Approximating the Bundled Crossing Number

Alan Arroyo [1] *Stefan Felsner* [2]

[1]IST Austria, Klosterneuburg, Austria
[2]Institut für Mathematik, Technische Universität Berlin, Germany

Abstract. Bundling crossings is a strategy which can enhance the readability of graph drawings. In this paper we consider good drawings, i.e., we require that any two edges have at most one common point which can be a common vertex or a crossing. Our main result is that there is a polynomial-time algorithm to compute an 8-approximation of the bundled crossing number of a good drawing with no *toothed hole*. In general the number of toothed holes has to be added to the 8-approximation. In the special case of circular drawings the approximation factor is 8, this improves upon the 10-approximation of Fink et al. [14]. Our approach also works with the same approximation factor for families of pseudosegments, i.e., curves intersecting at most once. We also show how to compute a $\frac{9}{2}$-approximation when the intersection graph of the pseudosegments is bipartite and has no toothed hole.

1 Introduction

The study of bundled crossings is a promising topic in Graph Drawing due to its practical applications in Network Visualization and the rich connections with related areas such as Topological Graph Theory. One of the mantras motivating the study of crossing numbers is that reducing

crossings can improve the readability of a drawing, leading to better representation of graphs. The study of bundled crossings provides an alternative way to assess readability by allowing crossings of a drawing to be bundled into regular grid-patterns with the goal of minimizing the number of bundled crossings instead of minimizing the number of individual crossings.

The *crossing number* of a graph G is the minimum integer $\mathrm{cr}(G)$ for which G has a drawing on the sphere with $\mathrm{cr}(G)$ crossings. Computing the crossing number of a graph is a notoriously hard problem. There are long standing conjectures regarding the crossing numbers of complete graphs [19, 4] and complete bipartite graphs [28]. Another family whose crossing numbers have been intensely studied are Cartesian products of two cycles $C_m \square C_n$. It is conjectured that their crossing number is $\mathrm{cr}(C_m \square C_n) = (m-2)n$ for, $3 \leq m \leq n$ [20]. Figure 1(left) indicates how to draw the product $C_m \square C_n$ with only $(m-2)n$ crossings, hence $\mathrm{cr}(C_m \square C_n) \leq (m-2)n$.
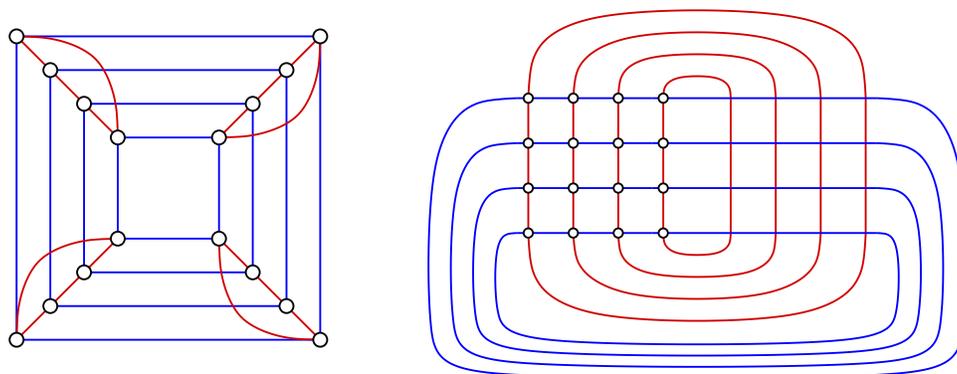

Figure 1: The crossing number of $C_4 \square C_4$ is 8 (left) but there is a drawing with 16 crossings which can be viewed as a single bundled crossing (right).

To define bundled crossings, we consider the *planarization* of a drawing D on the sphere, this is the plane graph obtained by replacing crossings by degree-4 vertices (we assume that the drawing is simple, i.e., a crossing point belongs to only two edges). A *bundled crossing* or *bundle* of D is a subgraph of the planarization of D isomorphic to an $n \times m$-grid graph ($n, m \geq 1$), whose vertices are exclusively crossings. A drawing of $C_4 \square C_4$ where all crossings can be assigned to a single bundle is shown in Figure 1(right).

A *bundling* of a drawing D is a partition of the crossings of D into disjoint bundles. The *bundled crossing number* $\mathrm{bc}(D)$ of D is the minimum number of bundles in any bundling of D, whereas the bundled crossing number $\mathrm{bc}(G)$ of a graph G is the minimum $\mathrm{bc}(D)$ taken over all drawings D of G. From the fact that $C_4 \square C_4$ is not planar and Figure 1, it follows that $\mathrm{bc}(C_4 \square C_4) = 1$. Indeed, $\mathrm{bc}(C_m \square C_n) = 1$ for $n, m \geq 3$.

1.1   Previous work

In his survey on crossing number problems [27] Schaefer suggests to consider bundlings of crossings. Alam et al. [1] were the first to study the problem from a graph drawing point of view. Later, Fink and coauthors considered the problem of computing $\mathrm{bc}(\cdot)$ both in the *free-drawing* variant, when a graph G is the input, and the goal is to compute $\mathrm{bc}(G)$, and in the *fixed-drawing* variant, where a simple drawing D is the input, and the goal is to find $\mathrm{bc}(D)$, i.e., to assign the crossings to as few

bundles as possible. In this work we will focus on the fixed-drawing variant for simple drawings which are *connected* in the sense that the boundary of every cell of the drawing is connected.

Fink et al. [14] showed that computing bc(·) is NP-hard in the fixed-drawing variant of the problem. The hardness of computing bc(·) in the free-drawing variant has been shown by Chaplick et al. [5]. An algorithm that computes a 10-approximation of bc(D) for circular drawings was presented by Fink et al. [14]; here a *circular drawing* of a graph is one where vertices are drawn on a circle and edges are drawn inside the circle, more precisely they consider circular drawings in which (0) there are no self-intersections of edges; (1) any two edges intersect at most once; (2) the intersection between any two edges is either a common vertex or a crossing; and (3) no three edges share a crossing. Our work was motivated by the question of whether bc(D) can be approximated on more general classes of drawings of graphs.

1.1.1   Related concepts and applications

Bundling the edges of a graph is a commonly used simplification technique in the visualization of large and dense graphs or networks. Together with van Wijk, Holten introduced the idea of edge bundling to the area of information visualization [21, 22]. Subsequently the concept was studied by many researchers, see for example [17, 18, 24, 26, 6, 13, 3, 25, 31, 30]. In contrast to the concept of the bundled crossing number most algorithms used in information visualization for edge bundling are of heuristic nature and do not provide any guarantees on the quality of the solution.

A related definition to bundled crossings are *block crossings*. This notion was studied in the context of layouts of metro maps [15] and storyline visualizations [29]. It can be viewed as the minimization of bundled crossings for a family of paths on a plane graph. Another approach to formalize the idea of edge bundling are so-called *confluent drawings*. Here, edges are drawn as continuous curves that are allowed to merge and split similar to switches in a train network. These drawings were introduced by Dickerson et al. [7] and subsequently studied by the graph drawing community [12, 10, 11, 16, 23, 2]. Recently, these drawings were also picked up in more applied contexts, using heuristic approaches [3, 31].

1.2   Our contribution

To prepare for the statement of our main results we first show how to reduce the problem of computing bc(·) for a graph drawing D to computing bc(·) for a *set of strings*. We then introduce special configurations called toothed-faces which play a special role in this work. In fact toothed-faces cause an additive term which interferes with the approximation factor in some of our results.

To a graph drawing D we associate a set $\mathcal{E}$ of strings obtained in two steps: first, delete all the uncrossed edges of D; second, for each edge e of D, remove a small bit of e at each endpoint to obtain a *string* (a closed arc). This results in a set $\mathcal{E}$ of strings which is the drawing of a matching, thus bc($\mathcal{E}$) is well-defined. Moreover, the bundlings of D are in one-to-one correspondence with the bundlings of $\mathcal{E}$, so often in this work we restrict ourselves to study bundlings of sets of strings.

In connection with the crossing number minimization problem it is natural to restrict the attention to simple drawings, in this context they are often referred to as *good drawings*. The set of strings associated with a good drawing turns out to be a *family of pseudosegments*, i.e., it is a set of strings with the property that no string self-intersects and no two strings cross more than once.

Any set of strings divides the plane into open regions called *faces*. A string *ends* in a face F if one of its endpoints is incident with the boundary of F. Before defining toothed-faces, keep in mind that the boundary of a face is not necessarily the same as the boundary of its closure. For

instance, Figure 2 shows two examples of faces where their boundaries include pieces of strings ending in the face while their closure is bounded by only four pieces of strings.

Figure 2: Examples of toothed-faces.

A *toothed-face* is a face F of a set of strings such that (1) at least one string ends in F; (2) the closure $\overline{F}$ is bounded by exactly four pieces of strings; and (3) all the crossings between the strings ending in F and the boundary of $\overline{F}$ occur in only two opposite string-pieces among the four string-pieces bounding $\overline{F}$. For each set of strings $\mathcal{E}$ we will let $t(\mathcal{E})$ be the number of toothed-faces. If $\mathcal{E}$ is associated to a drawing D, then we let $t(D) := t(\mathcal{E})$. The next is our first main result.

Theorem 1 *For a connected good drawing D with $t = t(D)$ there is a polynomial-time algorithm to compute a bundling with at most $8bc(D) + t$ bundles.*

The algorithm used in Theorem 1 is a simple greedy approach that has been known for some time. In the next section we will informally explain this greedy algorithm.

Our main contribution is to improve upon the 10-approximation of Fink et al. on two fronts. First, since circular drawings do not induce toothed-faces, Theorem 1 shows that the greedy approach produces an 8-approximation instead of a 10-approximation:

Corollary 1 *For circular drawings there is a polynomial-time algorithm to compute a bundling with at most $8bc(D)$ bundles.*

Theorem 1 also extends the guarantee of obtaining an approximate solution to the more general class of simple drawings. The proof of our results is based on the same greedy approach as the approximation algorithm of Fink et al. [14]; the difference is in the details of the analysis.

In a separate section we discuss further consequences from the proof of Theorem 1. In particular we show that by starting the greedy algorithm with a suitably chosen set of cut-segments it is possible to get rid of the additive t in the approximation. The cost for finding the initial set of segments, however, is an exponential function of t. The result is a FPT algorithm (FPT = fixed-parameter tractable).

Theorem 2 *For a connected good drawing D there is a FPT algorithm with parameter $t = t(D)$ which computes a bundling with at most $8bc(D)$ bundles.*

In our second main result we improve the approximation factor when the drawing D is as in Theorem 1 and has the additional property that the intersection graph of the edges of D is bipartite. In such case we call D a *bipartite instance*.

Theorem 3 *If D is a bipartite instance with $t = t(D)$ toothed-faces, then there is a polynomial-time algorithm to compute a bundling with at most $\frac{9}{2}bc(D) + \frac{1}{2}t$ bundles.*

1.3   An easy example: Bundling bi-laminar families of chords

In this section, in an informal approach, we consider a concrete example that captures many of the concepts that will be used in the later parts of the paper.

A *laminar family* of chords is a collection of pairwise disjoint chords in a circle such that for any three chords one of them separates the other two. A *bi-laminar instance* is a circle together with a red and a blue laminar family of chords such that any two chords have at most one crossing (Figure 3a).

A bi-laminar instance can be converted, by an appropriate crossing-preserving transformation, into a family of blue vertical chords and red horizontal chords drawn inside an *orthogonal polygon* P, i.e., a polygon whose edges are parallel to the x- and y-axis (Figure 3b). Moreover, the polygon and the chords can be chosen so that the chords are evenly spaced forming a regular grid inside P.
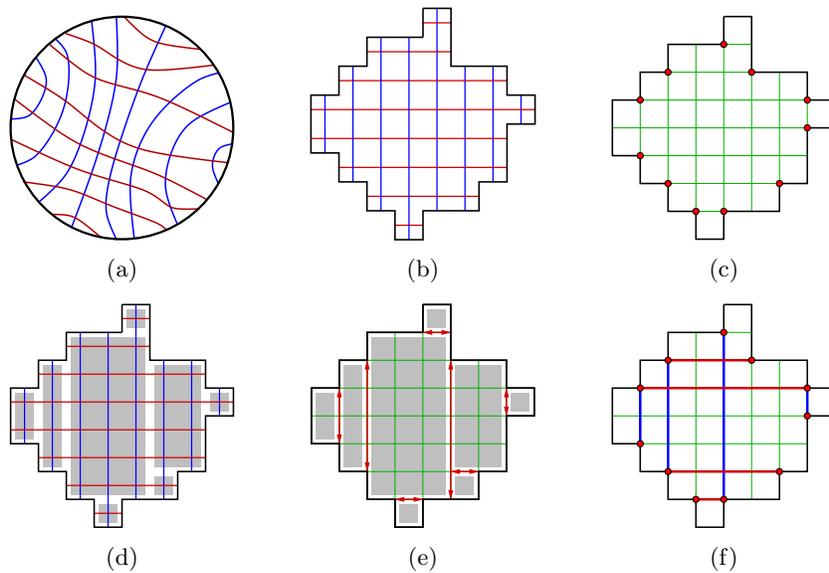

Figure 3: Bundling of a bi-laminar instance.

With the tranformed instance in hand, we can now subdivide the interior of P into squares, each of them containing a crossing in its center. We will refer to the green graph obtained from the union of P and the perimeter of the squares as the *dual net* of the instance (Figure 3c). A bundled crossing corresponds to a collection of squares in the dual net forming a *rectangle* (Figures 3d-3e); moreover, a *bundling* of the new instance corresponds to a partition of the squares of the dual net into rectangles. Any such partition is called a *rectangulation* of P, and a minimal bundling corresponds to a rectangulation using a minimum number of rectangles.

The problem of rectangulating an orthogonal polygon into the minimum number of rectangles was solved by at least three independent groups in the 80's, Eppstein [9] provides the relevant references. We next describe a polynomial-time algorithm for finding a minimal rectangulation.

The segments added to the polygon P to obtain a rectangulation are either horizontal or vertical (red bi-arrows in Figure 3e). In a rectangulation with R rectangles and S segments, the relation $R = S + 1$ holds. Therefore, the problem of finding a minimal rectangulation is equivalent to the problem of adding a set of segments of minimum cardinality that induces a rectangulation of P.

Each concave corner v of P (red vertices in Figure 3c) must be incident with at least one segment of any rectangulation. We call this the *requirement* at v (the requirement is related to what we later call the *exponent* of v). A *good segment* is a segment connecting two concave corners; it is called good because it satisfies the requirement of two corners. Figure 3f shows the given polygon together with all the good segments. The example shows that pairs of segments may cross or share an endpoint, we say that they are in *conflict*.

Ultimately the rectangulation problem for P reduces to studying a conflict graph C whose vertices are the possible good segments. Two good segments in C are joined by an edge, when they are in conflict. The conflict graph is bipartite, with blue segments on one side and red segments on the other. A minimum family of segments that yields a rectangulation corresponds to a maximum independent set I of C plus a set of segments covering the corners which are not incident with elements in I. Since computing a maximum independent set in a bipartite graph can be done in polynomial-time, the same holds true for computing a minimum family of segments rectangulating P.

Although we know how to find an exact solution for rectangulating P, we now describe how to find an approximate solution to illustrate the algorithm we will use to prove Theorem 1. Let S be a set of segments which is initially empty. Consider the concave corners of P one by one. If the current corner v is not incident to a segment in S choose a direction d (horizontal or vertical) and *shoot a segment* in direction d from v, i.e., let s_v be the segment with one end at v extending into the interior of P and ending at the first point which belongs to a segment in S or the boundary of P. This segment s_v is added to S. From the discussion it should be clear that for laminar families of chords this process yields a rectangulation that uses at most twice as many segments as an optimal one, i.e., it is a 2-approximation for the number of segments and also for the number of rectangles. We refer to this approach as the *greedy strategy*. Fink et al. [14] analyze this strategy in the setting where the input of the bundling problem is a circular drawing. They show that it yields a 10-approximation for the number of bundles. In Sections 4-6 we analyze the greedy strategy for the bundling problem for good drawings and show that this simple-minded strategy guarantees an 8-approximation (Theorem 1). In Section 7 we discuss how to benefit by deviating from the pure greedy strategy. In particular this leads to FPT algorithms with a slightly improved approximation quality.

In Section 8, we study biparte collections of strings; this is closer to the laminar family studied here. There we show that a slightly modified greedy strategy produces a solution which is a $\frac{3}{2}$-approximation for the number of segments needed to rectangulate P. The key is to first compute a set of segments of one color which maximizes a parameter called marginal gain. This set is extended using the greedy strategy until a rectangulation is obtained.

2   Strings and Nets

We now begin with the discussion of concepts and ideas which will eventually (at the send of Section 6) lead to the proof of Theorem 1, our main approximation result for the bundeled crossing number

We think of a bundling instance as a set of strings, i.e., as a set of simple curves in the plane. Throughout this paper, unless otherwise stated, we assume that the endpoints of any two strings are different and the union of the strings forms a connected set. A set $\mathcal{E}$ of strings is *grounded* in $\mathcal{B}$ (see Figure 4a) if $\mathcal{B}$ is a set of pairwise disjoint simple closed curves, such that (1) each string has its ends in the union of the boundary curves; (2) boundary curves only intersect the strings at

their ends; (3) each boundary curve contains at least one end of a string; and (4) no two boundary curves are incident with the same cell of $\mathcal{E} \cup \mathcal{B}$. The curves of such a family $\mathcal{B}$ are denoted as *boundary curves* of $\mathcal{E}$, in our figures we use the color blue for boundary curves.

One can always turn a set of strings into a grounded one by adding a single blue curve in each face where strings end (Figure 4a). Henceforth, unless otherwise stated, any set of strings we consider is grounded.

(a) strings and grounded strings   (b) strings and net   (c) net
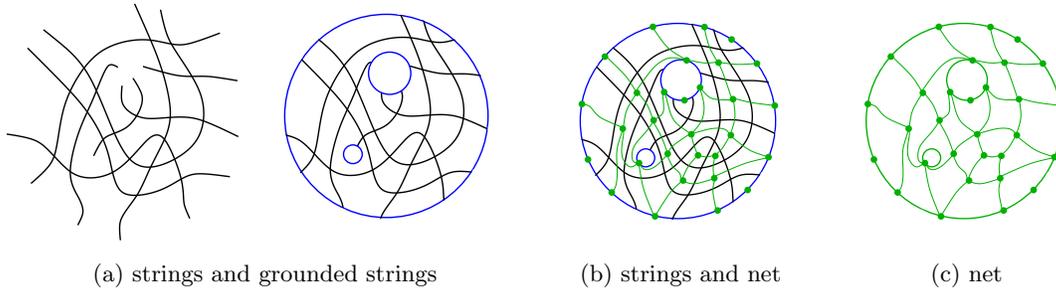
Figure 4: From a set of strings to its net.

To a set of strings $\mathcal{E}$ grounded in $\mathcal{B}$ we associate a plane graph which is called the *dual net* or just the *net*. The net $\mathcal{N}$ is obtained by placing a vertex in each cell which is not bounded by a single closed curve in $\mathcal{B}$ and by adding an edge between two vertices whenever their corresponding cells share a segment of a string connecting two consecutive intersection points. We will draw our dual net as in Figure 4c so that each vertex corresponding to a cell incident to a boundary curve is drawn on the boundary curve, and edges connecting two consecutive vertices in the same boundary curve are also drawn along the boundary curve. Note that boundary curves containing one or two ends of strings cause loops and double-edges in the net.

The faces of $\mathcal{N}$ come in two kinds: *boundary-holes*, defined as the faces bounded by the boundary blue curves; and faces of degree four which enclose a crossing, which will be called *squares*.

A *rectangle* in the dual net of a grounded set of strings is a set of squares which can be labeled with a vertex-set of an $n \times m$-grid, so that any two squares adjacent in the grid share an edge of their boundaries. A bundle of a set of strings corresponds to a rectangle in the dual net. Hence, a bundling of a set of strings corresponds to a *rectangulation*, i.e., a partition of the squares into rectangles. A rectangulation is *optimal* if it corresponds to a bundling with a minimum number of bundles.

3   Segments and Holes

In this section we consider a fixed set of grounded strings with its dual net $\mathcal{N} = (V, E)$. The *border* of $\mathcal{N}$ is the subgraph induced by vertices and edges drawn on boundary curves. Any vertex or edge of $\mathcal{N}$ not on the border is *interior*. A degree-4 interior vertex is called *regular* and any other interior vertex is a *vertex-hole*. A *hole* is a vertex or face of $\mathcal{N}$ that is either a vertex-hole or a boundary-hole.

A path of $\mathcal{N}$ is *straight* if all its inner vertices are regular and any two consecutive edges are opposite in the rotation at their common vertex. Since a single edge has no inner vertex it qualifies as straight path. A *cut-set* is a set $\mathcal{S}$ of interiorly-disjoint straight-paths where every end of a path

is either on a hole or in the interior of another path of $\mathcal{S}$. We refer to the elements of a cut-set as *segments*.

Given a rectangulation $\mathcal{R}$ of $\mathcal{N}$, an edge of E is *separating* if it belongs to the boundary of two squares belonging to different rectangles. A cut-set *delimits* $\mathcal{R}$ if its segments only include separating edges and each separating edge is included in a segment.

Given $\mathcal{R}$ it is easy to iteratively build a delimiting cut-set: The first segment s_1 is obtained by considering any edge e_1 in the set $E' \subseteq E$ of separating edges, and by maximally extending e_1 into a straight path of edges in E'. In the i-th step, s_i is obtained by maximally extending an edge e_i of E' which is not covered by $s_1 \cup \cdots \cup s_{i-1}$, i.e., $e_i \in E' \setminus E(s_1 \cup \cdots \cup s_{i-1})$, into a straight path that is interiorly-disjoint from $s_1, \ldots, s_{i-1}$. This is done until all the edges of E' are covered by segments.

If there is no regular vertex v whose four incident edges are separating, then the cut-set delimiting $\mathcal{R}$ is unique. Otherwise, for any such vertex v, one can choose the pair of opposite edges which belong to a common segment, the other two segments end at v. This binary choice at any such vertex v may give rise to exponentially many delimiting cut-sets. In practice we can choose and fix an arbitrary cut-set of $\mathcal{R}$. With this in mind, we can now refer to the *segments of $\mathcal{R}$* as the segments of the chosen cut-set delimiting $\mathcal{R}$.

Figure 5a shows a rectangulation delimited by a cut-set consisting of 10 segments and Figure 5b shows the corresponding bundling of the crossings with 6 bundles.
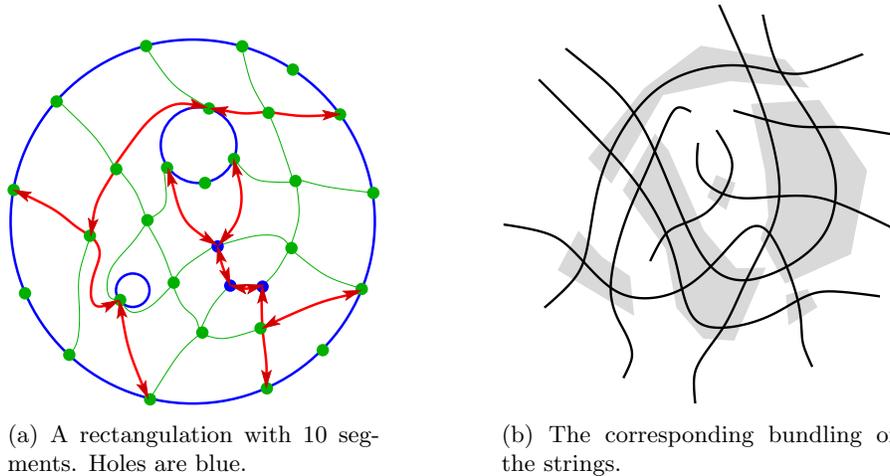
(a) A rectangulation with 10 segments. Holes are blue.

(b) The corresponding bundling of the strings.

Figure 5: A rectangulation of a net and the corresponding bundling of strings.

Next we relate the number of rectangles and the number of segments in a rectangulation. Let $H = H(\mathcal{N})$ denote the number of holes of $\mathcal{N}$.

Lemma 1 *In a net $\mathcal{N}$ with H holes the numbers R of rectangles and S of segments satisfy:*

$$R - S + H = 2. \tag{1}$$

To prove Lemma 1, we associate a cubic plane graph $\Gamma = \Gamma(\mathcal{R})$ to a rectangulation $\mathcal{R}$ of a net $\mathcal{N}$. The construction will be used again in Section 5.

Construction of Γ: First, we consider the subgraph $\mathcal{H}$ of $\mathcal{N}$ obtained from the union of the delimiting segments in $\mathcal{R}$ and the border of $\mathcal{N}$. Color the edges of $\mathcal{H}$ included in segments such

that for each segment all the edges on the segment have the same color and the colors used for different segments are distinct.

Next, we apply a local transformation at each vertex $v \in V(\mathcal{H})$: If v is a regular vertex, then we keep v unchanged unless $\deg_{\mathcal{H}}(v) = 4$. In this case we split v into two vertices v_1, v_2 joined by an edge as in Figure 6, where each v_i has degree 3 and the color of $v_1 v_2$ is the color of the unique segment having v in its interior.

Figure 6: Local transformation when v is regular and $\deg_{\mathcal{H}}(v) = 4$.

If v belongs to the border and $\deg_{\mathcal{H}}(v) \geq 4$ or if v is a vertex-hole of arbitrary degree we split v as follows. Let $\rho_{\mathcal{H}}(v) = (e_1, \ldots, e_d)$ be its rotation, i.e. the circular order of edges around v, in $\mathcal{H}$, so that if v is on the border edges e_1 and e_d also belong to the border. Subdivide each edge e_i by adding a degree-2 vertex v_i close to v in e_i. Remove v and all the edges $v_i v$ incident with v. Next, add a path $(v_1, \ldots, v_d)$ or a cycle $(v_1, \ldots, v_d, v_1)$ depending whether v is in the border or not.

Finally, we suppress degree-2 vertices, i.e., if v is a vertex with only two incident edges e_1, e_2, then we delete v and make e_1, e_2 a single edge. This yields Γ. Figure 7 shows an example.
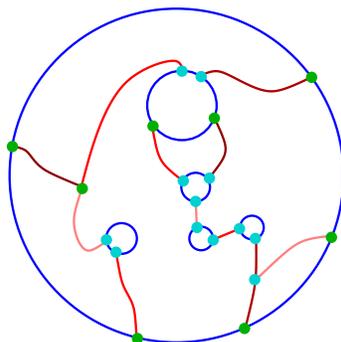

Figure 7: The graph Γ corresponding to the rectangulation of Figure 5a. Vertices obtained by splitting vertex-holes or vertices on boundaries are shown in light blue.

Proof: [Proof of Lemma 1] Given a rectangulation $\mathcal{R}$, let $\Gamma = \Gamma(\mathcal{R})$ as above. Each segment of the rectangulation corresponds to a monochromatic path in Γ and each vertex of Γ is an end-vertex of exactly one of them. Thus $|V(\Gamma)| = 2S$. As Γ is cubic, $E(\Gamma) = \frac{3}{2}|V(\Gamma)| = 3S$. Finally, as each face of Γ corresponds to either a rectangle or a hole, Γ has $R + H$ faces. Equation 1 now follows from Euler's formula. $\qquad \square$

Before concluding this section, we make some remarks about Γ that will be used later. We let Γ^* denote the plane dual of Γ and note that both Γ and Γ^* may in general have multi-edges.

Remark 1 *(i) Each face of Γ corresponds to a hole or to a rectangle.*

(ii) Γ^* is a plane triangulation, i.e., the boundary of each face consists of three edges.

(iii) Γ *and its planar dual* Γ^* *are simple graphs when every hole is incident with at least three segments of* $\mathcal{R}$ *(otherwise* Γ *has multi-edges and/or loops).*

(iv) *The vertices of* Γ^* *corresponding to holes form an independent set in* Γ^*.

4   Approximating the Number of Segments

Let $\mathcal{N} = (V, E)$ be a net and $B \subset E$ be the set of edges in the border of $\mathcal{N}$. An edge-set $E_0 \subseteq E$ *saturates* a vertex $v \in V$ if each angle induced by the edges of $E_0 \cup B$ at v contains a corner of at most two squares of $\mathcal{N}$, or, if v is regular and no edge of E_0 is incident with v. Moreover, E_0 *saturates* $\mathcal{N}$ if every vertex is saturated by E_0.

Naturally, the (edge-set of the) segments of a rectangulation saturate $\mathcal{N}$. The next lemma shows that saturation is also sufficient to induce a rectangulation.

Lemma 2 *A set of edges is saturating a net* $\mathcal{N}$ *if and only if it is an edge-set delimiting a rectangulation.*

Before proving Lemma 2, we observe that this lemma does not extend to more general systems of curves. In Figure 8a we show a system of five curves one of them being closed. The corresponding dual net is saturated by $\emptyset$ whereas $\emptyset$ does not induce a rectangulation of the net.

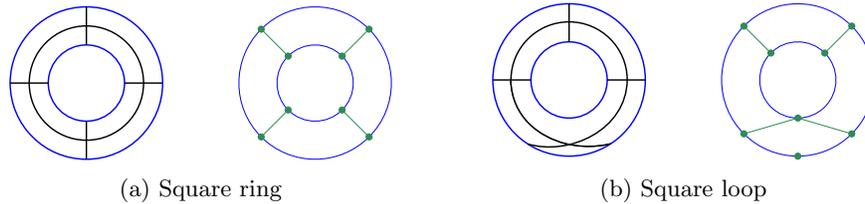
(a) Square ring   (b) Square loop

Figure 8

The dual nets of Figure 8a and 8b contain special configurations of squares forbidden in nets of strings: A *square-ring* is a circular sequence $(s_0, s_1, \ldots, s_m, s_0)$ of squares, where s_i is glued to s_{i-1} and s_{i+1} by using opposite sides of s_i. A *square-loop* is similar to a square-ring, except that one square is glued by using two consecutive sides instead of opposite sides.

Proof: [Proof of Lemma 2] A set of edges delimiting a rectangulation can be partitioned to make a set $\mathcal{S}$ of segments. If $\mathcal{S}$ is delimiting a rectangulation, then each vertex is saturated by the edges of $\mathcal{S}$ whence the set of edges of the segments is saturating.

Now let S be a saturating edge-set, let B be the set of boundary-edges, and let $U \subset E \setminus (B \cup S)$ be the set of interior edges of the net that are not covered by $S \cup B$. We proceed by induction on $|U|$. If $|U| = 0$, then each boundary edge of each square is either in the border or in S. In this case S induces a rectangulation.

Now let F be a face of the subgraph induced by $B \cup S$, such that F contains at least one edge from U. We have to show that F is a rectangle. Let $(e_1, \ldots, e_m)$ be a counterclockwise boundary walk of F (where e_i is the i-th edge encountered in the walk). Let $e_0 = e_m$ and define $W = (e_0, e_1, \ldots, e_m)$. Since S is saturating, at most one edge of U is pointing into the interior of F between any pair e_i and e_{i+1} of edges. A square of $\mathcal{N}$ is a *corner* of F, if it is contained in F and two of its boundary edges are consecutive in W. Consider the set S_W of squares incident to the

edges in W, if S_W contains less that two corner squares, then S_W forms square-ring (O corners) or a square-loop (1 corner). It follows that F has at least two corner squares.

Let s and s' be consecutive corners along W and let $u_1, \ldots, u_r$ be the sequence of edges pointing into F between s and s' along W and let u_0 and u_{r+1} be the edges opposite to u_1 and u_r on s and s' respectively. Note that u_0 and u_{r+1} belong to W. Let $s_0 = s$ and $s_r = s'$ and for $1 \leq i < r$ let s_i be the square incident to u_i and u_{i+1}. By relabeling W, we may assume that edge e_i is incident with s_i for $i = 0, \ldots, r$. We let e_i' be the edge opposite to e_i in s_i. If an e_i' edge is in U, then all the e_i' edges must be in U, or else a vertex incident with two e_i's, one in U and one not in U, would not be a saturated by S. To see this note that such a vertex x is incident to an edge not in U, and x has an angle which contains the corner of at least three squares.

If all the e_i' edges are in $B \cup S$, then $W = (e_1, \ldots, e_r, u_{r+1}, e_r', \ldots, e_1', u_0)$ and $F = \bigcup_{i=0}^{r} s_i$. In this case F is a rectangle. Otherwise all the e_i' edges are in U. In this case we add all the edges e_i' to S to obtain S'. Since S was saturating we easily conclude that S' is saturating as well. By induction S' delimits a rectangulation. One rectangle of this rectangulation is $F = \bigcup_{i=0}^{r} s_i$. Let F' be the rectangle covering the square on the other side of e_1'. For each i let s_i' be the square opposite of s_i at e_i'. The saturation property of the vertices on the path $e_1', \ldots, e_r'$ implies that all the interior vertices of this path are of degree four in the net with all incident edges in U. At the end vertices of the path we find that s_1' and s_r' are consecutive corner squares of F'. Hence the union of F and F' is a rectangle delimited by S. □

In Section 3 we have seen that from a set of edges delimiting a rectangulation we can build a set of segments. We call a set of segments saturating if the underlying set of edges is saturating. With this definition Lemma 2 has a corresponding version for segments: $\mathcal{S}$ is saturating if and only if $\mathcal{S}$ is delimiting a rectangulation.

The following definition will be crucial for finding lower bounds on the size of saturating families of segments which has implications on the number of rectangles.

Definition 1 (Exponent) *Given a net $\mathcal{N} = (V, E)$ and $v \in V$, the exponent of v is the minimum number of edges in an edge-set S_v saturating[1] v . Hence $exp(v) = 0$ if v is a regular vertex and*

$$exp(v) = \begin{cases} \left\lfloor \frac{deg_\mathcal{N}(v)}{2} \right\rfloor - 1 & v \text{ is in the border;} \\ \left\lceil \frac{deg_\mathcal{N}(v)}{2} \right\rceil & v \text{ is a vertex-hole.} \end{cases}$$

We let $exp(\mathcal{N}) := \sum_{v \in V} exp(v)$.

The greedy strategy: Let $v_1, \ldots, v_k$ be an arbitrary linear order on the vertices of $\mathcal{N}$ with positive exponent. The strategy is adding segments at vertices in the given order. When it comes to v_i some incident edges may already be contained in segments belonging to earlier vertices. Select a minimal set of edges not covered by the segments such that shooting segments from these edges results in an edge set saturating v_i. The number of segments introduced to saturate v_i is upper bounded by $exp(v_i)$. Here is an illustrating example: Let v_i be a vertex hole of degree 9 with incident edges $e_0, e_1, \ldots, e_8$ in clockwise order, $exp(v_i) = 5$. Suppose that when it comes to process v_i the edges e_0 and e_3 already belong to segments. Now the algorithm will shoot segments in the directions of e_5, e_7, and in the direction of either of e_1 or e_2, three segments in total.

---

[1]This is a local condition, we do not ask for a set S of edges saturating the net such that S_v is the subset of edges of S which contain v.

Henceforth, we will denote the number of rectangles and segments in an optimal rectangulation of $\mathcal{N}$ by R_{opt} and S_{opt}, respectively. Likewise, we let R_{greed} and S_{greed} be the number of rectangles and segments, respectively, obtained after a run of the greedy strategy in $\mathcal{N}$.

Observation 1 *The following hold true for a net $\mathcal{N}$:*

(i) $S_{greed} \leq exp(\mathcal{N})$;

(ii) $S_{opt} \geq \frac{1}{2} exp(\mathcal{N})$; and consequently

(iii) $S_{greed} \leq 2 \cdot S_{opt}$.

Proof: Item (i) directly follows from the definition of the greedy strategy. Since each segment of a rectangulation contains at most two vertices of positive exponent, (ii) holds because a segment has only two ends. □

5   Rectangles and Holes

In this section we collect a few facts about rectangles and holes that will be used in our approximations. It is important to observe that a greedy rectangulation approximates the optimal when R_{greed} is bounded by a constant factor of R_{opt}. The next observation already gives a related bound by adding the holes.

Observation 2 $R_{greed} \leq 2R_{opt} + H - 2$.

Proof: Apply Lemma 1 to both sides of Observation 1.(iii). □

Our task of approximating R_{opt} now reduces to understand under which circumstances $H = O(R_{opt})$. We begin with a general observation about triangulations:

Observation 3 *In a simple plane triangulation with n vertices, an independent set has size at most $\frac{2}{3}(n - 2)$.*

Proof: The degree of a vertex in a simple triangulation is at least 3. Hence, each vertex of the independent set is incident to at least three triangles. Each triangle sees at most one vertex from the independent set I. Hence we get $3|I| \leq |F| = 2(n - 2)$, whence $|I| \leq \frac{2}{3}(n - 2)$. □

Definition 2 *Given a rectangulation $\mathcal{R}$ of $\mathcal{N}$, let $\delta(\mathcal{R})$ denote the minimum number of segments of $\mathcal{R}$ incident to any hole of $\mathcal{H}$ ($\delta(\mathcal{R})$ is the same as the minimum degree among the vertices of $\Gamma^*(\mathcal{R})$ that represent holes). We let $\delta(\mathcal{N})$ be the minimum integer k for which there is an optimal rectangulation $\mathcal{R}_0$ of $\mathcal{N}$ with $\delta(\mathcal{R}_0) = k$.*

Note that when h is a vertex-hole, then $\deg_{\Gamma^*(\mathcal{R})}(h) \geq exp(h)$. For boundary-holes a corresponding lower bound is given by the sum of exponents of the incident vertices. Figure 9 shows that there are examples where $\delta(\mathcal{N})$ is much larger than given by these lower bounds.

Observation 4 *Let $\mathcal{R}_{opt}$ be an optimal rectangulation of a net $\mathcal{N}$ with $\delta(\mathcal{N}) \geq 2$. If $H_{>2} = H_{>2}(\mathcal{R}_{opt})$ is the number of holes incident to at least three segments of $\mathcal{R}_{opt}$, then $H_{>2} \leq 2 \cdot R_{opt} - 4$.*
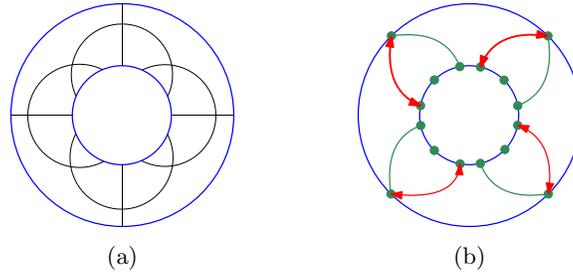
Figure 9: A set of strings (a) and its dual net (b). The red segments induce an optimal rectangulation, indeed $\delta(\mathcal{N}) = 4$. Since the exponents of all vertices at the inner hole are 0, the exponents only imply $\delta(\mathcal{N}) \geq 0$.

Proof: Let $\Gamma = \Gamma(\mathcal{R}_{opt})$ be the auxiliary 3-regular graph introduced in Section 3, page 440. For each hole incident to exactly two segments delete the hole and replace the two segments by a single one. The resulting 3-regular graph $\Gamma_{>2}$ has no loops and no multiedges. Hence, its dual $\Gamma^*_{>2}$ is a simple triangulation and the holes counted in $H_{>2}$ are an independent set in $\Gamma^*_{>2}$. From Observation 3 and Remark 1.(i) it follows that $H_{>2} \leq \frac{2}{3}|V(\Gamma^*_{>2}) - 2| = \frac{2}{3}(H_{>2} + R_{opt} - 2)$. Consequently $H_{>2} \leq 2 \cdot R_{opt} - 4$. □

If $\delta(\mathcal{N}) \geq 3$, then $H_{>2}(\mathcal{R}_{opt}) = H$ and the following directly follows from Observation 4.

Observation 5 *If $\delta(\mathcal{N}) \geq 3$, then $H \leq 2 \cdot R_{opt} - 4$.*

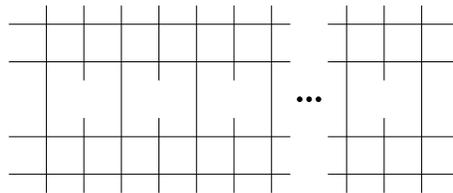## 5.1 Toothed-holes and toothed-faces

Figure 10: An example where $R_{opt} = 2$ and H is arbitrarily large.

In Observation 5 we saw that if all holes are incident with at least three segments, then H is $O(R_{opt})$. Unfortunately, it is not true in general that H is $O(R_{opt})$: Figure 10 shows that H can be arbitrarily large compared to R_{opt}. With the next lemma we prove that the unboundedness of H in terms of R_{opt} can always be attributed to the presence of toothed-faces.

Lemma 3 $H \leq 6R_{opt} + t(\mathcal{E}) - 4.$

The proof of this lemma is given at the end of the section. As we are now dealing with nets, let us introduce the dual counterpart of a toothed-face:

Definition 3 *A toothed-hole of $\mathcal{N}$ is a boundary-hole h for which all the vertices on its boundary satisfy one of the two conditions:(a) one vertex has degree 7 while the rest has degree 3; or (b) two vertices have degree 5 while the rest has degree 3. See Figure 11. We let $t(\mathcal{N})$ the number of toothed-holes in $\mathcal{N}$ (equal to $t(\mathcal{E})$ the number of toothed-faces in $\mathcal{E}$).*
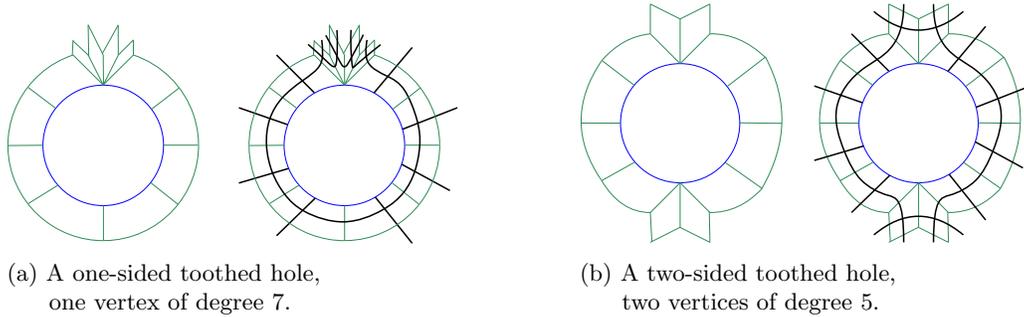
(a) A one-sided toothed hole,
one vertex of degree 7.

(b) A two-sided toothed hole,
two vertices of degree 5.

Figure 11: The two kinds of toothed-holes. Plain and with the local configutation of strings.

Given a rectangulation $\mathcal{R}$, a hole h is *incident to a corner* if there is a vertex v in the net which is a corner of a rectangle of $\mathcal{R}$ and either $h = v$ (vertex-hole) or v is a vertex on the boundary of h (boundary-hole).

Lemma 4 *Let $\mathcal{N}$ be the dual net of a set of pseudosegments with an optimal rectangulation $\mathcal{R}_{opt}$. Suppose that $H_2 = H_2(\mathcal{R}_{opt})$ is the number of holes incident to at most two segments in $\mathcal{R}_{opt}$. Then the following hold:*

(i) $H_2 \leq 4 \cdot R_{opt} + t(\mathcal{N})$; and

(ii) if $R_{opt} \geq 2$ then every hole is incident with at least two segments of $\mathcal{R}_{opt}$.

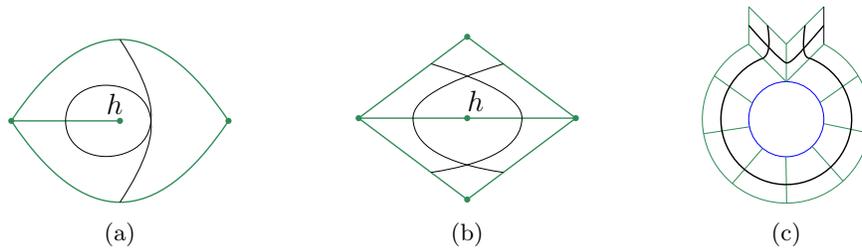(a)   (b)   (c)

Figure 12: Three forbidden situations in the dual net of an arrangement of pseudosegments

Proof: Let h be a hole incident with at most two segments but not incident with a corner. We will show that h is toothed. This is enough to prove (i) because then every hole counted in H_2 is either incident to a corner (and there are at most $4R_{opt}$ of them) or is toothed.

First, suppose by contradiction that h is a vertex-hole. How small can $\deg_{\mathcal{N}}(h)$ be? If $\deg_{\mathcal{N}}(h) = 1$, then the only square incident to h contains a self-intersecting string as shown in Figure 12a, such a string is not a pseudosegment. If $\deg_{\mathcal{N}}(h) = 2$, then two squares incident with v contain two pieces of strings forming a lense as shown in Figure 12b, this is not allowed for pseudosegments. If $\deg_{\mathcal{N}}(h) \geq 3$, then $\deg_{\mathcal{N}}(h)$ cannot be odd, as otherwise h would be incident to a corner. Hence $\deg_{\mathcal{N}}(h)$ must be even, and because h is a hole, $\deg_{\mathcal{N}}(h) \geq 6$. However, this implies that $\exp(h) \geq 3$, showing that h is incident with at least three segments, a contradiction. Thus h is a boundary hole.

Consider the vertices of $\mathcal{N}$ on the boundary ∂h of h. Since h is a vertex hole which is incident to no corner the vertices on ∂h have odd degree, indeed a vertex of even degree would be incident to an odd number of squares of the net, whence any saturating set of edges would induce a corner. Since h is incident to at most 2 segments, the sum of their exponents of the vertices on ∂h is at most 2. This restricts the degrees of the vertices on ∂h to be arranged as one of the following kinds: (a) exactly one vertex has degree 7 while the rest has degree 3; (b) two vertices have degree 5 while the rest has degree 3; (c) exactly one vertex has degree 5 while the rest has degree 3; (d) all vertices have degree 3. If either (a) and (b) occurs, then h is toothed; it remains to show that neither (c) nor (d) occurs.

If (d) occurs, then the squares incident with h would form a square-ring, which would correspond to a cyclically closed string, not a pseudosegment. If (c) occurs, then, as illustrated in Figure 12c, the squares incident to the vertices in h would induce a pair of strings that cross at least twice, a contradiction. Thus, h is toothed, ultimately implying (i).

Now we turn into proving (ii). For contradiction, suppose that $R_{opt} \geq 2$ and that $\mathcal{N}$ has a hole h incident with less than 2 segments. As shown in (i), it follows that h must be a boundary-hole. Since h is incident with less than 2 segments there is a rectangle r in the rectangulation whose boundary contains ∂h.

If h has no incident segments, then ∂h is a component of the boundary of a rectangle. This is is only possible if ∂h is the outer boundary-hole and all the pseudosegments form a single bundle, i.e., there is only one rectangle in the rectangulation. This contradicts the assumption that $R_{opt} \geq 2$.

Now suppose that h has a single incident segment s. There is a rectangle r in the rectangulation whose boundary contains ∂h and continues along s on both sides. Let $\sigma \in \mathcal{E}$ be the first string crossing s when s is oriented away from its end in h. The intersection of σ with the rectangle r can only consist of one connected piece, otherwise σ would have a selfintersection in r or have two crossings with some other pseudosegment. This, however, implies that σ is a closed loop and not a pseudosegment. Hence every hole is incident to at least two segments. □

Proof: [Proof of Lemma 3] If $R_{opt} = 1$, then Lemma 3 holds true because $\mathcal{N}$ is a grid and $H = 1$. Suppose $R_{opt} \geq 2$. Consider an optimal rectangulation $\mathcal{R}_{opt}$. From Lemma 4.(ii) we know that every hole is incident to at least two segments. Let H_2 be the number of holes incident with 2 segments and let $H_{>2} = H - H_2$ be the number of holes incident to at least three segments.

From Observation 4 we know that $H_{>2} \leq 2 \cdot R_{opt} - 4$. From Lemma 4.(i) we know the bound $H_2 \leq 4R_{opt} + t(\mathcal{N})$. Together this yields $H = H_2 + H_{>2} \leq 6R_{opt} + t(\mathcal{N}) - 4$. Finally substitute $t(\mathcal{N})$ by $t(\mathcal{E})$ to obtain the statement of the lemma. □

6   Approximations for the Number of Rectangles

The first proposition in this section states that the greedy strategy results in a 4-approximation when $\delta(\mathcal{N}) \geq 3$. Two families of strings whose elements have a dual net satisfying this condition are circular drawings with a bipartite sets of pseudosegments and triangle-free hyperbolic line arrangements. For references to triangle-free hyperbolic line arrangements we refer to Eppstein [9, Section 7] and his figure [8] in the Wikipedia article on circle graphs.

Proposition 1 *If $\delta(\mathcal{N}) \geq 3$, then $R_{greed} \leq 4 \cdot R_{opt} - 6$.*

Proof: Apply Observations 2 and 5. □

Condition $\delta(\mathcal{N}) \geq 3$ is restrictive as it forbids in a set of strings the existence of a cell bounded by three pieces of strings. The next lemma handles very general sets of strings at the expense of a larger approximation factor.

Lemma 5 *If $\mathcal{N}$ is the dual net of a set of pseudosegments, then $R_{greed} \leq 8R_{opt} + t(\mathcal{E}) - 6$.*

Proof: Apply Observation 2 and Lemma 3. □

For convenience we restate our main theorem which is an immediate corollary of our findings.

Theorem 1 *For a connected good drawing D with $t = t(D)$ there is a polynomial-time algorithm to compute a bundling with at most $8bc(D) + t$ bundles.*

Proof: Apply the greedy strategy and Lemma 3 to each connected component of the set of strings $\mathcal{E}$ associated to D to obtain the desired bundling. □

7   Discussion and FPT-Algorithms

In Section 4 we established the inequality $S_{greed} \leq 2 \cdot S_{opt}$. It is based on the observation that a segment can reduce the exponent of a net by at most two while each segment added in the greedy solution reduces the exponent of a net by at least one.

Let us review the notion of *adding segments to a net*. If we add a segment s to net $\mathcal{N}$ we decide to use s as a boundary for the rectangulation. In the primal set $\mathcal{E}$ of pseudosegments we can model this by breaking pseudosegments into two at each intersection with s. Applying this surgery to $\mathcal{E}$ makes s into a boundary hole, this hole is being merged with holes at the ends of s. See Figure 13 for an example. In this example the exponent of the vertex hole (left) is 3 and sum of the exponents of the vertices of the boundary hole (right) is 2, after adding the segment the exponents of the vertices of the new boundary hole add up to 3, i.e., adding the segment has reduced the exponent of the net by two. It may also happen, that a segment has both ends at the same hole, in this case the number of holes remains the same but the number of components increases by one.
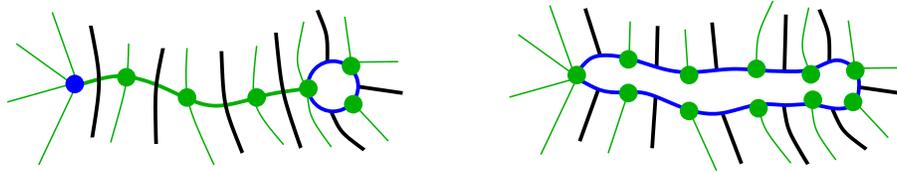

Figure 13: Adding a segment connecting two holes to a net yields the merge of the holes.

It is clear that in order to get a bundling of the crossings of an instance $\mathcal{E}$ with few bundles it is advisable to give preference to segments which reduce the exponent of the net by two. However, as we have already seen in our introductory example two such segments can cross whence they are in conflict and at most one of them can be added. Also, if two segments end at the same vertex v of the net and both reduce $\exp(v)$ by one it may happen that adding the two segments together also reduces $\exp(v)$ by only one. In the next section we introduce the notion of the *marginal gain* of a set of segments. From the discussion following Observation 6 it follows that finding a set of segments which maximizes the marginal gain is equivalent to finding one with S_{opt} segments, i.e., to finding an optimal solution. Since the ends of segments can also be on other segments the

number of segments which may potentially be part of a collection of maximal marginal gain can be very large. Still by looking at all the subsets of possible segments we can optimally solve the NP-hard problem with an FPT algorithm where the parameter is the number of segments which may be used by an optimal solution.

We now proceed to show our second theorem.

Theorem 2 *For a connected good drawing D there is a FPT algorithm with parameter $t = t(D)$ which computes a bundling with at most $8bc(D)$ bundles.*

Proof: Let h be a toothed-hole of $\mathcal{N}$. The exponent of h is two and there is a unique way of saturating the exponent with just two edges, see Figure 14.
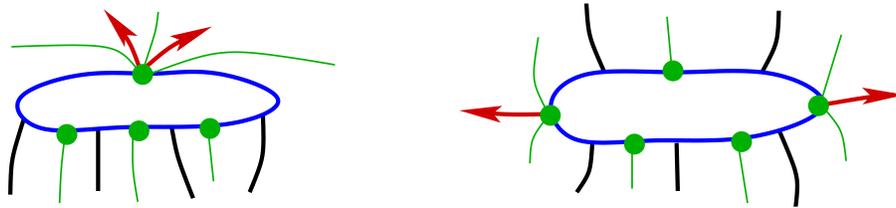

Figure 14: Saturating toothed-holes with two segments can only be done by using the segments starting with the red arrows.

For each toothed-hole consider these two segments maximally extended until they hit another hole and let $\mathcal{C}$ be the collection of these segments, clearly $|\mathcal{C}| \leq 2t$.

Among all possible ways of adding uncrossed segments from $\mathcal{C}$ in linear order, consider the way that maximizes $|X| - K_X$ where X is the set of segments that reduce the exponent by 2 when they are added in this linear order and K_X is the number of components of the net $\mathcal{N}_X$ obtained from $\mathcal{N}$ by applying the surgery for all segments of X. There is some function f such that this set X can be found in time $f(t)$ and that the size of $\mathcal{N}_X$ is at most twice the size of $\mathcal{N}$. By definition we have $\exp(\mathcal{N}_X) = \exp(\mathcal{N}) - 2|X|$. Let $\mathcal{S}'_{greed}$ be a greedy set of segments that saturates the exponents of $\mathcal{N}_X$, i.e., $X \cup \mathcal{S}'_{greed}$ is the set of segments of a rectangulation of $\mathcal{N}$, and let $\mathcal{S} = X \cup \mathcal{S}'_{greed}$. Imagine applying the surgery involved in adding all the segments of X one by one to $\mathcal{N}$. Each addition reduces the number of holes by one or increases the number of components by one. If H_X is the number of holes and K_X the number of components of $\mathcal{N}_X$, then $|X| = (H - H_X) + (K_X - 1)$, i.e., $H_X = H + K_X - |X| - 1$. The choice of X also implies that $|\mathcal{S}| = |X \cup \mathcal{S}'_{greed}| \leq \exp(\mathcal{N}) - |X|$. We let $R_{\mathcal{S}}$ denote the number of rectangles obtained with $\mathcal{S}$. Using the identity of Lemma 1 for each component of $\mathcal{N}_X$ we obtain $|\mathcal{S}'_{greed}| = R_{\mathcal{S}} + H_X - 2K_X$.

Now consider an optimal solution with a set $\mathcal{S}_{opt}$ of segments. Let T be the set of toothed-holes of $\mathcal{N}$ which are incident to only two of the segments of $\mathcal{S}_{opt}$ and let $\mathcal{S}_T$ be the set of segments of $\mathcal{S}_{opt}$ incident to a hole in T. Finally let $\mathcal{S}'_{opt} = \mathcal{S}_{opt} - \mathcal{S}_T$. The segments in $\mathcal{S}_T$ are partitioned into two sets Y and Z as follows. Starting from $\mathcal{N}$ we add the segments of $\mathcal{S}_T$ one by one and put them in Y if the exponent is reduced by two and in Z if the exponent is only reduced by one. Let $\mathcal{N}_Y$ be the net obtained after adding the segments of Y to $\mathcal{N}$ and let K_Y be its number of components. Let $\mathcal{N}_{YZ}$ be the net obtained after adding the segments of Z to $\mathcal{N}_Y$ and let K_{YZ} be its number of components. The choice of X and $K_{YZ} \geq K_Y$ yield $|X| - K_X \geq |Y| - K_Y \geq |Y| - K_{YZ}$.

Since segments in Z only reduce the exponent by one we have $\exp(\mathcal{N}) \leq 2|\mathcal{S}_{opt}| - |Z|$. Partition the segments in Z into the set Z_1 consisting of segments having their second end at a hole of $\mathcal{N}$

and Z_0 where the second end is on some other segment of $\mathcal{S}_{opt}$. If H_{YZ} is the number of holes and K_{YZ} is the number of components of $\mathcal{N}_{YZ}$, then $H_{YZ} = H + K_{YZ} - |Y| - |Z_1| - 1$. From an application of Lemma 1 we also get $|\mathcal{S}'_{opt}| = R_{opt} + H_{YZ} - 2K_{YZ}$.

Now we follow the scheme used for the proof of Theorem 1, $|\mathcal{S}| + |X| \leq \exp(\mathcal{N}) \leq 2|\mathcal{S}_{opt}| - |Z|$ whence

$$|\mathcal{S}'_{greed}| + 2|X| \leq \exp(\mathcal{N}) \leq 2|\mathcal{S}_{opt}| - |Z| = 2|\mathcal{S}'_{opt}| + 2|Y| + 2|Z| - |Z|.$$

This implies $0 \leq 2|\mathcal{S}'_{opt}| - |\mathcal{S}'_{greed}| + 2|Y| + |Z| - 2|X|$ and consequently

$$R_{\mathcal{S}} \leq 2R_{opt} + 2H_{YZ} - H_X + 2|Y| + |Z| - 2|X| - 4K_{YZ} + 2K_X.$$

Using the previously established indentity $H_X = H + K_X - |X| - 1$ and the identiy $H_{YZ} = H + K_{YZ} - |Y| - |Z_1| - 1$ for one of the two contributions of H_{YZ} in the previous inequality we get

$$R_{\mathcal{S}} \leq 2R_{opt} + H_{YZ} + |Y| + |Z_0| - |X| - 3K_{YZ} + K_X.$$

We have already established the inequality $|X| - K_X \geq |Y| - K_{YZ}$. Using this and $K_{YZ} \geq 0$ in the above formula we get $R_{\mathcal{S}} \leq 2R_{opt} + H_{YZ} + |Z_0|$. We claim that $H_{YZ} + |Z_0| \leq 6R_{opt}$. To see this we have to go into the proofs of the previous sections.

We consider $\mathcal{N}_{YZ}$. If $\mathcal{N}_{YZ}$ has several components the following argument can first be used for each component, then the addition of the result for the components yields the claim. From Observation 4 we obtain that the number of holes of $\mathcal{N}_{YZ}$ incident to at least 3 segments is upper bounded by $2R_{opt}$. Recall the proof of Lemma 4 where the key of the argument was to assign corners of the R_{opt} rectangles to holes. In the proof of the lemma it was shown that among the holes incident with only two segments only toothed-holes fail to have an assigned corner. Due to the addition of the segments in Y and Z there are no such toothed-holes whence $H_{YZ} \leq 6R_{opt}$. Now look at the holes of $\mathcal{N}_{YZ}$ which include a segment from Z_0. These are segments whose second end is on some other segment of $\mathcal{S}_{opt}$, hence, the segment and thus the hole is incident to at least two corners of rectangles. This shows that $H_{YZ} + |Z_0| \leq 6R_{opt}$ as claimed.

Altogether we have shown that $R_{\mathcal{S}} \leq 8 \cdot R_{opt}$. This completes the proof of Theorem 2.   □

8   Approximating bipartite instances

A set of strings is *bipartite* if the strings can be colored blue and red, so that the only crossings are between a blue and a red string. A dual net $\mathcal{N}$ is bipartite, if it is the net of a bipartite set of strings. In this section, $\mathcal{N} = (V, E)$ will always be a bipartite instance. The edges of $\mathcal{N}$ inherit a coloring from the strings when we color an edge crossed by a red string red and an edge crossed by a blue string blue. Note that on the boundary of each square of $\mathcal{N}$ and around each regular vertex of $\mathcal{N}$ the colors alternate. In particular every straight-path of $\mathcal{N}$ is either blue or red.

Our ultimate goal in this section is to prove Theorem 3. This theorem will follow easily from the approximation for the number of segements in such instanced.

8.1   A 3/2-approximation for segments in bipartite instances

Our goal in this section is to approximate the minimum number S_{opt} of segments needed to rectangulate a bipartite instance. By first selecting an initial set of good segments and then applying the greedy strategy to extend this set into a rectangulation, we will prove the following:

Theorem 4 *Let $\mathcal{N} = (V, E)$ be a bipartite dual net. Then there exists a cut-set A such that starting the greedy strategy from A produces a saturated cut-set $\mathcal{S}_A$ with at most $\frac{3}{2} \cdot S_{opt}$ segments.*

Definition 4 (Relative Exponent) *Given an edge-set $E_0 \subset E$ of $\mathcal{N} = (V, E)$, the exponent of v relative to E_0, denoted $exp_{E_0}(v)$, is the minimum number of edges at v that need to be added to E_0 so that the resulting edge-set saturates v. We let $exp_{E_0}(\mathcal{N}) := \sum_{v \in V} exp_{E_0}(v)$. We extend this notation to cut-sets $\mathcal{S}$, so that $exp_{\mathcal{S}}(v) := exp_{E_0}(v)$, where E_0 is the edge-set of the segments in $\mathcal{S}$.*

Remark 2 $exp(v) = exp_\emptyset(v)$.

We now introduce the *marginal gain* of a collection $\mathcal{S}$ of segments, measuring how good it is to use $\mathcal{S}$ as the start of a greedy strategy. Let the *marginal gain* of $\mathcal{S}$ be the following

$$g(\mathcal{S}) = \sum_{v \in V} (\exp(v) - \exp_{\mathcal{S}}(v)) - |\mathcal{S}|. \tag{2}$$

Observation 6 *Let $\mathcal{S}_0$ be a set of segments in a net $\mathcal{N}$. If $\mathcal{S}$ is an extension of $\mathcal{S}_0$ obtained from $\mathcal{S}_0$ by applying some steps of the greedy strategy, then $g(\mathcal{S}_0) \leq g(\mathcal{S})$.*
Moreover, if $\mathcal{S}$ induces a rectangulation, then $|\mathcal{S}| = exp(\mathcal{N}) - g(\mathcal{S}) \leq exp(\mathcal{N}) - g(\mathcal{S}_0)$.

Proof: By induction, it suffices to consider a 1-step greedy extension $\mathcal{S}_1$ of $\mathcal{S}_0$ with $\mathcal{S}_1 = \mathcal{S}_0 + s$. Greedy says that there is a vertex v_1 such that the exponent at v is decreased by one upon adding segment s while $\exp_{\mathcal{S}_1}(v) \leq \exp_{\mathcal{S}_0}(v)$ for all v. Therefore, $\sum_{v \in V} (\exp(v) - \exp_{\mathcal{S}_1}(v)) \geq \sum_{v \in V} (\exp(v) - \exp_{\mathcal{S}_0}(v)) - 1$. Since $|\mathcal{S}_1| = |\mathcal{S}_0| + 1$ this implies $g(\mathcal{S}_1) \geq g(\mathcal{S}_0)$. Finally, the moreover part follows from the fact that $\exp_{\mathcal{S}}(\mathcal{N}) = 0$ when $\mathcal{S}$ induces a rectangulation. $\square$

Observation 6 converts the problem of finding a cut-set with S_{opt} segments into the problem of maximizing g: Suppose that $\mathcal{S}_0^*$ maximizes g and let $\mathcal{S}^*$ be a greedy extension of $\mathcal{S}_0^*$ inducing a rectangulation. Observation 6 implies $g(\mathcal{S}^*) = g(\mathcal{S}_0^*)$, moreover

$$S_{opt} = |\mathcal{S}_{opt}| = \exp(\mathcal{N}) - g(\mathcal{S}_{opt}) \geq \exp(\mathcal{N}) - g(\mathcal{S}_0^*) = \exp(\mathcal{N}) - g(\mathcal{S}^*) = |\mathcal{S}^*|.$$

We do not know how to compute an optimum $\mathcal{S}_0^*$, however, to prove Theorem 4 it will be sufficient to find a monochromatic set A such that $g(A) \geq \frac{1}{2} g(\mathcal{S}_{opt})$.

The following considerations are motivated by the aim of finding a minimal set of blue segments of maximum marginal gain.

Let B be the set of all blue segments in $\mathcal{N}$. Since a blue segments will never hit another blue segment, all the segments in B connect two holes, they are the maximal straight-paths of blue edges. Any set $\mathcal{S}$ of blue segments is a subset of B.

We divide the vertices v of V with $\exp(v) > 0$ into two classes: a vertex v is *weak* if $\deg_{\mathcal{N}}(v)$ is odd, v is a boundary vertex of $\mathcal{N}$, and the two boundary edges at v are red. The rest of the vertices are *strong*[2]. In a bipartite instance every square of the net has the property that the colors red and blue alternate around the square. This alternation property also holds at vertices, in particular every vertex-hole is of even degree. Hence, the set of blue edges is a minimum saturating set for every vertex-hole. In the case of a vertex v incident to a boundary-hole the set of blue edges is a subset of a minimum saturating set unless the two boundary edges at v are red, i.e., v is a weak vertex.

---

[2] We remark that our definition of weak and strong depends on the color of choice (our current choice is blue). An analogous definition of weak and strong can be made with respect to the red color.

Let $\mathcal{S} \subset B$ be a set of blue segments and let v be a vertex with $\exp(v) > 0$ incident to k segments of $\mathcal{S}$. Then

$$\exp(v) - \exp_{\mathcal{S}}(v) = \begin{cases} k & v \text{ is strong} \\ \max(k-1, 0) & v \text{ is weak.} \end{cases} \tag{3}$$

The distinction between weak and strong is helpful to understand the marginal gain g. For instance, if we add a blue segment s between two strong vertices to a set $\mathcal{S}$, then the marginal gain is increased by 1. Moreover, $g(\{s\}) > 0$ if and only if s connects two strong vertices. So, it would be natural to think that adding segments between pairs of strong vertices is the only way to increase g. However, we will see that it is possible to add sets of segments $\mathcal{S}$ where no gain is produced from adding individual edges from $\mathcal{S}$, but only by adding $\mathcal{S}$ as a whole.

To facilitate the computation of g we consider an auxiliary multigraph $G_B = (V_B, E_B)$. The vertex-set V_B consist of the vertices of $\mathcal{N}$ with positive exponent. The edge-set E_B is the disjoint union $B \cup L$, where B are the blue segments in $\mathcal{N}$ (each joining its corresponding ends in V_B) and L is a set of loops, one at each strong vertex of $\mathcal{N}$. The subgraph of G_B induced by the edges in $\mathcal{S}$ is denoted as $G_B[\mathcal{S}]$. We ease our notation by letting $V(\mathcal{S}) = V(G_B[\mathcal{S}])$ and $\deg_{\mathcal{S}}(v) = \deg_{\mathcal{N}[\mathcal{S}]}(v)$. We remark that a loop at v contributes 2 to the degree of v. If $\mathcal{S} \subseteq B$, then let $\widehat{\mathcal{S}}$ be obtained from $\mathcal{S}$ by adding all the loops from L at the strong vertices in $V(\mathcal{S})$. The next lemma translates $g(\mathcal{S})$ as a function of G_B.

Lemma 6 *Given $\mathcal{S} \subseteq B$, then*

$$g(\mathcal{S}) = |\widehat{\mathcal{S}}| - |V(\mathcal{S})| - tc(\widehat{\mathcal{S}}) \tag{4}$$

where $tc(\widehat{\mathcal{S}})$ is the number of connected components in $G_B[\widehat{\mathcal{S}}]$ isomorphic to a tree.

Proof: We proceed by induction on $|\mathcal{S}|$. Let $f(\mathcal{S})$ be the right-hand side of Equation 4. For the base case note that $g(\emptyset) = 0 = f(\emptyset)$. Now consider a non-empty set $\mathcal{S} \subseteq B$. By induction assume that $g(\mathcal{S}') = f(\mathcal{S}')$ for every proper subset $\mathcal{S}' \subset \mathcal{S}$. Thus, to show $g(\mathcal{S}) = f(\mathcal{S})$ it is enough to find a proper subset $\mathcal{S}' \subset \mathcal{S}$ for which $g(\mathcal{S}) - g(\mathcal{S}') = f(\mathcal{S}) - f(\mathcal{S}')$.

First, suppose that $\mathcal{S}$ has a subset C such that $G_B[C]$ is a connected component in $G_B[\widehat{\mathcal{S}}]$ isomorphic to a cycle of length at least 3, i.e., all the vertices in $V(C)$ are weak. Then $g(\mathcal{S}) - g(\mathcal{S} \setminus C) = \sum_{v \in V(C)}(\exp_{\mathcal{S}}(v) - \exp_{\mathcal{S} \setminus C}(v)) + |C| = \sum_{v \in V(C)}(-1) + |C| = 0$. On the other hand, because $|C| = |V(C)|$, $f(\mathcal{S}) - f(\mathcal{S} \setminus C) = 0$. Therefore, assuming $g(\mathcal{S} \setminus C) = f(\mathcal{S} \setminus C)$ we obtain $g(\mathcal{S}) = f(\mathcal{S})$.

Second, suppose that $G_B[\widehat{\mathcal{S}}]$ has an edge xy for which x is a leaf. Let $s \in \mathcal{S}$ be the segment connecting x and y and let $\mathcal{S}' = \mathcal{S} - s$. Since x has no loop in $G_B[\widehat{\mathcal{S}}]$ it is weak, since s it the only segment in S incident to x we have $\exp_{\mathcal{S}'}(x) - \exp_{\mathcal{S}}(x) = 0$. Therefore, $g(\mathcal{S}) - g(\mathcal{S}') = \exp_{\mathcal{S}'}(y) - \exp_{\mathcal{S}}(y) - 1$. From Equation 3 it follows that $g(\mathcal{S}) - g(\mathcal{S}') = 0$ unless y is weak and a leaf, in this case $g(\mathcal{S}) - g(\mathcal{S}') = -1$. On the other hand $f(\mathcal{S}) - f(\mathcal{S}') = -tc(\widehat{\mathcal{S}}) + tc(\widehat{\mathcal{S}}')$. Hence, $g(\mathcal{S}) - g(\mathcal{S}')$ and $f(\mathcal{S}) - f(\mathcal{S}')$ are equal to 0 or -1 depending on whether y is a leaf in $G_B[\widehat{\mathcal{S}}]$ or not.

Finally, suppose that $G_B[\widehat{\mathcal{S}}]$ neither has leafs nor cycle components. In this case, we pick any $s \in \mathcal{S}$ and let $\mathcal{S}' = \mathcal{S} - s$. Since the endpoints x and y of s are not leafs, $\exp_{\mathcal{S}'}(z) - \exp_{\mathcal{S}}(z)$ is either 1 or 0 depending whether $z \in \{x, y\}$ or not. This implies that $g(\mathcal{S}) - g(\mathcal{S}') = 1$. Since $G_B[\widehat{\mathcal{S}}]$ neither has cycle components nor leafs, $tc(\widehat{\mathcal{S}}) = 0 = tc(\widehat{\mathcal{S}}')$ and $V(\mathcal{S}) = V(\mathcal{S}')$. Therefore $f(\mathcal{S}) - f(\mathcal{S}') = 1$, and we are done. $\qquad\square$

With the help of Lemma 6 we can now build examples of sets $\mathcal{S} \subseteq B$ with $g(\mathcal{S}) > 0$ but where segments in $\mathcal{S}$ are not necessarily between strong vertices. A *bicycle* is a connected graph with no degree-1 vertices that has exactly two cycles. Every bicycle is one of three kinds: a *theta-graph*, obtained from a cycle by adding a path connecting two vertices of the cycle; an *eight-graph*, obtained by gluing two disjoint cycles at exactly one vertex; or a *barbell*, obtained by joining two disjoint cycles by a path.

Note that $|\widehat{\mathcal{S}}| = \sum_v \deg_{\widehat{\mathcal{S}}}(v)/2 = \sum_v \deg_{\mathcal{S}}(v)/2 + \#(\text{strong vertices in } V(\mathcal{S}))$. If $\mathcal{S}$ induces a bicycle in G_B, then $|\mathcal{S}| = \frac{1}{2} \sum_v \deg_{\mathcal{S}}(v) = |V(\mathcal{S})| + 1$, therefore Eq. 4 implies $g(\mathcal{S}) > 0$.

Bicycles are the circuits of a matroid and although we will not use Matroid Theory here, some of the concepts that we define next are well known in this context.

A *pseudoforest* is a graph where each connected component has at most one cycle. Given the graph $G_B = (V_B, E_B)$, its *bicircular matroid* is the pair $M = (E_B, \mathcal{I})$ where $\mathcal{I}$ is the set of subsets of E_B inducing a pseudoforest in G_B. In the matroid context, the subsets of E_B in $\mathcal{I}$ are called *independent* while the subsets not in $\mathcal{I}$ are called *dependent*. Indeed, the bicycles of G_B are the minimal dependent sets of M, known as circuits. The *rank function* $\text{rk}(\cdot) : 2^{E_B} \to \mathbb{Z}_{\geq 0}$ assigns to each $\mathcal{S} \subseteq E_B$ the size $\text{rk}(\mathcal{S})$ of a maximum independent set of M contained in $\mathcal{S}$. The *nullity* of $\mathcal{S}$ is defined as $\text{null}(\mathcal{S}) = |\mathcal{S}| - \text{rk}(\mathcal{S})$.

We will use two basic properties of bicircular matroids whose simple proofs are left to the reader: (P1) $\text{rk}(\mathcal{S}) = |V(\mathcal{S})| - \text{tc}(\mathcal{S})$ where $\text{tc}(\mathcal{S})$ is the number of tree components of $G_B[\widehat{\mathcal{S}}]$; and (P2) $\text{null}(\mathcal{S}_1) \leq \text{null}(\mathcal{S}_2)$ for $\mathcal{S}_1 \subseteq \mathcal{S}_2$. The next observation follows from Lemma 6 and P1:

Observation 7 *For $\mathcal{S} \subseteq B$, $g(\mathcal{S}) = \text{null}(\widehat{\mathcal{S}}) - 2 \cdot \text{tc}(\widehat{\mathcal{S}})$.* $\qquad\qquad\square$

We are now ready for finding a blue set maximizing g:

Lemma 7 *There is a unique set $A \subseteq B$ such that for all $\mathcal{S} \subset B$, $g(\mathcal{S}) \leq g(A)$. Moreover, A can be obtained in $O(|E_B|)$ time.*

Proof: We start by pruning E_B, that is, we iteratively remove edges incident with leaves until no leaf remains. Afterwards, we remove all the connected components isomorphic to cycles. The resulting edge-set A is the one we are looking for.

Note that P1 implies that the nullity of a set does not change after removing a cycle component, or after pruning a leaf, this also holds when pruning the only edge of a tree. Thus, $g(A) = \text{null}(A) - 0 = \text{null}(E_B)$. Now Observation 7 and P2 imply that for $\mathcal{S} \subseteq B$, $g(\mathcal{S}) \leq \text{null}(\widehat{\mathcal{S}}) \leq \text{null}(E_B) = g(A)$. $\qquad\qquad\square$

We now turn our head back to the proof of Theorem 4 where also red segments are considered. Analogous to B, we let R be the set of red segments connecting vertices with positive exponent.

Given a saturating set of segments $\mathcal{S}$ (not necessarily monochromatic), a *blue-red split* is a partition $\mathcal{S}_B, \mathcal{S}_R$ of $\mathcal{S}$, where $\mathcal{S}_B$ are the blue segments and $\mathcal{S}_R$ are the red segments. Note that the straight-paths in $\mathcal{S}_B$ or in $\mathcal{S}_R$ do not necessarily have ends in holes.

Lemma 8 *Let $\mathcal{S}_B, \mathcal{S}_R$ be a blue-red split of a saturating cut-set $\mathcal{S}$. Then*

$$g(\mathcal{S}_B) + g(\mathcal{S}_R) \geq g(\mathcal{S}).$$

Proof: From the definition of $g(\cdot)$ (Eq. 2) and since $|\mathcal{S}_B| + |\mathcal{S}_R| = |\mathcal{S}|$ it is enough to show that for every v with $\exp(v) > 0$,

$$\exp_{\mathcal{S}}(v) + \exp(v) \geq \exp_{\mathcal{S}_B}(v) + \exp_{\mathcal{S}_R}(v). \tag{5}$$

Since $\mathcal{S}$ is saturating we have $\exp_{\mathcal{S}}(v) = 0$. Let $\mathcal{S}_R(v)$, $\mathcal{S}_B(v)$, and $\mathcal{S}(v)$ denote the segments in $\mathcal{S}_R$, $\mathcal{S}_B$, and $\mathcal{S}$ respectively which are incident to v. From the definition of the sets, $|\mathcal{S}_R(v)| + |\mathcal{S}_B(v)| = |\mathcal{S}(v)|$.

If v is strong in both colors, then from Eq. 3 we know $\exp_{\mathcal{S}_B}(v) = \exp(v) - |\mathcal{S}_B(v)|$ and $\exp_{\mathcal{S}_R}(v) = \exp(v) - |\mathcal{S}_R(v)|$. Hence, Eq. 5 reduces to $|\mathcal{S}_R(v)| + |\mathcal{S}_B(v)| \geq \exp(v)$ which is true because $|\mathcal{S}_R(v)| + |\mathcal{S}_B(v)| = |\mathcal{S}(v)|$ and $\mathcal{S}$ is saturating.

Now let v be weak in one of the colors and note that it is strong in the other. By symmetry we may assume that v is weak in red. If $\mathcal{S}_R(v) = \emptyset$ we have $\exp_{\mathcal{S}_R}(v) = \exp(v) - |\mathcal{S}_R(v)|$ as above. If $|\mathcal{S}_R(v)| > 0$, then $\exp_{\mathcal{S}_R}(v) = \exp(v) - |\mathcal{S}_R(v)| + 1$ and we have to verify $|\mathcal{S}(v)| = |\mathcal{S}_R(v)| + |\mathcal{S}_B(v)| \geq \exp(v) + 1$. For a vertex of the given kind, however, the unique saturating set of edges of size $\exp(v)$ is the set of all blue edges. Since $\mathcal{S}$ is saturating and contains a red segment at v the required inequality $|\mathcal{S}(v)| \geq \exp(v) + 1$ holds.   □

Proof: [Proof of Theorem 4] Let $\mathcal{S}$ be a saturating set of segments of minimum cardinality. Let $\mathcal{S}_B$, $\mathcal{S}_R$ be the blue-red split of $\mathcal{S}$. By possibly interchanging colors, we may assume that $g(\mathcal{S}_B) \geq g(\mathcal{S}_R)$. Lemma 8 implies that $g(\mathcal{S}_B) \geq \frac{1}{2}g(\mathcal{S})$.

Consider a set $\mathcal{S}'_B \subseteq B$, obtained by extending the straight-paths of $\mathcal{S}_B$ into elements of B. A segment of $\mathcal{S}'_B$ may contain multiple elements of $\mathcal{S}_B$, and also the exponent of each vertex relative to $\mathcal{S}'_B$ is at most its exponent relative to $\mathcal{S}_B$. Thus, $g(\mathcal{S}'_B) \geq g(\mathcal{S}_B)$.

Let $A \subset B$ be the unique minimal set of blue segments maximizing the gain, see Lemma 7, and let A^+ be a saturating extension of A which is obtained by the greedy strategy. Then Obs. 6 and the properties of A imply the first and second inequalities in $g(A^+) \geq g(A) \geq g(\mathcal{S}'_B) \geq g(\mathcal{S}_B) \geq \frac{1}{2}g(\mathcal{S})$.

Since $\mathcal{S}$ and A^+ are saturating $|\mathcal{S}| = \exp(\mathcal{N}) - g(\mathcal{S})$, and $|A^+| = \exp(\mathcal{N}) - g(A^+)$. From the above

$$|A^+| = \exp(\mathcal{N}) - g(A^+) \leq \exp(\mathcal{N}) - \frac{1}{2}g(\mathcal{S}) = \frac{1}{2}\exp(\mathcal{N}) + \frac{1}{2}|\mathcal{S}|$$

Finally, Obs. 1.(ii) tells us that $\exp(\mathcal{N})/2 \leq |\mathcal{S}|$ which yields $|A^+| \leq \frac{3}{2}|\mathcal{S}|$.   □

8.2   A 9/2-approximation for rectangles in bipartite instances

We are ready for the proof of the main theorem of this section.

Theorem 3 *If D is a bipartite instance with $t = t(D)$ toothed-faces, then there is a polynomial-time algorithm to compute a bundling with at most $\frac{9}{2}bc(D) + \frac{1}{2}t$ bundles.*

Proof: By Theorem 4, we can compute a rectangulation $\mathcal{R}$ with R rectangles, S segments, and $S \leq \frac{3}{2}S_{opt}$. Substitute both sides of this inequality using Eq. 1 to obtain $R \leq \frac{3}{2}R_{opt} + \frac{1}{2}H - 1$. Substitute H by $6R_{opt} + t - 4$ (Lemma 3) to obtain $R \leq \frac{9}{2}R_{opt} + \frac{1}{2}t - 3 \leq \frac{9}{2}R_{opt} + \frac{1}{2}t$.   □

9   Conclusion

In this paper we studied the bundled crossing number of connected good drawings and showed that the greedy strategy derived from the problem of rectangulating an orthogonal polygon leads to an 8-approximation (up to adding the number of toothed-faces in the drawing). Moreover, we improved this strategy for bipartite instances by considering an initial good set of segments.

We hope that the tools and the framework developed in this work will inspire more results about bundled crossings. We close with two open questions.

1. Is there a constant c guaranteeing that, for any simple drawing D, the greedy algorithm produces bundling with at most $c \cdot \mathrm{bc}(D)$ bundles? In other words, are toothed-faces relevant to approximate $\mathrm{bc}(D)$?

2. What is the computational complexity of computing $\mathrm{bc}(\mathcal{E})$ for bipartite instances?

References

[1] M. J. Alam, M. Fink, and S. Pupyrev. The bundled crossing number. In *Proc. GD 2016*, volume 9801 of *LNCS*, pages 399–412. Springer, 2016. `doi:10.1007/978-3-319-50106-2_31`.

[2] P. Angelini, M. A. Bekos, M. Kaufmann, P. Kindermann, and T. Schneck. 1-fan-bundle-planar drawings of graphs. *Theor. Comput. Sci.*, 723:23–50, 2018. `doi:10.1016/j.tcs.2018.03.005`.

[3] B. Bach, N. H. Riche, C. Hurter, K. Marriott, and T. Dwyer. Towards unambiguous edge bundling: Investigating confluent drawings for network visualization. *IEEE Trans. Vis. Comput. Graph.*, 23:541–550, 2017. `doi:10.1109/TVCG.2016.2598958`.

[4] L. Beineke and R. Wilson. The early history of the brick factory problem. *The Mathematical Intelligencer*, 32(2):41–48, 2010. `doi:10.1007/s00283-009-9120-4`.

[5] S. Chaplick, T. C. van Dijk, M. Kryven, J. Park, A. Ravsky, and A. Wolff. Bundled crossings revisited. In *Proc. GD 2019*, volume 11904 of *LNCS*, pages 63–77. Springer, 2019. `doi:10.1007/978-3-030-35802-0_5`.

[6] W. Cui, H. Zhou, H. Qu, P. C. Wong, and X. Li. Geometry-based edge clustering for graph visualization. *IEEE Trans. Vis. Comput. Graph.*, 14:1277–1284, 2008. `doi:10.1109/TVCG.2008.135`.

[7] M. Dickerson, D. Eppstein, M. T. Goodrich, and J. Y. Meng. Confluent drawings: Visualizing non-planar diagrams in a planar way. *J. Graph Algorithms Appl.*, 9:31–52, 2005. `doi:10.7155/jgaa.00099`.

[8] D. Eppstein. `https://en.wikipedia.org/wiki/File:Ageev_5X_circle_graph.svg`.

[9] D. Eppstein. Graph-theoretic solutions to computational geometry problems. In *Proc. WG 2009*, volume 5911 of *LNCS*, pages 1–16. Springer, 2009. `doi:10.1007/978-3-642-11409-0_1`.

[10] D. Eppstein, M. T. Goodrich, and J. Y. Meng. Delta-confluent drawings. In *Proc. GD 2005*, volume 3843 of *LNCS*, pages 165–176. Springer, 2005. `doi:10.1007/11618058_16`.

[11] D. Eppstein, D. Holten, M. Löffler, M. Nöllenburg, B. Speckmann, and K. Verbeek. Strict confluent drawing. *J. of Comp. Geom. (JoCG)*, 7:22–46, 2016. `doi:10.20382/jocg.v7i1a2`.

[12] D. Eppstein and J. A. Simons. Confluent Hasse diagrams. *J. Graph Algorithms Appl.*, 17:689–710, 2013. `doi:10.7155/jgaa.00312`.

[13] O. Ersoy, C. Hurter, F. V. Paulovich, G. Cantareiro, and A. Telea. Skeleton-based edge bundling for graph visualization. *IEEE Trans. Vis. Comput. Graph.*, 17:2364–2373, 2011. `doi:10.1109/TVCG.2011.233`.

[14] M. Fink, J. Hershberger, S. Suri, and K. Verbeek. Bundled crossings in embedded graphs. In *Proc. LATIN 2016*, volume 9644 of *LNCS*, pages 454–468. Springer, 2016. `doi:10.1007/978-3-662-49529-2_34`.

[15] M. Fink, S. Pupyrev, and A. Wolff. Ordering metro lines by block crossings. *J. Graph Algorithms Appl.*, 19(1):111–153, 2015. `doi:10.7155/jgaa.00351`.

[16] H. Förster, R. Ganian, F. Klute, and M. Nöllenburg. On strict (outer-)confluent graphs. In *Proc. GD 2019*, volume 11904 of *LNCS*, pages 147–161. Springer, 2019. `doi:10.1007/978-3-030-35802-0_12`.

[17] E. R. Gansner, Y. Hu, S. C. North, and C. E. Scheidegger. Multilevel agglomerative edge bundling for visualizing large graphs. In *Proc. PacificVis 2011*, pages 187–194. IEEE, 2011. `doi:10.1109/PACIFICVIS.2011.5742389`.

[18] E. R. Gansner and Y. Koren. Improved circular layouts. In *Proc. GD 2006*, volume 4372 of *LNCS*, pages 386–398. Springer, 2006.

[19] F. Harary and A. Hill. On the number of crossings in a complete graph. *Proceedings of the Edinburgh Mathematical Society*, 13(4):333–338, 1963. `doi:10.1017/S0013091500025645`.

[20] F. Harary, P. C. Kainen, and A. J. Schwenk. Toroidal graphs with arbitrarily high crossing numbers. *Nanta Math*, 6(1):58–67, 1973.

[21] D. Holten. Hierarchical edge bundles: Visualization of adjacency relations in hierarchical data. *IEEE Trans. Vis. Comput. Graph.*, 12:741–748, 2006. `doi:10.1109/TVCG.2006.147`.

[22] D. Holten and J. J. van Wijk. Force-directed edge bundling for graph visualization. *Comput. Graph. Forum*, 28:983–990, 2009. `doi:10.1111/j.1467-8659.2009.01450.x`.

[23] P. Hui, M. J. Pelsmajer, M. Schaefer, and D. Stefankovic. Train tracks and confluent drawings. *Algorithmica*, 47:465–479, 2007. `doi:10.1007/s00453-006-0165-x`.

[24] A. Lambert, R. Bourqui, and D. Auber. Winding roads: Routing edges into bundles. *Comput. Graph. Forum*, 29:853–862, 2010. `doi:10.1111/j.1467-8659.2009.01700.x`.

[25] A. Lhuillier, C. Hurter, and A. C. Telea. State of the art in edge and trail bundling techniques. *Comput. Graph. Forum*, 36(3):619–645, 2017.

[26] S. Pupyrev, L. Nachmanson, S. Bereg, and A. E. Holroyd. Edge routing with ordered bundles. *Comput. Geom.*, 52:18–33, 2016. `doi:10.1016/j.comgeo.2015.10.005`.

[27] M. Schaefer. The graph crossing number and its variants: a survey. *Electron. J. Comb.*, Dynamic Survey 21:90, 2013. `doi:10.37236/2713`.

[28] P. Turán. A note of welcome. *Journal of Graph Theory*, 1(1):7–9, 1977. `doi:10.1002/jgt.3190010105`.

[29] T. C. van Dijk, M. Fink, N. Fischer, F. Lipp, P. Markfelder, A. Ravsky, S. Suri, and A. Wolff. Block crossings in storyline visualizations. *J. Graph Algorithms Appl.*, 21(5):873–913, 2017. `doi:10.7155/jgaa.00443`.

[30] M. Wallinger, D. Archambault, D. Auber, M. Nöllenburg, and J. Peltonen. Edge-path bundling: A less ambiguous edge bundling approach. *IEEE Trans. Vis. Comput. Graph.*, 28(1):313–323, 2021. `doi:10.1109/TVCG.2021.3114795`.

[31] J. X. Zheng, S. Pawar, and D. F. M. Goodman. Further towards unambiguous edge bundling: Investigating power-confluent drawings for network visualization. *IEEE Trans. Vis. Comput. Graph.*, 27(2):2244–2249, 2019. `doi:10.1109/TVCG.2019.2944619`.