Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 27, no. 7, pp. 565–580 (2023)
DOI: 10.7155/jgaa.00634

# The Complexity of Angular Resolution

*Marcus Schaefer*[1]

[1]School of Computing
DePaul University
Chicago, Illinois 60604, USA

| Submitted: July 2022 | Reviewed: July 2023 | Revised: August 2023 |
|---|---|---|
| Accepted: August 2023 | Final: August 2023 | Published: August 2023 |
| Article type: Regular paper | Communicated by: M. Kaufmann | |

**Abstract.** The angular resolution of a straight-line drawing of a graph is the smallest angle formed by any two edges incident to a vertex. The angular resolution of a graph is the supremum of the angular resolutions of all straight-line drawings of the graph. We show that testing whether a graph has angular resolution at least $\pi/(2k)$ is complete for $\exists\mathbb{R}$, the existential theory of the reals, for every fixed $k \geq 2$. This remains true if the graph is planar and a plane embedding of the graph is fixed.

## 1 Introduction

In graph drawing we measure the readability of a visualization by many parameters, including area, crossing number, crossing resolution, curve complexity, and angular resolution [9, Chapter 55].

The *(vertex) angular resolution* of a straight-line drawing of a graph is the smallest angle formed by any two incident edges of the graph. The *(vertex) angular resolution* of a graph is the supremum of the angular resolutions of all straight-line drawing of the graph.[1] If $G$ is planar we can require the drawing to be an embedding, that is, crossing-free, and we can even require the embedding to be isomorphic to a given plane embedding of $G$; in general, the definition of angular resolution does not require planar drawings, even of planar graphs. For a recent survey on angular resolution, see [15].

In this paper we settle the computational complexity of the angular resolution problem and some of its variants. Determining the angular resolution is as hard as deciding the truth of existential quantified sentences over the real numbers, that is, the *existential theory of the reals*. The corresponding complexity class is called $\exists\mathbb{R}$. Section 1.2 contains some background on $\exists\mathbb{R}$.

*E-mail address:* mschaefer@cdm.depaul.edu (Marcus Schaefer)



[1]When studying crossing numbers one typically excludes drawings in which more than two edges intersect in a crossing; we do not do so here, e.g. see the rose drawings of the complete graph in Figure 4. Allowing multiple crossings does not change the angular resolution of a graph: we can slightly perturb the vertices in the drawing to remove any multiple crossings. Since the angular resolution of a graph is defined as a supremum, its value is not affected.

**Theorem 1** *The following two problems are $\exists\mathbb{R}$-complete for every fixed $k \geq 2$.*

(*i*) *Deciding whether a graph has a straight-line drawing with angular resolution at least $\pi/(2k)$.*

(*ii*) *Deciding whether a graph has angular resolution at least $\pi/(2k)$.*

*Both problems remain $\exists\mathbb{R}$-complete if the graph is planar and we ask for an embedding, and if the drawing or embedding is fixed (combinatorially).*

Why do we require angles of the form $\pi/(2k)$, what happens for $k = 1$, and why do we distinguish variants (*i*) and (*ii*)? We answer those questions next.

**Why $\pi/(2k)$?** Our $\exists\mathbb{R}$-hardness proof is based on right angles. We need the angular resolution to be of the form $\pi/(2k)$, where $k \geq 2$, so we can combine $k$ of these angles to form a right angle.

**Angular Resolution $\pi/2$.** The case $k = 1$ is different. If the embedding is fixed, then Tamassia [23] showed that the angular resolution problem for $\pi/2$ can be solved efficiently using network flows (and that result extends to fixed drawings by planarizing the drawing in a first step). For flexible embeddings, Formann, Hagerup, et al. [8] showed that deciding whether a graph of maximum degree 4 has a straight-line embedding with angular resolution $\pi/2$ is **NP**-hard.[2] By a result of Bodlaender and Tel [5], building on Tamassia's network flow model, having a straight-line drawing with angular resolution at least $\pi/2$ implies the existence of an isomorphic *rectilinear drawing*, that is, an orthogonal grid drawing (all angles are multiples of $\pi/2$ and vertices can be placed on a grid). It follows that if the drawing or embedding is not fixed, the angular resolution problem for $\pi/2$ is **NP**-complete.

**Variants (*i*) and (*ii*).** Theorem 1 distinguishes between problem variants (*i*) and (*ii*) because of a subtlety in the definition of the angular resolution of a graph: it is defined as a supremum, not a maximum. Formann, Hagerup, et al. [8] already observed that the maximum may not actually exist; they constructed a graph on 11 vertices which has a drawing with angular resolution $\pi/3-\varepsilon$ for every $\varepsilon > 0$, but not for $\pi/3$. Figure 1 shows that this same result still holds for $\pi/4$. As a consequence, problems (*i*) and (*ii*) in the theorem are not a priori equivalent. Bieker [4] showed that problem (*i*) can be tested in $\exists\mathbb{R}$, and we extend his proof to (*ii*), which is a bit surprising, since taking a supremum translates into a universal quantifier.

## 1.1 Related Problems

Just as angular resolution is defined for angles formed by edges at vertices, crossing resolution is defined for angles formed by edges at crossings. Specifically, a straight-line drawing of a graph is a *right-angle crossing (*RAC*)* drawing if all crossings occur at right angles. It would be natural to define the *crossing resolution* of a graph as the supremum of the crossing resolutions of all straight-line drawings of the graph, but I have not been able to locate this definition in the literature. Crossing resolution, and the related *graphs with junctions* model introduced in [19], will be the starting point for our $\exists\mathbb{R}$-completeness proof.

Another notion similar to angular resolution is the *planar slope number* of a graph, that is, the smallest number $k$ of slopes so that the graph has a planar straight-line drawing in which the line-segments require only $k$ different slopes. Hoffmann [10] showed that deciding whether a graph has planar slope number at most $k$ is $\exists\mathbb{R}$-complete.

[2] Pretty much the same construction also works for straight-line *drawings*.

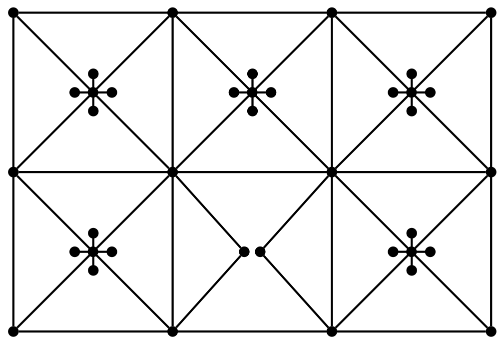

Figure 1: A graph with angular resolution $\pi/4$. The graph has a straight-line drawing with angular resolution $\pi/4 - \varepsilon$ for every $\varepsilon > 0$, but not with $\pi/4$. The two vertices in the lower middle square would be forced to overlap. For the proof note that two triangles that share an edge cannot be nested inside each other in a drawing with angular resolution $\pi/4$.

Also related is the question whether a graph has a straight-line drawing with a given rotation system. The *rotation* at a vertex is a cyclic permutation of edges incident to the vertex, and a *rotation system* for a graph specifies a clockwise rotation for each vertex. Testing whether a graph can be realized with a given rotation system is $\exists\mathbb{R}$-complete, a result due to Kynčl [11]. This remains true even if the maximum degree of the graph is bounded (by 131) as shown in [18]; the bounded-degree condition takes us a step closer to the angular resolution problem, since graphs with fixed angular resolution have bounded degree.

### 1.2 The Existential Theory of the Reals

The *existential theory of the reals* is the set of existentially quantified true sentences over the real numbers, $\mathsf{ETR} := \{(\exists x \in \mathbb{R}^k)\ \varphi(x)\}$, where $\varphi$ is a Boolean formula over the non-logical signature $(0, 1, +, \cdot)$. The complexity class $\exists\mathbb{R}$ is then defined as the set of all problems that polynomial-time many-one reduce to ETR, and $\exists\mathbb{R}$-hardness and $\exists\mathbb{R}$-completeness are defined in the usual way, see, for example [16]. $\exists\mathbb{R}$ captures the complexity of many problems in graph drawing and computational geometry, and the number of problems identified as being $\exists\mathbb{R}$-complete is steadily growing. Recent examples of $\exists\mathbb{R}$-hard problems include the art gallery problem [2], polygon coverings [1], area universality [7], and, outside graph drawing/computational geometry, continuous constraint satisfaction problems [13] and training neural networks [3].

In terms of traditional complexity classes, it is known that $\exists\mathbb{R}$ lies between **NP** [21] and **PSPACE** [6], and it is expected to lie strictly between the two. $\exists\mathbb{R}$-hardness implies **NP**-hardness, since **NP** $\subseteq \exists\mathbb{R}$, so all the angular resolution problems in Theorem 1 are **NP**-hard as well. Matoušek [12] gives an excellent survey (and introduction) to the existential theory of the reals.

## 2 Graphs with Junctions

We introduce two special types of vertices, called junctions, which come with restrictions on their rotation (the clockwise permutation of incident edges) and the angles between incident edges. The following definitions and illustrations are taken verbatim from [17].

- a $\top$-*junction* is a vertex $v$ which is incident to three special edges $e_1, e_2, e_3$. A straight-line drawing *respects* the $\top$-junction if there are right angles between $e_1$ and $e_2$ and $e_2$ and $e_3$ at $v$; additional edges at $v$ can occur, at any angle, between $e_1$ and $e_3$ (opposite of $e_2$),

- a $\times$-*junction* is a vertex $v$ which is incident to four special edges $e_1, e_2, e_3, e_4$. A straight-line drawing *respects* the $\times$-junction if the rotation of the special edges at $v$ is $e_1e_2e_3e_4$, or the reverse, and there are right angles between $e_i$ and $e_{i+1}$, for $1 \leq i \leq 3$; additional edges may occur, at any angle, inside one of the quadrants, e.g. between $e_3$ and $e_4$.

Figure 2 shows these junctions, and how we symbolize them in drawings.

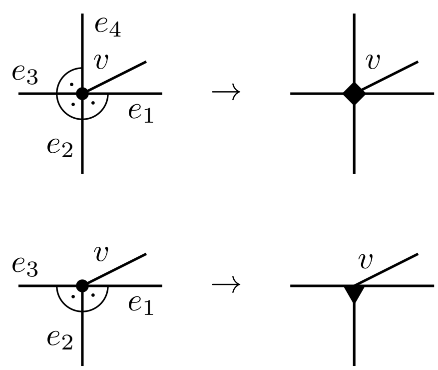


Figure 2: $\top$- and $\times$-junctions, and how we draw them in graphs using a ▼ and a ⧫. Each of the junctions is shown with one additional edge.

Two drawings of a graph (with or without junctions) are *isomorphic* if there is a homeomorphism of the plane (which may be orientation-reversing) that maps the graphs to each other.

We introduced drawings with junctions in [17] as an intermediate problem to show that RAC-drawability of graphs is $\exists\mathbb{R}$-complete. We can use the same problem in our reduction to the angular resolution problem.

**Theorem 2 (Schaefer [17, 19, Theorem 3])** *Given a graph $G$ with $\top$- and $\times$-junctions and a plane drawing $D$ of $G$, it is $\exists\mathbb{R}$-complete to decide whether $G$ has a drawing respecting all junction constraints. The problem remains $\exists\mathbb{R}$-complete even with the following restrictions:*

(*i*) *the only non-junction vertices in $G$ have degree 1, all $\times$-junctions have at most one additional edge, and all $\top$-junctions have at most two additional edges, and*

(*ii*) *if $G$ does have a straight-line drawing respecting all junction constraints, it has such a drawing $D_+$ which is isomorphic to $D$, and*

(*ii*-$\times$) *if a $\times$-junction has an additional edge, it forms an angle of $\pi/4$ in $D_+$ with the two edges it neighbors in the rotation, and*

(*ii*-$\top$) *if a $\top$-junction has two additional edges, they form a right angle in $D_+$.*

We can eliminate ⊤-junctions from Theorem 2 by simulating them with ×-junctions; we did the opposite when proving ∃ℝ-hardness of RAC-drawability in [19], where we simulated ×-junctions using ⊤-junctions.

**Corollary 3** *Given a graph $G$ with ×-junctions (with at most one additional edge each) and a planar drawing $D$ of $G$, it is ∃ℝ-complete to decide whether $G$ has a straight-line drawing respecting all junction constraints with angular resolution at least $\pi/4$. We can assume that*

($a$) *the only non-junction vertices in $G$ have degree* 1*,* 4*,* 5*, or* 6 *and*

($b$) *if $G$ does have a straight-line drawing respecting all junction constraints, it has such a drawing $D_+$ which is isomorphic to $D$ and has angular resolution at least $\pi/4$.*

**Proof:** By Theorem 2 we can assume that we are given a graph $G$ with ⊤- and ×-junctions, and a planar drawing $D$ of $G$ with $G$ and $D$ satisfying the restrictions stated in the theorem.

From $G$ we construct a graph $G'$ by replacing each ⊤-junction with the gadget shown in Figure 3; this gadget consists of six standard vertices with degrees ranging from 4 to 6 and thirty-two ×-junctions, as well as some vertices of degree 1, which are not explicitly shown, to cap the short stubs. We also extend the planar drawing $D$ of $G$ to a planar drawing $D'$ of $G'$. If $G$ has

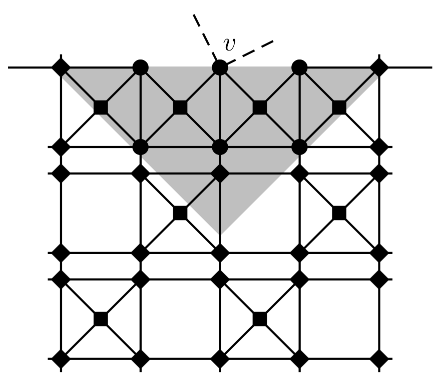


Figure 3: Simulating a ⊤-junction (shaded gray) using ×-junctions. There are six standard vertices, the corners of the two squares containing $v$, the remaining 32 vertices are ×-junctions. The short stubs are capped by degree-1 vertices (not shown), the long stubs connect to other gadgets or vertices.

straight-line drawing $D_+$ isomorphic to $D$, then $G'$ has a straight-line drawing $D'_+$ isomorphic to $D'$: we can simply replace the drawing of each ⊤-junction with the gadget shown in Figure 3. Condition ($a$) then follows from condition ($i$) of the theorem.

Condition $(ii)_\times$ of Theorem 2 tells us that the ×-junctions of $D_+$ have angular resolution at least $\pi/4$, so their counterparts in $D'_+$ still do. And the newly introduced ×-junctions (in each gadget) all have angular resolution at least $\pi/4$. This leaves the six vertices of degree 4 to 6 (all vertices of degree 1 having angular resolution $2\pi$); we see that all these vertices also have angular resolution at least $\pi/4$, the main vertex to check is $v$; if it is incident to two edges, which form a right angle (by $(ii)_\top$). And any edge incident to $v$ in $D_+$ lies above $e_1$ and $e_3$, which means these edges form an angle of at least $\pi/4$ with the two diagonal edges incident to $v$ in $D'_+$. This proves condition ($b$).

For the other direction, let us assume that $G'$ has a straight-line drawing $D_+$ with angular resolution at least $\pi/4$. We need to argue that the gadget shown in Figure 3 is drawn as shown. Let us label the twelve squares in the figure like a chessboard, so the bottom left square is $A1$ and the top right square is $D3$. We first note that $A1$, $B2$, $C1$, and $D2$ each consist of five $\times$-junction, forcing them to be geometric squares. This also forces $A2$, $B1$, $C2$ and $D1$ to be squares. Now $A3$ has to be a square (because the angular resolution is at least $\pi/4$, and its two vertices on the left side are $\times$-junctions), so its height and width are the same, making it a square. By a symmetric argument, $D3$ is a square. Now $A3$ has the same size as $A2$ which has the same size as $D2$, which has the same size as $D3$. The angular resolution of at least $\pi/4$ then forces $B3$ and $C3$ to have the same width and height (they are geometric squares with the top edge missing). This proves that the gadget is drawn as shown, including the one or two additional edges attached to $v$, which, because of the angular resolution, must lie above the line through the top of $A3$, $B3$, $C3$ and $D3$. This means, we can replace the whole gadget with a $\top$-junction, yielding a straight-line drawing $D_+$ of $G$.

In summary, we have shown that $G$ has a straight-line drawing $D_+$ isomorphic to $D$ if and only if $G'$ has a straight-line drawing $D'_+$ isomorphic to $D'$. Since testing the former is $\exists\mathbb{R}$-complete, by Theorem 2, the later is $\exists\mathbb{R}$-hard, even, as we showed, $G'$ and $D'_+$ satisfy conditions $(a)$ and $(b)$ of the corollary. $\square$

# 3 Proof of Theorem 1

## 3.1 $\exists\mathbb{R}$-membership

Showing that a problem lies in $\exists\mathbb{R}$ is typically straightforward, but here we face two obstacles: we have to express $\cos(\pi/(2k))$, and the definition of angular resolution in part $(ii)$ of the theorem hides a universal quantifier, because it is defined as a supremum. Both problems can be solved.

Bieker [4] shows membership in $\exists\mathbb{R}$ for problem $(i)$, but his proof is restricted to a type of angles that does not work for us. Following Bieker's proof, it is easy to construct for a given $n$-vertex graph $G$ a predicate $A_G(x, y, t)$ with $x, y \in \mathbb{R}^n$, and $t \in \mathbb{R}$ that expresses that if we place the $i$-th vertex of $G$ at position $(x_i, y_i)$, for all $i$, then we obtain a straight-line drawing of $G$ with angular resolution $\alpha$, where $0 \leq \alpha \leq \pi/2$ and $\cos(\alpha) \geq t$.

Also, it is easy to write predicates $\Pi_G(x, y)$ that express that if we place the $i$-th vertex of $G$ at position $(x_i, y_i)$, for all $i$, then we obtain a straight-line *embedding* of $G$, and a variant $\Pi'_D(x, y)$ that restricts the embedding to be isomorphic to a given plane embedding $D$ of $G$.

The values $\cos(\pi/(2k))$ are so-called trigonometric numbers, and it is known that all trigonometric numbers are algebraic, that is, for every $\pi/(2k)$, there is a nonzero polynomial $p$ with integer coefficients such that $p(\pi/(2k)) = 0$. As a matter of fact, the $k$-th Chebyshev polynomial (of the first kind) will do (this is probably a folklore result, but can be found in [24], for example). We can also pick an interval $(r, s)$, with $r, s \in \mathbb{Q}$, such that $\pi/(2k)$ is the only root of $p$ in $(r, s)$.

Then $G$ has a drawing with angular resolution at least $\pi/(2k)$ if and only if

$$(\exists x \in \mathbb{R}^n, y \in \mathbb{R}^n)(\exists t \in \mathbb{R})\ [p(t) = 0 \wedge (r < t < s) \wedge A_G(x, y, t)]$$

is true. This shows membership of problem $(i)$ in $\exists\mathbb{R}$.

For part $(ii)$ we can then express that $G$ has angular resolution at least $\pi/(2k)$ as

$$(\forall \varepsilon > 0)(\exists x \in \mathbb{R}^n, y \in \mathbb{R}^n)(\exists t \in \mathbb{R})\ [p(t) = 0 \wedge (r < t < s) \wedge A_G(x, y, t - \varepsilon)].$$

The formula is of the form $\forall\exists\mathbb{R}$, but we can eliminate the $\forall$ quantifier using a known result. Let $\Phi(\varepsilon) = (\exists x \in \mathbb{R}^n, y \in \mathbb{R}^n)(\exists t \in \mathbb{R})[p(t) = 0 \wedge (r < t < s) \wedge A_G(x, y, t - \varepsilon)]$. The formula $\Phi(\varepsilon)$ is monotone in $\varepsilon$ in the sense that if $\varepsilon' > \varepsilon > 0$, then $\Phi(\varepsilon)$ implies $\Phi(\varepsilon')$. Using [20, Lemma 4.1], we can then conclude that $(\forall \varepsilon > 0)[\Phi(\varepsilon)]$ is equivalent to $(\exists \varepsilon, z)[\psi(\varepsilon, z)]$ for some formula $\psi$ whose length is polynomial in the length of $A_G$ and $p$.[3] This shows that problem $(ii)$ lies in $\exists\mathbb{R}$.

If necessary, we can add $\Pi_G(x, y)$, or $\Pi'_D(x, y)$ if we are given a plane embedding $D$ of $G$, to the inner part of the formula for $(i)$ or $(ii)$. This yields $\exists\mathbb{R}$-membership for all remaining variants claimed in the theorem.

## 3.2 ∃ℝ-hardness

The theorem claims $\exists\mathbb{R}$-hardness of testing the angular resolution of $(i)$ a drawing, and $(ii)$ a graph. We will show part $(i)$ in this section and then see how it implies part $(ii)$ in Section 3.3.

Using Corollary 3 we are left with showing how to simulate $\times$-junctions using angular resolution constraints. We know that each $\times$-junction has at most one additional edge, and that the angular resolution of a straight-line drawing $D_+$ respecting the junctions in $D$ and isomorphic to $D$ is at least $\pi/4$.

We first complete the proof for angular resolution $\pi/4$ in Section 3.2.2, since the gadgets are simpler for this case. We then treat angular resolution $\pi/(2k)$ for $k > 2$ in Section 3.2.3. In the preparatory Section 3.2.1 we study some properties of drawings of the complete graph.

### 3.2.1 Rose Drawings

To build gadgets, we will work with a result essentially due to Formann, Hagerup, et al. [8]. They show that the complete graph $K_n$ has angular resolution $\pi/n$ and that this angular resolution is achieved by placing the $n$ vertices at the corners of a convex $n$-gon, let us call this the *(mystic) rose* drawing.[4]

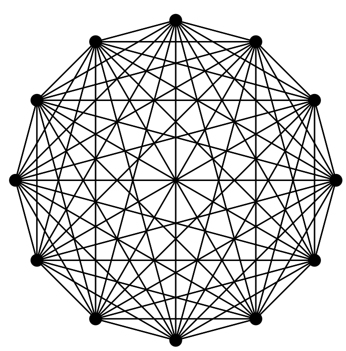

Figure 4: A mystic rose drawing of a $K_{12}$ with angular resolution $\pi/12$.

[3]Lemma 4.1 in [20] is stated in a parameterized version, with parameter $y \in \mathbb{R}^\ell$, but the proof only establishes the case $\ell = 0$; fortunately, that's the only case we need here.

[4]Both the drawings and the term are old, but it is not clear when the term was first applied to this type of drawing. The earliest instance of this usage I could find is in a book from 1906 [22, pp. 40, 64].

**Lemma 1** *Any straight-line drawing of a $K_n$ with angular resolution $\pi/n$ is a rose drawing.*

**Proof:** Let $P$ be the $k$-gon formed by the outer hull of the drawing, where $k \leq n$. Then the corners of $P$ must be vertices of $K_n$. The total inner angle of the polygon is $(k-2)\pi$, so at least one of the inner angles, say at $v$, is at most $(k-2)\pi/k$; since $v$ is incident to at least $n-2$ angles inside $P$, the angular resolution is at most $(k-2)\pi/(k(n-2))$. Then, by assumption, $(k-2)\pi/(k(n-2)) \geq \pi/n$, so $n(k-2) \geq k(n-2)$. Since $k \leq n$, this forces $n = k$, and all vertices of $K_n$ lie on $P$. Then each inner angle must be at least $(n-2)\pi/n$, but this is the most it can be, so each inner angle is exactly $(n-2)\pi/n$ which implies that $P$ is a regular $n$-gon. □

Let us call the interior of the region bounded by the outer polygon of a rose drawing of a $K_n$ the *inner region*.

**Lemma 2** *In a straight-line drawing with angular resolution $\pi/n$ of two $K_n$ which have at least one vertex in common, the inner regions of the two $K_n$ are disjoint, unless they are identical.*

In other words, if the drawings are not identical, they must look like one of the two illustrations shown in Figure 5.

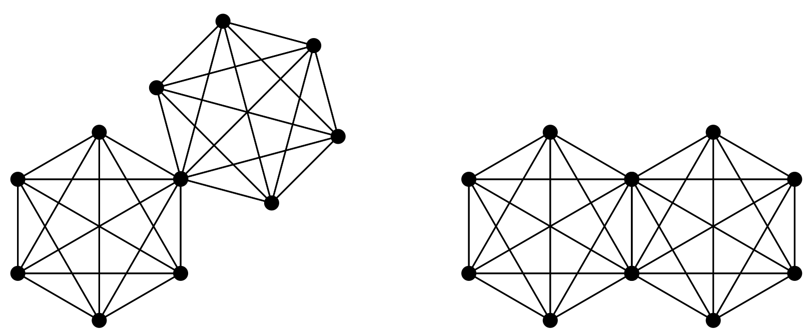

Figure 5: Two rose drawings of a $K_6$ intersecting in one vertex (on the *left*) and in two vertices (on the *right*).

**Proof:** Any two consecutive edges in a rose drawing of a $K_n$ form an angle of exactly $\pi/n$, so any other edge incident to a vertex of the rose drawing cannot lie in the inner region of the rose drawing, unless it overlaps an edge of that drawing which would lead to a vertex of one of the two $K_n$ lying in the interior of an edge of the other $K_n$ which we do not allow. Hence, if the two $K_n$ have exactly one vertex in common, their interior regions are disjoint. If the two $K_n$ have at least two vertices in common, then the common edge must lie on the circumference of both rose drawings. If the two $K_n$ have a third vertex in common, then their rose drawings are bounded by the same outer circle and must therefore be identical. Otherwise, the two $K_n$ have exactly two vertices in common, and their inner regions are disjoint. □

### 3.2.2 Angular Resolution $\pi/4$

We distinguish two cases, based on whether $D_+$ is required to be an embedding or not.

**Case 1: $D_+$ is not required to be an embedding.**

Consider the two gadgets shown in Figure 6. In straight-line drawings of these gadgets with angular resolution at least $\pi/4$, each $K_4$ is drawn as shown, by Lemma 1. Since the $K_4$s cannot overlap by Lemma 2, the five $K_4$ in the left gadget form a $\times$-junction without an additional edge, and the ten $K_4$ in the right gadget form a $\times$-junction with an additional edge. Both gadgets are drawn as shown, forcing the four (or five) edges incident to the gadget to leave at the required angles; note that for the $\times$-junction with an additional edge we do not *have* to enforce the angle, but it turns out to be easier to do so.

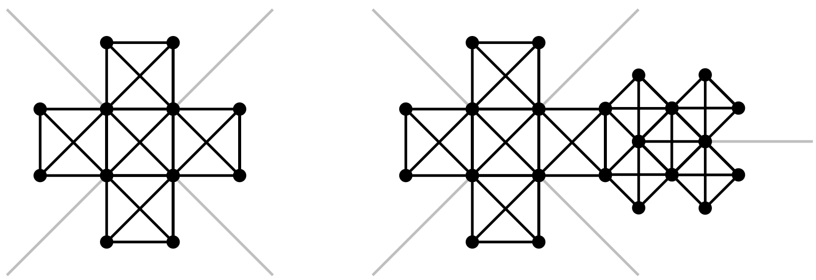

Figure 6: Angular gadgets simulating a $\times$-gadget without (*left*) and with (*right*) an additional edge.

Let $G'$ be the graph obtained from $G$ by replacing each $\times$-junction with the appropriate gadget, and let $D'$ be the corresponding drawing obtained from $D$. Then $G'$ has a straight-line drawing $D_+$ with angular resolution at least $\pi/4$ if and only if $G$ has a straight-line drawing respecting the junctions. If the straight-line drawing exists, then there is a straight-line drawing $D_+$ of $G'$ with angular resolution at least $\pi/4$ isomorphic to $D'$.

This completes the proof of $\exists\mathbb{R}$-hardness in Theorem 1 for the case that $D_+$ is not required to be an embedding.

**Case 2: $D_+$ has to be an embedding.**

The proof is similar to Case 1, we only modify the gadgets in Figure 6 by replacing each crossing with a dummy vertex. This turns each $K_4$ into a $W_4$, a wheel on five vertices. The argument in Lemma 1 also applies to $W_4$ as long as we know that the center of the wheel lies inside the 4-gon; but this has to be the case, since otherwise two of the outer vertices of the wheel would have to lie inside one of the triangles, forcing two right angles in that triangle, which is not possible. We conclude that a plane drawing of $W_4$ with angular resolution at least $\pi/4$ is a square with four spokes lying on the two diagonals. The argument from Case 1 then still applies, so we can conclude that the gadgets shown in Figure 6, modified to replace $K_4$ with $W_4$ work the same way as they do in the drawing case.

Hence, we can construct $G'$ and $D'$ as in Case 1, with the difference that $D'$ is a plane embedding. The conclusion then remains exactly the same, completing the $\exists\mathbb{R}$-hardness proof in Theorem 1 for the case that $D_+$ has to be an embedding.

### 3.2.3 Angular Resolution $\pi/(2k)$ for $k > 2$

We start with the embedding case, since it is easier for $k > 2$.

**Case 1: $D_+$ has to be an embedding.**
We base the $\times$-gadget on a modified wheel; in a drawing of a wheel $W_{4k}$ with angular resolution at least $\pi/(2k)$, the angles between neighboring spokes have to be exactly $\pi/(2k)$. Since the rotation system of a 3-connected graph, like the wheel, is uniquely determined in an embedding, we know that the angle between two spokes that are separated by exactly $k-2$ other spokes, is $\pi/2$. The spokes $ac$ and $bc$ in Figure 7 illustrate that situation. We modify the wheel, by making both $ac$ and $bc$ spokes in a $W_{2k} - v$, with centers $a$ and $b$, where $v$ is the vertex opposite $c$, as can be seen in Figure 7.

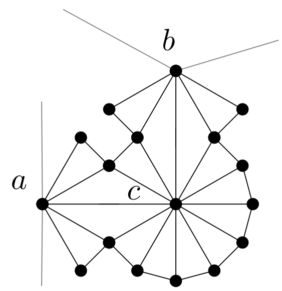


Figure 7: The modified wheel gadget $M_k$ for $k = 3$, with additional edges (in gray) attached to $a$ and $b$. Both $ac$ and $bc$ are spokes in a common $W_{12}$, and each is a spoke in a $W_6 - v$.

Additional edges attached to $a$ or $b$ can form an angle of at most $\pi$ (and at least $\pi/(2k)$, of course).

For the plane $\times$-gadget, we assemble three $M_k$ as shown in Figure 8. Each of the $a$- and $b$-vertices of the three gadgets is attached to two additional edges, four of which meet in a central vertex $c$, and the other four of which become $e_1$, $e_2$, $e_3$, and $e_4$. We connect each of the three gadgets by a separate edge to the center $c$, and attack two more $(W_{2k} - v)$-gadgets to the free $a$- and $b$-vertices, as shown in the upper/right quadrant of Figure 8.

The combination of the three $M_k$-gadgets enforces that $e_1$ and $e_3$, as well as $e_2$ and $e_4$, are collinear, and the two lines are orthogonal *as long as we know* that $e_1$, $e_2$, $e_3$ and $e_4$ lie on the outer face of the gadget. This we can enforce by connecting neighboring $\times$-gadgets. Figure 9 shows an example of a $\times$-junction with an additional edge between a degree-1 vertex to the left and another $\times$-junction on the right.

**Case 2: $D_+$ is not required to be an embedding.**
Let $Q_n$ be the gadget shown in Figure 10 on the left, that is, a 4-cycle $a$, $b$, $c$, $d$, in which one edge, $bc$, has been identified with a common edge of two $K_n$.

**Lemma 3** *In a straight-line drawing of $Q_n$ with angular resolution at least $\pi/n$, the inner angles at vertices $a$ and $d$ are exactly $\pi/n$.*

**Proof:** If the 4-gon $abcd$ is not simple (crossing-free), as shown on the right in Figure 10, then $ad$ crosses $bc$. But then the angle $\delta$ at $d$ is smaller than the angle $\angle(c, d, b)$ which is at most

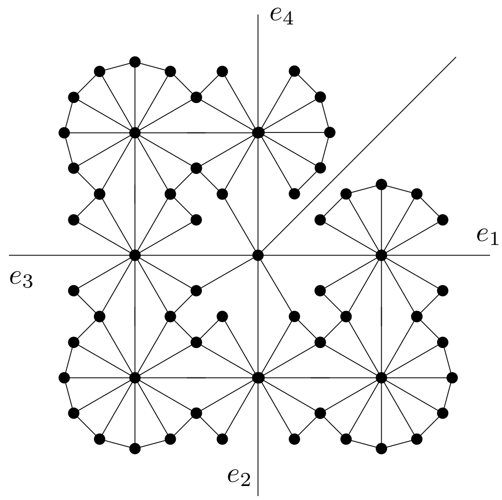


Figure 8: The plane ×-gadget replacing each ×-junction, pictured with the additional edge for $k = 3$.

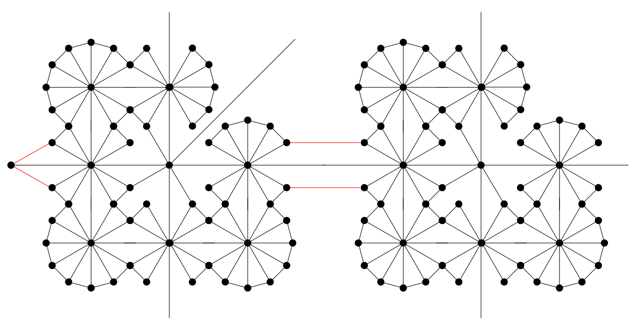

Figure 9: Connecting ×-gadgets to enforce rotation; connecting edges are drawn in red.

$\pi - (\pi - \pi/n) = \pi/n$ contradicting the assumption about the angular resolution. Hence, the 4-gon is simple.

Let $\alpha$, $\beta$, $\gamma$, $\delta$ be the inner angles at $a$, $b$, $c$ and $d$. Then $\alpha, \beta \in [\pi - \pi/n, \pi + \pi/n]$, because $ab$ and $cd$ have to lie in the exterior of the rose drawings. In a 4-gon we have $\alpha + \beta + \gamma + \delta = 2\pi$, so it follows that $\gamma + \delta \leq 2\pi - 2(\pi - \pi/n) = 2\pi/n$. Since $\gamma$ and $\delta$ have to be at least $\pi/n$, we conclude that $\gamma = \delta = \pi/n$ and therefore $\alpha = \beta = \pi - \pi/n$. □

With $Q_n$ as a building block to force angles of degree $\pi/n$ we can complete the construction, but we need to distinguish a couple of cases based on $k$ mod 4.

**Case 2.1.1:** $k = 0$ mod 4**.**

The first two cases are $k = 4$ and $k = 8$, and the ×-gadget for each, with additional edge shown, is pictured in Figure 11. We construct the gadget by taking four copies of $K_{2k}$. We choose a Hamiltonian cycle in each copy and identify the $k$-th edge of that cycle with the first edge of the

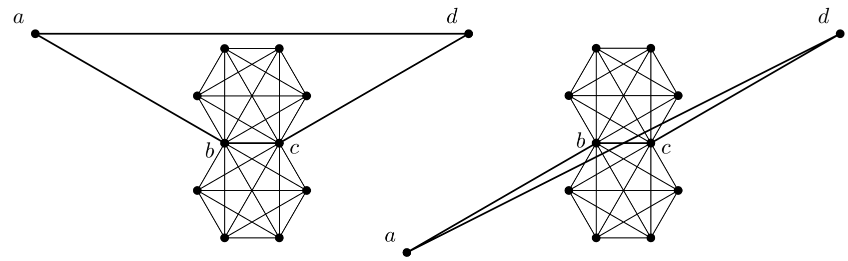


Figure 10: Drawings of gadget $Q_n$ for $n = 6$ with angular resolution $\pi/n$ *(left)*, and angular resolution less than $\pi/n$ *(right)*.

Hamiltonian cycle of the next $K_{2k}$. To all remaining edges of the Hamiltonian cycles we attach a $Q_{2k}$-gadget by identifying the edge with the *ad*-edge of the $Q_{2k}$-gadget.

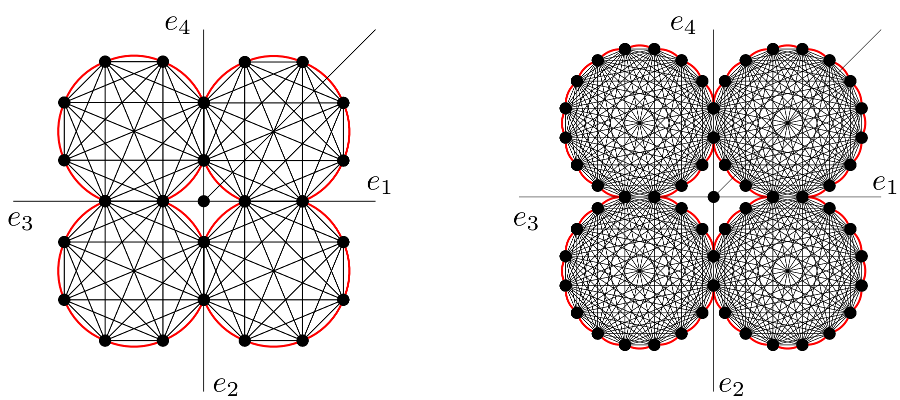


Figure 11: The ×-gadget replacing each ×-junction, pictured with the additional edge; the $Q_{2k}$-gadgets are symbolized by red arcs. *Left:* for $k = 4$, *right:* for $k = 8$.

Since the four $K_{2k}$ share edges, they all have the same size; the $Q_{2k}$-gadgets force which edges appear on the outer cycle of each $K_{2k}$ and in which order. This already implies that in a straight-line drawing with angular resolution at least $\pi/(2k)$, the four rose-drawings look as shown in Figure 11. Then the angles at which $e_1$, $e_2$, $e_3$ and $e_4$ leave the drawing are forced, and similarly for the four edges connecting to the central vertex. If there is an additional edge attached to it, the angle of that edge is not forced, but that was not required. It can be drawn as shown, however, forming an angle of $\pi/4$ with both $e_1$ and $e_4$, since the $Q_{2k}$-gadgets it intersects do not have vertices along that line (see Figure 10: there are no vertices along the axis of symmetry orthogonal to $bc$).

**Case 2.1.2:** $k = 2 \bmod 4$**.**

We treated the case $k = 2$ in Section 3.2.2, so the first interesting case is $k = 6$. The construction is similar to the case $k = 0 \bmod 4$, and for ×-junctions without additional edge, we can use the ×-gadget for that case. However, if there is an additional edge, that gadget no longer works, since the additional edge would pass through vertices of the gadget when it is drawn so as to bisect the right angle between the lines through $e_1$ and $e_4$. Instead of attaching the additional edge to the center vertex, we attach it to the outmost vertex of the $K_{2k}$ which the edge would otherwise intersect. At this point, we have to enforce that the edge leaves the gadget at the right angle, to simulate it starting at the center vertex. For this we combine $k$ of the $Q_{2k}$-gadgets as shown in Figure 12 to build a gadget $R_{2k}$ that forces a right angle. (Figure 12 illustrates the case $k = 3$, rather than $k = 6$, to keep the drawing readable.)

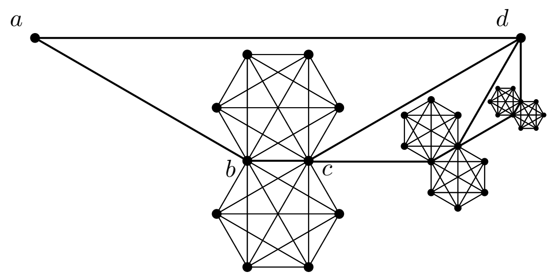


Figure 12: $R_6$: Using three $Q_6$ to force a right angle at vertex $d$.

We attach two $R_{2k}$-gadgets to a $K_{2k}$ by identifying the $ad$-edges of the $R_{2k}$-gadgets with two adjacent edges, the $d$-vertices of the gadgets becoming the shared vertex of the two $R_{2k}$-gadgets. In a straight-line drawing of angular resolution at least $\pi/(2k)$, an additional edge at the shared vertex must lie on a ray through the center of the rose drawing of the $K_{2k}$, see Figure 13.

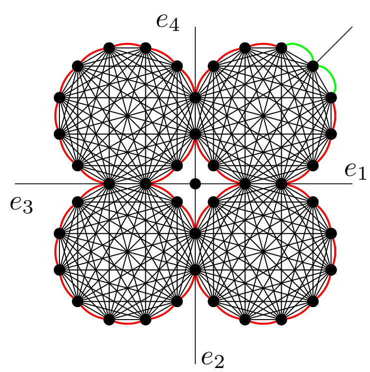


Figure 13: The $\times$-gadget for $k = 6$, pictured with the additional edge; the $Q_{2k}$-gadgets are symbolized by red arcs, and the two $R_{2k}$-gadgets by green arcs.

**Case 2.1.3:** $k = 1, 3 \bmod 4$**.**
The remaining two cases are similar to the cases we have already covered, except that the rose drawings of $K_{2k}$ in these cases do not contain two outer edges which lie on lines forming a right angle. This can be addressed in various ways; we arrange four $K_{2k}$ as shown in Figure 14. The two $K_{2k}$ in the top half and the bottom half have the same size, since they share an edge. Then the $K_{2k}$ at the bottom and the top also have the same size, since their distance from the same

points on the boundary is the same (the edges on the boundary, and their order, is forced by the $Q_{2k}$-gadgets).

Similar to the construction of $R_{2k}$, the gadget forcing a right-angle, we can build a gadget $A_{2k}$ that forces an angle of $\pi(1/2 - 1/k)$. We attach a copy to each side of the edge connecting the bottom two $K_{2k}$ to the center vertex. This forces the center vertex to be geometrically centered in a drawing with angular resolution at least $\pi/(2k)$.

We had to distinguish the cases $k = 0$ and $k = 2 \bmod 4$, since for $k = 2 \bmod 4$ there were vertices of the $\times$-gadget along the diagonal lines, blocking an additional edge starting at the central vertex. We do not have the same problem for $k = 1$ and $k = 3 \bmod 4$: in both cases, the additional edge does not pass through any $K_{2k}$- or $Q_{2k}$-vertices when drawn at a $\pi/4$-degree angle, since the $K_{2k}$- and $Q_{2k}$-vertices can be made to lie arbitrarily close to the center of the edge they are attached to, and the additional edge cuts through the attachment edge off-center, see Figure 14.

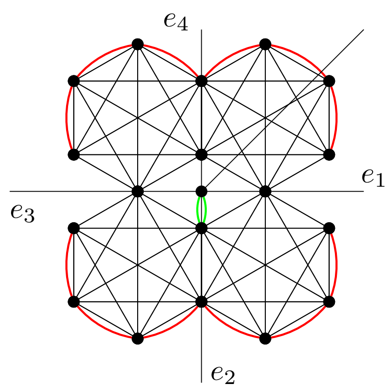


Figure 14: The $\times$-gadget for $k = 3$, pictured with the additional edge; the $Q_{2k}$-gadgets are symbolized by red arcs, the two $A_{2k}$-gadgets in green.

## 3.3 Angular Resolution of a Graph

At this point we have established part $(i)$ of Theorem 1. We still have to argue part $(ii)$. Recall that part $(ii)$ differs in considering the supremum of the angular resolutions of straight-line drawings of $G$. As we saw in Figure 1 this can make a difference, but only because as we go to the limit some vertex would overlap with another vertex or an edge it is not incident to. We claim that this cannot happen in our specific $\exists\mathbb{R}$-hardness construction: The gadgets we use to simulate junctions in the proof of part $(i)$ all have drawings with angular resolution exactly $\pi/(2k)$. So realizability in the limit could only be affected when combining multiple gadgets, which we do by identifying vertices and edges of gadgets. However, except for these explicit vertex- and edge-overlaps, the gadgets do not interfere with each other, or get close to each other. So a lack of realizability is due to the same vertex being forced to lie in different locations by different gadgets, not by a vertex getting close to another vertex or edge. This shows that for the graphs and drawings constructed in the proof of part $(i)$, the angular resolution equals the angular resolution achieved by the drawing $D_+$, so the supremum is a maximum, and part $(ii)$ is $\exists\mathbb{R}$-hard as well.

## 4 Open Questions

**Angles.** We have shown that angular resolution is $\exists\mathbb{R}$-complete for an infinite family of angles; our construction required angles of the form $\pi/(2k)$, for $k \geq 2$, so we could form right angles, which in turn allowed us to build on results from [19] on RAC-drawings. It's likely that the construction in [19] can be modified to work with other, fixed, angles, suggesting that angular resolutions of $\alpha/(2k)$ lead to $\exists\mathbb{R}$-hardness for all $0 < \alpha < \pi$, $k \geq 2$ as well (membership in $\exists\mathbb{R}$ will depend on $\alpha$, of course). On the other hand, showing angular resolution $\pi/3$ hard for $\exists\mathbb{R}$ looks more challenging.

**Degree Bounds.** For a specific angular resolution, we can ask whether the problem remains $\exists\mathbb{R}$-complete if we restrict the maximum degree; for example, for angular resolution $\pi/4$, our construction requires graphs of degree eight; can that be reduced? We cannot expect the maximum degree to be less than four in this case, since Mukkamala and Pálvöl [14] showed that all cubic graphs have straight-line drawings with angular resolution at least $\pi/4$ [14].

**Planar Graphs.** The proof of Theorem 1 does not cover the case where $G$ is planar, but we do not require the drawing of $G$ to be planar; we suspect that this variant of the problem is still $\exists\mathbb{R}$-hard, but we need stronger gadgets than the ones we saw in Section 3.2; it seems hard to find a sufficiently rigid gadget whose underlying graph is planar, even for angular resolution $\pi/4$.

## References


[1] M. Abrahamsen. Covering polygons is even harder. In *2021 IEEE 62nd Annual Symposium on Foundations of Computer Science—FOCS 2021*, pages 375–386. IEEE Computer Soc., 2022. doi:10.1109/FOCS52979.2021.00045.

[2] M. Abrahamsen, A. Adamaszek, and T. Miltzow. The art gallery problem is $\exists\mathbb{R}$-complete. *J. ACM*, 69(1):Art. 4, 70, 2022. doi:10.1145/3486220.

[3] D. Bertschinger, C. Hertrich, P. Jungeblut, T. Miltzow, and S. Weber. Training fully connected neural networks is $\exists\mathbb{R}$-complete. *ArXiv e-prints*, 2022. arXiv:2204.01368 (last accessed 5/24/2022).

[4] N. Bieker. Complexity of graph drawing problems in relation to the existential theory of the reals. Bachelor's thesis, Karlsruhe Institute of Technology, August 2020.

[5] H. L. Bodlaender and G. Tel. A note on rectilinearity and angular resolution. *J. Graph Algorithms Appl.*, 8(1):89–94, 2004. doi:10.7155/jgaa.00083.

[6] J. Canny. Some algebraic and geometric computations in pspace. In *STOC '88: Proceedings of the twentieth annual ACM symposium on Theory of computing*, pages 460–469, New York, NY, USA, 1988. ACM. doi:10.1145/62212.62257.

[7] M. G. Dobbins, L. Kleist, T. Miltzow, and P. Rzążewski. Completeness for the complexity class $\forall\exists\mathbb{R}$ and area-universality. *Discrete Comput. Geom.*, 2022. doi:10.1007/s00454-022-00381-0.

[8] M. Formann, T. Hagerup, J. Haralambides, M. Kaufmann, F. T. Leighton, A. Symvonis, E. Welzl, and G. J. Woeginger. Drawing graphs in the plane with high resolution. *SIAM J. Comput.*, 22(5):1035–1052, 1993. doi:10.1137/0222063.

[9] J. E. Goodman, J. O'Rourke, and C. D. Tóth, editors. *Handbook of discrete and computational geometry*. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, third edition, 2018. doi:10.1201/9781315119601.

[10] U. Hoffmann. On the complexity of the planar slope number problem. *J. Graph Algorithms Appl.*, 21(2):183–193, 2017. doi:10.7155/jgaa.00411.

[11] J. Kynčl. Simple realizability of complete abstract topological graphs in P. *Discrete Comput. Geom.*, 45(3):383–399, 2011. doi:10.1007/s00454-010-9320-x.

[12] J. Matoušek. Intersection graphs of segments and ∃ℝ. *ArXiv e-prints*, 2014. arXiv:1406.2636 (last accessed 6/10/2020). arXiv:1406.2636.

[13] T. Miltzow and R. F. Schmiermann. On classifying continuous constraint satisfaction problems. In *2021 IEEE 62nd Annual Symposium on Foundations of Computer Science—FOCS 2021*, pages 781–791. IEEE Computer Soc., 2022. doi:10.1109/FOCS52979.2021.00081.

[14] P. Mukkamala and D. Pálvölgyi. Drawing cubic graphs with the four basic slopes. In *Graph drawing*, volume 7034 of *Lecture Notes in Comput. Sci.*, pages 254–265. Springer, Heidelberg, 2012. doi:10.1007/978-3-642-25878-7_25.

[15] Y. Okamoto. Angular resolutions: around vertices and crossings. In *Beyond planar graphs—communications of NII Shonan meetings*, pages 171–186. Springer, Singapore, 2020. doi:10.1007/978-981-15-6533-5_10.

[16] M. Schaefer. Complexity of some geometric and topological problems. In *Graph drawing*, volume 5849 of *Lecture Notes in Comput. Sci.*, pages 334–344. Springer, Berlin, 2010. doi:10.1007/978-3-642-11805-0_32.

[17] M. Schaefer. RAC-drawability is ∃ℝ-complete. In *Graph drawing and network visualization. 29th international symposium, GD 2021, Tübingen, Germany, September 14–17, 2021. Revised selected papers*, pages 72–86. Cham: Springer, 2019. doi:10.1007/978-3-030-92931-2_5.

[18] M. Schaefer. Complexity of geometric $k$-planarity for fixed $k$. *J. Graph Algorithms Appl.*, 25(1):29–41, 2021. doi:10.7155/jgaa.00548.

[19] M. Schaefer. RAC-drawability is ∃ℝ-complete and related results. Unpublished manuscript (submitted), 2023.

[20] M. Schaefer and D. Štefankovič. Fixed points, Nash equilibria, and the existential theory of the reals. *Theory Comput. Syst.*, 60(2):172–193, 2017. doi:10.1007/s00224-015-9662-0.

[21] P. W. Shor. Stretchability of pseudolines is NP-hard. In *Applied geometry and discrete mathematics*, volume 4 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 531–554. Amer. Math. Soc., Providence, RI, 1991.

[22] E. L. Somervell. *A Rhythmic approach to mathematics*. George Philip & Son, 1906.

[23] R. Tamassia. On embedding a graph in the grid with the minimum number of bends. *SIAM J. Comput.*, 16(3):421–444, 1987. doi:10.1137/0216030.

[24] W. Watkins and J. Zeitlin. The minimal polynomial of $\cos(2\pi/n)$. *Amer. Math. Monthly*, 100(5):471–474, 1993. doi:10.2307/2324301.