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Simplifying Non-Simple Fan-Planar Drawings

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Abstract. A drawing of a graph is fan-planar if the edges intersecting a common edge a share a vertex A on the same side of a. More precisely, orienting a arbitrarily and the other edges towards A results in a consistent orientation of the crossings. So far, fan-planar drawings have only been considered in the context of simple drawings, where any two edges share at most one point, including endpoints. We show that every non-simple fan-planar drawing can be redrawn as a simple fan-planar drawing of the same graph while not introducing additional crossings. The proof is constructive and corresponds to a quadratic time algorithm. Combined with previous results on fan-planar drawings, this yields that n-vertex graphs having such a drawing can have at most $6.5n - 20$ edges and that the recognition of such graphs is NP-hard. We thereby answer an open problem posed by Kaufmann and Ueckerdt in 2014.

1 Introduction

In a *fan-planar* drawing of a graph, each edge a is either not involved in any crossing or its crossing edges $c_1, \ldots, c_k$ have a common endpoint A that is on a *common side* of a, i.e., orienting a

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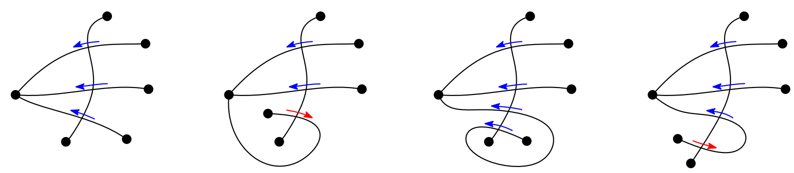
arbitrarily and the edges $c_1, \dots, c_k$ towards A results in a consistent orientation of the crossings on a (either a crosses each c_i from left to right at each crossing, or it crosses each c_i from right to left at each crossing); for illustrations refer to Figure 1. We call A the *special* vertex of a. All *graphs* in this paper are simple, that is, we do not allow parallel edges or self-loops. Hence, the vertex A is uniquely defined if $k \geq 2$. If $k = 1$, then A is an arbitrary endpoint of c_1.



(a) (b) (c) (d)

Figure 1: Drawings that are (a) simple and fan-planar, (b) simple and not fan-planar, (c) non-simple and fan-planar, and (d) non-simple and not fan-planar.

Previous literature is exclusively concerned with fan-planar drawings that are also *simple*, meaning that each pair of edges intersects in at most one point, which can be either an endpoint or a proper crossing. Simple drawings can be characterized in terms of two forbidden crossing configurations[1] (see Figure 2):

S1 Two adjacent edges cross.

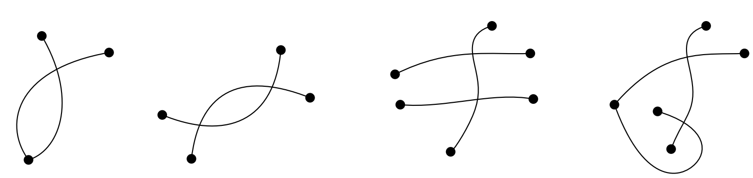
S2 Two edges cross at least twice.

Simple drawings that are fan-planar can be characterized in terms of two additional forbidden crossing configurations [19] (see Figure 2):

SF1 Two independent edges cross a common third edge.

SF2 Two adjacent edges cross a third edge a such that their common endpoint A is not on a common side of a.

In this paper, we study non-simple fan-planar drawings and how to turn them into simple fan-planar drawings.



(a) S1 (b) S2 (c) SF1 (d) SF2

Figure 2: Forbidden configurations in simple fan-planar drawings.

[1]In the literature, usually more obstructions are mentioned, which we exclude for *all* drawings (simple or not), see Section 2.

Previous and related work. A drawing is *k-planar* if each edge is crossed at most k times and a graph is *k-planar*, if it admits such a drawing [22]. A *k-quasiplanar* graph can be drawn such that no k edges mutually cross – such a drawing is called *k-quasiplanar* [2]. Kaufmann and Ueckerdt [19] introduced the notion of fan-planarity in 2014. They describe the class of graphs representable by simple fan-planar drawings[2] as somewhere between 1-planar graphs and 3-quasiplanar graphs. Indeed, every 1-planar graph admits a simple 1-planar drawing. Since such a drawing cannot contain configuration SF1 or SF2, it is fan-planar. Moreover, a simple fan-planar drawing cannot contain three mutually crossing edges and, therefore, it is 3-quasiplanar. Binucci et al. [10] have shown that for each $k \geq 2$ the class of graphs admitting simple k-planar drawings and the class of graphs admitting simple fan-planar drawings are incomparable. In contrast, every so-called optimal 2-planar graph can be drawn as a simple fan-planar drawing [8]. This follows from the fact that these graphs can be characterized as the graphs obtained by drawing a pentagram in the interior of each face of a pentangulation [8], which yields a fan-planar drawing. Angelini et al. [4] introduced a drawing style that combines fan-planarity with a visualization technique called edge bundling [16, 17, 24]. Each of their so-called 1-sided 1-fan-bundle-planar drawings represents a graph that is also realizable as a simple fan-planar drawing, but the converse is not true [4]. Brandenburg [11] examines *fan-crossing* drawings, where *all* edges crossing a common edge share a common endpoint (in particular, this implies that SF1 is forbidden), as well as *adjacency-crossing* drawings, where SF1 is the only obstruction, i.e., every *pair* of edges crossing a common edge has a common endpoint, though different pairs might have different common endpoints. Simple fan-planar drawings are somewhat opposite to simple *k-fan-crossing-free* [12] drawings, where no $k \geq 2$ adjacent edges cross another common edge.

The maximum number of edges in a simple fan-planar drawing on n vertices is upperbounded by $6.5n-20$ [19], which follows from the known density bounds for 3-quasiplanar graphs [1]. A better upper bound of $5n-10$ edges was claimed in a preprint [19]. However, the corresponding proof appears to be flawed. We spoke with the authors and they confirmed that the current version of their proof is not correct and that they do not see a simple way to fix it[3]. Kaufmann and Ueckerdt [19] described an infinite family of simple fan-planar drawings with $5n-10$ edges. The same lower bound also follows from the aforementioned connection to optimal 2-planar graphs [8].

The recognition of graphs realizable as simple fan-planar drawings is NP-hard [10]. The same statement also holds in the fixed rotation system setting [6], where the cyclic order of edges incident to each vertex is prescribed as part of the input. Consequently, efficient algorithms have only been discovered for special graph classes [6] and for restricted drawing styles [6, 9].

For a more comprehensive overview of previous work related to fan-planarity, we refer to a very recent survey article dedicated to fan-planarity due to Bekos and Grilli [7]. The study of fan-planarity also falls in line with the recent trend of studying so-called beyond-planar graph classes, whose corresponding drawing styles permit crossings in restricted ways only. Apart from k-planar [22], k-quasiplanar [2, 3], k-fan-crossing-free [12], fan-bundle-planar [4], fan-crossing [11], adjacency-crossing [19], and fan-planar [19] drawings, which have already been mentioned above, several other classes of beyond-planar graphs and their corresponding drawing styles have been

[2]In [19], these graphs are called *fan-planar*. We do not use this terminology to avoid mix-ups with the class of graphs admitting (not necessarily simple) fan-planar drawings.

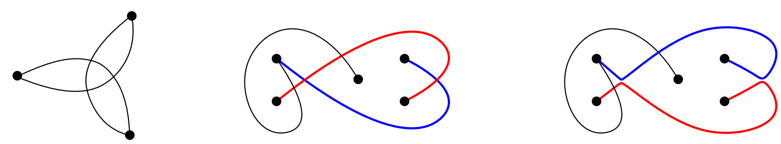
[3]More specifically, the statement and proof of [19, Lemma 1] are incorrect. A counterexample can be obtained by removing the edge g from the construction illustrated in Figure 15 (vertices R, B correspond to the vertices u, w in [19, Lemma 1]); for a formal description of the construction see Lemma 3.

After the submission of the preliminary version of our paper to GD'21, the authors of [19] have uploaded a new version [20] of their preprint in which they state a different definition of fan-planarity with an additional forbidden crossing configuration. We discuss this new definition in Section 4; also see [20, last paragraph of Section 1].

studied, e.g.: k-gap-planar drawings [5] (each crossing is assigned to one of the involved edges such that each edge is assigned at most k crossings), RAC-drawings [14] (straight-line drawings with right angle crossings), and many more. We refer to [15, 18] for recent surveys on beyond-planar graphs.

Contribution. A fan-planar drawing that is not simple may contain configuration S1. Configuration S2 is allowed in a partial sense: two edges may cross any number of times, but only if orienting them arbitrarily results in a consistent orientation of their crossings; cf. Figures 1(c) and (d). Recall that every simple fan-planar drawing is 3-quasiplanar. In contrast, Figure 3(a) depicts a non-simple fan-planar drawing that is not 3-quasiplanar, which suggests that graphs admitting non-simple fan-planar drawings are not necessarily 3-quasiplanar. Consequently, the density bound of $6.5n - 20$ [1] for 3-quasiplanar graphs does not directly carry over. However, the depicted graph is just a K_3, which can obviously be redrawn as a simple (fan-)planar drawing. This raises two very natural questions:

1. Is the largest number of edges in a n-vertex non-simple fan-planar drawing larger than the number of edges in any n-vertex simple fan-planar drawing?
2. Which non-simple fan-planar drawings can be redrawn as simple fan-planar drawings of the same graph?



(a) (b) (c)

Figure 3: (a) A non-simple fan-planar drawing that is not 3-quasiplanar. (b) A non-simple fan-planar drawing. Applying the standard procedure for simplifying configuration S2 yields the drawing in (c), which is not fan-planar since the black edge crosses two independent edges.

Question 1 is also mentioned as an open problem by Kaufmann and Ueckerdt [19]. Regarding question 2, we remark that the standard method for simplifying the configurations S1 and S2 does not necessarily maintain fan-planarity, see Figures 3(b) and (c). Consequently, it is not possible to argue inductively when exhaustively applying these operations in a naive fashion. As our main result, we answer both questions, thereby solving the open problem by Kaufmann and Ueckerdt:

Theorem 1 *Every non-simple fan-planar drawing can be redrawn as a simple fan-planar drawing of the same graph without introducing additional crossings.*

Moreover, there is an algorithm that, given a non-simple fan-planar drawing Γ*, constructs such a redrawing in* $O(n + k^2 + mk)$ *time, where* n*,* m*, and* k *denote the number of vertices, edges, and crossings of* Γ*, respectively.*

Combined with the aforementioned previous results regarding the density [1, 19] and the recognition complexity [10] of graphs realizable as simple fan-planar drawings, we obtain:

Corollary 1 *Every (not necessarily simple) fan-planar drawing realizes a 3-quasiplanar graph.*

Corollary 2 *Every (not necessarily simple) fan-planar drawing on* n *vertices has at most* $6.5n-20$ *edges.*

Corollary 3 *Recognizing graphs that admit (not necessarily simple) fan-planar drawings is NP-hard.*

We start with some basic terminology and conventions in Section 2. The algorithm for simplifying non-simple fan-planar drawings is described in Section 3. We conclude with a discussion and open problems in Section 4.

2 Terminology

In all drawings in this paper, edges are represented by simple curves. We assume no two edges touch, that is, meet tangentially. Further, we assume that no three edges share a common crossing and that edges do not contain vertices except their endpoints. Let Γ be a drawing of a graph G. A *redrawing* of Γ is a drawing of G. *Redrawing* an edge e in Γ refers to the process of obtaining a redrawing Γ' of Γ such that $(\Gamma - e) = (\Gamma' - e)$.

In the beginning of Section 1, we introduced the notion of special vertices for crossed edges. To streamline the arguments, we also assign an arbitrarily chosen *special* vertex to each uncrossed edge. Let e and f be edges that cross and let E be the special vertex of e. We define the i^{th} crossing of f with e as the i^{th} crossing between f and e encountered when traversing f from endpoint E. For example, in Figure 6(a), the first crossing of g with b is x and the second crossing is y. Lastly, to refer to the subarc of an edge e in between two crossings or endpoints a and b, we write one of $[a,b]/e, (a,b]/e, [a,b)/e, (a,b)/e$ depending on whether we want to include a and/or b into the subarc.

3 The Redrawing Procedure

We prove Theorem 1 by providing an algorithm that redraws the edges of a non-simple fan-planar drawing Γ to obtain a simple fan-planar drawing. It is based on three subroutines (Lemmata 1, 2 and 4), which can be iteratively applied to remove crossings between adjacent edges (configuration S1) and multiple crossings between pairs of edges (configuration S2). More specifically, the first procedure (Lemma 1) eliminates a particular type of adjacent crossings, namely, those that involve an edge that is incident to its special vertex. The second procedure (Lemma 2) removes multiple crossings between edge pairs. Both procedures reduce the overall number of crossings. Hence, they can be exhaustively applied to obtain a redrawing Γ' of Γ that does not contain multiple crossings between edge pairs and where adjacent crossings only involve edges that are not incident to their special vertices (Corollary 4). The procedure (Lemma 4) for removing these remaining crossings is quite involved and based on a structural analysis (Lemma 3) of the drawing Γ'.

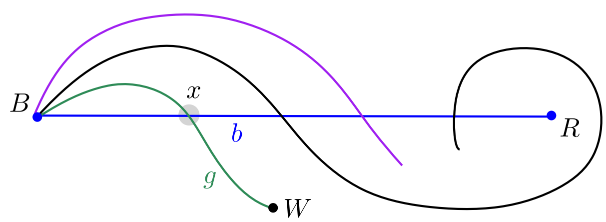
Algorithmic considerations. The standard data structure for encoding and traversing crossing-free drawings on surfaces is the well-known *doubly connected edge list* (DCEL); for a detailed description see [13]. A drawing with crossings can be interpreted as a crossing-free drawing by replacing each crossing with a dummy vertex of degree four. Thus, a drawing with crossings may also be encoded by means of a DCEL. We assume our given fan-planar drawing Γ, which represents some graph G, to be represented in this fashion. We also assume to be given an adjacency list of G in which each edge is explicitly represented[4] and such that each (half-)edge of our DCEL is equipped with a pointer to its corresponding edge in the adjacency list. Due to the introduction of the dummy vertices, each crossed edge $e = (A, B)$ is represented as a sequence of edges (or half-edge pairs) in the DCEL. We assume that the record for e in the adjacency list is equipped with pointers to the first and last edges of this sequence, which are incident to A and B, respectively. By means of a linear-time preprocessing, it is easy to equip each edge (record in the adjacency list) with a pointer to its special vertex.

The first procedure, for getting rid of some of the adjacent crossings, is very simple and is detailed in Lemma 1.

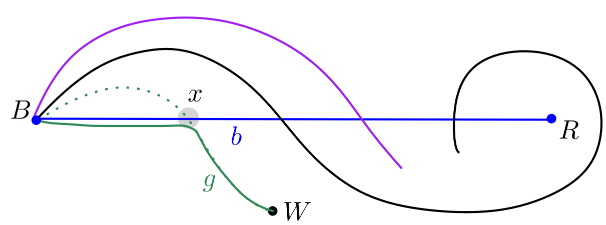
Lemma 1 *Let Γ be a non-simple fan-planar drawing. Let $b = (B, R)$ be an edge in Γ that is incident to its special vertex B. If b has at least one crossing, then one of its crossing edges can be redrawn such that*

- *one of the crossings on b is removed,*
- *the crossings in the resulting drawing form a proper subset of the crossings in the original drawing, and*
- *fan-planarity is maintained and the special vertices do not change.*

Given a pointer to b, the redrawing can be found in time that is linear in the number of removed crossings.



(a) Edge g is adjacent to b and crosses b.



(b) Redrawing of edge g.

Figure 4: Illustration of Lemma 1. If b is incident to its special vertex B, then all crossings on b are adjacent crossings. We redraw the edge g whose crossing x with b is closest to B along b. Redrawing $(B, x)/g$ along b cannot introduce any new crossings.

[4]I.e., we have a list of all vertices, each vertex is equipped with a list of its incident edges, and each edge has pointers to its two incident vertices.

Proof: Every edge that crosses b is incident to B. Traverse along the edge b from B to R, until an edge $g = (B, W)$ that crosses b is encountered. Let this crossing be x. We redraw the edge g to follow the drawing of b from B until x and then follow its previous drawing from x to W without crossing b at x. The rerouting is illustrated in Figure 4.

$[B, x)/b$ has no crossings by definition of x. Hence, the rerouting introduces no new crossings. In particular, no new crossings are introduced on g and, hence, fan-planarity is maintained. Finally, since the crossing between b and g at x is eliminated, the total number of crossings decreases.

Algorithmic considerations. By means of our data structures, the crossing x and the edge g can be determined in constant time. The redrawing operation is then easily carried out in the claimed runtime by traversing $[B, x]/g$ starting from B. □

We continue by describing the second procedure (Lemma 2), which eliminates crossings between pairs of edges (independent or adjacent) that cross more than once. For the proof, we require the following simple observation:

Proposition 1 *Let Γ be a non-simple fan-planar drawing. Let e and f be edges that cross at least three times in Γ and let E be the special vertex of e. Let $x_1, x_2, \ldots, x_k$ be $k \geq 3$ crossings of f and e such that x_i, where $1 \leq i \leq k$, is the i^{th} crossing of f with e. Then $x_1, x_2, \ldots, x_k$ appear in this order along e.*

Proof: Edge f must be incident to E. Let the other endpoint of f be G. Towards a contradiction, assume that the order of crossings is not as claimed and let i be the smallest index such that x_i, x_{i+1}, x_{i+2} appear in the order x_i, x_{i+2}, x_{i+1} along e. Consider the closed curve δ formed by the arcs $[x_i, x_{i+1}]/e$ and $[x_i, x_{i+1}]/f$, see Figure 5 for an illustration. Since the crossing x_{i+2} lies on $(x_i, x_{i+1})/e$, the arc $(x_{i+1}, G)/f$ must cross the curve δ. However, at the crossing between $(x_{i+1}, G)/f$ and δ that is closest to x_{i+1}, f either crosses itself or it crosses e such that the resulting crossings of e and f do not have a consistent orientation. We thus obtain a contradiction, and hence, the crossings x_i, x_{i+1}, x_{i+2} appear in this order along e. □

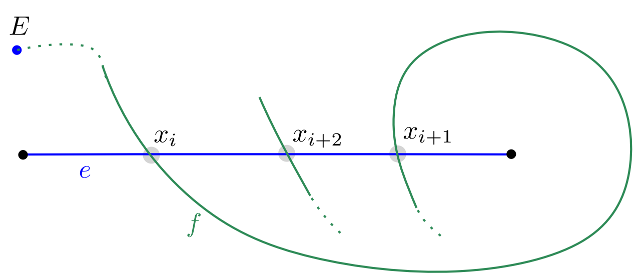


Figure 5: Illustration of Proposition 1: crossings of f with e.

Lemma 2 *Let Γ be a non-simple fan-planar drawing. Let $b = (X, Y)$ be an edge in Γ whose special vertex B is not incident to b. If edge b has multiple crossings with at least one other edge, then an edge that crosses b multiple times, say $g = (B, W)$ (where W could also be incident to b), can be redrawn such that*

- *one of the crossings between b and g is removed,*

- *there is an injective mapping that assigns each crossing of the redrawing to a crossing of the original drawing that involves the same edges,*
- *fan-planarity is maintained and the special vertices do not change.*

Given a pointer to b, the redrawing can be found in time that is linear in the number of crossings in Γ.

Proof: We start by describing a procedure to pick the edge that will be redrawn. We traverse b from X to Y, until the second crossing of an edge $g = (B, W)$ with b is encountered such that the first crossing of g with b appeared before its second crossing, i.e., the second crossing y with b is closer to Y than its first crossing x with b, see Figure 6(a). If no such edge exists, we exchange the roles of Y and X and repeat the procedure. We are guaranteed to find an edge g with the desired properties, since there is an edge crossing b multiple times.

So without loss of generality, assume that the edge g has its second crossing y with b closer to Y than its first crossing x. We then walk from y towards X along b until we encounter a crossing z between an edge p and b. The edge p must also be incident to B, the special vertex of b.

We can now describe the redrawing procedure; for illustrations see Figures 6(b) and 6(c). The edge g is redrawn to follow its previous drawing from W to y, cross b at y, follow $[y, z]/b$, and, finally, closely follow $[z, B]/p$ until we encounter g, which is either at B or at a crossing of g with p. If $[z, B]/p$ crosses the old drawing of g, then we closely follow $[z, B]/p$ until we encounter g, cross p at this point, and then follow the previous drawing of g until B.

Proposition 2 *The crossing z is the first crossing of p with b.*

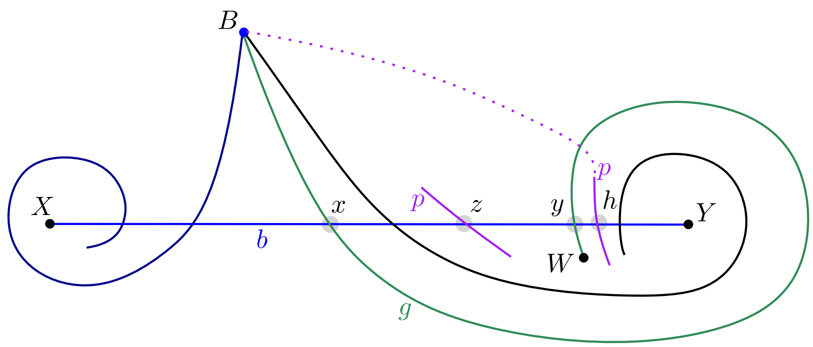
Proof: If $p = g$, then we claim that $x = z$, i.e., z is also the first crossing of $p(= g)$ with b. Assume otherwise that z is the i^{th} crossing of $p(= g)$ with b, where $i > 2$. Then the crossings x, y and z must appear in this order along b due to Proposition 1, which is a contradiction. Thus, z is indeed the first crossing of $p(= g)$ with b if $p = g$.

Now assume $p \neq g$ and assume that the crossing at z is the i^{th} crossing of p with b, where $i \geq 2$. We claim that the first crossing of p with b, say h, has to be located on $(z, Y)/b$. Otherwise, the second crossing of p with b would be located on $(h, z]/b$ due to Proposition 1 but in that case we would have chosen to redraw p, not g. More specifically, the first crossing of p with b, i.e. h, must be on $(y, Y)/b$ since $(z, y)/b$ has no crossings by construction; the situation is illustrated in Figure 6(a). Consider the closed curve δ formed by $[x, y]/b$ and $[x, y]/g$. Since p is incident to B and h is the first crossing of p with b, the arc of p starting from B must cross $(x, y)/g$ to enter the region enclosed by δ before p crosses b at h. After p crosses b at h, it has to cross δ again in order to cross z such that the crossing orientation of z and h is consistent. However, this second crossing with δ implies that the crossings of p with g or the crossings of p with b are not consistently oriented; a contradiction.

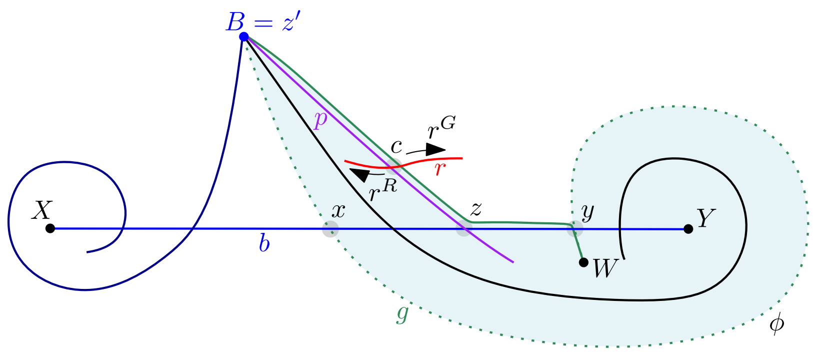
Hence, in any case, z is the first crossing of p with b. □

Proposition 3 *Redrawing g maintains fan-planarity. Moreover, there is an injective mapping that assigns each crossing on the redrawn part of g to a crossing on the replaced part of g that involves the same edges.*

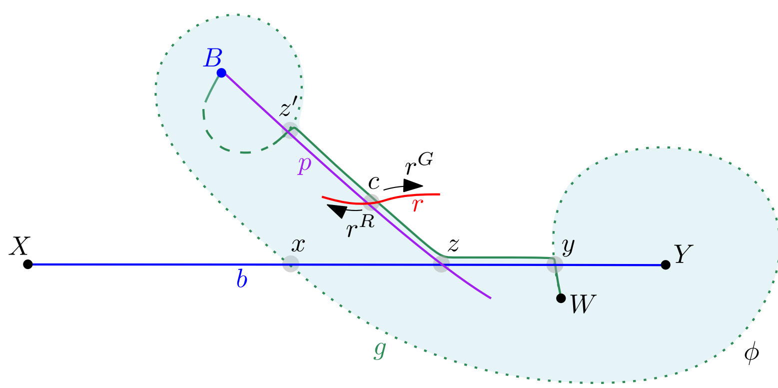
Proof: To show that fan-planarity is maintained, we have to prove that the crossings introduced along the redrawn edge g satisfy the conditions of fan-planarity. If $p = g$, then fan-planarity is



(a)



(b) Redrawing of edge g when $z' = B$.



(c) Redrawing of edge g when $z' \neq B$.

Figure 6: Edges b and g cross multiple times and the special vertex B of b is not incident to b. The figures do not reflect the case when W is incident to b. Any new crossing with the redrawn version of g involves an edge r crossing $(z, z')/p$, which has to cross the replaced part of g since it is incident to X or Y.

maintained since then the crossings of the redrawing of g are a subset of the crossings on its original drawing and the orientations of the crossings are preserved. So assume $p \neq g$. If the old drawing of g was crossed by $[B, z]/p$, then let z' be the first crossing between p and g that is encountered while traversing along $[B, z]/p$ from z to B. Otherwise, let $z' = B$.

Note that $[B, z]/p$ does not enter the side of the curve γ formed by the replaced part $[x, y]/g$ and $[x, y]/b$ that does not contain B. This is true for the two endpoints of $[B, z]/p$ by definition and fan-planarity. Moreover, by Proposition 2, $(B, z)/p$ does not cross b, so the only way to enter the region would be to cross the replaced part $[x, y]/g$, but to leave the region this part would have to be crossed again with an inconsistent orientation. Therefore, if z' is a crossing, then it cannot be on the arc $[x, y]/g$, which belongs to γ, nor on $[y, W]/g$ since this arc lies entirely on the side of γ that does not contain B by fan-planarity. Thus, the crossing z' must lie on the arc $[B, x]/g$ as illustrated in Figure 6(c). Moreover, the orientation of z' has to be as depicted: Towards a contradiction, assume otherwise. Since g and b cross p, they share a common endpoint, which has to be W since b is not incident to B. We have $W = Y$ since $[y, W]/g$ is entirely on the side of γ that contains Y and not X. However, this implies an inconsistent orientation of the crossings z' and z on p and we obtain a contradiction to fan-planarity.

Now, observe that new crossings introduced along g are on $(z', z)/g$ and the involved edges have to cross edge p as well. Let an edge r cross p at a point $c \in (z', z)/p$ (and the redrawn version of g nearby).

From Proposition 2, we know that z is the first crossing of p with b. This implies that b does not cross $(B, z)/p$, and thus, $r \neq b$. Let ϕ be the closed curve formed by the old drawing of $[z', y]/g$, the arc $[y, z]/b$, and the arc $[z', z]/p$, see Figure 6(b) and 6(c) for an illustration of ϕ. The edge r must cross ϕ by definition since c lies on ϕ. Since the second crossing of g with b is closer to Y than the first crossing and due to fan-planarity, Y and X have to lie on distinct sides of ϕ. At c, we split r into two parts. We use r^Y to denote the part that enters the side of ϕ that contains Y – the other part of r is denoted by r^X. The special vertex of p, say P, must be incident to edge b since b crosses p, and the edge r must be incident to P since r also crosses p. We distinguish two cases, namely $P = X$ and $P = Y$. We show that in both cases, r crosses the original arc $(z', y)/g$.

First, assume $P = X$. Since $P(= X)$ must be on a common side of p for r and b, the part of r that is incident to $P(= X)$ has to be r^Y. This implies that r^Y has to cross the curve ϕ by definition of r^Y. Let s be the crossing of r_Y with curve ϕ that is closest to c along r_Y. The crossing s cannot lie on the arc of p on ϕ, since then two crossings at c and s between the two edges r and p would result in an inconsistent crossing orientation. Further, s cannot lie on the arc of b on ϕ since this part of b is uncrossed by the definition of z. Hence, s must lie on the original $(z', y)/g$, i.e., along the part of ϕ formed by the old drawing of g. This implies that r crosses the original $(z', y)/g$.

It remains to consider the case that $P = Y$. Since $P(= Y)$ is on a common side of p for r and b, the part of r that is incident to $P(= Y)$ has to be r^X. The arguments why r crosses the original $(z', y)/g$ are analogous to those used in the case $P = X$.

We have shown that r crosses the original $(z', y)/g$. The corresponding crossing is eliminated when redrawing g, and a crossing between r and g is introduced after the redrawing. Hence, even though the redrawn version of g crosses r on $(z', z)/g$, the number of crossings does not increase. Moreover, the orientation of the crossings between r and g is consistent with the orientation of the crossings in the redrawn version, i.e., fan-planarity is maintained. □

The described redrawing of g eliminates the crossing between g and b at x. No special vertices were changed during the procedure. Moreover, by Proposition 3, the remaining statements of Lemma 2 are fulfilled.

Algorithmic considerations. We begin with the following preprocessing: let e be an edge that crosses b. We traverse e from B towards its other endpoint and equip each encountered crossing with b with a number such that the i^{th} crossing of e with b is equipped with the number i. This process is repeated for each edge that crosses b. The entire preprocessing can be carried out in time that is linear in the total number of crossings in Γ. Equipped with this information, the edge g and its crossings x and y can now be found in time that is linear in the number of crossings on b. Finally, the actual redrawing operation can easily be carried out in time that is linear in the number of crossings on g. □

Equipped with Lemmata 1 and 2, we can apply the following normalization to the drawing:

Corollary 4 *Let Γ be a non-simple fan-planar drawing. There is a fan-planar redrawing Γ' of Γ such that*

- *no two edges cross more than once in Γ';*
- *no edge is incident to its special vertex; and*
- *Γ' does not have more crossings than Γ.*

Proof: The redrawing procedures guaranteed by Lemma 2 and 1 decrease the number of crossings. Hence, they can be exhaustively applied to Γ to obtain a drawing Γ' with no more crossings than Γ such that Γ' does not satisfy the precondition of Lemma 1 or 2. In particular, if an edge e is incident to its special vertex, all edges crossing e must be adjacent to e. If there is such an edge, Lemma 1 is applicable. If there is no edge crossing e, we may choose a new special vertex for e, which is not incident to it. Hence, Γ' has the desired properties. □

It is not always feasible to use the previously described edge-rerouting strategies to eliminate adjacent crossings between edges that are not incident to their special vertices, e.g., consider Figure 7. In the following lemma, we deal with some unproblematic cases and characterize the remaining, more challenging, configurations in terms of a sequence of conflicting edges.

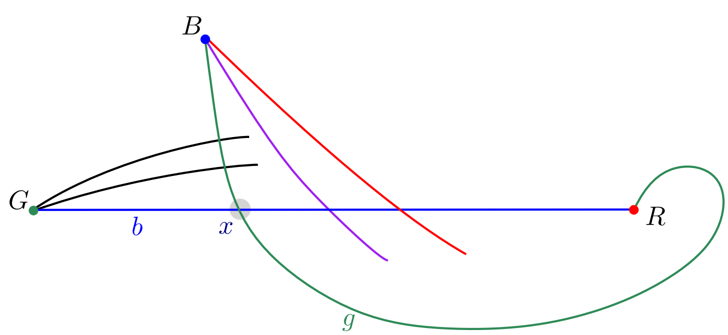


Figure 7: The redrawing strategy corresponding to Lemma 1 cannot be applied to remove the adjacent crossing x since b and g are not incident to their special vertices.

Lemma 3 *Let Γ be a non-simple fan-planar drawing in which no two edges cross more than once and such that no edge is incident to its special vertex. Let $b = (G, R)$ and $g = (R, B)$ be adjacent edges which cross each other at x.*

We can redraw g such that

- *the crossing x is removed,*

- *the crossings in the resulting drawing form a proper subset of the crossings in the original drawing, and*
- *fan-planarity is maintained and the special vertices do not change,*

or, alternatively, we can determine a sequence of edges $r_0, b_1, r_2, b_3, r_4, \ldots, r_k$ *such that the edges* $b, g, r_0, b_1, r_2, b_3, r_4, \ldots, r_k$ *are pairwise distinct and the following properties are satisfied (we call the edges* r_i *"red" and the edges* b_i *"black"; for an illustration, see Figure 8, as well as Figure 15, which also depicts* r_k*):*

I1 B *is incident to the red edges and* R *is incident to the black edges.*

I2 B *is the special vertex of the black edges and* R *is the special vertex of the red edges.*

I3 For any odd i*, the first crossing* x_{i+1} *of* b_i *when traversed from* R *towards its other endpoint is with* r_{i+1}*. For any even* $i < k$*, the first crossing* x_{i+1} *of* r_i *when traversed from* B *towards its other enpoint is with* b_{i+1}*.*

I4 r_0 *crosses* b_1 *but no other black edge.* b *crosses* r_0 *and* r_k *but no other red edges.*

I5 For the purposes of the final two invariants, we define $q_{-1} = b$*. For* $0 \leq i < k$*, let* α_i *be the closed curve defined by* g*, the arc of* q_i *and the arc of* q_{i-1}*, where* $q \in \{r, b\}$*, that connect* R, B *and* x_i*. Let* Γ_i *be the drawing induced by the edges* b*,* $g, r_0, b_1, r_2, \ldots, q_i$*.*

The curve α_i *is simple and bounds a region* f_i *in* Γ_i *that contains only* G*, an arc of* b *that connects* G *to a point* $x \in \alpha_i$ *and, possibly, an arc of* r_0 *that connects* G *to* α_i*, in its interior.*

I6 For $0 < i < k$*,* $f_i \subset f_{i-1}$ *and* $f_{i-1} \setminus f_i$ *is an empty triangular face in* Γ_i *bounded by the following three arcs:*

- *the arc of* q_i *between* x_i *and the special vertex of* q_{i-1}*,*
- $[x_i, x_{i-1}]/q_{i-1}$*,*
- *the arc of* q_{i-2} *between* x_{i-1} *and the special vertex of* q_{i-1}

where $q \in \{r, b\}$*.*

Given a pointer to x*, the redrawing or the sequence can be obtained in time that is linear in the number of crossings in* Γ*.*

Remark 1 *Note that invariant I5 implies that in* Γ_i*,* g *crosses only* b *and possibly* r_0*. Moreover, the arcs of* q_i *and* q_{i-1} *connecting* R *and* B *via* x_i *are uncrossed in* Γ_i*.*

Proof: It follows from the preconditions that B is the special vertex of b and G is the special vertex of g. We will construct the sequence of edges recursively.

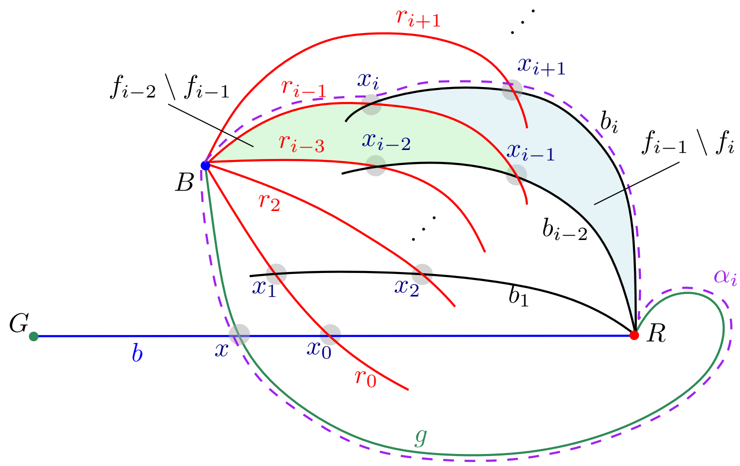


Figure 8: An example of the sequence of edges described in Lemma 3. The face f_i is the unbounded region delimited by the dashed curve, the face $f_{i-1} \setminus f_i$ is depicted in blue and the face $f_{i-2} \setminus f_{i-1}$ is depicted in green.

Base case.

For the induction base case, we show how to determine r_0 and b_1 such that all invariants are satisfied with respect to r_0. For b_1, we will only establish the invariants I1, I4, I5 and I6.

We traverse from R along b until we encounter an edge r_0 that crosses b and denote its crossing by x_0. If $x_0 = x$ and, hence, $r_0 = g$, we can redraw the part of g that leads from R to x along b such that the crossing at x is removed. Moreover, since the redrawn part is crossing-free, the set of crossings decreases and fan-planarity is maintained. Hence, if $x_0 = x$, the statement of the lemma holds.

So assume that $x_0 \neq x$. It follows that, $r_0 \neq g$ since g cannot cross b multiple times. Moreover, $r_0 \neq b$ since edges are realized as simple curves. Since r_0 intersects b, it is incident to B.

Now, we traverse r_0 from B towards x_0 until we encounter a crossing x_1 with an edge b_1. If $x_1 = x_0$ and, hence, $b_1 = b$, we redraw g along $(R, x_0)/b$ and $(x_0, B)/r_0$. The redrawn version of g is crossing-free. Hence, we have eliminated at least one crossing (namely x) while maintaining fan-planarity and, thus, the statement of the lemma holds if $x_1 = x_0$.

So assume that $x_1 \neq x_0$. It follows that b_1 is distinct from b since b has no multiple crossings with r_0. Moreover, $b_1 \neq r_0$ since edges are simple curves. Finally, we show that $b_1 \neq g$. In fact, we prove the following stronger claim:

Proposition 4 *The arc $(B, x_0)/r_0$ cannot cross g.*

Proof: Assume otherwise and consider the closed curve γ formed by the parts of g and b that lead from R to x. Both G and B are on the same side of γ since there are no multiple crossings. Since r_0 intersects g, it follows that r_0 is incident to the special vertex G of g. Let $(B, x_0)/r_0$ be denoted by r_0^B and $(G, x_0)/r_0$ be denoted by r_0^G. Since B must lie on the same side of b with respect to the two crossings x and x_0, the arc r_0^G must be the arc on the side of γ that does not contain B, refer to Figure 9 for an illustration. However, for r_0^G to be incident to G, r_0^G must cross γ, thereby crossing b or g a second time (recall that r_0^B intersects g by assumption), arriving at a contradiction. □

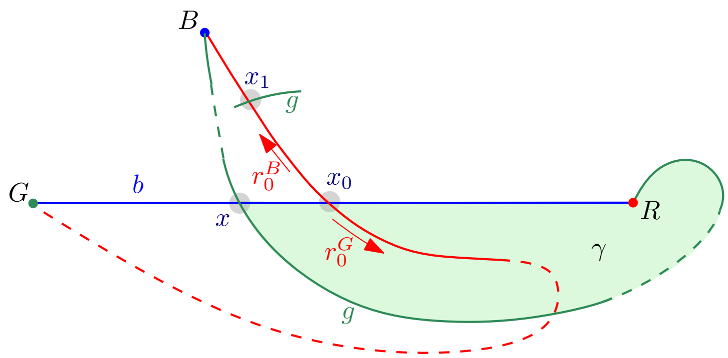


Figure 9: If $b_1 = g$, r_0^G must lie on the side of γ that does not contain B, and cannot cross γ a second time.

In particular, Proposition 4 implies $b_1 \neq g$, as claimed. Thus, we have determined two edges r_0 and b_1 such that b, g, r_0, b_1 are pairwise distinct. It remains to show that the desired invariants hold. We have already established that r_0 is incident to B (since it intersects b) and, thus, I1 is satisfied for r_0.

Since b_1 and b cross r_0, it follows that b_1 shares an endpoint with b, which is the special vertex of r_0. Accordingly, we consider two cases. First, assume that the special vertex of r_0 is G, which is illustrated in Figure 10. Consider the closed curve α_0 described by g, the arc $(B, x_0)/r_0$ and the arc $(R, x_0)/b$. By Proposition 4 and the fact that there are no multiple crossings, the curve α_0 is indeed simple. Orient b and b_1 towards G. Since the resulting orientation of the crossings x_0 and x_1 has to be consistent, it follows that the part of b_1 that connects x_1 with G has to intersect α_0. More specifically, since there are no multiple crossings, it needs to intersect g in some point z. We now redraw g along the part of b that connects R with x_0 and the part of r_0 that connects x_0 with B. The redrawn version of g only has crossings on $(B, x_0)/g$. In particular, it crosses b_1 at x_1, but the orientation of this crossing is consistent with the orientation of z in the original drawing of g. The same argument applies for all other intersections on g, which are all on $(x_0, x_1)/g$. Consequently, we introduce no additional crossings, eliminate the crossing x, and maintain fan-planarity. Hence, the statement of the lemma holds if the special vertex of r_0 is G.

It remains to consider the case where the special vertex of r_0 is R and, hence, b_1 is incident to R. It follows that invariant I1 is satisfied for b_1 and invariant I2 is satisfied for r_0.

Invariant I3 for r_0 is satisfied by construction (and for b_1 there is nothing to show). Invariant I4 is also satisfied for r_0 and b_1 by construction.

The edge r_0 cannot cross b or b_1 a second time. If it crosses g, then it is incident to G, the special vertex of g, see Figure 10. In any case, this implies invariant I5 for Γ_0.

We observe that b_1 cannot cross b or g since this would imply that b_1 is incident to B or G (the special vertex of b and g, respectively) and hence b_1 is parallel to g or b, respectively. Moreover, b_1 cannot cross r_0 a second time. Hence, the part of b_1 that leads from R to x_1 is crossing-free in the drawing Γ_1. Together with invariant I5 for Γ_0, the invariant I5 holds for Γ_1 and invariant I6 holds, which concludes the base case.

Inductive Step:

Now, assume the first $j+1$ edges, $r_0, b_1, \ldots, q_j$, have been determined and $j < k$. We assume all invariants hold for $r_0, \ldots q_{j-1}$. Additionally, we assume that I1, I4, I5 and I6 hold for q_j.

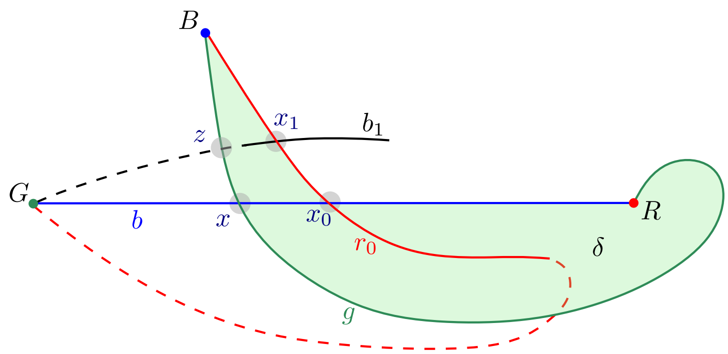


Figure 10: r_0 can be incident to G. b_1 is drawn as if G is the special vertex of r_0.

We will now determine the edge q_{j+1}. We need to prove the invariants I2 and I3 for q_j and the invariants I1, I4, I5 and I6 (and I2 and I3 if $j + 1 = k$) for q_{j+1}.

We distinguish two cases depending on whether q_j is red or black. Both cases feel and are structured similarly, but there are several subtle differences, which is why we decided to treat them separately.

Case 1: $q_j = r_j$

For an illustration refer to Figure 11. If the edge r_j has no crossings on $(B, x_j)/r_j$, then we could redraw g along this part of r_j and $(x_j, R]/b_{j-1}$. The redrawn version of g would then be uncrossed by invariant I3 for b_{j-1} and the lemma is proved.

Otherwise r_j has at least one crossing on $(B, x_j)/r_j$. We determine edge b_{j+1} as follows: traverse along r_j from B towards x_j until we encounter the first edge that crosses r_j. We denote this edge by b_{j+1} and its crossing with r_j by x_{j+1}. Invariant I5 implies that b_{j+1} is distinct from the edges of Γ_j. Let the arc of b_{j+1} that exits the region f_j at x_{j+1} and enters region $f_{j-1} \setminus f_j$ be denoted by b^i_{j+1}.

Invariant I3 for r_j is satisfied by construction. To prove the remaining invariants, we establish several propositions. First we prove a proposition for invariant I2 for r_j and invariant I1 for b_{j+1}.

Proposition 5 *The arc b^i_{j+1} cannot leave the region $f_{j-1} \setminus f_j$ anymore after entering it at x_{j+1}.*

Proof: To exit the region $f_{j-1} \setminus f_j$, arc b^i_{j+1} must cross r_j, b_{j-1} or r_{j-2}. Arc b^i_{j+1} cannot cross r_j again, and also cannot cross r_{j-2} since the arc of r_{j-2} in this region is uncrossed by invariant I3. Hence, b^i_{j+1} must cross b_{j-1} to exit the region $f_{j-1} \setminus f_j$.

Since edge b_{j+1} now crosses r_j and then b_{j-1}, the two edges r_j and b_{j-1} must have a vertex in common. Since $b_{j-1} \neq g$, the edge b_{j-1} is not incident to B. Similarly, r_j is not incident to R since $r_j \neq g$. Thus, the common vertex of r_j and b_{j-1} cannot be B or R, and must be Z, the other vertex of b_{j-1} (and r_j). However, Z lies on a different side of b_{j+1} at crossing x_{j+1}, when compared to the crossing with b_{j-1}, which is a contradiction to fan-planarity. Hence b^i_{j+1} cannot leave the region $f_{j-1} \setminus f_j$. □

The following corollary to proposition 5 establishes invariants I1 for b_{j+1} and I2 for r_j, because R is the only vertex incident to both b_{j-1} and b_{j+1}.

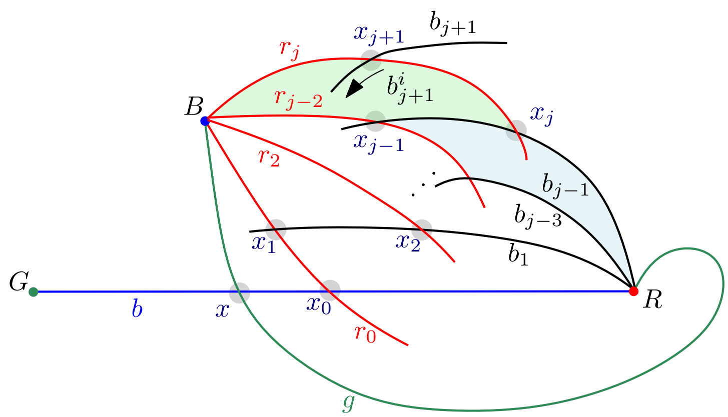


Figure 11: Determining b_{j+1}.

Corollary 5 *Edge b_{j+1} is incident to R, but not G.*

Proof: By Proposition 5 the endpoint at the end of arc b^i_{j+1} lies in $f_{j-1} \setminus f_j$. Thus, by invariant I6, this endpoint is not shared with b_{j-1}. However, since r_j crosses both the edges b_{j-1} and b_{j+1}, b_{j-1} and b_{j+1} must share an endpoint. This common endpoint must be R since the other endpoint of b_{j+1} is not incident to b_{j-1}. Thus, b_{j+1} is incident to R.

By invariant I6, the vertex G is not located in $f_{j-1} \setminus f_j$ and $G \neq R$, hence b_{j+1} is not incident to G. □

Next, we prove invariant I4.

Proposition 6 *The edge b_{j+1} does not cross edge r_0.*

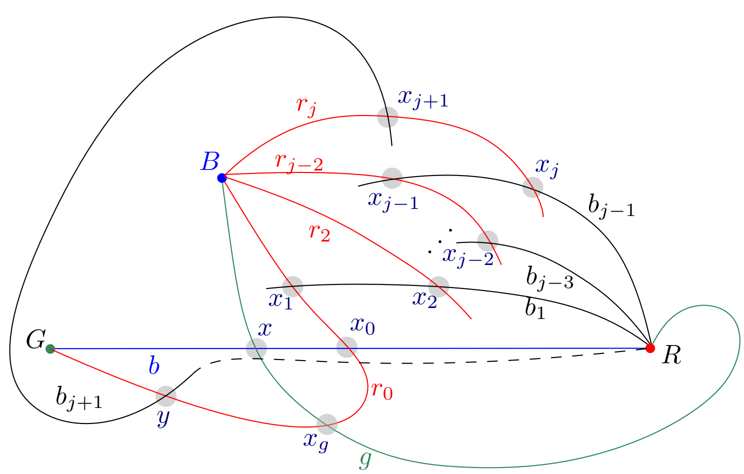


Figure 12: Illustrates the case that b_{j+1} crosses r_0 directly after x_{j+1}.

Proof: Towards a contradiction, assume that b_{j+1} crosses the edge r_0. The arc b^i_{j+1} cannot cross r_0 since r_0 does not pass through $f_{j-1} \setminus f_j$ by invariant I6. Therefore r_0 has to cross $(R, x_{j+1})/b_{j+1}$

at a crossing y. We know b_{j+1} is not incident to G nor B, so it doesn't cross g nor b_{j-1}. Therefore this arc lies entirely in f_j, since none of the arcs b_{j-1}, r_j and g can be crossed by it. r_0 leaves $f_1 \supset f_j$ when crossing b_1 at x_1. r_0 has to cross α_1 again in order to re-enter f_1. To cross α_1 again, r_0 has to cross g, since it cannot cross itself nor b_1 again. However, if r_0 crosses g at a crossing x_g, the other end point of r_0 must be G. The edge r_0 thus will enter f_1 again after crossing g at x_g, crosses b_{j+1} at y and then ends at G. Consider the arc of b_{j+1} that starts at y and ends at R. After crossing r_0 at y, this arc enters the triangle x, x_g, G bounded by b, g and r_0, see Figure 12 for an illustration. However, it is impossible for this arc to exit the triangle since none of the edges bounding the triangle can be crossed (again) by b_{j+1} as shown above, and thus the arc of b_{j+1} cannot be incident to R, a contradiction. □

Lastly, to prove invariants I5 and I6, we have to prove the following proposition.

Proposition 7 *The arc $(R, x_{j+1})/b_{j+1}$ is uncrossed in the drawing Γ_{j+1}.*

Proof: Towards a contradiction assume that $(R, x_{j+1})/b_{j+1}$ is crossed in Γ_{j+1}. Let y be the crossing on $(R, x_{j+1})/b_{j+1}$ that is closest to x_{j+1}. We claim that y does not lie on α_j. First, the crossing y cannot lie on r_j as this would imply that b_{j+1} crosses r_j twice. Second, the crossing y cannot lie on b_{j-1} since this would imply that b_{j+1} is incident to B and, hence, $b_{j+1} = g$, which contradicts the fact that b_{j+1} is distinct from the edges in Γ_j. Finally, the crossing y can also not lie on g and since the special vertex of g is G, and this would imply that b_{j+1} is incident to G and hence, $b_{j+1} = b$, which is again a contradiction. Hence, y does not lie on α_j, as claimed.

It follows that y is located in the interior of f_j. By invariant I5, it follows that y lies on b or r_0. The former is excluded by the fact that b_{j+1} is not incident to B and the latter is excluded by Proposition 6. □

So the arc $(R, x_{j+1})/b_{j+1}$ is uncrossed in the drawing Γ_{j+1}. Further, the arc $(B, x_j)/r_j$ was uncrossed in Γ_j as noted in Remark 1. Since b_{j+1} is the only new edge introduced for Γ_{j+1}, the arcs $(x_j, x_{j+1})/r_j$ as well as $(x_{j+1}, B)/r_j$ are uncrossed in Γ_{j+1}. The latter can be used in conjunction with the above proposition to conclude that invariant I5 is maintained: to see this, recall that by Corollary 5, b_{j+1} is not incident to G and, hence, it cannot cross g since the special vertex of g is G. Therefore, no additional edges cross g while extending the subdrawing from Γ_j to Γ_{j+1}.

Invariant I5 can be combined with the fact that the arc of b_{j-1} from R to x_j is uncrossed by invariant I3 to conclude that the triangular region $f_j \setminus f_{j+1}$ is indeed empty and invariant I6 is established. This concludes the proof of the lemma in the case when $q_j = r_j$.

Case 2: $q_j = b_j$

For an illustration see Figure 13. If $(R, x_j)/b_j$ has no crossings, then we can redraw g along it and $(x_j, B)/r_{j-1}$. The redrawn version of g is then be uncrossed by invariant I3 and the lemma is proved. So assume that $(R, x_j)/b_j$ is crossed.

We determine the edge r_{j+1} as follows: traverse along b_j from R towards x_j until we encounter the first edge that crosses b_j. We denote this edge by r_{j+1} and its crossing with b_j by x_{j+1}. Invariant I5 implies that r_{j+1} is distinct from the edges of Γ_j. Let the arc of r_{j+1} which exits the region f_j at x_{j+1} and enters region $f_{j-1} \setminus f_j$ be denoted by r^i_{j+1}.

Invariant I3 is true for b_j by construction. To prove invariant I2 for b_j and invariant I1 for r_{j+1}, we use the following proposition.

Proposition 8 *The arc r^i_{j+1} cannot leave the region $f_{j-1} \setminus f_j$ anymore after entering it at x_{j+1}.*

Proof: To exit the region $f_{j-1} \setminus f_j$, arc r^i_{j+1} must cross b_j, r_{j-1} or b_{j-2}. Arc r^i_{j+1} cannot cross b_j again, and also cannot cross b_{j-2} since the arc of b_{j-2} in this region is uncrossed by invariant I3. Hence, r^i_{j+1} must cross r_{j-1} to exit the region $f_{j-1} \setminus f_j$.

Since edge r_{j+1} now crosses b_j and then r_{j-1}, the two edges b_j and r_{j-1} must have a vertex in common. Since $r_{j-1} \neq g$, the edge r_{j-1} is not incident to R. Similarly, b_j is not incident to B since $b_j \neq g$. Thus, the common vertex of b_j and r_{j-1} cannot be B or R, and must be Z, the other vertex of r_{j-1} (and b_j). However, Z lies on a different side of r_{j+1} at crossing x_{j+1}, when compared to the crossing with r_{j-1}, which is a contradiction to fan-planarity. Hence r^i_{j+1} cannot leave the region $f_{j-1} \setminus f_j$. □

The following corollary of Proposition 8 implies invariants I1 for r_{j+1} and I2 for b_j, since B is the only vertex that r_{j-1} and r_{j+1} have in common.

Corollary 6 *Edge r_{j+1} is incident to B, but not G.*

Proof: By Proposition 8 the endpoint at the end of arc r^i_{j+1} lies in $f_{j-1} \setminus f_j$. Thus, by invariant I6, this endpoint is not shared with r_{j-1}. However, since b_j crosses both the edges r_{j-1} and r_{j+1}, they must share an endpoint. This common endpoint must be B since the other endpoint of r_{j+1} is not incident to r_{j-1}. Thus, r_{j+1} is incident to B.

By invariant I6, the vertex G is not located in $f_{j-1} \setminus f_j$ and $G \neq B$, hence r_{j+1} is not incident to G. □

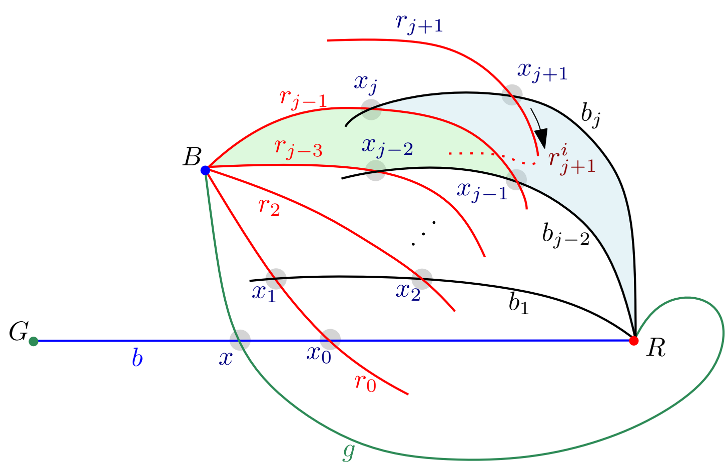


Figure 13: Determining edge r_{j+1}.

To prove the remaining invariants, we require the following proposition.

Proposition 9 *If $j + 1 = k$, then all invariants hold. Otherwise, the arc $(x_{j+1}, B)/r_{j+1}$ is uncrossed in the drawing Γ_{j+1}.*

Proof: Assume that $(x_{j+1}, B)/r_{j+1}$ is crossed in Γ_{j+1}. Let y be the crossing on $(x_{j+1}, B)/r_{j+1}$ that is closest to x_{j+1}.

If y lies on the edge b, then invariant I4 is satisfied with $j + 1 = k$. Invariants I3, I5 and I6 are void in this case. Invariant I1 has already been proved. Finally, invariant I2 is true for r_{j+1}, because it crosses b_j and b, both of which are incident to R, so R has to be its special vertex.

Assume y lies on the closed curve α_j. Recall that the curve α_j is formed by the edge g, the arc $[R, x_j]/b_j$ and the arc $[x_j, B]/r_{j-1}$. The crossing y must lie on g since r_{j+1} cannot cross the arc of b_j again and the arc of r_{j-1} is uncrossed due to invariant I3. Recall that the special vertex of g is G. This implies that r_{j+1} must be incident to G since the edge g is crossed by r_{j+1} if y lies on g. Thus, the other endpoint of r_{j+1}, which is the endpoint of arc r^i_{j+1}, must be G. For arc r^i_{j+1} to end in G, it must cross the curve α_j. The arc r^i_{j+1} cannot cross the arc of r_{j-1} along α_j since the arc of r_{j-1} along α_j is uncrossed by invariant I3, and it cannot cross the arc of b_j nor g again.

Hence, the crossing y is not on the curve α_j, which implies that y lies in the interior of f_j. However, the interior of f_j contains only an arc of b and possibly an arc of r_0 due to invariant I5.

Edge r_{j+1} cannot be incident to R, since otherwise r_{j+1} would be an edge that is parallel to g. Therefore r_{j+1} cannot cross r_0 whose special vertex is R.

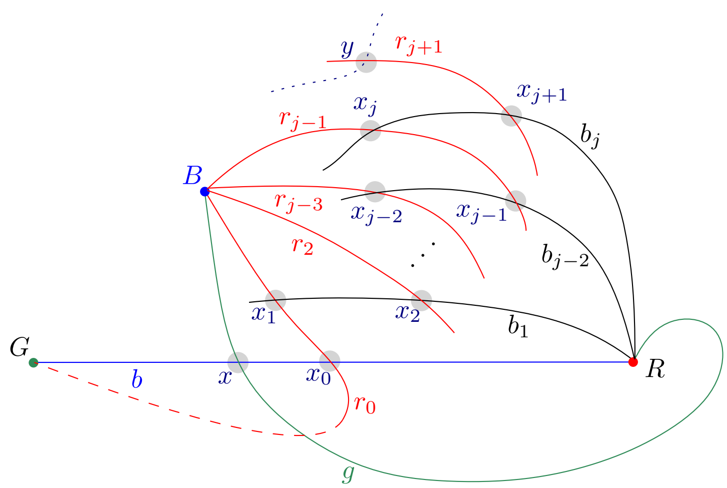


Figure 14: Illustration of the case that $(B, x_{j+1})/r_{j+1}$ is crossed.

This proves the proposition that $(B, x_{j+1})/r_{j+1}$ is uncrossed in the drawing Γ_{j+1}, unless $j+1 = k$. □

Assume $r_{j+1} \neq q_k$. Then $(B, x_{j+1})/r_{j+1}$ is uncrossed in the drawing Γ_{j+1}. Further, the arc $(x_j, R)/b_j$ was uncrossed in Γ_j by invariant I6. Since r_{j+1} is the only new edge introduced for Γ_{j+1}, the arcs $(x_j, x_{j+1})/b_j$ as well as $(x_{j+1}, R)/b_j$ are uncrossed. The latter can be used in conjunction with the above proposition to conclude that invariant I5 is maintained: to see this, recall that r_{j+1} cannot be incident to G by Corollary 6. Consequently, r_{j+1} cannot cross g since the special vertex of g is G. Therefore, no additional edges cross g when extending the drawing from Γ_j to Γ_{j+1}.

Invariant I5 can be combined with the fact that the arc $(B, x_j)/r_{j-1}$ is uncrossed by invariant I3 to conclude that the triangular region $f_j \setminus f_{j+1}$ is empty and invariant I6 is established.

Since $(B, x_{j+1})/r_{j+1}$ is uncrossed, it does not intersect b. On the other hand, by Proposition 8, the arc of r^i_{j+1}, which enters $f_{j-1} \setminus f_j$ at x_{j+1}, cannot leave this region, and $f_{j-1} \setminus f_j$ does not contain any part of b by invariant I6. This finally proves invariant I4, and concludes the proof of the lemma in the case where $q_j = b_j$.

Algorithmic considerations. The inductive proof directly corresponds to a procedure for determining the sequence of edges (along which g is possibly redrawn) in an iterative fashion. When appending a red edge r_{j+1} to the sequence, we need to check whether it is the final edge of the sequence (i.e., whether $j + 1 = k$). To this end, we need to check whether the arc $(x_{j+1}, B)/r_{j+1}$ crosses b, which can be done in time that is linear in the number of crossings on r_{j+1}. Since the edges of our sequence are pairwise distinct, the total cost for performing all these checks is linearly bounded by the number of crossings in Γ. Disregarding the time to perform these checks, the time to append an edge to the sequence is constant. Since the length of the sequence is linearly bounded by the number of crossings in Γ, it can be determined in the claimed runtime. Finally, the redrawing operation (if applicable) is also easy to carry out in the claimed runtime. □

Now that we concluded the proof of Lemma 3, we have all the tools to prove Lemma 4.

Lemma 4 *Let Γ be a non-simple fan-planar drawing in which no two edges cross more than once and such that no edge is incident to its special vertex. If there is an edge $b = (G, R)$ in Γ that crosses an edge g at x and g is incident to R, then we can redraw an edge such that*

- *the crossing x is removed,*
- *the crossings in the resulting drawing form a proper subset of the crossings in the original drawing, and*
- *fan-planarity is maintained and the special vertices do not change.*

Given a pointer to x, the redrawing can be determined in time that is linear in the number of crossings in Γ.

Proof: Let $b = (G, R)$ and $g = (B, R)$ be two adjacent edges which cross at x. Due to the precondition, their common endpoint is not the special vertex of either of the edges. Thus, the special vertices of b and g must be B and G, respectively. We apply Lemma 3 on x. If g can be redrawn using Lemma 3, this concludes the proof of Lemma 4. Assume that g cannot be redrawn. Then we can determine a sequence of edges $r_0, b_1, r_2, \ldots, r_k$ with the properties described in Lemma 3. We now describe how the edge b can be redrawn to eliminate the crossing x while maintaining fan-planarity and decreasing the overall number of crossings.

Let the other endpoint of edge r_k be W. By invariant I4, r_k has a crossing with edge b. First assume this crossing occurs on $(x_k, W)/r_k$, i.e., after r_k enters the triangular region $f_{k-2} \setminus f_{k-1}$ at x_k. Since b does not enter this region, r_k has to leave it. It cannot cross b_{k-1} again, nor can it cross r_{k-2}, because it is not incident to its special vertex R (note that $W \neq R$ since otherwise r_k would be parallel to g). Finally, it cannot cross b_{k-3}, because this is the part of b_{k-3} that is uncrossed by invariant I3. Hence, the crossing of r_k and b cannot lie on $(x_k, W)/r_k$ and must instead lie on $(B, x_k)/r_k$. In this case, we claim that edge b can be redrawn. Redraw edge b to follow g from R until x, and then follow its previous drawing from x until G while avoiding crossing g at x, as illustrated in Figure 15. We now prove that this redrawing does not introduce any new crossings on b.

Proposition 10 *Redrawing b does not introduce any new crossings on b.*

Proof: Assume a new crossing with an edge p is introduced on b by the redrawing operation. Since the redrawn part of b is parallel to a part of g, the edge p crosses g as well. Consequently, edge p is incident to G, the special vertex of g.

Consider the closed curve δ formed by the arcs $[x_k, B]/r_k$, $[B, x_{k-1}]/r_{k-2}$, and $[x_{k-1}, x_k]/b_{k-1}$. The edge b crosses r_k exactly once, does not cross r_{k-2} due to invariant I4, and also does not cross b_{k-1} since the special vertex of b_{k-1} is B due to invariant I2 and b is not incident to B. This

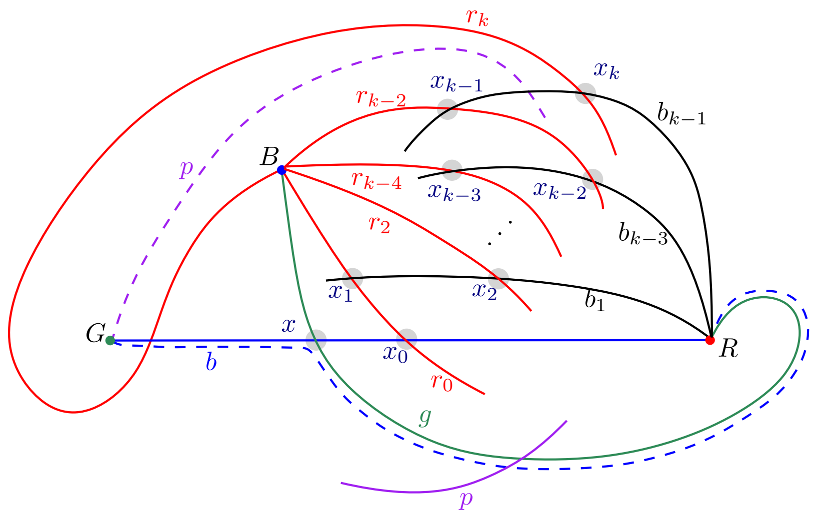


Figure 15: Redrawing of edge b.

implies that b crosses δ exactly once and thus R and G have to lie on distinct sides of δ. This is illustrated in Figure 15. Edge g does not cross any of the edges on the boundary of δ since b is the only edge crossed by g by Remark 1 except possibly for r_0, and even if $r_{k-2} = r_0$ the part of r_{k-2} on δ is still uncrossed by invariant I3, and therefore g is contained in the same side of δ as its endpoint R.

The edge p crosses g and is incident to G, and thus must cross the curve δ since g and G lie on distinct sides of δ. Edge p cannot be incident to R since then p would be parallel to b. Since R is the special vertex of r_k and r_{k-2} and p is not incident to R, p cannot cross the edges r_k and r_{k-2}. Hence, p must cross edge b_{k-1} to cross the curve δ. Then the other endpoint of p must be B, the special vertex of b_{k-1}. However, the part of p connecting G with b_{k-1} is on the same side as $[B, x_{k-1}]/r_{k-2}$. Consequently, the part of p connecting B to b_{k-1} and $[B, x_{k-1}]/r_{k-2}$ lie on distinct sides of b_{k-1}, which contradicts fan-planarity. Overall, we have shown that p cannot cross δ and, by extension, it cannot cross g; a contradiction. □

The only redrawn edge is b and no new crossing is introduced on b, which ensures that fan-planarity is maintained. Additionally, we eliminate the crossing x, which decreases the total number of crossings in the drawing.

Algorithmic considerations. Lemma 3 can be applied in the claimed runtime. If it does not redraw the edge g, then the redrawing operation on b is now easily carried out in the claimed runtime. □

We already described in the beginning of Section 3 how Lemmata 1–4 can be combined to obtain the desired redrawing. We now formally summarize the proof and discuss the corresponding algorithm.

Proof: [of Theorem 1] We begin with a purely combinatorial proof and then discuss the algorithmic aspects.

Let Γ be a non-simple fan-planar drawing. By Corollary 4 we can find a fan-planar redrawing Γ' of Γ such that

- no two edges cross more than once in Γ',
- if two adjacent edges cross in Γ', then their common endpoint is not the special vertex of either of the two edges;, and
- Γ' does not have more crossings than Γ.

If Γ' still contains adjacent crossings, we can apply Lemma 4 to obtain a fan-planar redrawing of Γ' with fewer crossings. Since the number of crossings is finite, we can iterate this procedure to eventually obtain a simple fan-planar drawing.

Algorithmic considerations. As described in the beginning of Section 3, we begin by preprocessing the given drawing in $O(n+m+k)$ time to determine the special vertex of each edge. We have established that the desired redrawing can be found by exhaustively applying Lemmata 1, 2, and 4 (recall that Corollary 4 is established by exhaustively applying Lemmata 1 and 2). Algorithmically, we do so in three phases, where Phases 1, 2, and 3 are concerned with applying Lemmata 1, 2, and 4, respectively.

In the beginning of Phase 1, we pick an arbitrary edge b and check whether it is incident to its special vertex. If so, we repeatedly apply Lemma 1 until b is crossing-free; see Figure 16. We then assign an arbitrary non-incident vertex as the new special vertex of b. This procedure is now repeated for the remaining edges. Disregarding the time to execute Lemma 1, the time to handle one edge is $O(1)$, so $O(m)$ time in total. Each application of Lemma 1 takes time that is linear in the number of removed crossings. Hence, the total time for all applications of Lemma 1 is bounded by $O(k)$, and the total runtime for Phase 1 is $O(n+m+k)$. Applying Lemma 1 does not change special vertices. Hence, during Phase 1, no edge *becomes* incident to its special vertex. Moreover, in Phases 2 and 3, we will not change special vertices at all (in particular, applying Lemma 2 or 4 does not change special vertices). Hence, after handling an edge in Phase 1, it will remain non-incident to its special vertex for the remainder of the algorithm. In particular, this implies that at the end of the last phase of our algorithm, Lemma 1 can indeed not be applied to any edge.

In the beginning of Phase 2, we pick an arbitrary edge b. We repeatedly apply Lemma 2 until there is no edge that crosses b multiple times. Note that the precondition of Lemma 2 is satisfied since Lemma 1 is not applicable. We repeat the process with the remaining edges. Each application of Lemma 2 removes at least one crossing in $O(k)$ time. Hence, the total time spent for applying Lemma 2 is bounded by $O(k^2)$. Disregarding the time to execute Lemma 2, the time to handle one edge is $O(k)$ (to check whether it has multiple crossings with some edge), so $O(mk)$ in total. Therefore, the total time for Phase 2 is $O(n+k^2+mk)$. In Phases 2 and 3, no new crossings are introduced. In particular, applying Lemma 2 or 4 does not introduce any new crossings. Hence, after handling an edge in Phase 2, it will remain without multiple crossings for the remainder of the algorithm. In particular, this implies that at the end of the last phase of our algorithm, Lemma 2 can indeed not be applied to any edge.

In the beginning of Phase 3, we pick an arbitrary crossing x and check if its two edges are adjacent. If so, we apply Lemma 4 to remove x (and possibly other crossings). Note that the preconditions of Lemma 4 are satisfied since Lemma 1 and 2 are not applicable (cf. Corollary 4). We repeat this procedure for the remaining crossings. Disregarding the time to apply Lemma 4, the time to handle a crossing is $O(1)$. Each application of Lemma 4 removes at least one crossing in $O(k)$ time. Hence, the total time spent for applying Lemma 4 and the total time for Phase 3 is bounded by $O(n+m+k^2)$. Applying Lemma 4 does not introduce any new crossings. Therefore, at the end of Phase 3, each of the remaining crossings involves independent edges. Consequently, at the end of Phase 3, Lemma 4 can indeed not be applied.

Altogether, we have shown that Lemmata 1, 2, and 4 can be applied exhaustively in $O(n + k^2 + mk)$ time, which proves the claim. □

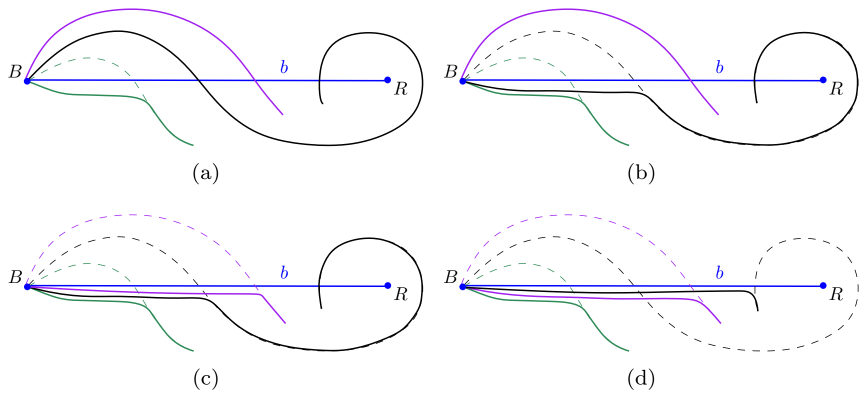


Figure 16: Illustration of applying Lemma 1 to remove all adjacent crossings along an edge.

4 Discussion

Recall that the original definition of fan-planarity dates back to 2014 [19]. As already mentioned (in a footnote) in Section 1, the authors of [19] very recently uploaded a new version [20] of their preprint, in which they provide the following new definition of fan-planarity, which is quite different from the original version (which is used in our paper): they state that a simple drawing is fan-planar if and only if it does not contain the forbidden crossing configurations SF1, SF2, and SF3; see Figure 17. Configurations SF1 and SF2 were already used to characterize simple fan-planar drawings according to the original definition (see Section 1). The new additional configuration SF3 is to be understood as a drawing in the plane rather than on the sphere, i.e., it is sensitive to the choice of the outer face. Consequently, the fan-planarity of a simple drawing according to the new definition now also depends on the choice of the outer face. In particular, configuration SF3 becomes a (simple) fan-planar drawing (according to the new definition), when exchanging the role of the outer and inner face.

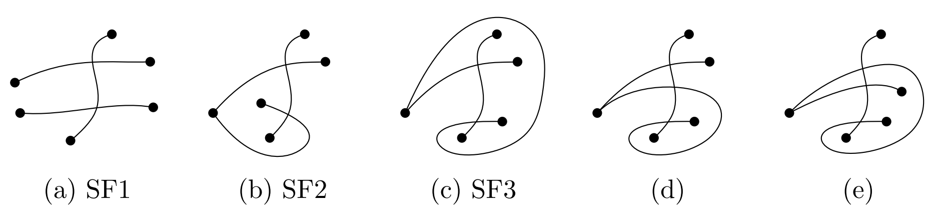


Figure 17: (a,b,c) Forbidden configurations in simple fan-planar drawings according to the updated definition in [20]. (d,e) Non-simple drawings reminiscent of SF3.

The original definition of fan-planarity applied to both simple and non-simple drawings. In contrast, it appears less straight-forward to generalize the new definition to non-simple drawings:

the original definition was phrased in terms of consistent crossing orientations. While this property is still implied for *simple* drawings without configurations SF1, SF2, and SF3, it is not clear whether it should be part of the definition of non-simple fan-planar drawings. Moreover, as soon as edges are allowed to cross multiple times, one quickly ends up with examples that are reminiscent of SF3 even if the crossing orientations are consistent, see Figures 17(d) and (e). Consequently, it is not clear how our results fit into the context of the new definition.

In terms of the original definition, it might be an interesting question when a non-simple fan-planar drawing can be redrawn into a simple fan-planar drawing without changing the rotation system. Pammer [23] proved that the rotation system of the drawing of K_4 illustrated in Figure 18 cannot be realized as a simple drawing. This immediately implies that this rotation system cannot be realized as a simple fan-planar drawing of K_4. Hence, we observe that it is not always possible to redraw a non-simple fan-planar drawing into a simple fan-planar drawing with the same rotation system.

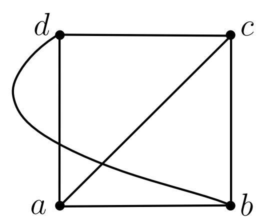


Figure 18: A non-simple fan-planar drawing of K_4

In Figures 3(b) and (c), we illustrated that a single step of the traditional methods for removing configurations S1 and S2 is not guaranteed to preserve fan-planarity. In fact, it can give rise to an arbitrarily high number of conflicts to fan-planarity (and a same number of new conflicts to simplicity). Consequently, it is not possible to prove Theorem 1 by arguing inductively when applying these procedures iteratively in a naive fashion. Nevertheless, it could still be true that the exhaustive application of the traditional simplification procedures always results in a simple fan-planar drawing, thereby giving rise to an alternative proof of Theorem 1. It would be interesting to confirm or disprove this statement.

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