2]^1$, $\Delta = 1$. By Lemma 14, we simply reduce by one unit the value of spirality of the unique child whose non-negative rectilinear spirality set is $[1,2]^1$ (each other child of ν has non-negative rectilinear spirality set $[0]$).

Suppose $\Sigma^+_{\nu,\rho_i} = [1,M]^2$ or $\Sigma^+_{\nu,\rho_i} = [0,M]^2$. We have that Δ is even. By Lemma 14, each child of ν is either jump-2 or trivial. Iterate over all $j = 1,\dots,s$ and for each j decrease both σ_{μ_j} and Δ by the value $\min\{\Delta, 2M_i\}$, which is always even, until $\Delta = 0$.

Finally, suppose $\Sigma^+_{\nu,\rho_i} = [0,M]^1$. By Lemma 14, ν has at least one jump-1 child. First, iterate over all jump-1 children of ν. For any such child μ_j, decrease both σ_{μ_j} and Δ by the value $\min\{\Delta, 2M_i\}$ until either $\Delta = 0$ or all jump-1 children have been considered. Note that this iterative step is not be applicable to a jump-1 child μ_j whose set is $[1,2]^1$ when $\Delta = 2$. In this case, we apply the following strategy:

- If ν has at least one jump-2 child or a child whose set is $[1]$, we decrement the spirality value of this child by two units.
- If a jump-1 child μ_k of ν have been processed before μ_j, reduce by three units the spirality of μ_j and increase by one unit the spirality of μ_k.
- Otherwise, there is at least another jump-1 μ_k that has not yet been processed. We reduce by one unit both the spirality of μ_j and the spirality of μ_k.

If after the procedure above $\Delta = 0$, we are done. Otherwise, $\Delta > 0$. If Δ is even, the desired value of spirality for ν is obtained by decreasing the spirality of the jump-2 children or the trivial children with non-negative spirality set $[1]$ (if any), as done in the previous case. If Δ is odd, we increment by one unit the spirality of an arbitrarily chosen jump-1 child, and then we reach the desired value of spirality for ν by decreasing the spirality of the jump-2 children or the trivial children as before.

With the procedure above, we can process in $O(1)$ time each child of an S-node and thus all S-nodes are processed in $O(n)$ time. □

The next theorem extends the result of Lemma 19 to independent-parallel partial 2-trees that are not necessarily biconnected.

Theorem 4 *Let G be an n-vertex independent-parallel partial 2-tree. There exists an $O(n)$-time algorithm that tests whether G is rectilinear planar, and that computes a rectilinear representation of G if the test is positive.*

Proof: We can design a testing algorithm based on the same strategy as the one in Theorem 1. Namely, Phase 2 of the testing algorithm in Theorem 1, which takes $O(n)$ time, is performed in the same way, without any change. As for Phase 1 (i.e., the pre-processing phase), we need to slightly revise it in order to reduce the computation time from $O(n^2)$ to $O(n)$. More precisely:

- Step 1 is performed exactly as described in Theorem 1. It takes $O(n)$ time. We recall that this step enhances each block B_j of G with a gadget for each cutvertex that requires a reflex-angle constraint or an external reflex-angle constraint. It can be seen that, for each block B_j, the block B'_j obtained from B_j after the addition of the gadgets remains an independent-parallel SP-graph.

- Step 2 is performed by applying on each block B'_j the testing algorithm of Lemma 19, which takes in total $O(n)$ time. Notice that, differently from Theorem 1, in this step each spirality set is succinctly described in $O(1)$ space (see Theorem 3), hence we do not explicitly store the values of the leftmost and rightmost external angles that can be assigned to each pole of a component for each admitted value of spirality.

- Step 3 has to be revised to lower its complexity from $O(n^2)$ to $O(n)$. Namely, as in the proof of Theorem 1, for each distinct configuration of the cutvertex-nodes incident to a block-node $\beta(B_j)$ of the BC-tree, we decide its corresponding Boolean local label, based on the output of the previous step and on whether the configuration requires an external angle constraint at a cutvertex of B_j or not. If the configuration is such that all cutvertex-nodes incident to $\beta(B_j)$ are children of $\beta(B_j)$ (which models the case when $\beta(B_j)$ is the root of the BC-tree), there is no external angle constraints on the cutvertices of B_j, hence the local label is true if and only if B'_j was rectilinear planar in Step 2. Consider vice versa a configuration such that $\chi(c)$ is the parent of $\beta(B_j)$, for a cutvertex c in B_j. If B'_j was not rectilinear planar in Step 2, the local label for the configuration is false. However, if B'_j was rectilinear planar in Step 2, we must check whether it remains rectilinear planar with the additional external angle-constraint on c. If there is an external reflex-angle-constraint or a non-right-angle constraint on c, we use a strategy similar to the proof in Theorem 1, while we adopt a different argument for handling an external flat-angle constraint on c. More precisely.

  (*i*) If there is an external reflex-angle-constraint on c, similar to the proof of Theorem 1, we just consider the output of the testing algorithm of Step 2 restricted to the SPQ*-tree of B'_j whose reference chain is the path of length four of the reflex-angle gadget for c. The local label is set to true if and only if the test for this rooted tree was positive, as it equals to say that B_j is rectilinear planar with c on the external face and with a reflex angle on the external face. This takes $O(1)$ time for the given configuration, and therefore $O(n)$ over all configurations of the cutvertex-nodes incident to B_j.

  (*ii*) If there is an external non-right-angle constraint on c, we know that $\deg(c|B_j) = 2$. Similarly to Theorem 4, we restrict the output of the testing algorithm of Step 2 to the only root ρ of the SPQ*-tree whose reference chain π contains c. Denote by ℓ the length of π and let s and t be the two poles of π. Since c is not allowed to have a $90°$ angle on the external face, the spirality σ_ρ is restricted to take values in the range $[-(\ell-1),(\ell-2)]$, instead of $[-(\ell-1),(\ell-1)]$ ($\sigma_\rho = (\ell-1)$ corresponds to having a $90°$ angle on the external face at all degree-2 vertices of π). Hence, for each candidate value of spirality of ρ in the interval $[-(\ell-1),(\ell-2)]$ we check in $O(1)$ time whether there is a value $\sigma_\nu \in \Sigma_{\nu,\rho}$ such that $\sigma_\nu - \sigma_\rho = 4$. In the positive case, we set the local label for the configuration to true, otherwise we set this label to false. This test takes $O(\ell)$ time for the given configuration. Since the sum of the length of all possible reference chains for B'_j is $O(n_{B_j})$, the procedure takes $O(n_{B_j})$ over all configurations of the cutvertex-nodes incident to B_j.

  (*iii*) Finally, suppose that there is an external flat-angle constraint on c, which implies that $\deg(c|B'_j) = 3$. We have to check whether B'_j remains rectilinear planar when we choose as reference chain of the SPQ*-tree one of the tree chains incident to c, requiring that

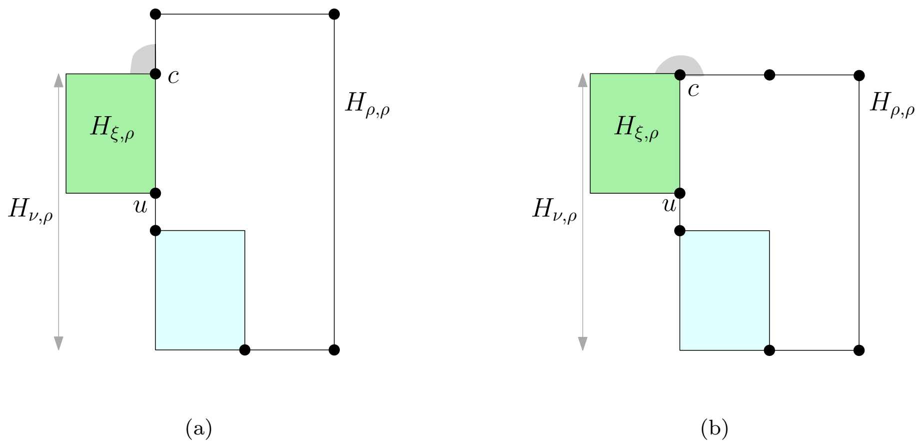


Figure 12: Transformation that guarantees a flat external angle (shaded) at c.

the external angle at c is larger than 90°. Let π be any of the three reference chains incident to c in B'_j and let ρ be the Q*-node corresponding to π. Note that, vertex c is necessarily a pole of a P-node ξ with two children. Also, denoted by ν the root child, we have that either ξ coincide with ν or ξ is a child of ν. There are two cases.

Case 1: π is not a single edge, i.e., the length of π is at least two. In this case we claim that B'_j is rectilinear planar without any additional constraint on c if and only if B'_j is rectilinear planar with the external flat-angle constraint on c. Indeed, assume that B'_j is rectilinear planar without constraints on c. As we proved in Lemma 18 there exists an unconstrained rectilinear representation H of B'_j such that, denoted by σ'_ν the spirality of $H_{\nu,\rho}$ in H and by σ'_ρ the spirality of $H_{\rho,\rho}$ in H, we have $\sigma'_\rho \in \Sigma^+_{\rho,\rho}$, $\sigma'_\nu \in \Sigma^+_{\nu,\rho}$, and $\sigma'_\nu + \sigma'_\rho = 4$. Also, by Lemma 11, we can assume that the internal angle at c in the parallel-component $H_{\xi,\rho}$ is a 90° angle. Namely, let u and c be the two poles of $H_{\xi,\rho}$. By Lemma 11, we can exclude that both the internal angles at u and c in $H_{\xi,\rho}$ are flat angles and, if the internal angle at c is a flat angle and the internal angle at u is a right angle, we can transform the representation by exchanging the values of the internal angles at u and c without changing the spirality of $H_{\xi,\rho}$ (recall that both u and c have degree three, as ξ is a P-node with two children and B'_j is independent-parallel SP). If the external angle at c in H is a flat angle then we are done. Otherwise, we can easily transform H into another rectilinear representation of B'_j with a flat angle at c on the external face by simply increasing the spirality of $H_{\xi,\rho}$ (and therefore of $H_{\nu,\rho}$) by one unit and decreasing the spirality of $H_{\rho,\rho}$ by one unit (see Fig. 12). This is always possible, as $\sigma'_\rho \geq 0$ and π is not a single edge.

Case 2: π is a single edge. Since $\sigma_\rho = 0$ is the only spirality value admitted by ρ, we have to check whether ν admits spirality $\sigma_\nu = 4$ when we impose that the external angle at c is flat. With the notation used in the statement of Lemma 3, we require that $\alpha^l_c = 0$.

– Suppose first that ν coincides with ξ (i.e., ξ is the root child). In this case, we can perform the test in $O(1)$ time by considering the $O(1)$ possible configurations of ν that satisfy the relationships of Lemma 3, and by checking whether there is at least one configuration such that $\sigma_\nu = 4$ and $\alpha^l_c = 0$.

– Suppose vice versa that ν and ξ do not coincide. In this case, ν is an S-node, which represents the series composition of the pertinent graph of ξ with the union of all pertinent graphs of the siblings of ξ. For the sake of simplicity, denote by $\hat{\nu}$ a dummy S-node that represents the union of the pertinent graphs of the siblings of ξ. Also denote by $\Sigma^+_{\hat{\nu},\rho}$ the set of non-negative spirality values admitted by $\hat{\nu}$. With the same approach as in the proof of Lemma 15, we can compute $\Sigma^+_{\hat{\nu},\rho}$ in $O(1)$ time by removing the contribution of $\Sigma^+_{\xi,\rho}$ from $\Sigma^+_{\nu,\rho}$. Let $\hat{M}$ be the maximum value of spirality in $\Sigma^+_{\hat{\nu},\rho}$.

Assume first that $\hat{M} \le 4$. For each candidate spirality value $\sigma_{\hat{\nu}} \in [-4, 4]$, we can check whether there exists a value $\sigma_\xi \in \Sigma_{\xi,\rho}$ such that $\sigma'_\xi + \sigma'_{\hat{\nu}} = 4$ and $\alpha^l_c = 0$. This can be done in $O(1)$ time through the relationships of Lemma 3.

Assume now that $\hat{M} > 4$. In this case we can restrict to test whether one of these three following configurations of spirality values σ_ξ for ξ and $\sigma_{\hat{\nu}}$ for $\hat{\nu}$ holds: (a) $\sigma_\xi = 1$ and $\sigma_{\hat{\nu}} = 3$; (b) $\sigma_\xi = 0$ and $\sigma_{\hat{\nu}} = 4$; (c) $\sigma_\xi = 2$ and $\sigma_{\hat{\nu}} = 2$. Indeed, suppose that there exists a rectilinear representation of G with $\alpha^l_c = 0$ and with spirality values σ'_ξ and $\sigma'_{\hat{\nu}}$ for ξ and $\hat{\nu}$, respectively. If σ'_ξ and $\sigma'_{\hat{\nu}}$ are odd, then, by Lemma 10, and since $\hat{M} > 4$, then ξ and $\hat{\nu}$ also admit two values σ_ξ and $\sigma_{\hat{\nu}}$ that satisfy configuration (a). If σ'_ξ and $\sigma'_{\hat{\nu}}$ are both even and if ν admits spirality 0, then ξ and $\hat{\nu}$ also admit two values σ_ξ and $\sigma_{\hat{\nu}}$ that satisfy configuration (b). Otherwise, σ'_ξ and $\sigma'_{\hat{\nu}}$ are both even and $\Sigma^+_{\xi,\rho} = [1,2]^1$; this means that we are already in configuration (c) with $\sigma_\xi = \sigma'_\xi = 2$ and $\sigma_{\hat{\nu}} = \sigma'_{\hat{\nu}} = 2$. Since we can test in constant time whether one of the three possible configurations (a), (b), and (c) holds, also the case $\hat{M} > 4$ is handled in $O(1)$ time.

If the test is positive, the construction algorithm is exactly the same as in Theorem 1, which takes $O(n)$ time. □

5 Final Remarks

We proved that rectilinear planarity can be tested in $O(n^2)$ time for general partial 2-trees and in $O(n)$ time for independent-parallel SP-graphs. Establishing a tight bound on the complexity of rectilinear planarity testing algorithm for partial 2-trees remains an open problem. A pitfall to achieve $O(n)$-time complexity in the general case is that, in contrast with the independent-parallel SP-graphs, the spirality set of a component may not exhibit a regular behavior. See for example Figs. 13 and 14. In particular, the component in Fig. 13 is a series-composition of two parallel-components, each having only one possible rectilinear representation, up to a rotation/flipping. To construct the set of spirality values admitted by a rectilinear representation of the whole component, one can use the relationship of Lemma 1, which establishes that the spirality value of a series-composition is the sum of the spirality values of its children. A similar analysis can be done for the component in Fig. 14.

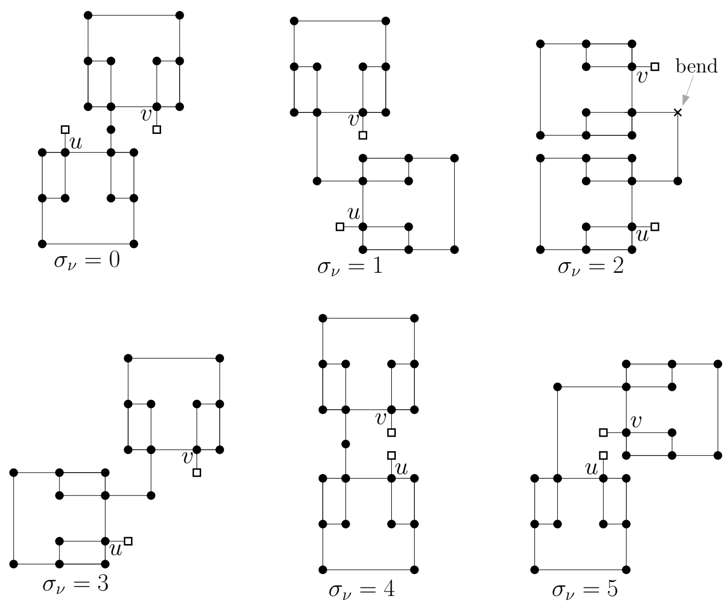


Figure 13: Component that admits non-negative spiralities 0, 1, 3, 4, 5. Spirality 2 needs a bend (×).

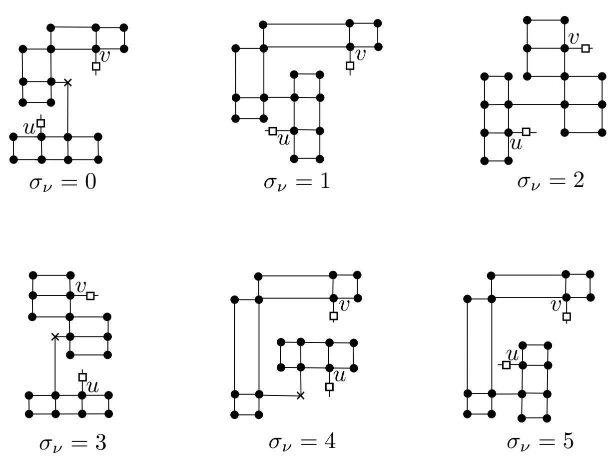


Figure 14: Component that admits non-negative spiralities 1, 2, 5; other values require a bend (×).

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