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Small Point-Sets Supporting Graph Stories

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Abstract. In a graph story the vertices enter a graph one at a time and each vertex persists in the graph for a fixed amount of time ω, called viewing window. At any time, the user can see only the drawing of the graph induced by the vertices in the viewing window and this determines a sequence of drawings. For readability, we require that all the drawings of the sequence are planar. For preserving the user's mental map we require that when a vertex or an edge is drawn, it has the same drawing for its entire life. We study the problem of computing the whole sequence of drawings by mapping the vertices only to $\omega + k$ given points, where k is as small as possible. We show that: (i) The problem does not depend on the specific set of points but only on the size of the point set; (ii) the problem is NP-hard (even when k is a given constant) and it is FPT when parameterized by $\omega + k$; (iii) for $k = 0$ there are families of graph stories that can be drawn independent of ω, but also families that cannot be drawn even when ω is small; (iv) there are families of graph stories that cannot be drawn for any fixed k and families of graph stories that can be realized only when k is larger than a certain threshold.

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1 Introduction

In this paper we address "graph stories", a model introduced by Borrazzo et al. in [7] as a framework for exploring temporal data. In a *graph story* the vertices enter a graph one at a time and persist in the graph for a fixed amount of time ω, called the *size of the viewing window*. At any time, the user can see only the drawing of the graph induced by the vertices in the viewing window and this determines a sequence of drawings. This problem is inspired by applications where data are produced in streaming and need to be visualized by time frames. For readability, all the drawings of the sequence are required to be planar. For preserving the user's mental map, we require that when a vertex or an edge is drawn, it keeps the same drawing for its entire life. Also, in order to limit the constraints, we allow the edges to be represented as Jordan arcs, rather than requiring that they are drawn as polylines with a limited number of bends.

A different setting has been recently studied in [5], where there is no viewing window and each vertex persists in the graph until all its neighbors have been seen.

Graph stories are related to a rich body of literature devoted to the visualization of dynamic graphs (surveys can be found in [3, 21]). One of the main classification criteria of dynamic graph problems is whether the story is fully known in advance (*offline model*) or if only one event, i.e., the introduction or the removal of a vertex, is known at a time (*online model*). In this respect, our contribution falls in the offline model. A third intermediate category (*look-ahead model*) is when a small chunk of the incoming events is known in advance to the drawing algorithms. The events are also a classification criterion, as they may refer to vertices, edges, or both. Finally, further constraints may regard the timings of the events, the more common being that they occur one at a time at regular intervals and that the incoming objects have a fixed lifetime as in the case of graph stories. In some cases, the order of the events is constrained to correspond to a specific kind of visit of the graph.

Several results focus on dynamic trees. In [4], it is shown how to draw a tree in $O(\omega^3)$ area where the model is online, the incoming objects are edges that arrive in the order of an Eulerian tour of the tree and whose straight-line drawing persists for a fixed lifetime ω. In [12], a small look ahead on the sequence of vertices is used to add one vertex at a time to the drawing of an infinite tree. This is a good trade-off between the readability of the layout and the stability of the part of the graph shared by two consecutive drawings, in the sense that the drawing is perturbed as little as possible during the story. In [27], a sequence of trees (their union, though, may be an arbitrary graph) is completely known in advance; vertices and edges can change their position during the animation and can have an arbitrary lifetime (i.e., they can be removed and introduced). The purpose is to pursue aesthetic criteria commonly adopted for tree drawings which are described, for example, in [24].

Only few papers deal with more complex families of graphs. For instance, in [18] a stream of edges enters the drawing and never leaves it, forming an outerplanar graph that has to be drawn according to an online model, moving the previously drawn vertices by a polylogarithmic distance. In [10] the drawings of several families of graphs are updated as vertices and edges enter and leave the current graph according to the online model.

Only feebly related to our setting is the literature about dynamic planarity [14, 15, 19, 23, 25], where the model is online and the planar embedding of the graph is allowed to change. When the embedding has to be preserved, instead, planarly adding a stream of edges with a fixed lifetime is NP-complete even for the offline model [11]. Also, related to dynamic graph drawing is the use of geometric simultaneous embedding [6, 8], which can be used to model temporal graphs.

Concerning graph stories, Borrazzo et al. [7] address the setting where all the drawings of the story are straight-line and planar, and where vertices do not change their position once drawn. It is shown that graph stories of paths and trees can be drawn on a $2\omega \times 2\omega$ and on an $(8\omega+1) \times (8\omega+1)$ grid, respectively. Further, there exist graph stories of planar graphs that cannot be drawn straight-line within an area that is only a function of ω.

Contribution. We study the problem of drawing a graph story by mapping the vertices only to $\omega+k$ given points, where k is as small as possible. We consider the offline model, in which the whole graph is known in advance, and we assume that the edges are represented as Jordan arcs. We refer to this problem as *realizability testing of graph stories*. The idea of using a limited set of predefined points on which the vertices have to be mapped is inspired by the model in [4]. From a practical point of view, this setting keeps the visual complexity of the drawing low throughout the whole story (as it bounds the number of displayed elements in each time frame) and it limits the candidate positions for a new vertex that enters the drawing, thus helping the user to follow the story with respect to a setting in which many points (or even an infinite number of points) can be used. In particular, when $k = 0$, the next vertex that enters the drawing will occupy the same position as the vertex that leaves the drawing, thus the user can totally predict where the new vertex will appear. Notice that, this scenario may also introduce a readability problem, since distinguishing the vertex that enters from the one leaving the viewing window may be difficult. However, to counteract this issue, one may introduce in the representation an intermediate step where the position of the removed vertex is left empty or where other strategies are used to emphasize the change. From a theoretical perspective, limiting the vertex placement to a predefined set of points can be regarded as an extreme condition in terms of usable resources, and helps to investigate the complexity of the problem before considering a more relaxed scenario. Nonetheless, as it will be shown in the paper, the realizability of a story does not depend on the specific set of points that are chosen, but rather on the number of usable points. Namely, our contribution is as follows:

- We show that the realizability of graph stories is a topological problem, which depends only on the size of the set of points (not on their positions). We also give a characterization of realizable graph stories based on the concept of "compatible embeddings" (Section 3).
- We prove that the realizability of graph stories is an NP-complete problem, even if k is a given constant, and it belongs to FPT when parameterized by the size $\omega + k$ of the point set (Section 4).
- We study the realizability of *minimal* graph stories, i.e., stories for which $k = 0$ (Section 5). In particular, we show that: (i) Every minimal graph story of an outerplanar graph is realizable; (ii) for every $\omega \geq 5$ there exist minimal graph stories of series-parallel graphs that are not realizable; (iii) all minimal graph stories with $\omega \leq 5$ whose graph does not contain K_5 are realizable if we are allowed to redraw at most one edge at each vertex arrival; and (iv) minimal graph stories with $\omega \leq 5$ are always realizable for planar triconnected cubic graphs.
- Finally, we show that there are families of graph stories that are not realizable for any fixed k and families of graph stories that are realizable only when k is larger than a certain threshold (Section 6).

2 Basic Definitions

A *drawing* Γ of a graph $G = (V, E)$ maps each vertex of V to a distinct point of the plane and each edge of E to a Jordan arc connecting its end-vertices; Γ is *planar* if no two edges intersect

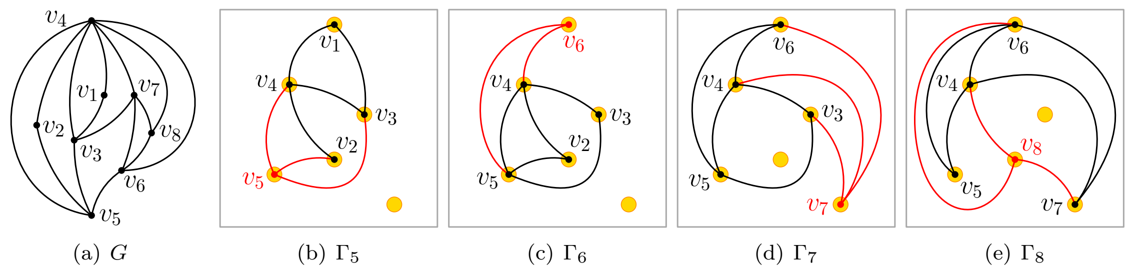


Figure 1: A realization of a graph story $\mathcal{S} = (G, 5, 1, \tau)$ on a set P of 6 points. The points of P are represented as yellow disks. For each Γ_i ($5 \leq i \leq 8$), vertex v_i and its incident edges are red.

except at common endpoints. Given a planar drawing Γ of G, the set of circular orders of the edges incident to each vertex is called a *rotation system*. Drawing Γ subdivides the plane into connected regions called *faces*. The unbounded face of Γ is the *external face*. Walking on the (not necessarily connected) border of a face f of Γ so to keep f to the left determines a set, called the *boundary of f*, of circular lists of alternating vertices and edges. Each list describes a (not necessarily simple) cycle, which can also consist of an isolated vertex: Each edge of G occurs either once in exactly two circular lists of different face boundaries or twice in the circular list of one face boundary.

Two planar drawings of G are *equivalent* if they have the same rotation system, face boundaries, and external face. An equivalence class of planar drawings of G is a *planar embedding of G*. Note that, if G is connected then each face boundary consists of exactly one circular list; in this case an embedding of G is fully specified by its rotation system and by its external face. If G is equipped with a planar embedding ϕ, it is a *plane* graph; a planar drawing Γ of G is *embedding-preserving* if $\Gamma \in \phi$. If G' is a subgraph of G and Γ' is the restriction of Γ to G', the planar embedding ϕ' of Γ' is the *restriction of ϕ to G'*.

Definition 1. A *graph story* is a tuple $\mathcal{S} = (G, \omega, k, \tau)$ where: (*i*) $G = (V, E)$ is an n-vertex graph; (*ii*) $\omega \leq n$ is a positive integer, called the *size of the viewing window*; (*iii*) k is a non-negative integer, called the *number of extra points*; and (*iv*) $\tau = \langle v_1, v_2, \ldots, v_n \rangle$ is a linear ordering of the vertices of G (i.e., $v_i \in V$ is the vertex at position i according to τ).

Let $G_i = (V_i, E_i)$ denote the subgraph of G induced by all vertices v_j such that $\max\{1, i - \omega + 1\} \leq j \leq i$. Observe that, if $i \leq \omega$ then G_i consists of the i vertices $\{v_1, v_2, \ldots, v_i\}$; otherwise G_i consists of the ω vertices $\{v_{i-\omega+1}, v_{i-\omega+2}, \ldots, v_i\}$. In other words, G_i is the subgraph induced by v_i and by the (up to) $\omega - 1$ vertices of G that precede v_i in τ. For each i, we say that v_i *enters* the viewing window at time i, and for each $i \in \{\omega + 1, \ldots, n\}$, we say that $v_{i-\omega}$ *leaves* the viewing window at time i.

Definition 2. A *realization* of a graph story $\mathcal{S} = (G, \omega, k, \tau)$ *on a set* P of $\omega + k$ points is a sequence of drawings $\mathcal{R} = \langle \Gamma_1, \Gamma_2, \ldots, \Gamma_n \rangle$ with the following two properties:

(R1) Γ_i ($1 \leq i \leq n$) is a planar drawing of G_i, where distinct vertices of V_i are mapped to distinct points of P;

(R2) the restrictions of Γ_{i-1} and of Γ_i ($2 \leq i \leq n$) to their common subgraph $G_{i-1} \cap G_i$ are identical.

Observe that the realization can be defined similarly also considering a final part of the story where the vertices fade out, as in [7].

Figure 1 shows a realization of a graph story $\mathcal{S} = (G, 5, 1, \tau)$ on a set of 6 points. A graph story $\mathcal{S}$ is *realizable* if there exists a set P of $\omega + k$ points such that $\mathcal{S}$ admits a realization on P. Since the planarity of all graphs G_i is necessary for realizability, from now on we consider graph stories that satisfy this requirement.

Remark 1 (Edge Visibility)**.** We assume that G only consists of *visible edges*, i.e., edges (v_i, v_j) such that $|i - j| < \omega$. Indeed, if $|i - j| \geq \omega$, (v_i, v_j) can be ignored, as it never appears in a realization. Our assumption has two implications: (i) G has vertex-degree at most $2\omega - 2$ (every G_i has vertex-degree at most $\omega - 1$); and (ii) G has *bandwidth* at most $\omega - 1$ and hence *pathwidth* at most $\omega - 1$ [20] (the set of bags of this decomposition is $\{V_1, V_2, \dots, V_n\}$).

Remark 2 (Minimality)**.** Clearly, if a graph story $\mathcal{S} = (G, \omega, k, \tau)$ is realizable, every other story $\mathcal{S}' = (G, \omega, k', \tau)$ with $k' > k$ is realizable too. Hence, a natural scenario is when the number of extra points k is zero. We call such a story *minimal* and we denote it as $\mathcal{S} = (G, \omega, \tau)$. For a minimal graph story, Property R2 of Definition 2 implies that each vertex v_i with $\omega + 1 \leq i \leq n$ is mapped to the same point as $v_{i-\omega}$, thus the mapping of the whole realization is fully determined by the mapping of Γ_ω (i.e., of the first ω drawings of the realization).

3 Geometry and Topology of Graph Stories

The following lemma shows that the realizability of graph stories is in essence more a topological problem than a geometric problem (recall that edges are represented as Jordan arcs).

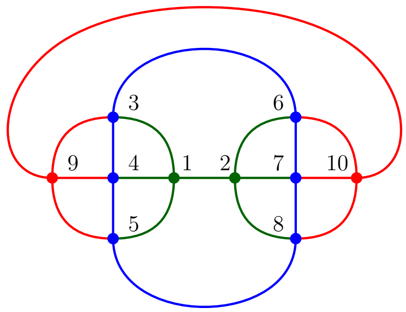
Lemma 1. A graph story $\mathcal{S} = (G, \omega, k, \tau)$ is realizable on a set of points P, with $|P| = \omega + k$, if and only if it is realizable on any set of points P' with $|P'| = |P|$.

Proof: Let $\mathcal{R} = \langle \Gamma_1, \Gamma_2, \dots, \Gamma_n \rangle$ be a realization of $\mathcal{S}$ on $P = \{p_1, p_2, \dots, p_{\omega+k}\}$. We show how to construct a realization $\mathcal{R}'$ of $\mathcal{S}$ on a given arbitrary set of points P' starting from $\mathcal{R}$. The realization $\mathcal{R}$ implicitly defines a function $\rho(\cdot)$ that for each edge e of G gives the Jordan arc $\rho(e)$ used to represent e. Let J be the codomain of ρ, i.e., the set of Jordan arcs used by $\mathcal{R}$. We remark that J is a set and not a multiset, in the sense that if two edges e and e' of G are mapped to the same Jordan arc, then the curve $\rho(e) = \rho(e')$ is present in J only once. Without loss of generality, we may assume that any two Jordan arcs c and c' of J have a finite intersection, i.e., either they do not cross, or they cross on a finite number of points. In fact, if two curves c and c' shared a portion of a curve, we could perturb one of them, say c', so that c' is drawn at an arbitrarily small distance from c until it crosses c or diverges from it.

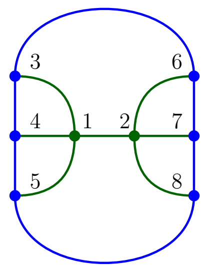
Starting from P and J, we construct a multigraph $\mathcal{M}$: For each point $p_i \in P$, with $i = 1, 2, \dots, \omega + k$, $\mathcal{M}$ has a vertex v_i. For each Jordan arc $c \in J$ with endpoints p_i and p_j, $\mathcal{M}$ has an edge between v_i and v_j. Observe that the Jordan arcs in J also provide a (non-planar) drawing $\Gamma(\mathcal{M})$ of $\mathcal{M}$, where each pair of edges crosses a finite number of times. We planarize $\mathcal{M}$ according to $\Gamma(\mathcal{M})$ by replacing crossings with dummy vertices. Further, we subdivide multiple edges of $\mathcal{M}$ with degree-2 dummy vertices in order to obtain a plane graph $\mathcal{G}$.

We draw $\mathcal{G}$ by preserving its planar embedding on the set of points P' plus an arbitrary set of additional points to host the planarization and subdivision dummy vertices; to this purpose we could use one of the algorithms described in [2, 22]. Let $\Gamma(\mathcal{G})$ be the obtained planar drawing of $\mathcal{G}$.

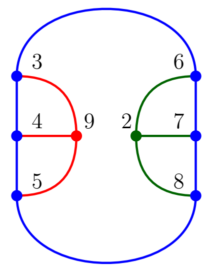
Now observe that a vertex v of G corresponds to a point of P, which in turn is associated with a vertex of $\mathcal{M}$ and with a vertex of $\mathcal{G}$, which is drawn on a point of P'. Also, an edge e of G



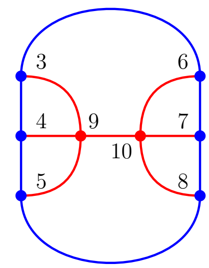
(a) Graph G with unique embedding ϕ



(b) Drawing of G_8



(c) Drawing of G_9



(d) Drawing of G_{10}

Figure 2: An illustration for Lemma 2, which states that there exists a realizable minimal graph story of a planar graph not admitting a supporting embedding. Observe that ϕ_8 is the restriction of ϕ to G_8, while ϕ_9 and ϕ_{10} are not the restrictions of ϕ to G_9 and G_{10}, respectively.

corresponds to a Jordan arc $\rho(e)$ in J, which in turn corresponds to an edge of $\mathcal{M}$ and to a simple path π in $\mathcal{G}$ between the two points of P' corresponding to the endpoints of e. Hence, we can define a function $\rho'(e)$ which gives, for each edge e of G, a Jordan arc that is the concatenation of the curves used in $\Gamma(\mathcal{G})$ to draw the path π.

Finally, observe that the Jordan arcs $\rho(e)$ and $\rho(e')$ of two edges e and e' of G cross if and only if the corresponding paths in $\mathcal{G}$ share an intermediate vertex and, hence, if and only if $\rho'(e)$ and $\rho'(e')$ cross. $\square$

Supporting embeddings. It is natural to ask whether for every realizable graph story where G is planar, there exists a planar embedding of G such that each drawing of the realization preserves this embedding. We formalize this concept and show that this is not always the case. Let $\mathcal{S}$ be a story whose graph G is planar. A *supporting embedding* for $\mathcal{S}$ is a planar embedding ϕ of G such that $\mathcal{S}$ admits a realization $\langle\Gamma_1,\ldots,\Gamma_n\rangle$ where the embedding of Γ_i is the restriction of ϕ to G_i $(i=1,\ldots,n)$.

Lemma 2. There exists a minimal graph story $\mathcal{S}=(G,\omega,\tau)$ such that: (i) G is planar; (ii) $\mathcal{S}$ is realizable; and (iii) $\mathcal{S}$ does not admit a supporting embedding.

Proof: We produce a minimal graph story $\mathcal{S}=(G,\omega,\tau)$ such that G admits a single planar embedding ϕ (up to a flip and up to the choice of the external face) and such that in any realization of $\mathcal{S}$ there is at least one embedding ϕ_i of G_i that is not the restriction of ϕ to G_i. Consider the graph story $\mathcal{S}=(G,8,\tau)$ where G is the graph depicted in Figure 2(a) and τ is given by the indices of the vertices of G. Observe that G is triconnected, and choose, without loss of generality, the embedding depicted in Figure 2(a) for the realization of $\mathcal{S}$. The restriction of such an embedding to G_8 provides the embedding ϕ_8 of G_8. A drawing according to ϕ_8 of G_8 is depicted in Figure 2(b). The drawings of G_9 and G_{10} are obtained by deleting v_1 and v_2, respectively, and adding v_9 and v_{10} in the positions where v_1 and v_2 are drawn in the drawing of G_8. Figures 2(c) and 2(d) show the obtained embeddings ϕ_9 and ϕ_{10}. It can be easily seen that both ϕ_9 and ϕ_{10} are not the restriction of ϕ to G_9 and G_{10}, respectively. $\square$

Characterizing realizable graph stories. We now give a characterization of a realizable graph story $\mathcal{S}=(G,\omega,k,\tau)$ in terms of a sequence of "compatible embeddings". To this aim, we give a generalization of the definition of planar embedding that associates with each face a weight representing how many of k notable points are inside such a face.

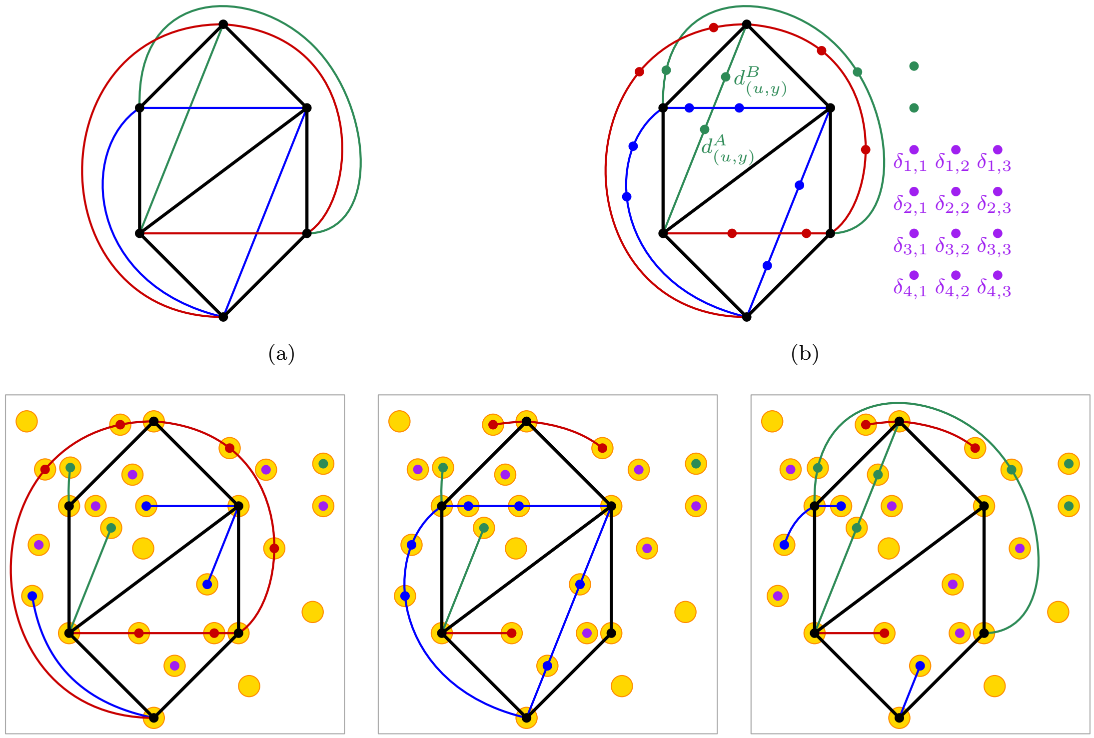
A *face-k-weighted planar embedding ϕ of a planar graph H* is a planar embedding of H together with a non-negative integer, called *weight*, for each face of ϕ such that the sum of all weights is k. The *removal of a vertex v* from a face-k-weighted planar embedding ϕ of H produces a face-$(k+1)$-weighted planar embedding ϕ^{-v} of $H \setminus v$ such that the planar embedding of ϕ^{-v} is the restriction of the planar embedding of ϕ to $H \setminus v$ and the weights of the faces are changed as follows: (i) all the faces in common between ϕ and ϕ^{-v} have the same weight in ϕ^{-v} as in ϕ, and (ii) the new face of ϕ^{-v} resulting by the removal of v has a weight that is one plus the sum of the weights of the faces of ϕ incident to v.

Let $\mathcal{S}$ be a graph story and let ϕ_i be a face-k-weighted planar embedding of G_i, for $i \in \{\omega, \dots, n\}$. Two face-$k$-weighted planar embeddings ϕ_{i-1} and ϕ_i, with $i = \omega+1, \dots, n$, are *compatible* if removing $v_{i-\omega}$ from ϕ_{i-1} produces the same face-$(k+1)$-weighted planar embedding of $G_{i-1} \cap G_i$ as removing v_i from ϕ_i.

Lemma 3. A graph story $\mathcal{S} = (G, \omega, k, \tau)$ is realizable if and only if there exists a sequence $\langle \phi_\omega, \phi_{\omega+1}, \dots, \phi_n \rangle$ of face-k-weighted planar embeddings for the graphs $\langle G_\omega, G_{\omega+1}, \dots, G_n \rangle$, such that ϕ_{i-1} and ϕ_i are compatible ($\omega + 1 \leq i \leq n$).

Proof: Suppose first that $\mathcal{S}$ is realizable and let $\mathcal{R} = \langle \Gamma_1, \dots, \Gamma_n \rangle$ be a realization of $\mathcal{S}$. Consider two consecutive drawings Γ_{i-1} and Γ_i of G_{i-1} and G_i, respectively, for $\omega+1 \leq i \leq n$. Define a face-k-weighted planar embedding ϕ_{i-1} of G_{i-1} (ϕ_i of G_i, respectively), where the planar embedding is that of Γ_i (Γ_{i+1}, respectively) and the weight of each face is the number of unused points that the face contains in the drawing Γ_i (Γ_{i+1}, respectively). By Property R2 of Definition 2, the restrictions of Γ_{i-1} and Γ_i to $G_{i-1} \cap G_i$ is the same drawing, comprehensive of the positions of the $k+1$ unused points. We use the $k+1$ unused points of $G_{i-1} \cap G_i$ to define a face-$(k+1)$-weighted planar embedding $\phi^\cap$ of $G_{i-1} \cap G_i$, where the planar embedding is the one of $\Gamma_i \cap \Gamma_{i+1}$ and the weight of each face is the number of unused points it contains in the drawing $\Gamma_i \cap \Gamma_{i+1}$. It is immediate to see that removing vertex $v_{i-\omega}$ from ϕ_{i-1} as well as removing vertex v_i from ϕ_i produces in both cases $\phi^\cap$, i.e., ϕ_{i-1} and ϕ_i are compatible.

Suppose vice versa that there exists a sequence of face-k-weighted planar embeddings $\langle \phi_\omega, \phi_{\omega+1}, \dots, \phi_n \rangle$ such that any two consecutive face-k-weighted planar embeddings in the sequence are compatible. Let Γ_ω be any planar drawing of G_ω and let P be the set of points of Γ_ω corresponding to the vertices of G_ω plus k unused points arbitrarily distributed inside the faces of Γ_ω according to the weights of ϕ_ω. For each $i = 1, \dots, \omega - 1$, define Γ_i as the restriction of Γ_ω to G_i. For each $i = \omega + 1, \dots, n$, by the compatibility of ϕ_i with ϕ_{i-1}, we have that removing vertex $v_{i-\omega}$ from Γ_i yields a drawing $\Gamma^\cap$ of $G^\cap = G_{i-1} \cap G_i$ that has the same face-$(k+1)$-weighted embedding $\phi^\cap = \phi_{i-1} \setminus v_{i-\omega} = \phi_i \setminus v_i$ of $G^\cap$. We construct Γ_i from $\Gamma^\cap$ by inserting v_i inside the face of $\Gamma^\cap$ corresponding to the face of $\phi^\cap$ that is generated by the removal of v_i from ϕ_i. Also, we planarly insert each edge connecting v_i to each of its neighbors according to embedding ϕ_i, without changing the starting drawing and by leaving on each generated face f the number of unused points that corresponds to the weight of f in ϕ_i (for example using the technique in [9], where the unused points are regarded as isolated vertices). The sequence $\langle \Gamma_1, \Gamma_2, \dots, \Gamma_n \rangle$ satisfies Properties R1 and R2, i.e., it is a realization of $\mathcal{S}$ on P. □



(a) (b)

(c) Γ_ω (d) $\Gamma_{\omega+2\tilde{\omega}}$ (e) $\Gamma_{\omega+4\tilde{\omega}}$

Figure 3: (a) A drawing Γ' of a `yes` instance $\{G'_1, G'_2, G'_3\}$ of Sunflower SEFE. The edges of $E'_\cap$ are black; the edges of E'_1, E'_2, and E'_3 are red, blue, and green, respectively. (b) A drawing of graph G of the story $\mathcal{S} = (G, \omega, 4, \tau)$ constructed from the instance of Figure 3(a). The vertices of D^A_1 (D^A_2, D^A_3, resp.) and D^B_1 (D^B_2, D^B_3, resp.) are red (blue, green, resp.); the vertices of Δ_i ($1 \leq i \leq 4$) are purple. (c) Γ_ω. (d) $\Gamma_{\omega+2\tilde{\omega}}$. (e) $\Gamma_{\omega+4\tilde{\omega}}$. The points of P are represented as yellow disks.

4 Realizability Testing of Graph Stories

We first prove that testing whether a graph story is realizable is NP-hard for any given integer $k \geq 0$ (Theorem 1). Then we prove the that the problem is in FPT when parameterized by $\sigma = \omega + k$ (Theorem 2). Note that, by Theorem 1 we can exclude that the problem is in FPT when parameterized by k only, unless $\mathsf{P} = \mathsf{NP}$.

Theorem 1. For any integer $k \geq 0$, testing the realizability of a graph story $\mathcal{S} = (G, \omega, k, \tau)$ is NP-hard.

Proof: We use a reduction from the Sunflower SEFE problem, which is defined as follows. Let $G'_1, G'_2, \ldots, G'_l$ be graphs having the same vertex-set V' such that each edge in the union of all graphs belongs either to only one of the input graphs or to all the input graphs. The Sunflower SEFE problem asks whether there exist l planar drawings $\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_l$ of $G'_1, G'_2, \ldots, G'_l$, respectively, such that: (i) each vertex of V' is mapped to the same point in every drawing Γ'_i ($1 \leq i \leq l$);

(ii) each edge that is common to all the input graphs is represented by the same simple curve in the drawings of all such graphs. In other words, the problem asks whether there exists a drawing Γ' of $G'_1 \cup G'_2 \cup \cdots \cup G'_l$ such that two edges cross only if they do not belong to the same graph G'_i. We remark that the SUNFLOWER SEFE problem is a variant of the SEFE problem, which allows an edge to belong to an arbitrary number of input graphs. Both the SEFE and the SUNFLOWER SEFE problems are NP-complete for $l \geq 3$ [1, 17, 26].

Starting from an instance of the SUNFLOWER SEFE problem with $l = 3$, we construct a non-minimal graph story $\mathcal{S} = (G = (V, E), \omega, k, \tau)$ as follows; refer to Figure 3 for an example with $k = 4$. Let G'_1, G'_2, and G'_3 be the input graphs of SUNFLOWER SEFE having vertex-set V', let E'_i be the set of edges that belong only to graph G'_i $(1 \leq i \leq 3)$, and let $E'_\cap$ be the set of edges that belong to all the input graphs. Without loss of generality, we can assume that $|E'_1| \geq |E'_2| \geq |E'_3|$.

We now show how we define sets V and E. For every graph G'_i $(1 \leq i \leq 3)$, we subdivide each edge e of E'_i with two vertices d^A_e and d^B_e and we add them to two sets D^A_i and D^B_i, respectively; see, e.g., $d^A_{(u,y)}$ and $d^B_{(u,y)}$ in Figure 3(b). We add the three edges obtained by subdividing e to a set E''_i. If needed, we enrich sets D^A_2, D^B_2, D^A_3, and D^B_3 with isolated vertices so that all the sets D^X_i have the same cardinality $\tilde{\omega} = |E'_1|$ (note that $\tilde{\omega} = |D^A_1| = |D^B_1|$), with $1 \leq i \leq 3$ and $X \in \{A, B\}$; see, e.g., the green isolated vertices in Figure 3(b). Also, we create four sets Δ_j $(1 \leq j \leq 4)$ of $\tilde{\omega}$ isolated vertices $\delta_{j,1}, \ldots, \delta_{j,\tilde{\omega}}$; see, e.g., the purple isolated vertices in Figure 3(b). We define the set V of vertices of G as $V = V' \cup D^A_1 \cup D^B_1 \cup D^A_2 \cup D^B_2 \cup D^A_3 \cup D^B_3 \cup \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$ and the set E of edges of G as $E = E'_\cap \cup E''_1 \cup E''_2 \cup E''_3$. We set the size ω of the viewing window as $\omega = |V'| + 6\tilde{\omega}$. Finally, we suitably define τ in such a way that the vertices of the various subsets of V appear in the following order: $\langle D^A_1, \Delta_1, D^A_2, \Delta_2, D^A_3, V', D^B_1, \Delta_3, D^B_2, \Delta_4, D^B_3 \rangle$.

Observe that $\mathcal{S}$ can be constructed in $O(|V'|)$ time and that $\omega \in O(|V'|)$. We now show that $\mathcal{S}$ is realizable if and only if the triplet $\{G'_1, G'_2, G'_3\}$ is a `yes` instance of SUNFLOWER SEFE. Refer to Figures 3(c) to 3(e) for an example.

$(\Rightarrow)$ Let P be a set of $\omega + k$ points. If $\mathcal{S}$ admits a realization on P, there exists a sequence of drawings $\langle \Gamma_1, \Gamma_2, \ldots, \Gamma_{|V|} \rangle$ that satisfies Properties R1 and R2.

Drawing Γ_ω is induced by all vertices of D^A_1, Δ_1, D^A_2, Δ_2, D^A_3, V', and D^B_1. This drawing is crossing-free by Property R1 and its restriction to the vertices of V', D^A_1, and D^B_1 is a subdivision of a drawing of G'_1. Similarly, drawing $\Gamma_{\omega+2\tilde{\omega}}$ is induced by all vertices of D^A_2, Δ_2, D^A_3, V', D^B_1, Δ_3, and D^B_2; its restriction to the vertices of V', D^A_2, and D^B_2 is a subdivision of a (planar) drawing of G'_2. Finally, drawing $\Gamma_{\omega+4\tilde{\omega}} (= \Gamma_{|V|})$ is induced by all vertices of D^A_3, V', D^B_1, Δ_3, D^B_2, Δ_4, and D^B_3; its restriction to the vertices of V', D^A_3, and D^B_3 is a subdivision of a (planar) drawing of G'_3.

By Property R2, Γ_ω, $\Gamma_{\omega+2\tilde{\omega}}$, and $\Gamma_{\omega+4\tilde{\omega}}$ are such that their restrictions to their common subgraph are identical. Since V' belongs to all these drawings, the edges in $E'_\cap$ are drawn identically in Γ_ω, $\Gamma_{\omega+2\tilde{\omega}}$, and $\Gamma_{\omega+4\tilde{\omega}}$.

$(\Leftarrow)$ If the triplet $\{G'_1, G'_2, G'_3\}$ is a `yes` instance of SUNFLOWER SEFE, there exists a drawing Γ' of $G'_1 \cup G'_2 \cup G'_3$ such that two edges cross only if they do not belong to the same graph G'_i $(1 \leq i \leq 3)$.

The set P that we use for the realization of $\mathcal{S}$ is the union of the following sets: (i) A set P' consisting of the points of Γ' to which the vertices of V' are mapped, thus $|P'| = |V'|$; (ii) a set P^A_i (P^B_i, resp.) consisting of an arbitrarily chosen point p^A_e ($p^B_e \neq p^A_e$, resp.) for each edge e of E'_i $(1 \leq i \leq 3)$, such that p^A_e (p^B_e, resp.) belongs to the simple curve representing e in Γ', thus $|P^A_i| = |E'_i|$ ($|P^B_i| = |E'_i|$, resp.); (iii) a set P_d consisting of $2(|E'_1| - |E'_2|) + 2(|E'_1| - |E'_3|)$ additional points; (iv) a set P_k of k additional points. Note that, $|P| = |V'| + 6|E'_1| + k = |V'| + 6\tilde{\omega} + k = \omega + k$. We now prove that $\mathcal{S}$ admits a realization on P.

Recall that $V = V' \cup D^A_1 \cup D^B_1 \cup D^A_2 \cup D^B_2 \cup D^A_3 \cup D^B_3 \cup \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$ and that τ is such that the vertices appear in the following order: $\langle D^A_1, \Delta_1, D^A_2, \Delta_2, D^A_3, V', D^B_1, \Delta_3, D^B_2, \Delta_4, D^B_3 \rangle$.

Observe that, by construction, each non-isolated vertex of D_i^X can be uniquely associated with a distinct point of P_i^X, where $1 \leq i \leq 3$ and $X \in \{A, B\}$. Namely, for each edge $e \in E_i'$, we associate vertex d_e^A to p_e^A and vertex d_e^B to p_e^B. Drawing Γ_ω is such that:

- Each vertex of D_1^A is mapped to the corresponding point of P_1^A;
- the vertices of Δ_1 are distributed on all the points of P_2^B and on $|E_1'| - |E_2'|$ points of P_d;
- each vertex of D_2^A is mapped to the corresponding point of P_2^A;
- the vertices of Δ_2 are distributed on all the points of P_3^B and on $|E_1'| - |E_3'|$ points of P_d;
- each vertex of D_3^A is mapped to the corresponding point of P_3^A;
- the vertices of V' are mapped on the points of P';
- each vertex of D_1^B is mapped to the corresponding point of P_1^B.

The edges of Γ_ω are the portions of the edges of Γ' between points of P to which the vertices of Γ_ω are mapped. Γ_ω is planar, since it is a subdivision of the planar subgraph of Γ' induced by the edges of $E_1' \cup E_\cap'$, plus some degree-1 vertices (the ones in D_2^A and D_3^A), plus the isolated vertices in Δ_1 and Δ_2. Also, for $j = 1, 2, \ldots, \omega - 1$, define Γ_j as the restriction of Γ_ω to G_j. This implies that the sequence $\langle \Gamma_1, \Gamma_2, \ldots, \Gamma_\omega \rangle$ satisfies Properties R1 and R2.

Drawing $\Gamma_{\omega+\tilde{\omega}}$ is obtained by replacing each vertex of D_1^A with the isolated vertices of Δ_3. The edges of $\Gamma_{\omega+\tilde{\omega}}$ are the portions of the edges of Γ' between points of P to which the vertices of $\Gamma_{\omega+\tilde{\omega}}$ are mapped. It is easy to see that Properties R1 and R2 are satisfied by the sequence $\langle \Gamma_\omega, \Gamma_{\omega+1}, \ldots, \Gamma_{\omega+\tilde{\omega}} \rangle$.

In the drawing $\Gamma_{\omega+2\tilde{\omega}}$, the vertices of D_2^B replace the isolated vertices of Δ_1, which were distributed on the points of P_2^B and on $|E_1'| - |E_2'|$ points of P_d. The edges of $\Gamma_{\omega+2\tilde{\omega}}$ are the portions of the edges of Γ' between points of P to which the vertices of $\Gamma_{\omega+2\tilde{\omega}}$ are mapped. $\Gamma_{\omega+2\tilde{\omega}}$ is planar, since it is a subdivision of the planar subgraph of Γ' induced by the edges of $E_2' \cup E_\cap'$, plus some degree-1 vertices (the ones in D_1^B and D_3^A), plus the isolated vertices in Δ_2 and Δ_3. Also, for $j = \omega + \tilde{\omega}, \omega + \tilde{\omega} + 1, \ldots, \omega + 2\tilde{\omega} - 1$ define Γ_j as the restriction of $\Gamma_{\omega+2\tilde{\omega}}$ to G_j. This implies that the sequence $\langle \Gamma_{\omega+\tilde{\omega}}, \Gamma_{\omega+\tilde{\omega}+1}, \ldots, \Gamma_{\omega+2\tilde{\omega}} \rangle$ satisfies Properties R1 and R2.

The drawing $\Gamma_{\omega+3\tilde{\omega}}$ is obtained by replacing each vertex of D_2^A with the isolated vertices of Δ_4. The edges of $\Gamma_{\omega+3\tilde{\omega}}$ are the portions of the edges of Γ' between points of P to which the vertices of $\Gamma_{\omega+3\tilde{\omega}}$ are mapped. It is easy to see that Properties R1 and R2 are by the sequence $\langle \Gamma_{\omega+2\tilde{\omega}}, \Gamma_{\omega+2\tilde{\omega}+1}, \ldots, \Gamma_{\omega+3\tilde{\omega}} \rangle$.

In the drawing $\Gamma_{\omega+4\tilde{\omega}} (= \Gamma_{|V|})$, the vertices of D_3^B replace the isolated vertices of Δ_2, which were distributed on the points of P_3^B and on $|E_1'| - |E_3'|$ points of P_d. The edges of $\Gamma_{\omega+4\tilde{\omega}}$ are the portions of the edges of Γ' between points of P to which the vertices of $\Gamma_{\omega+4\tilde{\omega}}$ are mapped. $\Gamma_{\omega+4\tilde{\omega}}$ is planar, since it is a subdivision of the planar subgraph of Γ' induced by the edges of $E_3' \cup E_\cap'$, plus some degree-1 vertices (the ones in D_1^B and D_2^B), plus the isolated vertices in Δ_3 and Δ_4. Also, for $j = \omega + 3\tilde{\omega}, \omega + 3\tilde{\omega} + 1, \ldots, \omega + 4\tilde{\omega} - 1$ define Γ_j as the restriction of $\Gamma_{\omega+4\tilde{\omega}}$ to G_j. This implies that the sequence $\langle \Gamma_{\omega+3\tilde{\omega}}, \Gamma_{\omega+3\tilde{\omega}+1}, \ldots, \Gamma_{\omega+4\tilde{\omega}} \rangle$ satisfies Properties R1 and R2.

Observe that the obtained realization of $\mathcal{S}$ is such that, for each drawing Γ_i ($i = 1, \ldots, |V|$) no vertex is mapped to a point of P_k.

We finally show that $\mathcal{S}$ admits a realization on a set of $\omega + k$ points if and only if it admits a realization on a set of ω points. Clearly, if $\mathcal{S}$ admits a realization on ω points, it also admits a realization on a set of $\omega + k$ points. For the other direction, suppose by contradiction that $\mathcal{S}$

admits a realization on a set of $\omega + k$ points and that it does not admit a realization on any set of ω points. Note that in each drawing Γ_i ($i = 1, 2, \ldots, |V|$) of the realization, there are k points to which no vertex is mapped. Also, observe that $|D_1^A \cup D_2^A \cup D_3^A \cup V' \cup D_1^B \cup D_2^B \cup D_3^B| = \omega$. Hence, there are at least k points to which only vertices of $\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$ are mapped in some set of drawings of the realization. These vertices are isolated and thus they could have been mapped (without creating crossings) to points to which vertices of $D_1^A \cup D_2^A \cup D_3^A \cup V' \cup D_1^B \cup D_2^B \cup D_3^B$ are mapped in some set of drawings of the realization. This implies that the k extra points could have not been used, and thus that $\mathcal{S}$ admits a realization on a set of ω points. □

Theorem 2. Let $\mathcal{S} = (G, \omega, k, \tau)$ be a graph story and let n be the number of vertices of G. There exists an $O(n \cdot 2^{(4\sigma+1)\log_2 \sigma})$-time algorithm that tests whether $\mathcal{S}$ is realizable, where $\sigma = \omega + k$.

Proof: For each subgraph G_i ($i = \omega, \ldots, n$), let $\mathcal{E}_i = \{\phi_i^1, \phi_i^2, \ldots, \phi_i^{s_i}\}$ be the set of all planar face-k-weighted embeddings of G_i. We construct a DAG (directed acyclic graph) D as follows: (i) For each element $\phi_i^j \in \mathcal{E}_i$ ($i = \omega, \ldots, n$ and $j = 1, \ldots, s_i$), D has a node v_i^j corresponding to ϕ_i^j. (ii) For each pair of elements ϕ_i^j and ϕ_{i+1}^r ($\omega \leq i \leq n-1; 1 \leq j \leq s_i; 1 \leq r \leq s_{i+1}$), D contains a directed edge (v_i^j, v_{i+1}^r) if and only if ϕ_i^j and ϕ_{i+1}^r are compatible face-k-weighted embeddings.

Each set $\mathcal{E}_i$, with $i = \omega, \ldots, n$, defines a distinct *layer* of vertices of D, called *layer i*. By construction, each vertex of layer i can only have outgoing edges towards vertices of layer $i+1$ (if $i < n$) and incoming edges from vertices of layer $i-1$ (if $i > \omega$). We complete the construction of D by adding a dummy source s connected with outgoing edges to all vertices of layer 1 and a dummy sink t connected with incoming edges to all vertices of layer n. By Lemma 3, we have that $\mathcal{S}$ is realizable if and only if there is a directed path Π from s to t in D. In fact Π (if any) consists of exactly one vertex per layer, and the sequence of its vertices from layer ω to layer n corresponds to a sequence of compatible face-k-weighted planar embeddings of $\langle G_\omega, G_{\omega+1}, \ldots, G_n\rangle$.

We now analyze the time complexity of the given algorithm. The number $|\mathcal{E}_i|$ of nodes in layer i equals the number of distinct face-k-weighted embeddings of G_i. This number is the product of two factors: the number π of possible planar embeddings of G_i times the number ρ of ways you have to distribute k units of weight among the faces of each planar embedding of G_i. If G_i is connected, π is upper bounded by the number of possible rotation systems for G_i, i.e., $\pi = O(\omega!)$. If G_i is not connected, for each rotation system we also have to consider all possible ways of arranging a component inside the face of some other component. Since the number of faces is $O(\omega)$, this leads to $\pi = O(\omega! \cdot \omega^\omega)$ planar embeddings, which can be increased to $O(\omega! \cdot (\omega+k)^\omega)$. Observe that this space of planar embeddings is actually computable at the same cost using SPQR-trees and BC-trees for describing the planar embeddings of each connected component and using an inclusion tree for describing the inclusion relationships among the different plane connected components. The number ρ of ways you have to distribute k units of weight among the faces of a planar embedding of G_i can be obtained (using the stars and bars metaphor popularized by [16]) as $\rho = \binom{\omega-1+k}{\omega-1} = \frac{\omega}{\omega+k}\binom{\omega+k}{\omega}$. Since $\binom{n}{m} = O(\frac{n^m}{m!})$, we have $\rho = O(\frac{\omega}{\omega+k} \cdot \frac{(\omega+k)^\omega}{\omega!})$, which can be increased to $\rho = O(\frac{(\omega+k)^\omega}{\omega!})$. Hence, $\pi \cdot \rho = O(\omega! \cdot (\omega+k)^\omega \cdot \frac{(\omega+k)^\omega}{\omega!}) = O((\omega+k)^{2\omega}) = O(2^{2\omega \log_2(\omega+k)})$.

Since we have $O(n)$ layers, generating the vertex set of D takes $O(n \cdot \pi \cdot \rho)$. The number of edges of D is $O(n \cdot (\pi \cdot \rho)^2)$, and checking whether we have to add an edge between two vertices of consecutive layers of D can be done in $O(\omega) = O(\omega + k)$ time, as we need to test the compatibility of the embeddings corresponding to the two vertices. Hence, generating the edge set of D takes $O((\omega+k) \cdot n \cdot (\pi \cdot \rho)^2)$ time. Finally, checking whether a directed path from s to t exists in D takes linear time in the size of D. It follows that the whole testing algorithm takes $O((\omega+k) \cdot n \cdot (\pi \cdot \rho)^2) = O(n \cdot 2^{\log(\omega+k)} \cdot 2^{4(\omega+k)\log_2(\omega+k)}) = O(n \cdot 2^{(4\sigma+1)\log_2 \sigma})$ time. □

Theorems 1 and 2 imply the following corollary.

Corollary 1. For any integer $k \geq 0$, testing the realizability of a graph story $\mathcal{S} = (G, \omega, k, \tau)$ is NP-complete.

Also, fixing $k = 0$ in the statement of Theorem 2, we have the following.

Corollary 2. Let $\mathcal{S} = (G, \omega, \tau)$ be a minimal graph story and let n be the number of vertices of G. There exists an $O(n \cdot 2^{(4\omega+1)\log_2 \omega})$-time algorithm that tests whether $\mathcal{S}$ is realizable.

5 Minimal Graph Stories

We now turn our attention to minimal graph stories. As observed in Remark 2, if a graph story $\mathcal{S} = (G, \omega, k, \tau)$ is realizable, every other story $\mathcal{S}' = (G, \omega, k', \tau)$ with $k' > k$ is also realizable. This observation naturally motivates the investigation of the extremal case $k = 0$, i.e., when we have a minimal graph story. In particular, we want to study which minimal graph stories can be realized for relatively small values of ω, i.e., keeping the visual complexity of the layout low. As it will be clarified, although the realizability problem is trivial for values of ω up to four, this problem becomes immediately more difficult for larger values of ω. More precisely, if $\omega \leq 4$ every minimal graph story is easily realizable, independent of G and of τ, and even if G is not a planar graph; it is enough to use any predefined planar drawing of the complete graph K_4 as a support for each Γ_i ($i = 1, \ldots, n$). On the contrary, establishing which minimal graph stories are realizable when $\omega \geq 5$ is more challenging. In the following, we show that every graph story is realizable if G is outerplanar (Theorem 3), while if G is a series-parallel graph this is not always the case, even if $\omega = 5$ (Lemma 4). Nonetheless, for $\omega = 5$ we prove that every minimal graph story is realizable if G is a planar triconnected cubic graph (Theorem 4); we recall that a graph is *cubic* if all its vertices have degree three. We finally prove, in Section 5.1, that stories of partial 2-trees (which include series-parallel graphs) are always realizable for $\omega = 5$ if we are allowed to "reroute" at most one edge per time (a formal definition is given later); this result is an implication of a more general result for stories with $\omega = 5$ (Theorem 5). Lemma 4 and Theorem 5 together close the gap on the realizability of minimal graph stories of partial 2-trees when $\omega = 5$.

For a story of an outerplanar graph, we show that any outerplanar embedding is a supporting embedding. We recall that a supporting embedding for a graph story is a planar embedding of the graph such that each drawing of the realization of the story preserves this embedding.

Theorem 3. Let G be an outerplanar graph such that all its edges are visible. Every minimal graph story $\mathcal{S} = (G, \omega, \tau)$ is realizable. Also, any outerplanar embedding of G is a supporting embedding for $\mathcal{S}$.

Proof: Let ϕ be any outerplanar embedding of G, and let ϕ_i be the restriction of ϕ to G_i ($1 \leq i \leq n$). Consider any two consecutive planar embeddings ϕ_{i-1} and ϕ_i, for $\omega + 1 \leq i \leq n$. Since ϕ_i and ϕ_{i-1} are restrictions of the same planar embedding of G, their restrictions to $G_i \cap G_{i-1}$ determine the same set F of faces. Also, both v_i and $v_{i-\omega}$ lie in the plane region corresponding to the external face of F. Hence, ϕ_{i-1} and ϕ_i are compatible and, by Lemma 3, $\mathcal{S}$ is realizable. □

Lemma 4. For any $\omega \geq 5$, there exists a minimal graph story $\mathcal{S} = (G, \omega, \tau)$ such that G is a series-parallel graph and $\mathcal{S}$ is not realizable.

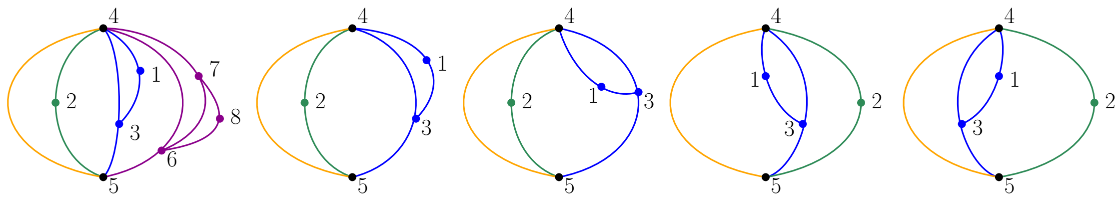


Figure 4: (a) A minimal graph story of a series-parallel graph that is not realizable. (b),(c),(d),(e) the four combinatorial embeddings of G_5.

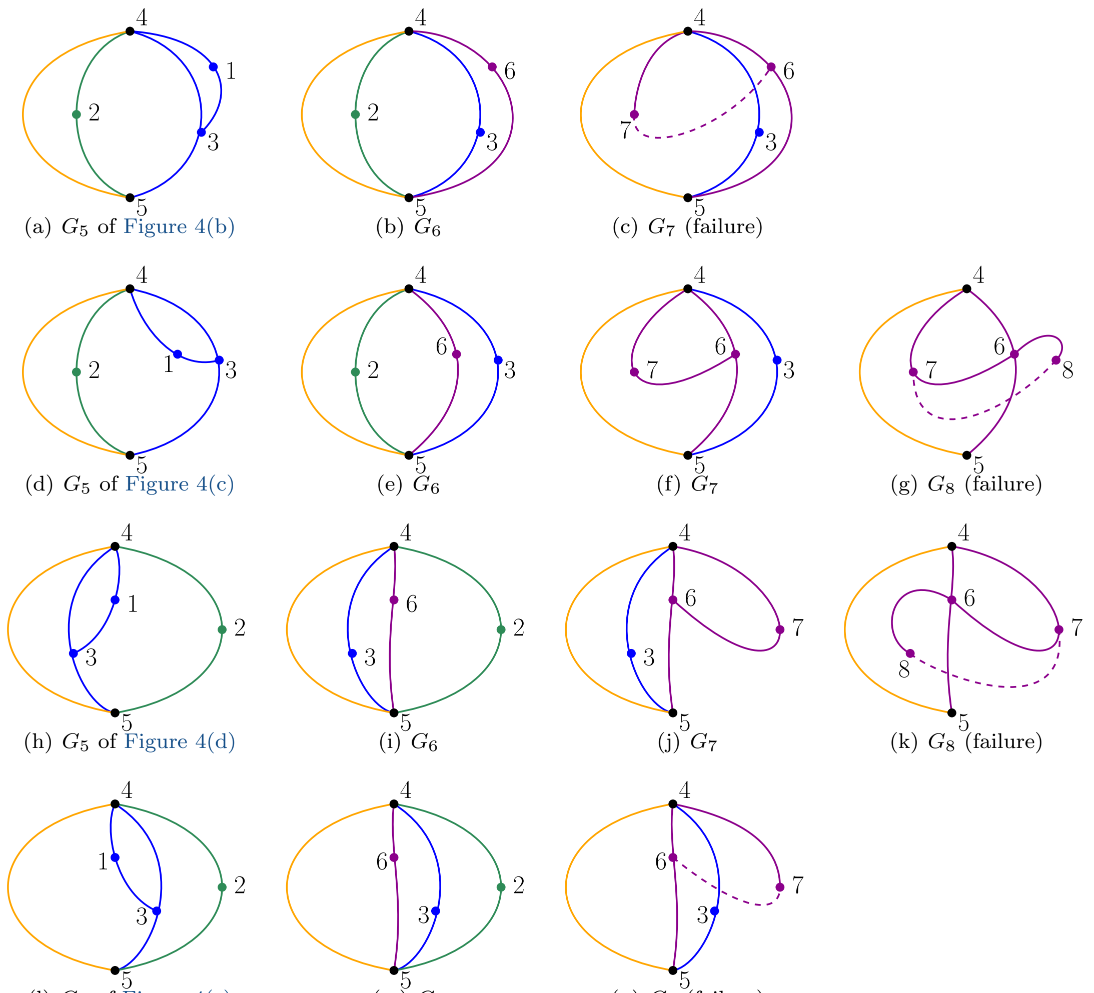
Proof: We first prove the statement for $\omega = 5$, and then extend the result to any $\omega > 5$. Consider the instance $\mathcal{S} = (G, 5, \tau)$ in Figure 4(a), where the vertices are labeled with their subscript in the sequence $\tau = \langle v_1, v_2, \ldots, v_8 \rangle$. Graph G_5 admits one of the four embeddings in Figures 4(b) to 4(e). Observe that, in all four cases either cycle $3, 4, 5$ separates 6 from 7 in G_7 (Figures 5(c) and 5(n)), or cycle $4, 5, 6$ separates 7 from 8 in G_8 (Figures 5(g) and 5(k)).

To extend the result to any $\omega > 5$, we modify the above described instance. Consider the instance $\mathcal{S}' = (G', \omega', \tau')$, where $\omega' > 5$, G' is obtained from G by adding $\omega - 5$ vertices in a path between v_4 and v_5, and $\tau' = \langle v'_1, v'_2, \ldots,\ v'_{8+\omega-5} \rangle$ is such that for $p = 1, 2, \ldots, 5$, we have $v'_p = v_p$; for $q = 6, 7, \ldots, \omega$ we have that v'_q is a vertex of the added path; and for $r = 1, 2, 3$ we have $v'_{\omega+1} = v_{5+r}$. Observe that $G'_5 = G_5$. From G'_6 to G'_ω vertices $v'_6, \ldots v'_\omega$ are added to G'_5. Neglecting the added path, for $r = 1, 2, 3$, graph $G'_{\omega+r} = G_r$ and the same non-planarity configurations of the graph story $\mathcal{S}$ occur. □

The next result that we present is described in Theorem 4. Before giving the proof, we need further definitions and terminology. Let G be a graph and $\mathcal{S} = (G, \omega, \tau)$ be a minimal graph story. A *basic element* of G is either a vertex or an edge. Two basic elements of G are *coeval* if they both belong to the same graph G_i, for some $1 \leq i \leq n$; observe that by the edge visibility assumption (i.e., Remark 1), every two adjacent vertices of G are coeval.

We will consider both planar and non-planar embeddings of graphs in the plane. Unless otherwise specified, with the term "embedding" we refer to a non-planar embedding, where we interpret each crossing as a dummy vertex. Given an embedding ϕ of G and a cycle C of G, we say that a point lies *inside* C if it belongs to one of the bounded plane regions delimited by C; note that C may cross itself many times. We denote by $G(C)$ the union of C and the subgraph of G that lies inside C, i.e., the vertices that lie inside C and all the edges incident to two vertices that lie inside C. A *critical cycle* of G in ϕ is a cycle C such that there exists a vertex $v \in G(C) \setminus C$ that is coeval with every vertex and every edge of C. In this case we say that C is *critical* for v. Of course, this does not imply that C and v belong to the same G_i, for some $1 \leq i \leq n$. See, for example, the story with $\omega = 5$ depicted in Figure 6(b), where the red cycles C_1 and C_2 are both critical for, respectively, v_4 and v_{18}. A *good embedding* of G is an embedding with no critical cycle and such that no two coeval edges cross. Figure 6(c) shows the same graph as the one of Figure 6(b), but where the embedding is a good embedding.

The following lemma shows that the existence of a good embedding is a sufficient condition for the realizability of a minimal graph story. The proof immediately follows from Lemma 3, assuming the number of extra points k is equal to zero. Indeed, for each $1 \leq i \leq n$, a good embedding ϕ of G induces an outerplanar embedding ϕ_i of G_i, and any two consecutive embeddings ϕ_{i-1} and ϕ_i are compatible for $\omega + 1 \leq i \leq n$.



(a) G_5 of Figure 4(b) (b) G_6 (c) G_7 (failure)

(d) G_5 of Figure 4(c) (e) G_6 (f) G_7 (g) G_8 (failure)

(h) G_5 of Figure 4(d) (i) G_6 (j) G_7 (k) G_8 (failure)

(l) G_5 of Figure 4(e) (m) G_6 (n) G_7 (failure)

Figure 5: Tentative realizations of the story of Figure 4(a) starting from the embeddings of Figures 4(b) to 4(e). They all lead to a failure.

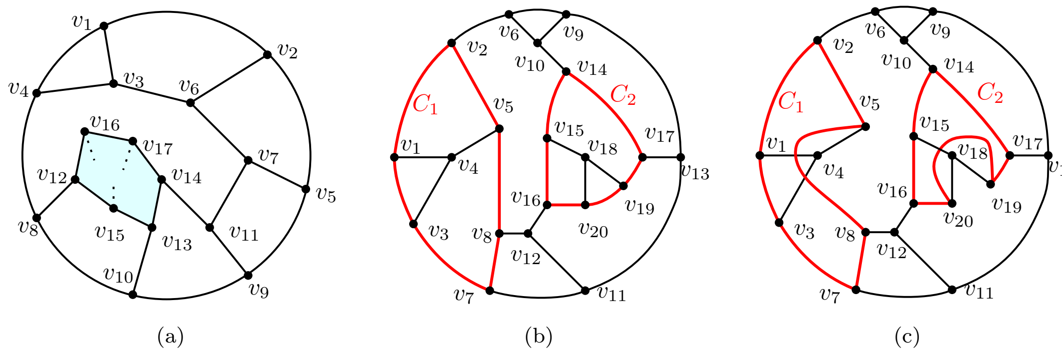


Figure 6: (a) A planar triconnected cubic graph. Notice that it can be extended in the highlighted face. (b) A planar embedding of G that is not a good embedding, since it contains two critical cycles C_1 and C_2. (c) A good embedding of G.

Lemma 5. If G admits a good embedding ϕ, then $\mathcal{S}$ is realizable.

We now show that minimal graph stories with $\omega \leq 5$ are always realizable for planar triconnected graphs.

Theorem 4. Let G be an n-vertex planar triconnected cubic graph such that all its edges are visible. Every minimal graph story $\mathcal{S} = (G, 5, \tau)$ is realizable. A sequence of compatible planar embeddings for $\mathcal{S}$ can be found in $O(n)$ time.

Proof: Our goal is to prove Claim 5, which claims that a planar triconnected cubic graph always admits a good embedding. Then, the proof follows immediately by Lemma 5 and Claim 5. In order to prove Claim 5 we need intermediate results, stated by the next four claims. A critical cycle is *internal* if it is not the boundary cycle of the external face. Claim 1 shows how to construct an embedding of G having only internal critical cycles and where the crossings have specific properties.

Claim 1. There always exists an embedding ϕ' of G that satisfies the following properties: (i) The boundary of the external face is a crossing-free cycle of G which is not a critical cycle; (ii) every two crossing edges are not coeval and one of them is an edge e^* incident to v_1.

Proof: First, observe that since G is cubic triconnected and since every edge is incident to two coeval vertices, there is no face of G incident to the vertices in the set $\{v_1, v_2, v_3, v_4, v_5\}$.

We first show that there exists a face f of G incident to v_1 and to at least three vertices of the set $\{v_2, v_3, v_4, v_5\}$. Suppose that v_1 is not adjacent to v_2. Refer to Figure 7(a). In this case v_1 is adjacent to v_3, v_4, and v_5. Also, v_2 has to be adjacent to at least two vertices v_i and v_j among v_3, v_4, and v_5. Since in a triconnected cubic plane graph different from K_4 any 4-cycle is a face, there exists a face formed by the edges (v_1, v_i), (v_i, v_2), (v_2, v_j), and (v_j, v_1). Suppose now that v_1 is adjacent to v_2. Refer to Figure 7(b). There exist $i, i', i'' \in \{3, 4, 5\}$ such that v_1 is adjacent to v_i and $v_{i'}$; $v_{i''}$ is adjacent to v_2. Since G is cubic: There exists a face f incident to the path $\{v_{i''}, v_2, v_1\}$; either (v_1, v_i) or $(v_1, v_{i'})$ is incident to f.

Consider the planar embedding ϕ of G when f is the external face of G. Let $a, b, c, d \in \{2, 3, 4, 5\}$ and assume that v_a, v_b, and v_c are adjacent to v_1. Refer to Figure 7(c), where $a = 2$, $b = 3$, $c = 5$, and $d = 4$. We have that two among these vertices, say v_a and v_b, are in the external face. Hence,

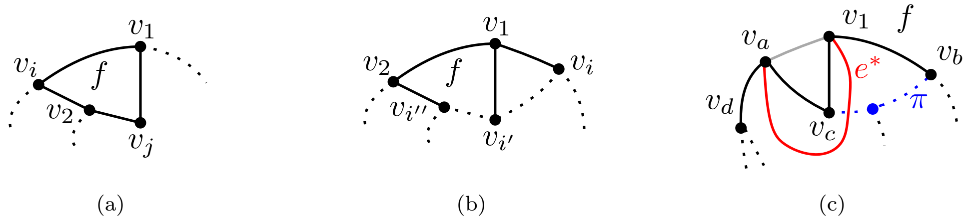


Figure 7: Illustration for Claim 1.

v_c is not in the external face, otherwise v_1, v_c would have been a separation pair. Vertex v_d is in the external face due to the choice of f. If $(v_c, v_a) \in E$ and $(v_c, v_b) \in E$, v_a, v_b form a separation pair. Hence, there exists a path π connecting v_c to either v_a or v_b containing only edges not coeval with v_1. Say that π connects v_c to v_b. Notice that, since $(v_1, v_c) \in E$, $e^* = (v_1, v_a) \in E$ is incident to f and to an internal face incident to v_c. We can change ϕ so that e^* crosses an edge in π and v_c is incident to the external face. We call ϕ' the new embedding.

Since v_i for $i = 1, \ldots, 5$ is part of the external cycle, every internal vertex of G is not coeval with v_1 and, consequently, Property (i) holds. Since we changed the embedding only with respect to e^*, Property (ii) holds for ϕ'. ■

The next claims prove relevant properties of the internal critical cycles.

Claim 2. For every internal critical cycle C of G, $6 \leq |C| \leq 8$.

Proof: Since G is cubic triconnected and since $G(C) \neq C$, we have $|C| \geq 6$. Since every vertex is coeval with at most 8 vertices, we have $|C| \leq 8$. ■

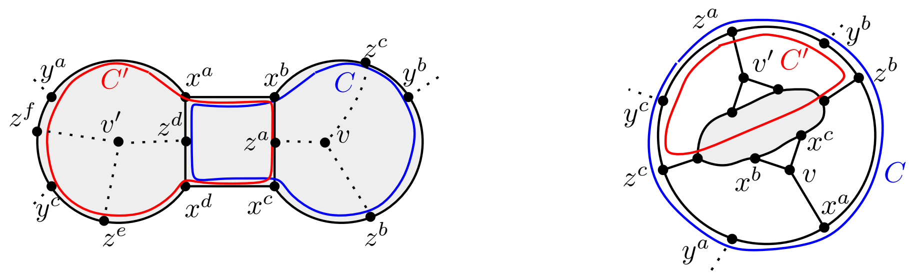
Claim 3. An internal critical cycle C is critical for exactly one vertex.

Proof: Suppose that there are two vertices v and v' for which C is critical. Since every vertex is coeval with at most 8 vertices and since v and v' are distinct vertices, in this case it is not possible that $|C| > 6$. If $|C| = 6$, there are two possibilities: (1) v and v' are the only two vertices in $G(C) \setminus C$. In this case, there are exactly two vertices x and y of C not adjacent to vertices in $G(C) \setminus C$. Vertices x, y are a separating pair. (2) Otherwise, in C there are exactly three vertices incident to vertices of $G(C) \setminus C$. Let v'' be another vertex in $G(C) \setminus C$. Vertex v'' is incident to v and v', but it cannot be coeval with both of them (the only vertices coeval with both v and v' are the ones of C). In Cases (1) and (2) we have a contradiction. ■

Claim 4. Let C be an internal critical cycle for vertex v. Given any other critical cycle C' for a vertex v', we have $(G(C') \setminus C') \cap (G(C) \setminus C) = \emptyset$.

Proof: By Claim 3, C is not critical for v' and C' is not critical for v. Suppose, by contradiction, that the statement does not hold. We consider two different cases: (i) $v' \notin C$ and $v \notin C'$; (ii) $v' \in C$ or $v \in C'$.

Case (i). Suppose first that $C \notin G(C')$ and $C' \notin G(C)$; see, e.g., Figure 8(a). In this case, since G does not have vertices of degree 4, there are at least four vertices $x^a, x^b, x^c, x^d \in C \cap C'$ such that x^i ($i \in \{a, b, c, d\}$) is incident to one edge in $C \cap C'$, one in C' and not in C, and one in C and not in C'. The external cycle of $C \cup C'$ has at least three vertices y^a, y^b, and y^c, adjacent to vertices outside it. Notice that, since G is cubic, two vertices x^i and y^j, with $i \in \{a, b, c, d\}$



(a) $C \notin G(C')$ and $C' \notin G(C)$ (b) $C \in G(C')$

Figure 8: Illustration for Claim 4, Case (i).

and $j \in \{a, b, c\}$ cannot coincide. Also, in $C \cup C'$ there are three other vertices z^a, z^b, and z^c connected to v with three disjoint paths, due to the fact that G is triconnected. Similarly, $C \cup C'$ contains three other vertices z^d, z^e, and z^f connected to v' with three disjoint paths. Notice that it is possible to choose z^i ($i \in \{a, \ldots, f\}$) such that it does not coincide with any: z^j ($j \neq i$ and $i \in \{a, \ldots, f\}$); x^k ($k \in \{a, \ldots, d\}$); y^g ($g \in \{a, b, c\}$). Hence, $|C \cap C'| \geq 4$ and $|C \cup C'| \geq 13$. It follows that either $|C| > 8$ or $|C'| > 8$, a contradiction. For example, in Figure 8(a) we have $|C'| = 9$.

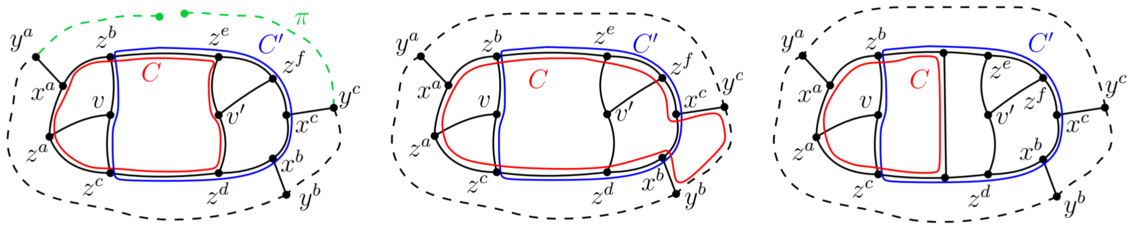
Suppose $C' \in G(C)$; see, e.g., Figure 8(b). Vertex v is coeval with: Its three adjacent vertices x^a, x^b, x^c; at least three vertices y^a, y^b, y^c in C adjacent to three vertices outside $G(C)$; at least three vertices z^a, z^b, z^c in C adjacent to three vertices in $G(C')$. Notice that if there are less than three vertices of C connected to vertices of $G(C')$, then the graph is not triconnected. In the figure, we have $z^a, z^b, z^c \in C \cap C'$. If, for example, C and C' are disjoint, we have that z^a, z^b, z^c are in C (and not in C'). We have that v is coeval with 9 vertices. A contradiction. The case $C \in G(C')$ is analogous, where the role of v, C and v', C' is inverted.

Case (ii). Refer to Figure 9(a). Observe that, since G is planar cubic triconnected, the external cycle of $C \cup C'$ has the following distinct vertices:

- (at least) three vertices x^a, x^b, and x^c adjacent to a vertex outside the cycle, y^a, y^b, and y^c, respectively.
- six other vertices z^a, z^b, z^c and z^d, z^e, z^f, such that z^a, z^b, z^c (z^d, z^e, z^f) are connected with three disjoint paths to v (v'), respectively.

Notice that one between x^a, x^b, x^c has to be contained in C (resp. C') and not in C' (resp. C), otherwise two between z^a, z^b, z^c (resp. z^d, z^e, z^f) form a separation pair. We consider two subcases: (*ii.a*) $v \in C'$ and $v' \in C$; (*ii.b*) $v \in C'$ and $v' \notin C'$ (the case $v \notin C'$ and $v' \in C'$ is analogous).

Case (ii.a): Suppose $v \in C'$ and $v' \in C$. Refer to Figure 9(a). In this case at least two between z^a, z^b, z^c, say z^b and z^c, and two between z^d, z^e, z^f, say z^d and z^e, are in $C \cap C'$. Cycle C (resp. C') contains z^a, z^b, and z^c (z^d, z^e, and z^f). Hence, either $|C'| = |C| = 8$, or $|C'| = 8$ and $|C| = 7$, or $|C'| = 7$ and $|C| = 8$. We assume $|C'| = 8$ and $|C| = 7$, the other two cases can be solved similarly. Since $|C| = 7$ and $|C'| = 8$: $z^a \in C$ and $z^a \notin C'$; $z^f \in C'$ and $z^f \notin C$; $x^a \in C$ and $x^a \notin C'$; $x^c, x^b \in C'$ and $x^c, x^b \notin C$. We show that, in this case, G contains an edge incident to



(a) $v \in C'$, $v' \in C$ (b) $v \in C'$, $v' \notin C$: Case (1) (c) $v \in C'$, $v' \notin C$: Case (2)

Figure 9: Illustration for Claim 4, Case (ii).

two vertices that are not coeval. This would be a contradiction. Let $\tau(u) = i$ for any vertex $u = v_i$ of G. Suppose, w.l.o.g., $\tau(v) < \tau(v')$.

We show that $\tau(y^a) < \tau(v)$. We have that C' contains all the vertices coeval with v'.

Since $\tau(v) < \tau(v')$ and v and v' are coeval ($v \in C'$ and $v' \in C$), for any u such that $\tau(u) > \tau(v)$ and u is coeval with v, we have that u and v' are coeval. Since x^a is coeval with v and not coeval with v', we have $\tau(x^a) < \tau(v)$. If $\tau(y^a) > \tau(v)$, since x^a and v are coeval, we have that y^a and v are coeval. In this case, y^a would be coeval with v', but this is not possible since all the vertices coeval with v' are in C'. Hence, $\tau(y^a) < \tau(v)$.

We now show that $\tau(y^c) > \tau(v')$. Notice that, similarly to the previous case, since $\tau(v) < \tau(v')$ and since v and v' are coeval, for any u such that $\tau(u) < \tau(v')$ and u is coeval with v', u and v are coeval. Let w be the vertex coeval with v and not in C (recall that we assumed $|C| = 7$). We have that w can be contained in at most one of the two paths connecting y^a to x^b and x^c that are outside $C \cup C'$ and containing y^b and y^c, respectively. Assume, w.l.o.g., that such path π connecting y^a to x^c does not contain w. Hence, $w \neq x^c$ and, consequently, x^c and v are not coeval. It follows that $\tau(x^c) > \tau(v')$. If $\tau(y^c) < \tau(v')$, since x^c and v' are coeval, we have y^c and v' are coeval. In this case, y^c would be coeval with v, but this is not possible since this implies $y^c = w$ but that contradicts the fact that $w \notin \pi$. Hence, $\tau(y^c) > \tau(v')$.

We have $\tau(v) < \tau(v')$, $\tau(y^a) < \tau(v)$, $\tau(y^c) > \tau(v')$. Any vertex u coeval with v such that $\tau(u) > \tau(v)$ is in $G(C) \cup G(C')$ or it is w. Recall that π connects y^a to y^c and does not contain w. Hence, π contains an edge incident to two vertices that are not coeval. A contradiction.

Case (ii.b): Suppose $v \in C'$ and $v' \notin C$. There are two cases: (1) $G(C') \subset G(C)$; (2) $G(C') \not\subset G(C)$, $G(C) \not\subset G(C')$, and $G(C') \cap G(C) \neq \emptyset$.

(1) Refer to Figure 9(b). In this case we can choose z^e and z^d such that C contains either: z^e or z^d, if C' contains only one between x^c and x^b; or, otherwise, both of them, as in the figure. Since $G(C') \subset G(C)$, C contains x^b, x^c, y^b, and y^c. Hence $|C| \geq 9$ and by Claim 2 C is not critical.

(2) Refer to Figure 9(c). There is a path (in the figure, an edge) such that v and v' are not in the same face. In this case the value of $|C'|$ increases by two with respect to its value in Case (ii.a) and $|C'| \geq 9$ ($|C'| = 9$ if one between x^b and x^c is in C and not in C, $|C'| = 10$ otherwise). Hence $|C'| \geq 9$ and by Claim 2 C' is not critical.

In both Cases (i) and (ii), we have a contradiction. ■

Finally, we have all the ingredients to prove Claim 5.

Claim 5. Graph G always admits a good embedding.

Proof: Let ϕ be any embedding of G with the properties described by Claim 1. We have that the

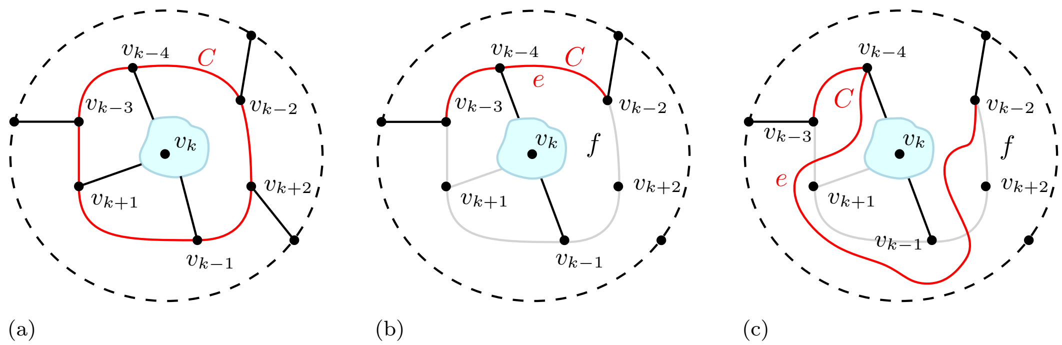


Figure 10: Illustration for Claim 5, `Part I`. Critical cycle C does not contain other critical cycles.

external cycle of G is not a critical cycle. In order to prove the claim, we consider all the critical cycles of G and we modify ϕ in order to make them not critical, one by one. In `Part I` of the proof we consider critical cycles that do not contain other critical cycles. In `Part II`, we will show how to handle the rest of the cycles. During the first two parts of the proof we change ϕ with respect to some edges. In `Part III` we show that these edges do not cross in the final embedding.

`Part I:` Let C be a critical cycle not containing other critical cycles. Let v_k be the vertex for which C is critical. By Claim 3, this vertex is unique. Since any vertex v_i such that $i \leq 3$ or $i \geq n-2$ is coeval with less than 6 vertices, by Claim 2 we can assume $3 < k < n-2$.

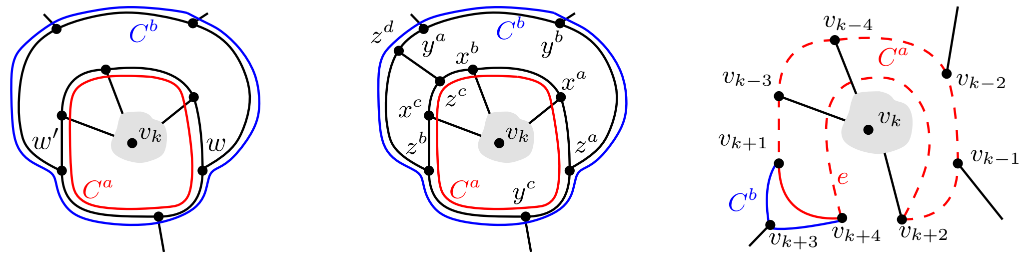
Suppose $|C| = 6$. See Figure 10(a). Cycle C always contains a vertex v_i such that at least two vertices in C are not coeval with v_i. In particular, if C contains a vertex v_j such that $j \in \{k-4, k+4\}$, $i = j$. Otherwise, $C = \{v_{k-3}, v_{k-2}, v_{k-1}, v_{k+1}, v_{k+2}, v_{k+3}\}$ and $i = k-3$ (or $i = k+3$).

Consider the 6 faces adjacent to C, three internal and three external. Vertex v_i is not coeval with some edge incident to 4 of the 6 vertices of C, hence, the subgraph of $G(C)$ consisting only of vertices coeval with v_i consists of two faces and at least one of them, say f, is incident to all the vertices of C. See Figure 10(b). Let $e = (v_i, v_{i'})$ be an edge of C and assume that, going from v_i to $v_{i'}$, v is on the right (left) of e. We can change the embedding around the vertices of e so that e goes through f such that v is on the left (right) of e. Cycle C is not critical for v anymore and we did not introduce crossings between two coeval edges. See Figure 10(c).

Suppose $|C| = 7$ or $|C| = 8$. In this case there is always a vertex v_i that has 3 and 4 vertices of C that are not coeval with v. In particular, we can always choose v_i such that either $i = k-4$ or $i = k+4$. Hence, we always have that the remaining vertices coeval with v_i are 4, as in the previous case, and we can prove these two new cases with the same considerations.

`Part II:` Let C^a and C^b be two cycles such that $C^a \in G(C^b)$ critical for v^a and v^b, respectively. By Claim 4 and Claim 3, C^a and C^b are critical for the same vertex $v = v^a = v^b$. Hence, by Claim 2 and since v is coeval with 8 vertices, $6 \leq C^a \cap C^b \leq 7$.

Suppose, w.l.o.g., that $C^b \not\subset C^a$. We first show that $C^a \subset C^b$. Suppose, by contradiction, $C^a \not\subset C^b$. Since the graph is triconnected, in this case there is an edge connecting the vertices of C^a not contained in C^b to the vertices of C^b not in C^a, otherwise there are two vertices w and w' that are a separation pair. See Figure 11(a). Refer now to Figure 11(b). Graph $C^a \cup C^b$ contains the following vertices: Three vertices x^a, x^b, x^c connected with a path to v; three vertices y^a, y^b, y^c connected to the graph outside $G(C^a \cup C^b)$; (at least) four vertices z^a, z^b, z^c, z^d incident to the three edges connecting C^a to C^b, that have to be different from the previous vertices. Hence,



(a) (b) (c)

Figure 11: Illustration for Claim 5, `Part II`. Critical cycle C^b contains critical cycle C^a.

$|C^a \cup C^b| \geq 10$ and v is coeval with 10 vertices, which is a contradiction.

We have $C^a \subset C^b$ and, consequently, C^a is composed of vertices of C^b and chords connecting two vertices of C^b. See Figure 11(c). We can apply the approach of `Part I`, with the difference that if v_i is adjacent to one of these chords, we change the embedding with respect to the edge e incident to v_i in C^a that is not the chord. In Figure 11(c), the edges in common between C^a and C^b are dashed.

`Part III`: Notice that in `Parts I` and `II` we changed the embedding with respect to exactly one edge e (or simply, we "moved one edge e") for each critical cycle C. Also, after this operation, e crosses only edges in $G(C)$. Hence, by Claim 4, no two edges that we moved in `Parts I` and `II` cross. We have that, for any edge e that we moved, e is never incident to v_k such that $k \leq 5$, since there exist no critical cycle of G such that the vertex v_i with maximum i in C is k. Hence, by Claim 1, no edge moved in `Parts I` and `II` crosses the edge moved while processing the external face.

In `Parts I` and `II` we processed one by one each critical internal cycle in order to make it not critical, without creating crossings between coeval edges. We did it starting from an embedding where the external face was not critical. Hence, the obtained embedding is a good embedding. ■

The theorem holds by Lemma 5 and Claim 5. Concerning the computational time, choosing an embedding where the external face is not critical can be done in $O(1)$ time as described in the proof of Claim 1. Computing all the critical cycles can be done in $O(n)$ time. Fixing one by one each one of them, as described in the proof of Claim 5, takes linear time. This is sufficient by Lemma 5. □

5.1 Rerouting

Since Property R2 of Definition 2 is a strict requirement, one can think of relaxing it by allowing a partial change of the drawing of $G_{i-1} \cap G_i$ when vertex v_i enters the viewing window. Let Γ be a planar drawing of G, (u, v) be an edge of G incident to two distinct faces f and f' of Γ, and p be a point of the plane inside face f; see Figure 12(a). *Rerouting* (u, v) *with respect to* p consists of planarly redrawing (u, v) such that u and v keep their positions and p lies inside f'; see Figure 12(b). The obtained drawing has the same planar embedding as Γ. Note that this operation "moves" a single point from f to f' by redrawing e. If e is rerouted with respect to a set Q of points, all inside f, we say that e is rerouted with respect to Q.

Figure 12: Rerouting edge (u, v) with respect to point p.

An *h-reroute realization* of $\mathcal{S} = (G, \omega, k, \tau)$ *on a set* P of $\omega + k$ points ($h \geq 0$) is a sequence $\langle \Gamma_1, \Gamma_2, \ldots, \Gamma_n \rangle$ satisfying Property R1 of Definition 2 and such that the restriction of Γ_i to $G_{i-1} \cap G_i$ ($2 \leq i \leq n$) is obtained from the restriction of Γ_{i-1} to $G_{i-1} \cap G_i$ by rerouting at most h distinct edges each with respect to a subset Q of P. $\mathcal{S}$ is *h-reroute realizable* if it has an h-reroute realization on a set of $\omega + k$ points.

The next theorem characterizes the set of graph stories $\mathcal{S} = (G, 5, \tau)$ that are 1-reroute realizable. It properly includes those stories whose G is planar.

Theorem 5. Let G be a graph such that all its edges are visible. Every minimal graph story $\mathcal{S} = (G, 5, \tau)$ is 1-reroute realizable if and only if G does not contain K_5.

Proof: Suppose first that G contains K_5. Since all the edges of G are visible, each edge appears in some G_i ($1 \leq i \leq n$). This implies that there must be an index $j \in \{1, \ldots, n\}$ for which G_j contains all vertices of the K_5, i.e., G_j coincides with K_5. Indeed, if there were a pair of vertices of the K_5 that never appear in the same G_i, the edge connecting these two vertices would not be visible throughout any realization of the graph story. Therefore, since G_j is non-planar, G is not h-reroute realizable, for any $h \geq 0$.

Suppose vice versa that G does not contain K_5. This implies that each subgraph G_i ($1 \leq i \leq n$) does not contain K_5 and hence it is planar. To prove that $\mathcal{S}$ is 1-reroute realizable, we show that it admits a 1-reroute realization on any arbitrarily chosen set P of 5 points. Let Γ_4 be a planar drawing of G_4 on P. Let p be the point of P to which no vertex of G_4 is mapped, let f be the face of Γ_4 that contains p, and let $N(v_5)$ be the set of neighbors of v_5 in G_5. If the boundary of f has four vertices, then v_5 can be mapped to p and it can be connected to all its neighbors without creating edge crossings, so to obtain a planar drawing Γ_5 of G_5. If the boundary of f has three vertices, mapping v_5 to p and connecting it to its neighbors may create an edge crossing. To avoid this crossing, it is possible to reroute an edge of the boundary of f with respect to p such that p lies inside a face whose boundary contains all vertices in $N(v_5)$. Such an edge always exists because the faces of Γ_4 are pairwise adjacent. More precisely, if G_4 is not K_4, then there must be a face f' of Γ_4 (adjacent to f) such that f' contains all vertices of G_4; in this case, we can reroute any edge e shared by f and f' so that p lies inside f'. If G_4 is K_4, then $|N(v_5)| \leq 3$, as G does not contain K_5. Also, there is a face f' of Γ_4 that contains all vertices of $N(v_5)$; as before, we can reroute any edge e shared by f and f' so that p lies inside f'. After the rerouting operation, v_5 can be mapped to p and connected to all its neighbors without creating edge crossings, so to obtain a planar drawing Γ_5 of G_5. This procedure can be applied for each pair of graphs G_{i-1} and G_i ($5 < i \leq n$): Γ_i is obtained by mapping v_i to the same point p of P to which v_{i-5} is mapped in Γ_{i-1}, by rerouting at most one edge with respect to p. $\square$

6 Lower Bounds for Non-minimal Graph Stories

In this section we study lower bounds on the number of extra points needed for the realizability of graph stories. More precisely, we provide a lower bound for stories whose graphs are nested triangulations and $\omega = 9$ (Theorem 6), and then for stories of series-parallel graphs with $\omega \geq 8$ (Theorem 7). To prove the first result, we start with the following technical lemma.

Lemma 6. Let $\mathcal{S} = (G, \omega, k, \tau)$ be a realizable graph story. Suppose that: (i) G contains vertex-disjoint cycles $C_1, \ldots, C_h$ such that $C_{j-2}, C_{j-1}, C_j \in G_{i_j}$, with $j = 3, \ldots, h$, $i_{j-1} < i_j$ and $i_j - i_{j-1} < \omega$; (ii) in all planar embeddings of G_{i_j}, C_{j-1} separates C_{j-2} from C_j. We have that $\sigma = \omega + k \in \Omega(h)$.

Proof: Let $\mathcal{R} = \langle \Gamma_1, \ldots, \Gamma_n \rangle$ be a realization of $\mathcal{S}$ and let σ_i be the total number of points used by $\mathcal{R}$ to draw $\langle \Gamma_1, \ldots, \Gamma_i \rangle$. Without loss of generality, assume that in all planar embeddings of G_{i_j}, cycle C_j is outside C_{j-1}, which is outside C_{j-2}. Also, observe that a cycle has at least 3 vertices.

The proof is by induction on j, showing that the points used by $\mathcal{R}$ to represent vertices in $\{C_1, \ldots, C_{h-1}\}$ are contained in the region of the plane delimited by C_h and that $\sigma_h \geq 9 + 3(h-1)$. As a base case, consider any planar drawing Γ_{i_3} of graph G_{i_3}. It contains vertex-disjoint cycles C_1, C_2, and C_3. Hence, by hypothesis the region of the plane delimited by C_3 contains the points used to represent C_1 and C_2. Also, $\mathcal{R}$ uses at least $\sigma_{i_3} \geq 9$ points for the drawings of $\langle \Gamma_1, \ldots, \Gamma_{i_3} \rangle$.

As for the inductive case, consider graph G_{i_j} and any of its drawings Γ_{i_j}. By the inductive hypothesis, the points used by $\mathcal{R}$ to represent vertices in $\{C_1, \ldots, C_{i_{j-1}-1}\}$ are contained in the region of the plane delimited by $C_{i_{j-1}}$. Also, the realization $\mathcal{R}$ uses $\sigma_{i_{j-1}} \geq 9 + 3((j-1)-1)$ points for drawings $\langle \Gamma_1, \ldots, \Gamma_{i_{j-1}} \rangle$. Drawing Γ_{i_j} contains cycles $C_{i_{j-2}}$, $C_{i_{j-1}}$, and C_{i_j}. By hypothesis, we have that cycle C_{i_j} is outside $C_{i_{j-1}}$, that is outside $C_{i_{j-2}}$ in Γ_{i_j}. There exist at least three vertices that belong to C_{i_j} and are outside $C_{i_{j-1}}$. Therefore, the points used by $\mathcal{R}$ to represent vertices in $\{C_1, \ldots, C_{i_j}\}$ are contained in the region of the plane delimited by C_{i_j} and that $\sigma_{i_j} \geq \sigma_{i_{j-1}} + 3 \geq 9 + 3(j-1)$. □

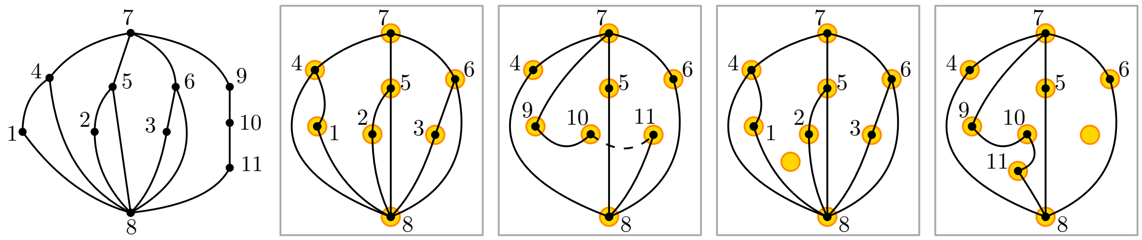
Theorem 6 generalizes a previous result given in the literature [7, Theorem 1]; as anticipated at the beginning of the section, its proof exploits Lemma 6. Let $n = 3h$, for an integer $h \geq 1$. An *n-vertex nested triangles graph* G contains the vertices and edges of the 3-cycle $C_i = (v_{i-2}, v_{i-1}, v_i)$, for $i = 3, 6, \ldots, n$, plus the edges (v_i, v_{i+3}), for $i = 1, 2, \ldots, n-3$. For $n \geq 6$, G is triconnected, thus it has a unique planar embedding (up to the choice of the external face) [28].

Theorem 6. Let $\mathcal{S} = (G, 9, k, \tau)$ be a realizable graph story such that G is a $3h$-vertex nested triangles graph, where τ is given by the indices of the vertices of G. Any realization of $\mathcal{S}$ has $k \in \Omega(n)$, where $n = 3h$ is the number of vertices of G.

Proof: Consider the vertex-disjoint cycles $C_1 = (v_1, v_2, v_3)$, $C_2 = (v_4, v_5, v_6), \ldots, C_i = (v_{3i-2}, v_{3i-1}, v_{3i}), \ldots, C_h = (v_{3h-2}, v_{3h-1}, v_{3h})$. We have that G_9 contains cycles C_1, C_2, and C_3. Also, in any planar embedding of G_9 we have that C_2 separates C_1 from C_3. More generally, for $j = 3, 4, \ldots, h$ graph G_{3j} contains cycles C_{j-2}, C_{j-1}, and C_j, and in any planar embedding of G_{3j} we have that C_{j-1} separates C_{j-2} from C_j. By Lemma 6, we have that $\omega + k \in \Omega(h) \in \Omega(n)$. Since ω is a constant, we have that $k \in \Omega(n)$. □

While Theorem 6 exploits the uniqueness of the embedding of G, the next result provides lower bounds also for graphs that have several planar embeddings.

Theorem 7. Let G be a series-parallel graph such that all its edges are visible. For any $\omega \geq 8$, there exists a graph story $\mathcal{S} = (G, \omega, k, \tau)$ that is not realizable for $k < \lfloor \frac{\omega}{2} \rfloor - 3$.



(a) Graph G (b) G_8 for $k = 0$ (c) G_{11} for $k = 0$ (failure) (d) G_8 for $k = 1$ (e) G_{11} for $k = 1$

Figure 13: Illustration for Theorem 7. Case $\omega = 8$. Graph G and drawings of G_8 and G_{11}.

Proof: We first prove the statement for $\omega = 8$, and then we extend the result to any $\omega > 8$.

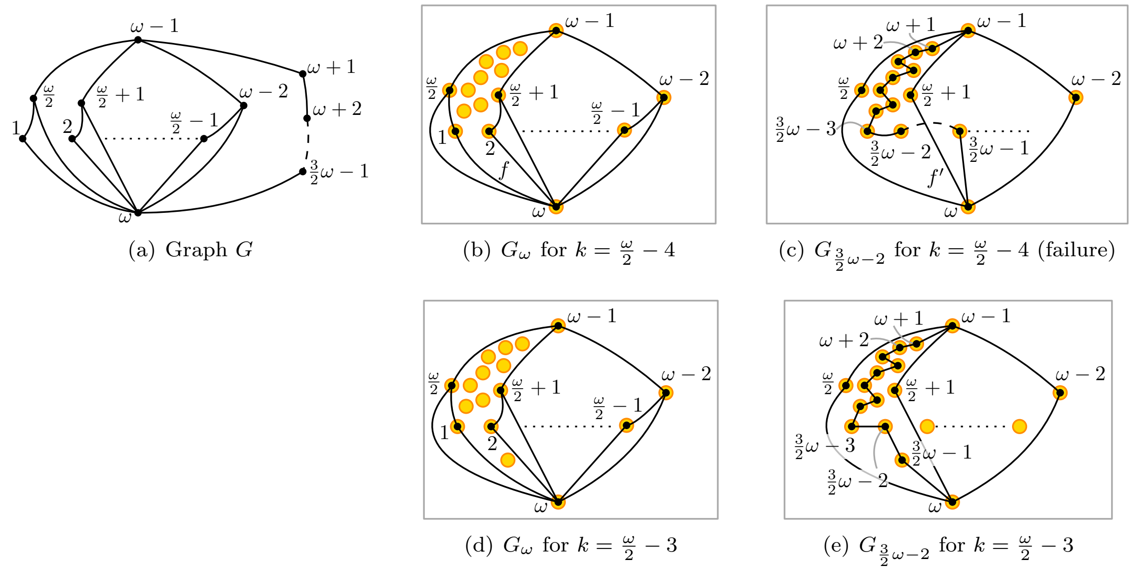
Consider the instance $\mathcal{S} = (G, 8, 0, \tau)$ in Figure 13(a), where the vertices are labeled with their subscript in the order $\tau = \langle v_1, v_2, \dots, v_{11}\rangle$. Graph G is a parallel composition of four components, three of which are a series of an edge and a triangle, which we call *flags*, and the other one is a path of length four. Observe that, in any planar embedding of $G_\omega = G_8$ at most two among v_1, v_2, and v_3 can be incident to the same face (see, e.g., Figure 13(b)). Graph G_{11} contains the paths $(v_7, v_4, v_8), (v_7, v_5, v_8), (v_7, v_6, v_8)$, and $(v_7, v_9, v_{10}, v_{11}, v_8)$. Since v_9, v_{10}, and v_{11} are mapped to the points where v_1, v_2 and v_3 are mapped, respectively, it is not possible to obtain a planar embedding of G_{11} (see, e.g., Figure 13(c)). Thus, $\mathcal{S}$ does not admit a realization.

To prove that $\mathcal{S} = (G, 8, k, \tau)$ is realizable for $k \geq \lfloor \frac{\omega}{2} \rfloor - 3 = 1$, suppose that v_1 and v_2 are drawn on the same face f and there is an extra point p inside f. In this case $\mathcal{S}$ is realizable, and G_8 and G_{11} are drawn as in Figures 13(d) and 13(e).

Consider now the case in which $\omega > 8$ and even. Graph G is similar to the one described in the previous case, but it has $\frac{\omega}{2} - 1$ parallel flags and a path of length $\frac{\omega}{2}$, as shown in Figure 14(a). In any planar embedding of G_ω, at most two vertices of the sequence $v_1, \dots, v_{\frac{\omega}{2}-1}$ can be inside the same cycle $v_\omega, v_i, v_{i+1}, v_{\omega-1}$ ($i = \frac{\omega}{2}, \dots, \omega - 2$). If $k = \frac{\omega}{2} - 4$, in G_ω the number of free points (i.e., points to which no vertex is mapped) is equal to the number of flag components minus 3. Let f be a face shared by two vertices of the sequence $v_1, \dots, v_{\frac{\omega}{2}-1}$, say v_1 and v_2, and assume that all the k free points are inside f (see, e.g., Figure 14(b)). For $G_{\omega+1}, \dots, G_{\frac{3}{2}\omega-2}$ all the vertices of the path can be drawn using the k extra points, as they are incident to the same face. Note that $G_{\omega+1}$ does not contain vertex v_1 and $G_{\omega+2}$ does not contain vertex v_2, thus face f has been merged with the faces of the flags comprehending v_1 and v_2, and this new face f' contains the two points to which v_1, v_2 were mapped. In $G_{\frac{3}{2}\omega-1}$, vertex $v_{\frac{3}{2}\omega-1}$ should be drawn in f' so to maintain planarity, but all the points in f' have been used to draw the $\frac{\omega}{2} - 2$ vertices of the path and the k free points are inside other faces (at most two are in the same face). Thus (G, ω, k, τ) is not realizable for $k = \frac{\omega}{2} - 4 < \lfloor \frac{\omega}{2} \rfloor - 3$ (see, e.g., Figure 14(c)). Note that with one more point on f', i.e., $k = \frac{\omega}{2} - 3 = \lfloor \frac{\omega}{2} \rfloor - 3$, $\mathcal{S}$ is realizable; see Figures 14(d) and 14(e).

Finally, consider the case in which $\omega > 8$ and odd. Again, graph G is similar to the one of the previous case, but there are $\frac{\omega+1}{2} - 1$ parallel flags, two of which are on the same parallel component, thus creating a *double flag*, and the path has length $\frac{\omega+1}{2}$, as shown in Figure 15(a).

In any planar embedding of G_ω, at most three vertices of the sequence $v_1, \dots, v_{\frac{\omega+1}{2}-1}$ can be inside the same cycle $v_\omega, v_i, v_{i+1}, v_{\omega-1}$ ($i = \frac{\omega+1}{2}, \dots, \omega - 2$), due to the presence of the double flag. Note that at most one face can be shared by more than two vertices of the sequence. If



(a) Graph G

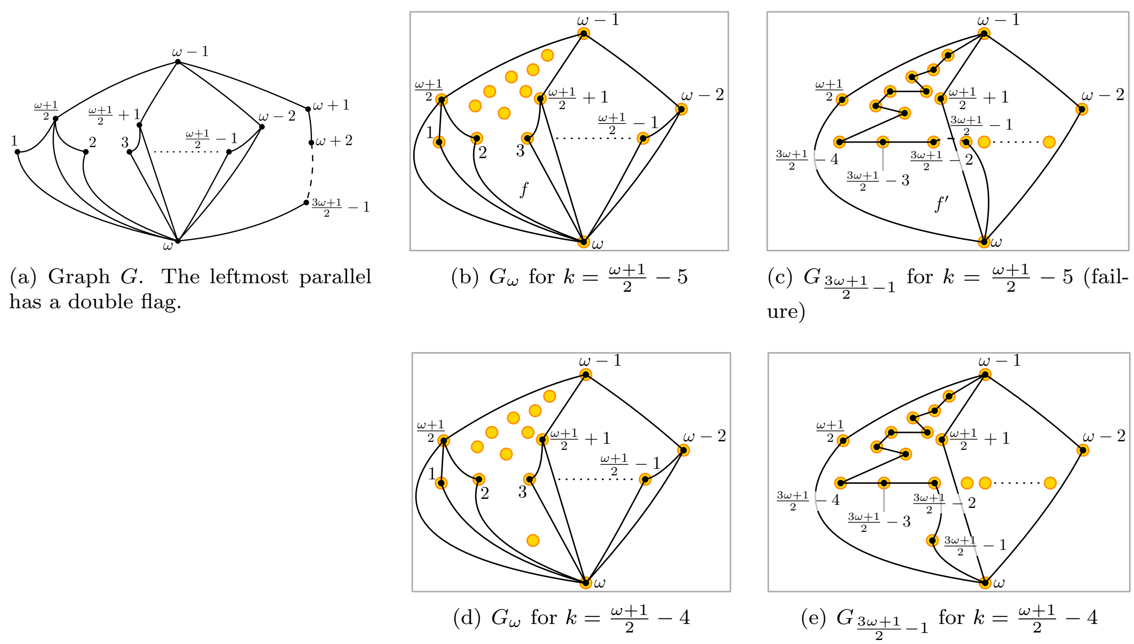
(b) G_ω for $k = \frac{\omega}{2} - 4$

(c) $G_{\frac{3}{2}\omega-2}$ for $k = \frac{\omega}{2} - 4$ (failure)

(d) G_ω for $k = \frac{\omega}{2} - 3$

(e) $G_{\frac{3}{2}\omega-2}$ for $k = \frac{\omega}{2} - 3$

Figure 14: Illustration for Theorem 7. Case $\omega > 8$ even. Graph G and drawings of G_ω and $G_{\frac{3}{2}\omega-2}$.



(a) Graph G. The leftmost parallel has a double flag.

(b) G_ω for $k = \frac{\omega+1}{2} - 5$

(c) $G_{\frac{3\omega+1}{2}-1}$ for $k = \frac{\omega+1}{2} - 5$ (failure)

(d) G_ω for $k = \frac{\omega+1}{2} - 4$

(e) $G_{\frac{3\omega+1}{2}-1}$ for $k = \frac{\omega+1}{2} - 4$

Figure 15: Illustration for Theorem 7. Case $\omega > 8$ odd. Graph G and drawings of G_ω and $G_{\frac{3\omega+1}{2}-1}$.

$k = \frac{\omega+1}{2} - 5$, in G_ω there is a number of free points (where no vertex is drawn) equal to the number of flag components minus 4. Let f be the face shared by two vertices of the sequence $v_1, \ldots, v_{\frac{\omega}{2}-1}$, say v_2 and v_3, one of which is part of the double flag containing also v_1, and assume that all the k free points are inside f (see, e.g., Figure 15(b)). For $G_{\omega+1}, \ldots, G_{\frac{3}{2}\omega-4}$ all the vertices of the path can be drawn using the k extra points, as they are incident to the same face. Note that $G_{\omega+1}$ does not contain vertex v_1, $G_{\omega+2}$ does not contain vertex v_2, and $G_{\omega+3}$ does not contain vertex v_3, thus the face f has been merged with the faces of the flags comprehending v_1, v_2, and v_3, and this new face f' contains the three points to which v_1, v_2, and v_3 were mapped. In $G_{\frac{3\omega+1}{2}-1}$, vertex $v_{\frac{3\omega+1}{2}-1}$ should be drawn in f' so to maintain planarity, but all the points in f' have been used to draw the $\frac{\omega+1}{2} - 2$ vertices of the path and the k free points are inside other faces (at most two are in the same face). Thus (G, ω, k, τ) is not realizable for $k = \frac{\omega+1}{2} - 5 < \lfloor \frac{\omega}{2} \rfloor - 3$ (see Figure 15(c)). Note that with one more point on f', i.e., $k = \frac{\omega+1}{2} - 4 = \lfloor \frac{\omega}{2} \rfloor - 3$, $\mathcal{S}$ is realizable; see Figures 15(d) and 15(e). □

7 Final Remarks and Open Problems

We conclude with some open research directions.

- Theorem 1 implies that the realizability testing of graph stories is paraNP-hard[1] when parameterized by k. On the other hand, Theorem 2 proves that the problem is in FPT when parameterized by $\omega + k$. For non-minimal graph stories, it remains open to establish the complexity of the realizability problem when parameterized by ω alone. The proof of Theorem 1 relies on the presence of isolated vertices. It would be interesting to extend the result to the case of connected or biconnected graphs.
- About minimal graph stories, we showed that for $\omega \geq 5$ there are stories of series-parallel graphs that are not realizable. For $k = 1$, the smallest ω for which we have a non-realizable story of a series-parallel graph is 10. What about the realizability of series-parallel graphs for $k = 1$ and $5 \leq \omega \leq 9$?
- We showed that every minimal graph story with $\omega = 5$ is 1-reroute realizable if and only if the graph does not contain K_5. Is any (minimal) graph story h-reroute realizable for h being a constant or a sublinear function of ω?
- We considered the scenario where vertices enter and exit one at a time. It would be interesting to study the case when ℓ vertices enter and exit at each time step ($2 \leq \ell \leq \omega$).
- We studied graph stories in the offline model for dynamic graphs. It would be interesting to extend the study to the online model or to the look-ahead model.
- Considering the result of Lemma 1, we focused on the topology of the problem. It remains open to study the complexity of realizing the graph story when the sequence of compatible embeddings is given.

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[1] A problem is paraNP-hard with respect to a certain parameter if it is NP-hard already for a constant value of the parameter.

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