Planar Confluent Orthogonal Drawings of 4-Modal Digraphs

Sabine Cornelsen ⓘ *Gregor Diatzko* ⓘ

Department of Computer and Information Science,
University of Konstanz, Germany

Abstract. In a *planar confluent orthogonal drawing (PCOD)* of a directed graph (digraph) vertices are drawn as points in the plane and edges as orthogonal polylines starting with a vertical segment and ending with a horizontal segment. Edges may overlap in their first or last segment, but must not intersect otherwise. PCODs can be seen as a directed variant of Kandinsky drawings or as planar L-drawings of subdivisions of digraphs. The maximum number of subdivision vertices in any edge is then the *split complexity*. A PCOD is *upward* if each edge is drawn with monotonically increasing y-coordinates and *quasi-upward* if no edge starts with decreasing y-coordinates. We study the split complexity of PCODs and (quasi-)upward PCODs for various classes of graphs.

1 Introduction

We consider *plane digraphs*, i.e., planar directed graphs with a fixed planar embedding and a fixed outer face. Directions of edges in node-link diagrams are usually indicated by arrow heads. Since this might cause clutter at vertices with high indegree, Angelini et al. [4] proposed *L-drawings* in which each edge is drawn with a 1-bend orthogonal polyline starting with a vertical segment at the tail. A plane digraph can only have an L-drawing without crossings if it is *4-modal*, where a plane digraph is *k-modal* if in the cyclic order around a vertex there are at most k pairs of consecutive edges that are neither both incoming nor both outgoing. However, not every 4-modal digraph admits a planar L-drawing. This motivates to extend the model to drawings with more than one bend per edge.

In a *planar confluent orthogonal drawing (PCOD)* of a digraph, vertices are represented as points in the plane with distinct x- and y-coordinates and each edge is represented as an orthogonal

---

E-mail addresses: sabine.cornelsen@uni-konstanz.de (Sabine Cornelsen) gregor.diatzko@uni-konstanz.de (Gregor Diatzko)
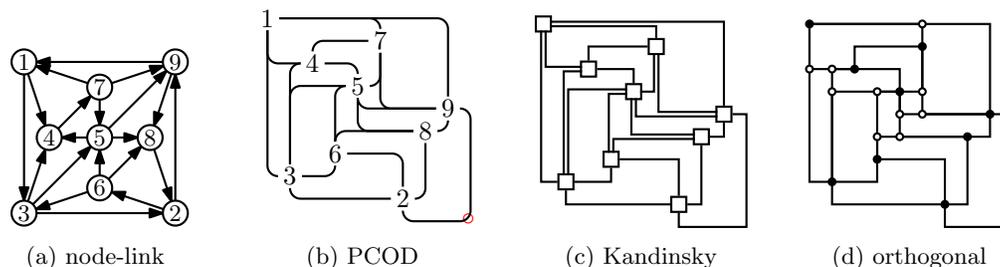
Figure 1: Different representations of a 4-modal irreducible triangulation.

polyline starting with a vertical segment at the tail and ending with a horizontal segment at the head. Distinct edges may overlap in a first or last segment, but must not intersect otherwise. For better readability bends have distinct coordinates and are drawn with rounded corners. A plane digraph has a PCOD if and only if it is 4-modal.

A PCOD of a digraph G corresponds to a planar L-drawing of a subdivision of G. The number of subdivision vertices on an edge is its *split complexity*. See the red encircled vertex in Figure 1b for the subdivision vertex. Since each edge starts with a vertical segment and ends with a horizontal segment, the number of bends on an edge is odd. An edge with split complexity k has $2k + 1$ bends. The *split complexity* of a PCOD is the maximum split complexity of any edge. The PCOD in Figure 1b has split complexity one. A *planar L-drawing* [4, 16] is a PCOD of split complexity zero. If the embedding is not fixed, then it is NP-complete to decide whether a digraph admits a planar L-drawing [16]. Every 2-modal digraph without 2-cycles has a planar L-drawing [3].

A PCOD of a digraph corresponds to a Kandinsky drawing [20] of the underlying undirected graph with the only difference that edges partially overlap instead of being drawn in parallel with a small gap. See Figure 1c. While every simple planar graph has a Kandinsky drawing with one bend per edge [15], deciding whether a multigraph has a Kandinsky drawing with one bend per edge [15] or finding the minimum number of bends in a Kandinsky drawing of a plane graph [13] is NP-hard. For the bend-minimization problem in the Kandinsky model there are 2-approximization algorithms [5, 19] and heuristics [6].

Among the results for orthogonal drawings of undirected graphs where edges must not overlap, we mention three: With one exception, every plane graph of maximum degree four admits an orthogonal drawing with at most two bends per edge [10]. In a bend-minimum drawing, however, there might have to be an edge with a linear number of bends [27]. An orthogonal drawing with the minimum number of bends can be computed by means of a min-cost flow approach [26] even if an upper bound on the number of bends per edge must be respected.

In an *upward-planar* drawing of a digraph, each vertex is drawn as a point, each edge is drawn as a y-monotone curve between the drawings of its end vertices, and the drawings of two edges must not cross except in common endpoints. A digraph is *upward-planar* if and only if it has an upward-planar drawing. A PCOD is *upward* if each edge is drawn with monotonically increasing y-coordinates. A *plane st-graph*, i.e., a plane acyclic digraph with a single sink and a single source, both on the outer face, is always upward-planar; moreover, it has an upward-planar L-drawing if and only if it admits a so-called bitonic *st*-ordering [16]. Since it suffices to subdivide the edges of a plane *st*-graph at most once in order to obtain a digraph that admits a bitonic *st*-ordering [1, 22], it follows that every plane *st*-graph admits an upward PCOD with split complexity one. Moreover, the minimum number of bends in an upward PCOD of a plane *st*-graph can be determined in linear

time. In general, a digraph admits an upward-planar L-drawing, if and only if it is a subgraph of a plane st-graph admitting a bitonic st-ordering [2]. Not every 2-modal tree admits an upward-planar L-drawing [2].

In a *quasi-upward-planar drawing* [7] edges must be strictly monotonically increasing in y-direction in a small vicinity around the end vertices. A digraph has a 2-modal embedding if and only if it admits a quasi-upward-planar drawing. Every 2-modal graph without 2-cycles admits a quasi-upward planar drawing with at most two bends per edge and the maximum number of bends on any edge in such drawings can be minimized utilizing a min-cost flow approach [12]. We call a PCOD *quasi-upward* if no edge starts with decreasing y-coordinates.

Our Contribution. We show that PCODs of 4-modal trees have split complexity zero (Theorem 2), split complexity two is sufficient (Theorem 4) and sometimes necessary (Theorem 3) for PCODs of 4-modal digraphs with parallel edges or loops, while split complexity one suffices for 4-modal *irreducible triangulations* (Theorem 5), i.e., internally triangulated plane graphs without separating triangles and with an outer face of degree 4. Every upward-planar digraph admits an upward PCOD. Moreover, split complexity one suffices for upward PCODs of upward-plane digraphs (Theorem 6) and for quasi-upward PCODs of 2-modal digraphs without 2-cycles (Theorem 8). We show that for plane st-graphs both the split complexity and the total number of bends can be minimized simultaneously in linear time (Theorem 6); however testing whether an arbitrary directed acyclic graph admits a PCOD with split complexity zero is NP-complete (Theorem 7). Using an ILP, we conducted experiments that suggest that every simple 4-modal digraph without separating 2-cycles admits a PCOD with split complexity one (Section 8). Constant split complexity is not to be expected for bend-minimum PCODs (Theorem 1).

2 Preliminaries

Consider an edge e in a PCOD. We call a bend on e *independent* if it is the second, fourth, etc. bend on e. Since the number of bends on any edge is odd in a PCOD, the definition of independent bends does not depend on the direction of the edge. The drawing of a PCOD is determined by the coordinates of the vertices and the independent bends. Considering a PCOD as an L-drawing of a subdivision, the independent bends correspond to the subdivision vertices. The split complexity of an edge is the number of its independent bends. The total number of bends equals the number of edges plus twice the number of independent bends. The top, left, bottom, and right side of a vertex is its *North, West, South*, and *East port*, respectively.

Two consecutive incident edges of a vertex v are a *switch* if both edges are incoming or both outgoing edges of v. A graph is *k-connected* if it contains at least $k + 1$ vertices and removing any $k - 1$ vertices yields a connected graph. A 2-connected graph is also called *biconnected*. A *separating triangle* of a plane graph is a triangle such that both its interior and its exterior contain at least one vertex. Observe that a plane triangulated graph is 4-connected if and only if it does not contain any separating triangles.

An *st-ordering* of a biconnected (undirected) graph $G = (V, E)$ with an edge between two distinct vertices s and t is a bijection $\pi : V \to \{1, \ldots, |V|\}$ such that $\pi(s) = 1$, $\pi(t) = |V|$, and each vertex $v \in V \setminus \{s, t\}$ has neighbors u and w with $\pi(u) < \pi(v) < \pi(w)$. Let now $G = (V, E)$ be a plane st-graph. If (v, v_i), $i = 1, \ldots, k$ are the outgoing edges of a vertex v from left to right then $S(v) = \langle v_1, \ldots, v_k \rangle$ is the *successor list* of v. A *bitonic st-ordering* of G is a bijection $\pi : V \to \{1, \ldots, |V|\}$ such that $\pi(u) < \pi(v)$ for $(u, v) \in E$ and $S(v) = \langle v_1, \ldots, v_k \rangle$ is *bitonic* for each

vertex v, i.e., there is a $1 \leq h \leq k$ such that $\pi(v_i) < \pi(v_{i+1})$, $i = 1, \ldots, h-1$ and $\pi(v_i) > \pi(v_{i+1})$, $i = h, \ldots, k-1$. The successor list $S(v) = \langle v_1, \ldots, v_k \rangle$ contains a *valley* with transitive edges $(v, v_{\ell-1})$ and (v, v_{r+1}) if there is a directed v_ℓ-$v_{\ell-1}$-path and a directed v_r-v_{r+1}-path for some $1 < \ell \leq r < k$. A plane st-graph admits a bitonic st-ordering if and only if it does not contain a valley [22].

Incremental Orthogonal Drawings.   The proof of Theorem 4 is inspired by an approach of Biedl and Kant [10] for incrementally constructing an orthogonal drawing of a planar undirected graph $G = (V, E)$ with maximum degree four. We briefly summarize the approach for biconnected graphs here. First, an edge $\{s, t\}$ on the outer face of G is chosen, an st-ordering π is computed, and the edges are oriented from lower to greater st-numbers. For $i = 1, \ldots, |V|$ let $v_i = \pi^{-1}(i)$. By the st-ordering, it follows that for $i = 1, \ldots, |V|$ the vertex v_i is in the outer face of the subgraph G_i of G induced by the vertices $v_1, \ldots, v_i$. Now the vertices are processed in the order $v_1, \ldots, v_{|V|}$. In principle, the st-number $\pi(v)$ of a vertex v is its y-coordinate. An edge where one endpoint is already processed is assigned a column. Depending on the indegree, the x-coordinate of a vertex is determined by its incoming edges as indicated in Figure 2. In order to guarantee planarity, columns for new outgoing edges are inserted near the currently processed vertex. Again, see Figure 2 for the different cases.
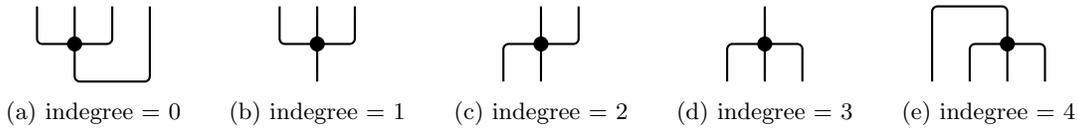

(a) indegree = 0   (b) indegree = 1   (c) indegree = 2   (d) indegree = 3   (e) indegree = 4

Figure 2: Cases by indegree in incremental orthogonal drawings.

3   Confluent Orthogonal Representation

Let Γ be a PCOD of a plane digraph G. We call a bend *covered* if it is contained in the drawing of another edge. We associate an orthogonal drawing of a plane graph G_Γ with Γ as follows [3]: Replace every covered bend in Γ by a dummy vertex. The result is shown in Figures 1d and 3b, where dummy vertices are drawn as empty circles. A *zig-zag* is a pair of uncovered bends on an edge, one with a left turn, and one with a right turn. E.g., on the edge (u, v) in Figure 3a there is a zig-zag, while on the edge (u, w) there is both a left and a right turn, but the left turn is covered, so there is no zig-zag. Since the number of bends in an orthogonal drawing can always be reduced by eliminating zig-zags, we will also do so in PCODs (see Figure 3c) and, thus, the ordering of left- and right-turns at uncovered bends of an edge will not matter. Since planar (confluent) orthogonal drawings can be stretched independently in x- and y-directions, it is algorithmically often easier not to work with actual x- and y-coordinates, but rather with the shape of the faces in terms of bends on the edges and angles at the vertices. See also [20, 26].

A *confluent orthogonal representation* R of a plane digraph $G = (V, E)$ is a set of circular lists $H(f)$, one for each face f of G. The elements of $H(f)$ are tuples $r = (e, v, a, s, b)$ associated with each edge e incident to f in counter-clockwise order.

(a) v is the end vertex of e traversed immediately before e.

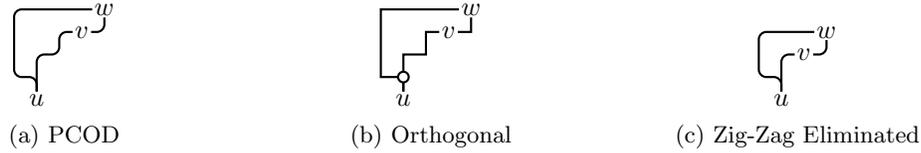(a) PCOD   (b) Orthogonal   (c) Zig-Zag Eliminated

Figure 3: Eliminating zig-zags.

(b) $a \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ is the angle at v between e and its predecessor on f. It is a multiple of π if and only if it describes an angle at a switch: If a describes an angle at a switch, then either both edges are incoming at v and, thus, attached to the South or the North port of v or both edges are outgoing at v and, thus, attached to the West or the East port of v. In either case a is an integer multiple of π. Analogously, a is $\pi/2$ or $3\pi/2$ if it does not describe the angle at a switch.

(c) s is the number of left turns (when traversing e starting from v) at bends on e.

(d) $b \in \{L, N, R\}$ represents a covered bend on the segment of e incident to v, if any, with (L) a left bend, (R) a right bend, or (N) no such bend. Observe that only one endpoint of the respective segment is a bend since the other endpoint is a vertex.

Let r_p be the predecessor of r in $H(f)$. If r is not clear from the context, we denote the entries by $e[r], v[r], a[r], s[r], b[r]$. For each edge, there are two tuples containing it in a confluent orthogonal representation, one for each of the two faces incident to the edge. Let $\bar{r}$ be the (other) entry containing $e[r]$. Observe that $s[\bar{r}]$ is the number of right turns at bends on $e[r]$ we make when traversing $e[r]$ starting from $v[r]$. A confluent orthogonal representation is *feasible* if it fulfills the following.

(i) The rotation $\sum_{r \in H(f)} (2 - a[r]/\frac{\pi}{2} + s[r] - s[\bar{r}])$ of a face f is -4 if f is the outer face and 4 otherwise. (ii) The angular sum $\sum_{r;v[r]=v} a[r]$ around a vertex v is 2π. (iii) If $b[r] = L$ or $b[\bar{r}] = R$, then $s[r] \geq 1$, and if both $b[r] = L$ and $b[\bar{r}] = R$, then even $s[r] \geq 2$. This ensures that covered bends are counted by s and that covered bends adjacent to the head or the tail of an edge must be distinct. (iv) The *bend-or-end property* [20] has to be fulfilled, i.e., if $a[r] = 0$, then $b[r] = R$ or $b[\overline{r_p}] = L$. (v) The total number of bends $s[r] + s[\bar{r}]$ on $e[r]$ is odd.

From a Representation to a PCOD. In order to construct a PCOD from a feasible confluent orthogonal representation R of a plane digraph G, we transform G into a graph G_R of maximum degree 4 and a feasible orthogonal representation R' without angles of magnitude 0 or 2π. Using compaction for orthogonal representations [26] on G_R then yields a PCOD or a $\pi/2$-rotation of a PCOD in linear time. The idea for the construction of G_R is analogous to the construction of G_Γ from a PCOD Γ and is as follows: Consider a vertex $v \in V$ and let $e_1, \dots, e_k$ be a maximum sequence of consecutive edges around v with angles of magnitude 0. For $i = 1, \dots, k$, let r_i be the entry with $e[r_i] = e_i$ and $v[r_i] = v$. Due to Property iv, there is some m with $1 \leq m \leq k$ such that $b[r_j] = L$ for all j with $j < m$ and $b[r_j] = R$ for all j with $j > m$. We subdivide the segment of e_m that is incident to v with $k - 1$ vertices $v_1, \dots, v_{m-1}, v_k, \dots, v_{m+1}$ in this order, starting from v. We attach $e_j, j \neq m$ to v_j instead of v. The representation R' is updated accordingly.

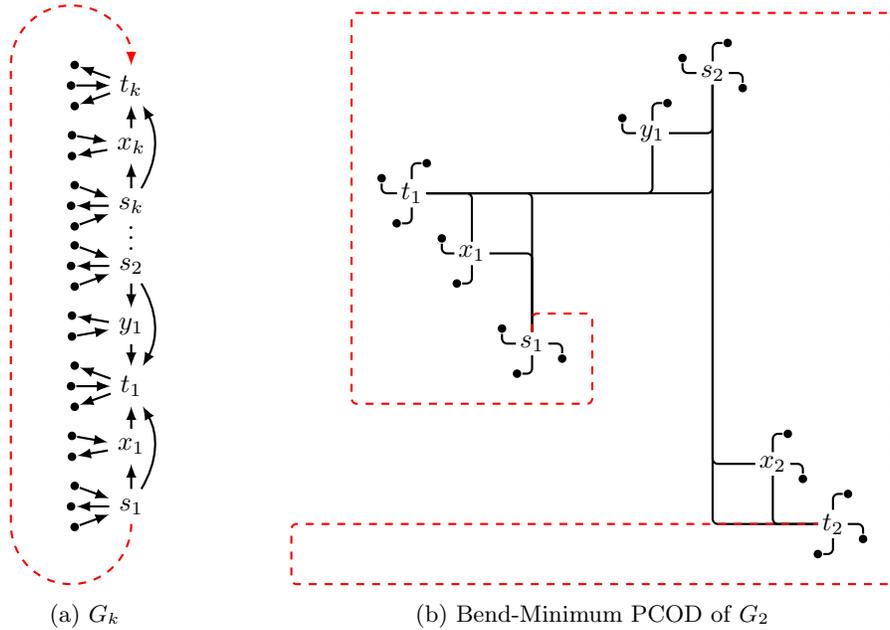(a) G_k (b) Bend-Minimum PCOD of G_2

Figure 4: Graphs with a linear number of bends in any bend-minimum PCOD.

4   Some Initial Results

We start with an example that shows that a bend minimum PCOD might require an edge whose split complexity is linear in the number of vertices. Together with Theorem 4, this implies in particular that the split complexity and the number of bends cannot be minimized simultaneously.

Theorem 1 *There is a family G_k, $k > 0$ of 4-modal digraphs with $14k - 3$ vertices and $16k - 4$ edges such that in any bend-minimum PCOD of G_k there is an edge with split complexity at least $k + 2$.*

Proof: Consider the digraphs G_k indicated in Figure 4a. Let e be the red dashed edge. Let P_k be the path $s_1, x_1, t_1, y_1, \ldots, s_k, x_k, t_k$ of length $4k - 2$ in G_k that is drawn vertically in Figure 4a. Consider a planar L-drawing of $G_k - e$ in which all edges of P_k (traversed from s_1 to t_k) bend to the left and the edge incident to s_1 is to the top of s_1. Such a drawing for G_2 is indicated in Figure 4b, where $k = 2$. Since all vertices of P_k are 4-modal this uniquely determines the drawing of P_k and also of the transitive edges of P_k. In order to preserve the embedding, e can only be inserted into the drawing with split complexity at least $k + 2$.

Consider a PCOD of G_k with fewer bends on e. Since all vertices are 4-modal, the rotation of the cycle C composed of P_k and e can only be maintained, if the number of bends on at least one edge of P_k, say (s_i, x_i), is increased. But then we also must increment the number of bends on an edge (s_i, t_i) to maintain the rotation of the face bounded by the edges $(s_i, t_i), (s_i, x_i), (x_i, t_i)$. Thus, for each independent bend less on e the total number of bends increases by at least 2.   □

Even though not every 2-modal tree has an upward-planar L-drawing [2], every 4-modal tree has a planar L-drawing, despite its fixed embedding.
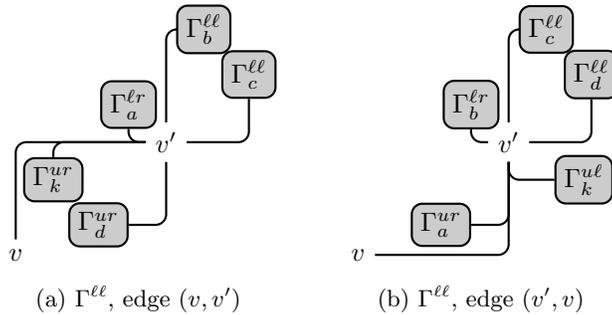
(a) $\Gamma^{\ell\ell}$, edge (v, v') (b) $\Gamma^{\ell\ell}$, edge (v', v)

Figure 5: Constructing a tree from its subtrees.
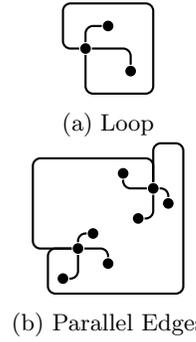
(a) Loop

(b) Parallel Edges

Figure 6: Multigraphs with split complexity two.

Theorem 2 *Every 4-modal tree has a planar L-drawing. Moreover, such a drawing can be constructed in linear time.*

Proof: Let T be a 4-modal tree and let v be a leaf of T. We show by induction on the number m of edges that we can draw T as a planar L-drawing Γ^α such that v is in the corner α (lower left ($\ell\ell$), lower right (ℓr), upper left ($u\ell$), upper right (ur)) of the bounding box of Γ^α. We give the details for $\Gamma^{\ell\ell}$; the other cases are analogous.

If $m = 1$, draw v at $(0,0)$ and its neighbor at $(1,1)$. If $m > 1$, let the neighbor of v be v', and let the connected components of $T - v'$ be $v, T_1, \ldots, T_k$ in clockwise order around v'. See Figure 5. Let T_0 be the subtree consisting of v only. Each tree $T_i + v'$, $i = 0, \ldots, k$ has at most $m - 1$ edges and the leaf v'; therefore, by the inductive hypothesis, we can construct planar L-drawings Γ_i^α, $\alpha \in \{\ell\ell, \ell r, u\ell, ur\}$ of $T_i + v'$ with v' in the respective corner of the bounding box. W.l.o.g. let v be the tail of the edge connecting v and v', see Figure 5a. Let $1 \le a \le b \le c \le d \le k$ such that $T_{d+1}, \ldots, T_k, T_0, T_1, \ldots, T_a$ and $T_{b+1}, \ldots, T_c$ are connected to v' by an incoming edge and $T_{a+1}, \ldots, T_b$ and $T_{c+1}, \ldots, T_d$ by outgoing edges. Choose $\Gamma_0^{ur}, \Gamma_1^{\ell r}, \ldots, \Gamma_a^{\ell r}, \Gamma_{a+1}^{\ell\ell}, \ldots, \Gamma_c^{\ell\ell}, \Gamma_{c+1}^{ur}, \ldots, \Gamma_k^{ur}$ for $T_i + v'$, $i = 1, \ldots, k$. Finally, merge the drawings of the subtrees at v'. The other case where v is the head of the edge connecting it to v' is shown in Figure 5b.

In order to compute a confluent orthogonal representation, using dynamic programming, only $O(\deg(v'))$ steps are required for each vertex v'. Thus, the total time complexity is linear. □

5   Multi-Graphs

In this section, we prove that split complexity two is always sufficient, even if there are loops or parallel edges. We start with two plane multi-graphs that also require split complexity two.

Theorem 3 *There are 4-modal multigraphs that need split complexity at least two in any PCOD.*

Proof: Consider the digraph containing a loop in Figure 6a or the digraph containing two parallel edges in Figure 6b. The incident 4-modal vertices on the one hand and the rotation of the outer face on the other hand, imply that the loop and one of the two parallel edges, respectively, must have split complexity two. In the case of the loop, the angle at the vertex in the outer face is
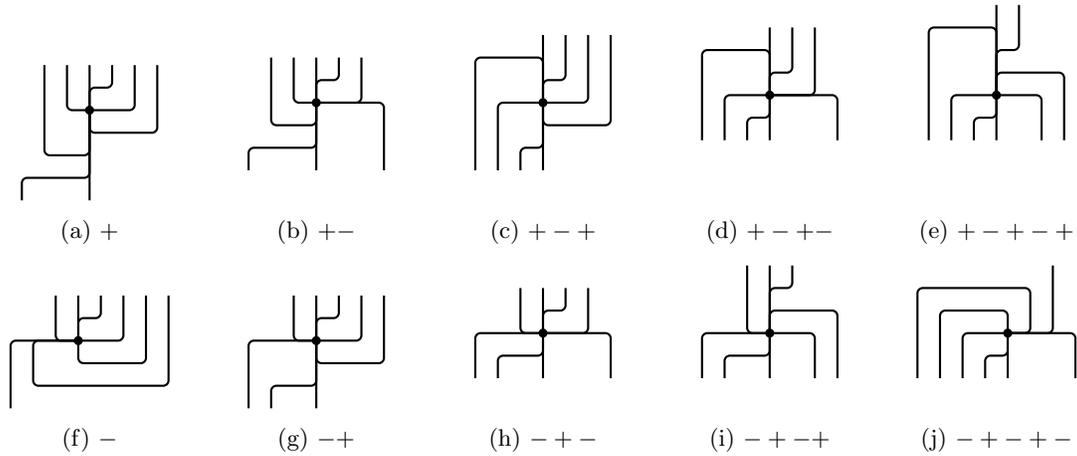
Figure 7: Drawings around v_k in the proof of Theorem 4. A $+$ represents outgoing edges from below. A $-$ represents incoming edges from below.

convex. Since the rotation of the outer face is -4, it follows that five concave bends (i.e., with angle $3\pi/2$ in the outer face) on the loop are needed. In the case of two parallel edges, the angles at the vertices in the outer face have magnitude 0. Thus, the two edges together must have eight concave bends. Since each edge has an odd number of bends, there must be an edge with five bends. □

Theorem 4 *Every 4-modal multigraph has a PCOD with split complexity at most two. Moreover, such a drawing can be computed in linear time.*

Proof: The approach is inspired by [10]. Subdivide each loop. Let the resulting digraph be G. Then make the digraph biconnected maintaining its 4-modality [3, Lemma 3]. Now compute in linear time [14] an st-ordering $v_1, \ldots, v_n$ of this biconnected graph G' (without taking into account the direction of the edges). Iteratively add the vertices with increasing y-coordinates in the order of the st-ordering, maintaining a column for each edge for which one of its end vertices is already drawn while the other one is not.

Let v_k be a vertex. An edge e incident to v_k is incident to v_k *from below* if e has an end vertex that is before v_k in the st-ordering. Let $e_1, \ldots, e_j$ be the sequence of edges incident to v_k from below as they appear from left to right. Since v_k is 4-modal, $e_1, \ldots, e_j$ can be divided into at most five subsequences of edges consisting only of incoming $(-)$ or only of outgoing $(+)$ edges of v_k. Depending on the arrangement of these subsequences, we assign the bends around v_k. E.g., consider the Case "$+-$" in Figure 7b, i.e., among the edges incident to v_k from below, there are first some outgoing edges, followed by some incoming edges. All outgoing edges from below are attached to the South port, while all incoming edges from below are attached to the East port. Mind that all outgoing edges except one need two bends near v_k. Consider now the edges incident to v_k to later vertices in the st-ordering. By 4-modality, there can be at most some incoming edges, followed by some outgoing, some incoming and again some outgoing edges in counter-clockwise order around v_k. We attach them to the East, North, West, and South port of v_k, respectively. The edges from below determine the position of v_k. More precisely, in all cases except Case "$-$" in Figure 7f, i.e.,

in particular in the case considered here, v_k inherits the x-coordinate of its rightmost outgoing edge from below.

See Figure 7 for the routing of the edges from below and the possible edges to later end vertices in the other cases. For v_1, we choose the assignment according to Figure 7a or Figure 7f. After all vertices are placed, we remove edges that are in G' but not in G. If x is a vertex that was inserted into a loop, we reroute the two incident edges near x such that the incoming edge of x has exactly one bend near x and the outgoing edge has no bend near x. Finally, we eliminate zig-zags.

By the st-ordering, the columns of the edges incident to v_k from below are consecutive among the edges for which exactly one of the end vertices is already drawn [10]. This implies planarity if the columns for the new edges are inserted directly next to v_k. For each edge e, there are at most two bends near the tail of e and at most three bends near the head of e. Consider now a 2-cycle $(v, x), (x, v)$ replacing a loop at v. Since the subdivision vertex x is incident to exactly one incoming and one outgoing edge, it follows that near x there is no bend on (x, v) and one bend on (v, x). If (x, v) does not have three bends near v then in total there are at most six bends on the loop, namely the four bends near v plus one bend near x plus the bend on x. Since the number of bends on an edge must be odd, there are only five. Consider now the case that (x, v) has three bends near v (Figures 7f and 7j). If in addition (v, x) has two bends near v, then there are seven bends on the loop. However, in this case, there is a zig-zag on (v, x) formed by the bend near x and the second bend near v. Thus, after eliminating zig-zags, the split complexity is at most two. See Figure 8 for an example.   □
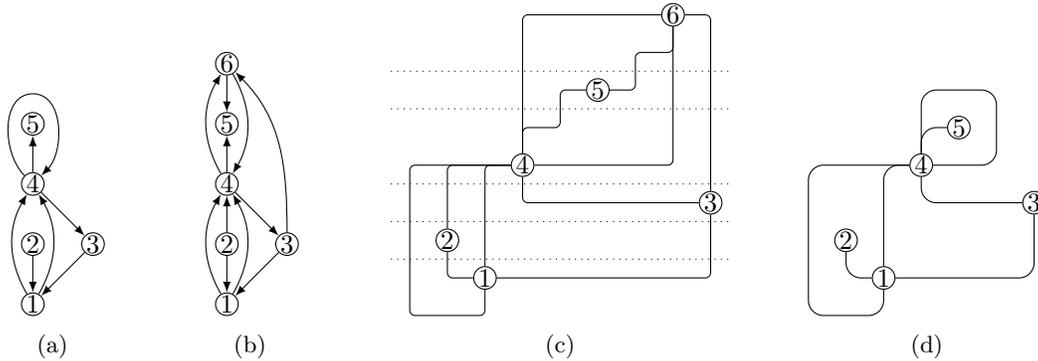

Figure 8: (a) A multigraph, (b) with subdivided loops, augmented to biconnectivity, (c) drawn as a PCOD, (d) with dummy edges and vertices removed and zig-zags eliminated. With each subsequent vertex, the drawing up to the next dotted line is computed.

6   Irreducible Triangulations

Recall that an *irreducible triangulation* is an internally triangulated graph with an outer face of degree four that does not contain any separating triangles. We prove that every 4-modal digraph whose underlying undirected graph is an irreducible triangulation has a PCOD with split complexity at most one.

Motivated by the approaches in [3, 11], we use *rectangular duals*, a contact representation of an irreducible triangulation $G = (V, E)$ with the following properties. Each vertex $v \in V$ is

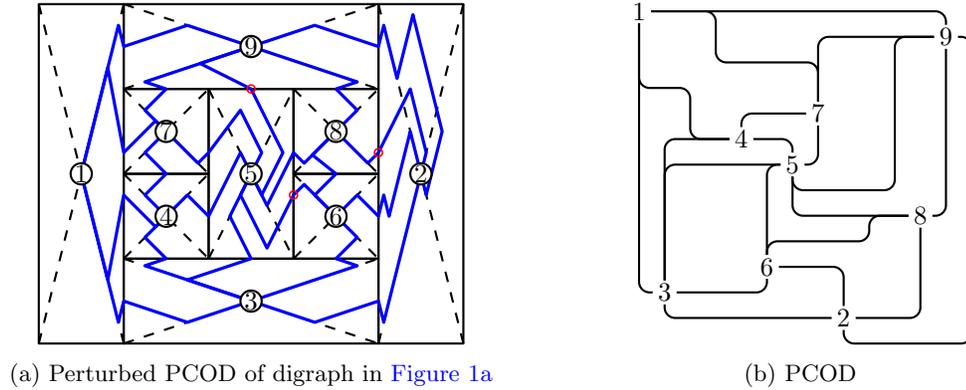(a) Perturbed PCOD of digraph in Figure 1a   (b) PCOD

Figure 9: Perturbed PCOD and corresponding PCOD after zig-zag elimination. Red encircled bends are due to the change of the coordinate system and not proper.

represented by an internally disjoint axis-parallel rectangle $R(v)$. Two rectangles touch if and only if the respective vertices are adjacent in G. Moreover, no four rectangles representing a vertex meet at the same point and $\bigcup_{v \in V} R(v)$ is a rectangle. See the rectangles in Figure 9a. A rectangular dual for an irreducible triangulation can be computed in linear time [8, 9, 23, 24].

Given a rectangular dual, we perturb the coordinate system such that in each rectangle the axes correspond to the diagonals. The *perturbed x-axis* is the diagonal containing the bottommost-leftmost point of the rectangle, the other diagonal is the *perturbed y-axis*. A *perturbed orthogonal polyline* is a polyline such that in each rectangle the segments are parallel to one of the axes. A bend of a perturbed orthogonal polyline at the boundary of two rectangles is a *proper bend* if among the two incident segments one is parallel to a perturbed x-axis and the other parallel to a perturbed y-axis. Bends inside a rectangle are always proper. In a *perturbed PCOD* each vertex v is drawn at the center of $R(v)$. An edge (u, v) is a perturbed orthogonal polyline in $R(u) \cup R(v)$ between u and v starting with a segment on the perturbed y-axis in $R(u)$ and ending with a segment on the perturbed x-axis in $R(v)$. The drawing of (u, v) must have at least one bend in the interior of both $R(u)$ and $R(v)$ and must cross the boundary of $R(u)$ and $R(v)$ exactly once. Distinct edges may overlap in a first or last segment, but must not intersect otherwise. No two bends have the same coordinates. See Figure 9a. The *North port* of v is the port above and to the left of the center of R_v. The *West, South, and East ports* are the other ports in counter-clockwise order.

Almost identically to the arguments in [3, Lemma 5], we obtain that a perturbed PCOD yields a confluent orthogonal representation where the number s of left turns counts only proper bends.

Lemma 1 *Let Γ be a perturbed PCOD of an irreducible triangulation G. There exists a PCOD Γ' of G such that for each edge e the number of left (right) turns on e in Γ' equals the number of left (right) turns at proper bends on e in Γ.*

Proof: Let G be an irreducible triangulation and let Γ be a perturbed PCOD of G. In a first step, we construct a PCOD Γ'' of G. We will then show that zig-zag elimination yields a PCOD with the desired property.

Similar to the construction on Page 526, we construct a graph G' of maximum degree 4 from Γ by replacing each bend and each intersection with the boundary of a rectangle by a dummy vertex. See Figure 10b. Now each edge is contained in exactly one rectangle. Let R_e be the rectangle
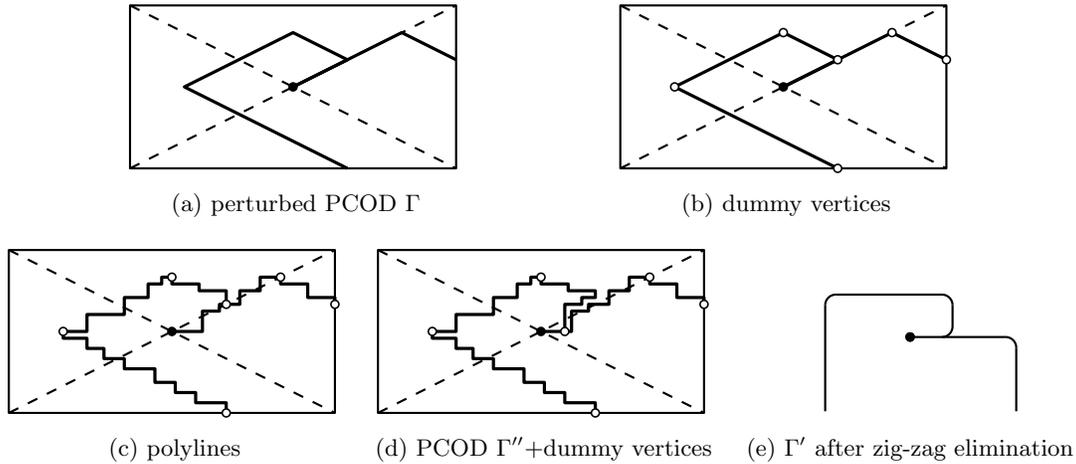
(a) perturbed PCOD Γ

(b) dummy vertices

(c) polylines

(d) PCOD Γ″+dummy vertices

(e) Γ′ after zig-zag elimination

Figure 10: From a perturbed PCOD to a PCOD. Only the part of the graph within a rectangle of the rectangular dual is shown.

containing e. We replace each edge e of G' with an orthogonal polyline close to e satisfying the following properties. (a) No two polylines of two edges cross. (b) The polyline of no edge is self-intersecting. (c) Each polyline contains as many left bends as right bends. (d) If e is incident to the top left (top right, bottom right, bottom left, respectively) of its tail then its polyline leaves its tail to the top (right, bottom, left) and enters its head from the bottom (left, top, right). See Figure 10c.

We obtain a drawing that fulfills all properties of a PCOD of G except that edges might overlap in a prefix or a suffix that need not be a straight-line segment. In order to remedy this, we split the edges on their first segment and route them close to each other in the order in which they appear in the embedding around their common end vertex. This results in the PCOD Γ″. See Figure 10d. Since the number of left turns equals the number of right turns on each polyline in Γ″, we can remove all bends on any polyline by zig-zag elimination which yields the drawing Γ′. See Figure 10e. Observe that the proper bends of the original drawing Γ still correspond to bends of the constructed drawing and have the same turns (left or right). Also observe that in Γ′ there is only a bend at a dummy vertex inserted at the border of a rectangle if there was a proper bend there in Γ. □

By the next theorem, we can derive a PCOD with split complexity one from a suitable perturbed PCOD after zig-zag elimination. See Figure 9b.

Theorem 5 *Every 4-modal irreducible triangulation has a PCOD with split complexity at most one; and such a drawing can be computed in linear time.*

Proof: Let G be an irreducible triangulation. We construct a rectangular dual for G ignoring edge directions. Routing the edges inside any rectangle independently, we then construct a perturbed PCOD that yields a confluent orthogonal representation with split complexity at most one after zig-zag elimination.

Let v be a vertex of G. For a side s of $R(v)$ let u_i, $i = 1, \ldots, k$ be the adjacent vertices of v in counter-clockwise order such that s and $R(u_i)$ intersect in more than a point. Let e_i be the edge

(a) mod(s) = 5   (b) with zig-zag   (c) in first

(d) mod 4   (e) mod 4   (f) mod 4   (g) mod 4   (h) mod 4

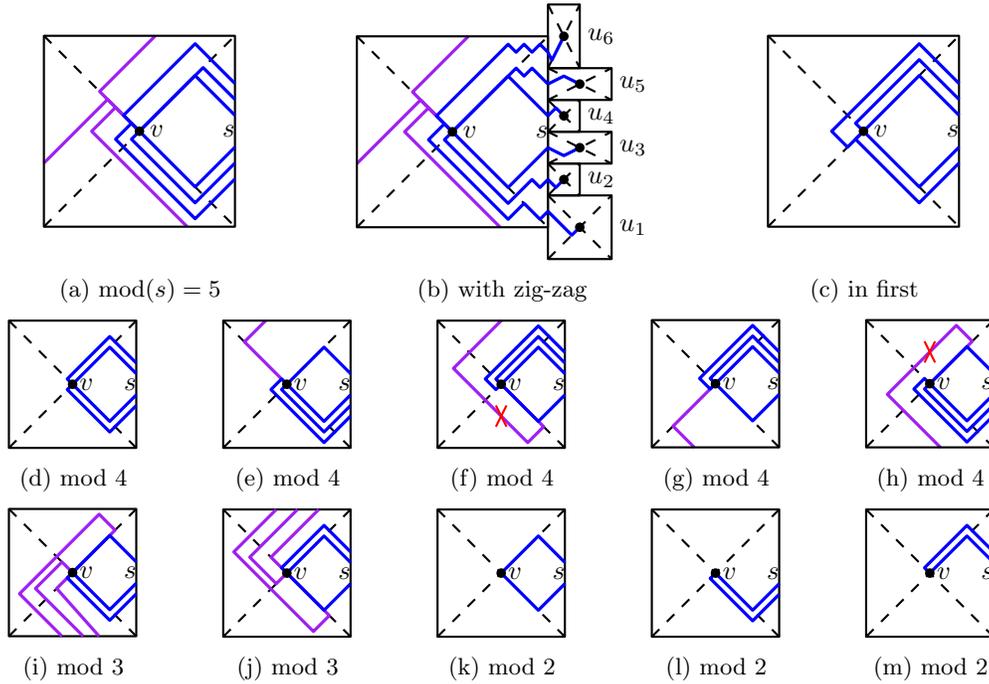(i) mod 3   (j) mod 3   (k) mod 2   (l) mod 2   (m) mod 2

Figure 11: Routing the edges in a perturbed PCOD.

between v and u_i. Consider the division of $\langle e_1, \ldots, e_k \rangle$ into *mono-directed classes*, i.e., maximal subsequences such that any two edges in a subsequence are either both incoming or both outgoing edges. Let the modality $\mathrm{mod}(s)$ of s be the number of these subsequences. Since G is 4-modal we have $\mathrm{mod}(s) \leq 5$. Assume now that s is a side of $R(v)$ with maximal modality. Assume without loss of generality that s is the right-hand side of $R(v)$.

$\mathrm{mod}(s) = 5$. If e_1 is an outgoing edge of v, assign the mono-directed classes of edges crossing s from bottom to top in this order to (i) the North port bending three times to the left, (ii) to the West port bending twice to the left, (iii) to the South port bending once to the left, (iv) to the East port bending once to the right, and (v) to the North port bending twice to the right. Route the edges as indicated in blue in Figure 11a to s. By adding zig-zags, it is always possible to route an edge e_i between v and u_i in such a way that the parts of e_i in $R(v)$ and $R(u_i)$ meet in s. See Figure 11b. Edges crossing other sides of R_v are all outgoing edges of v and are assigned to the North port, bending once or twice in the direction of the side where they leave R_v. See the purple edges in Figure 11a. If e_1 is an incoming edge, start analogously with the West port. See Figure 11c.

$\mathrm{mod}(s) \in \{1, \ldots, 4\}$. The assignment of edges to ports and the routing of the edges are contained in the drawing of the case $\mathrm{mod}(s) = 5$. See the blue edges in the second and third row in Figure 11. We make again sure that an edge to a side of $R(v)$ with modality one has at most two bends in the interior of $R(v)$. In order to do so, we have to take special care if $\mathrm{mod}(s) = 4$ and the bottommost edge is an outgoing edge of v. Let s_t and s_b be the top and bottom side, respectively, of $R(v)$. If

(a) proper bend on s (b) $b_e(v) \geq 3$ implies ...   (c) ... $b_e(u) \leq 2$
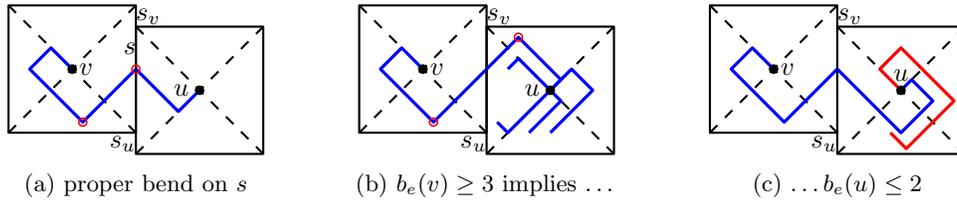
Figure 12: Eliminating zig-zags to reduce the number of bends per edge to three.

v is incident to an outgoing edge crossing s_t, we opt for the variant in Figure 11e and otherwise (including the case that v is incident to an incoming edge crossing s_b) for the variant in Figure 11g. No particular care has to be taken if $\mathrm{mod}(s) = 2$ (Figures 11l and 11m).

Let now e be an edge between two vertices u and v. We consider the number $b_e(u)$ and $b_e(v)$ of bends on e in $R(u)$ and $R(v)$, respectively, after eliminating zig-zags. We assume without loss of generality that $b_e(u) \leq b_e(v)$. Recall that then $1 \leq b_e(u) \leq b_e(v) \leq 3$. Let s_u and s_v be the sides of $R(u)$ and $R(v)$, respectively that contain the intersection s of $R(u)$ and $R(v)$.

We have to show that up to zig-zags there are in total at most three bends on e. This is clear if $b_e(u) = b_e(v) = 1$. Assume now that $b_e(v) \geq 2$. Since the number of proper bends on e is odd it follows that the bend on s is proper if and only if $b_e(u) + b_e(v)$ is even. In this case the bend on s bends in opposite direction as the next bend in $R(u)$ and $R(v)$ (otherwise e does not cross s). Since $b_e(v) \geq 2$, the proper bend of e on s and the next bend of e in $R(v)$ form a zig-zag and can be eliminated. See Figure 12a. Thus, if $b_e(u) + b_e(v) \leq 4$ then there are at most three bends on e after zig-zag elimination.

It remains to consider the case that $b_e(v) = 3$ and $b_e(u) \geq 2$. This implies $\mathrm{mod}(s_v) > 1$ and e is in the first or last mono-directed class among the edges crossing s_v. We assume without loss of generality that s_v is the right side of R_v and that e is in the bottommost mono-directed class. See Figure 12b. It follows that e is an outgoing edge of v and thus, an incoming edge of u. Since $b_e(u) \geq 2$ it follows that e is attached to the East port of u. Assume first that $b_e(u) = 2$. Then the bends of e in $R(u)$ are in opposite direction as the bends of e in $R(v)$. Thus, there is at least one zig-zag consisting of a bend in $R(u)$ and a bend in $R(v)$. After eliminating this zig-zag there are only three bends left.

Assume now that $b_e(u) = 3$. This is only possible if $\mathrm{mod}(s_u) \geq 2$. Hence, $R(u)$ is the topmost or bottommost rectangle incident to the right of $R(v)$. Since e is in the bottommost class with respect to s_v, it must be the bottommost one. Thus, $R(v)$ is the topmost neighbor to the left of $R(u)$. Moreover, since $\mathrm{mod}(s_u) \geq 2$ there must be a port of u other than the East port that contains an edge e' (red edge in Figure 12c) crossing s_u. But e' would have to bend at least four times in the interior of $R(u)$, which never happens according to our construction.   □

7 (Quasi-)Upward-Planar Drawings

In this section, we discuss PCODs in which each edge starts with increasing y-coordinates. We first consider edges with y-monotone curves.

Theorem 6 *Every upward-plane digraph admits an upward PCOD with split complexity at most one. Moreover, for plane st-graphs both the split complexity and the total number of bends can be minimized simultaneously in linear time.*
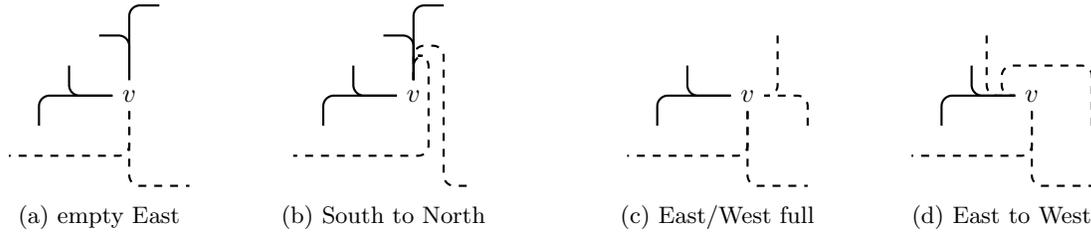
(a) empty East   (b) South to North   (c) East/West full   (d) East to West

Figure 13: From a planar L-drawing of a 2-modal graph to a quasi-upward PCOD.

Proof: Let G be an upward-plane digraph. Then G can be augmented to a plane st-graph by adding edges [18, Theorem 4.3]. Subdividing each edge once yields a plane st-graph with a bitonic st-ordering [22, Lemma 7] and thus with an upward-planar L-drawing [16, Theorem 3]. This corresponds to an upward PCOD of split complexity one for G.

If G is a plane st-graph, it can be decided in linear time whether G has an upward-planar L-drawing [16, Theorem 6] and thus an upward PCOD of split complexity zero. Otherwise, the minimum number of edges that has to be subdivided in order to obtain a digraph that has a bitonic st-ordering can be computed in linear time [1, Theorem 1]. Thus, a PCOD with the minimum number of bends among all upward PCODs of G with split complexity one can be computed in linear time. Observe that the total number of bends cannot be reduced by increasing the split complexity, since the subdivision of edges is only performed in order to break one of the transitive edges in a valley.   □

With a similar argument as in the proof of the previous theorem, we obtain that it is hard to decide whether a given directed acyclic graph (DAG) admits an upward-planar L-drawing if it is not yet known whether it is upward-planar.

Theorem 7 *It is NP-complete to decide whether a DAG admits an upward-planar L-drawing, i.e., an upward-planar PCOD with split complexity zero.*

Proof: If a graph admits an upward-planar L-drawing, then it has one on a quadratic grid. This implies containment in NP. In order to prove NP-hardness, we use that testing whether a DAG admits an upward-planar drawing is NP-complete [21]. Let G be a DAG and let G' be the DAG that is obtained from G by subdividing each edge exactly once. Then, on one hand, G must obviously be upward-planar if G' admits an upward-planar L-drawing. On the other hand, if G is upward-planar then G can be extended to a planar st-graph G_{st} [18, Theorem 4.3]. Splitting each edge of G_{st} once yields a digraph G'_{st} with a bitonic st-ordering [1, Theorem 1], and, thus an upward-planar L-drawing [16, Theorem 3]. Thus, the subgraph G' of G'_{st} admits an upward-planar L-drawing.   □

At least in the absence of 2-cycles, we can guarantee split complexity two for PCODs in which edges start with increasing y-coordinates, even for not necessarily upward-plane 2-modal digraphs.

Theorem 8 *Every 2-modal digraph without 2-cycles admits a quasi-upward PCOD with split complexity at most one. Moreover, such a drawing can be computed in linear time.*

Proof: Let G be a 2-modal graph without 2-cycles. G has a planar L-drawing [3], say Γ. Process the vertices v of G top-down in Γ. If there are edges attached to the South port of v, we reroute them such that they are attached to the North port. Consider first the case that at least one

among the East or the West port—say the East port—of v does not contain edges; see Figure 13a. Then we can reroute the edges as indicated in Figure 13b. Edges attached to the South port of v in Γ are now attached to the North port of v and get two additional bends near their tail.

Assume now that both the East and the West port of v contain edges (Figure 13c). Then, by 2-modality, no edge is attached to the North port of v. Those edges incident to the East port that bend upward are reattached to the West port without adding any additional bend. This is possible since these edges have already been rerouted near their other end vertex and two new bends have been inserted. So there are enough bends for the bend-or-end property. The rotation of the faces and the angular sum around the vertices are also maintained. The edges incident to the East port bending downward are rerouted from their original drawing to the West port with two new bends. See Figure 13d. Now we can reroute the edges attached to the South port as in the first case.

An edge attached to the South port in Γ gets at most two new bends near its tail; an edge attached to the North port at most two new bends near its head. Thus, in the end each edge has at most three bends, i.e., split complexity 1. $\qquad\square$

8 Experiments using an ILP

In order to support the conjecture that each simple 4-modal digraph admits a PCOD with split complexity at most one, we conducted some experiments. Based on the definition of a confluent orthogonal representation and on the fact that each 4-modal graph has a PCOD with split complexity at most two (Theorem 4), we use an ILP in order to find PCODs with minimum split complexity.

For each entry r in the confluent orthogonal representation, we use the variables $s_r^{(1)}, s_r^{(2)}, b_r^L, b_r^R \in \{0,1\}$, $a_r \in \{0,1,2\}$, and $e_r \in \{0,1\}$ as follows:

- $a[r] = a_r \cdot \pi + \begin{cases} 0 & \text{if } e[r] \text{ and } e[\overline{r_p}] \text{ is a switch at } v[r] \\ \frac{\pi}{2} & \text{otherwise} \end{cases}$

- $e_r = 1$ if and only if the number $s[r]$ of left turns on $e[r]$ traversed from $v[r]$ is odd. The variables $s_r^{(i)}, s_{\overline{r}}^{(i)}$, $i = 1,2$ represent a first and a second pair of bends on $e[r]$, where $s_r^{(i)} = 1$ if the two bends are left turns on $e[r]$ traversed from $v[r]$. I.e.,

$$s[r] = 2(s_r^{(1)} + s_r^{(2)}) + e_r.$$

  Recall that by Theorem 4 there are at most five bends on any edge in a PCOD with minimum split complexity.

- b_r^L and b_r^R together represent $b[r]$, namely $b_r^L = 1$ if and only if $b[r] = L$ and $b_r^R = 1$ if and only if $b[r] = R$.

In order to ensure consistency, we add the following constraints:

$$s_r^{(1)} + s_{\overline{r}}^{(1)} \le 1 \qquad\qquad s_r^{(2)} + s_{\overline{r}}^{(2)} \le 1 \qquad\qquad b_r^L + b_r^R \le 1$$

Property i and Property ii of feasible confluent orthogonal representations are straightforwardly formulated as linear constraints. Property iii translates to

$$s[r] - b_r^L - b_{\overline{r}}^R \ge 0.$$

Property iv can be formulated as

$$a_r + b_r^R + b_{\overline{r_p}}^L \geq 1 \text{ if } e[r] \text{ and } e[\overline{r_p}] \text{ is a switch at } v[r].$$

In order to fulfill Property v, we require

$$e_r + e_{\overline{r}} = 1.$$

The objective function is

$$\min \sum_r \left(s_r^{(1)} + |E| \cdot s_r^{(2)} \right).$$

This way, the number of edges with five bends is minimized first, then that of edges with three bends.

In a practical implementation, one might fix $s_r^{(2)} = 0$ at first to check for a solution with split complexity one and only allow $s_r^{(2)} \in \{0, 1\}$ if no such solution exists.

We generated sample graphs as follows. Since each simple 4-modal graph without 2-cycles can be extended to a triangulated 4-modal graph [3], we first sampled several thousand upward-planar triangulations for various numbers $n \leq 500$ of vertices with two different methods: sampling (a) undirected triangulations uniformly at random [25] orienting the edges according to an *st*-ordering [14] and (b) with an OGDF method [17]. Then we flipped the direction of each edge with probability 0.5 maintaining 4-modality. Finally, we added as many 2-cycles as 4-modality allowed. The resulting digraphs contained $(\frac{3}{4} \pm \frac{1}{4})n$ separating triangles, roughly n 2-cycles, but no separating 2-cycles. All digraphs had split complexity one.

9 Conclusion and Future Work

We examined the split complexity of PCODs of various graph classes. In particular, we have shown that every 4-modal digraph admits a PCOD with split complexity two even if it contains loops and parallel edges and that split complexity two is sometimes necessary. For simple digraphs, we made a first step, by proving that every 4-modal irreducible triangulation admits a PCOD with split complexity one. It still remains open whether split complexity one suffices for all simple 4-modal digraphs. Experiments suggest that this could very well be true. It would also be interesting to know whether the minimum split complexity or the minimum number of bends in a PCOD or a quasi-upward PCOD can be efficiently determined in the case of a given 4-modal, 2-modal, or upward-planar embedding, respectively, as well as in the case when no embedding is given. While we have shown that it is NP-complete to decide whether a directed acyclic graph admits an upward-planar PCOD with split complexity zero, the complexity of determining the minimum split complexity or the minimum number of bends in an upward-planar PCOD is still open for upward-planar DAGs with or without a given upward-planar embedding.

References

[1] P. Angelini, M. A. Bekos, H. Förster, and M. Gronemann. Bitonic st-orderings for upward planar graphs: The variable embedding setting. In I. Adler and H. Müller, editors, *WG 2020*, volume 12301 of *LNCS*, pages 339–351. Springer, 2020. doi:10.1007/978-3-030-60440-0_27.

[2] P. Angelini, S. Chaplick, S. Cornelsen, and G. Da Lozzo. On upward-planar L-drawings of graphs. In S. Szeider, R. Ganian, and A. Silva, editors, *MFCS 2022*, volume 241 of *LIPIcs*, pages 10:1–10:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. `doi:10.4230/LIPIcs.MFCS.2022.10`.

[3] P. Angelini, S. Chaplick, S. Cornelsen, and G. Da Lozzo. Planar L-drawings of bimodal graphs. *JGAA*, 26(3):307–334, 2022. `doi:10.7155/jgaa.00596`.

[4] P. Angelini, G. Da Lozzo, M. Di Bartolomeo, V. Di Donato, M. Patrignani, V. Roselli, and I. G. Tollis. Algorithms and bounds for L-drawings of directed graphs. *Int. J. Found. Comput. Sci.*, 29(4):461–480, 2018. `doi:10.1142/S0129054118410010`.

[5] W. Barth, P. Mutzel, and C. Yıldız. A new approximation algorithm for bend minimization in the Kandinsky model. In M. Kaufmann and D. Wagner, editors, *GD 2006*, volume 4372 of *LNCS*, pages 343–354. Springer, 2007. `doi:10.1007/978-3-540-70904-6_33`.

[6] M. A. Bekos, M. Kaufmann, R. Krug, and M. Siebenhaller. The effect of almost-empty faces on planar Kandinsky drawings. In E. Bampis, editor, *SEA 15*, volume 9125 of *LNCS*, pages 352–364. Springer, 2015. `doi:10.1007/978-3-319-20086-6_27`.

[7] P. Bertolazzi, G. Di Battista, and W. Didimo. Quasi-upward planarity. *Algorithmica*, 32(3):474–506, 2002. `doi:10.1007/s00453-001-0083-x`.

[8] J. Bhasker and S. Sahni. A linear algorithm to find a rectangular dual of a planar triangulated graph. *Algorithmica*, 3:247–278, 1988. `doi:10.1007/BF01762117`.

[9] T. Biedl and M. Derka. The (3,1)-ordering for 4-connected planar triangulations. *JGAA*, 20(2):347–362, 2016. `doi:10.7155/jgaa.00396`.

[10] T. Biedl and G. Kant. A better heuristic for orthogonal graph drawings. *Comp. Geom.*, 9(3):159–180, 1998. `doi:10.1016/S0925-7721(97)00026-6`.

[11] T. Biedl and D. Mondal. A note on plus-contacts, rectangular duals, and box-orthogonal drawings. Technical Report arXiv:1708.09560v1, Cornell University Library, 2017. `doi:10.48550/arXiv.1708.09560`.

[12] C. Binucci, E. Di Giacomo, G. Liotta, and A. Tappini. Quasi-upward planar drawings with minimum curve complexity. In H. C. Purchase and I. Rutter, editors, *GD 2021*, volume 12868 of *LNCS*, pages 195–209. Springer, 2021. `doi:10.1007/978-3-030-92931-2_14`.

[13] T. Bläsius, G. Brückner, and I. Rutter. Complexity of higher-degree orthogonal graph embedding in the Kandinsky model. In A. S. Schulz and D. Wagner, editors, *ESA 2014*, volume 8737 of *LNCS*, pages 161–172. Springer, 2014. `doi:10.1007/978-3-662-44777-2_14`.

[14] U. Brandes. Eager *st*-ordering. In R. Möhring and R. Raman, editors, *ESA 2002*, volume 2461 of *LNCS*, pages 247–256. Springer, 2002. `doi:10.1007/3-540-45749-6_25`.

[15] G. Brückner. Higher-degree orthogonal graph drawing with flexibility constraints. Bachelor thesis, Department of Informatics, KIT, 2013. Available at https://i11www.iti.kit.edu/_media/teaching/theses/ba-brueckner-13.pdf.

[16] S. Chaplick, M. Chimani, S. Cornelsen, G. Da Lozzo, M. Nöllenburg, M. Patrignani, I. G. Tollis, and A. Wolff. Planar L-drawings of directed graphs. In F. Frati and K.-L. Ma, editors, *GD 2017*, volume 10692 of *LNCS*, pages 465–478. Springer, 2018. `doi:10.1007/978-3-319-73915-1_36`.

[17] M. Chimani, C. Gutwenger, M. Jünger, G. W. Klau, K. Klein, and P. Mutzel. The open graph drawing framework (OGDF). In R. Tamassia, editor, *Handbook on Graph Drawing and Visualization*, chapter 17, pages 543–569. Chapman and Hall/CRC, 2013. URL: https://cs.brown.edu/people/rtamassi/gdhandbook/.

[18] G. Di Battista and R. Tamassia. Algorithms for plane representations of acyclic digraphs. *Theor. Comput. Sci.*, 61:175–198, 1988. `doi:10.1016/0304-3975(88)90123-5`.

[19] M. Eiglsperger. *Automatic Layout of UML Class Diagrams: A Topology-Shape-Metrics Approach*. PhD thesis, Eberhard-Karls-Universität zu Tübingen, 2003. URL: https://publikationen.uni-tuebingen.de/xmlui/handle/10900/48535.

[20] U. Fößmeier and M. Kaufmann. Drawing high degree graphs with low bend numbers. In F. J. Brandenburg, editor, *GD 1995*, volume 1027 of *LNCS*, pages 254–266. Springer, 1996. `doi:10.1007/BFb0021809`.

[21] A. Garg and R. Tamassia. On the computational complexity of upward and rectilinear planarity testing. *SIAM J. Comput.*, 31(2):601–625, 2001. `doi:10.1137/S0097539794277123`.

[22] M. Gronemann. Bitonic *st*-orderings for upward planar graphs. In Y. Hu and M. Nöllenburg, editors, *GD 2016*, volume 9801 of *LNCS*, pages 222–235. Springer, 2016. `doi:10.1007/978-3-319-50106-2_18`.

[23] X. He. On finding the rectangular duals of planar triangular graphs. *SIAM J. on Computing*, 22(6):1218–1226, 1993. `doi:10.1137/0222072`.

[24] G. Kant and X. He. Two algorithms for finding rectangular duals of planar graphs. In J. van Leeuwen, editor, *WG 1993*, volume 790 of *LNCS*, pages 396–410. Springer, 1994. `doi:10.1007/3-540-57899-4_69`.

[25] D. Poulalhon and G. Schaeffer. Optimal coding and sampling of triangulations. *Algorithmica*, 46(3):505–527, 2006. `doi:10.1007/s00453-006-0114-8`.

[26] R. Tamassia. On embedding a graph in the grid with the minimum number of bends. *SIAM J. on Computing*, 16(3):421–444, 1987. `doi:10.1137/0216030`.

[27] R. Tamassia, I. G. Tollis, and J. S. Vitter. Lower bounds for planar orthogonal drawings of graphs. *Information Processing Letters*, 39(1):35–40, 1991. `doi:10.1016/0020-0190(91)90059-Q`.