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Arrangements of orthogonal circles with many intersections

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Abstract. An arrangement of circles in which circles intersect only in angles of $\pi/2$ is called an *arrangement of orthogonal circles*. We show that in the case that no two circles are nested, the intersection graph of such an arrangement is planar. The same result holds for arrangement of circles that intersect in an angle of at most $\pi/2$.

For the case where circles can be nested we prove that the maximal number of edges in an intersection graph of an arrangement of orthogonal circles lies in between $4n - O(\sqrt{n})$ and $\left(4 + \frac{5}{11}\right) n$, for n being the number of circles. Based on the lower bound we can also improve the lower bound for the number of triangles in arrangements of orthogonal circles to $(3 + \frac{5}{9})n - O(\sqrt{n})$.

1 Introduction

A collection of n circles in the plane, is called an *arrangement of orthogonal circles* if any two intersecting circles intersect orthogonally. Here, we call an intersection orthogonal, if the tangents at the intersection point form an angle of $\pi/2$. By definition circles cannot touch in an arrangement of orthogonal circles and all circles have positive radius.

A natural object that arises from an arrangement of orthogonal circles is its intersection graph. A graph G is a *(geometric) intersection graph* if its vertices can be realized by a set of geometric objects, such that two objects intersect if and only if their corresponding vertices form an edge in G. Thus, for an arrangement of orthogonal circles $\mathcal{A}$ we define its intersection graph $G(\mathcal{A})$ as the graph, whose vertices correspond to the circles in $\mathcal{A}$ and two vertices are adjacent, if and only if the associated circles intersect in $\mathcal{A}$. The graph $G(\mathcal{A})$ is called an *orthogonal circle intersection graph*.

Arrangements of orthogonal circles and their intersection graphs were recently introduced by Chaplick et al. [4]. Here it was shown that the intersection graph of n orthogonal circles contains

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at most $7n$ edges. Furthermore, it is NP-hard to test whether a graph is an orthogonal unit circle intersection graph. Chaplick et al. also provide bounds for the maximal number of digonal, triangular and quadrilateral cells in arrangements of orthogonal circles.

Previous results and related work. General (nonorthogonal) arrangements of circles or disks have been studied extensively before. Giving a complete overview over the results in this field is out of scope for this article. We will hence only mention a few selected results. For the special case where all circles have the same radius the intersection graphs are known as *unit disk graphs*. For general arrangements of circles or balls the recognition problems for the corresponding intersection graphs are usually hard (for example for unit disk graphs [3]). We refer the reader to the survey of Hlinený and Kratochvíl [10] for more information. Other work focused on bounding the number of small faces in arrangements of circles [1] or about the circleability of topologically described arrangements [9, 12].

Note that we can have general circle arrangements in which all circles pairwise intersect. Thus, the density of the intersection graph can be $\Theta(n^2)$, although many graphs are not intersection graphs of circle arrangements [15] (for example every graph containing $K_{3,3}$ as a subgraph [10]). Hence, asking for the maximum density for intersection graphs in this setting is not an interesting question.

If the circles are allowed to only intersect pairwise in one point, then the intersection graph is called a *contact graph* and the corresponding arrangement is a circle packing. Due to the famous Andreev–Koebe–Thurston circle packing theorem [2, 14] the disk contact graphs coincide with the planar graphs. One direction of the circle packing theorem is obvious, a planar straight-line drawing of the contact graph can be derived by placing the vertices at the disk centers. A related result is due to Alon et al. [1]. A *lune* is a digonal cell in an arrangement of circles. If we restrict the intersection graph of the (general) circle arrangement to intersections that are formed by lunes (we call this the *lune-graph*) then also in this setting we can obtain a planar straight-line drawing by placing the vertices at the circle centers.

Every arrangement of orthogonal circles with the same radius can be turned into a unit circle packing by shrinking the circle size by a factor of $\sqrt{2}/2$, but there are unit disk contact graphs that are not intersection graphs of an arrangement of orthogonal circles [4].

A well established quality criteria for drawing graphs is to avoid crossings. However, crossings with large angles are considered less problematic [11]. For this reason graphs that can be drawn with right-angle crossings are considered an interesting class from a graph drawing perspective. This kind of drawing is known as RAC-drawing. It was shown that graphs that have straight-line RAC-drawings have at most $4n - 10$ edges, for $n \geq 4$ being the number of vertices [6]. Recently, this approach was carried over to graphs that admit drawings with circular arcs that can intersect at right angles only, called arc-RAC graphs. Chaplick et al. showed that arc-RAC graphs can have at most $14n - 12$ edges and there are such graphs with $4.5n - O(\sqrt{n})$ edges [5].

Orthogonal circle arrangements can also be seen as circular arc drawings (of 4-regular multigraphs) with perfect angular resolution. Such drawings are known as Lombardi drawings and have been studied deeply [7, 8, 13].

Results. We prove bounds for the maximal number of edges in an intersection graph of an arrangement of n orthogonal circles. We show an upper bound of $\left(4 + \frac{5}{11}\right) n$ and present a lower bound of $4n - O\left(\sqrt{n}\right)$. As a crucial intermediate result we show that in the case of arrangements without nested circles, the intersection graph is planar. In particular, (in a similar vein to disk contact graphs and lune graphs) we obtain a planar straight-line drawing by placing the vertices at the centers of the corresponding circles. As an immediate consequence we get that for arrangements of nonnested orthogonal circles the intersection graph has at most $3n-6$ edges. We can refine the analysis to improve this bound to $3n-8$. This bound is tight, since we can show a matching lower bound. Our lower bound constructions can be slightly modified to also improve the lower bounds for the maximal number of triangular cells in arrangements of orthogonal circles to $(3+5/9)n-O(\sqrt{n})$. However, there is still a gap to the upper bound of $4n$ triangular cells [5].

Some of our results for nonnested arrangements hold also in a more general setting. If we require that the angles of intersections between intersecting circles are at most $\pi/2$ (in the sense that two touching circles would have an intersection angle of 0), we can also show that the resulting intersection graph has at most $3n-6$ edges.

Organization. We first prove in Section 2 that the circle intersection graphs are planar in the nonnested case if the intersection angle between any pair of circles is acute. This case includes orthogonal nonnested circle arrangments. In order to prove our results we rely on basic properties of so-called Apollonian circles, which we prove along the way to keep the presentation self-contained, but these observations were known before [16, Chapter 2]. In Section 3 we extend our ideas to nested orthogonal circle arrangements and prove the upper bound. In Section 4 we discuss lower bound constructions.

2 Bounds for acute nonnested arrangements

Although our main object of our interest are orthogonal circle arrangements, the results in this section hold in large parts in a more general setting. If two circles intersect properly, their tangents at either intersection point subdivide the plane into four polyhedral cones. If the angle of the cone that contains the intersection of the associated disks is acute, we call the intersection of the circles an *acute intersection*. See Figure 1 for a reference. Moreover, we call call a circle arrangement *acute* if all intersecting circles have acute intersections. Note that every orthogonal circle arrangement is also an acute circle arrangement.

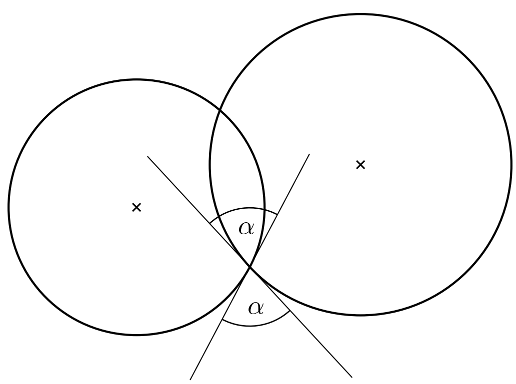

Figure 1: Two circles with an acute intersection witnessed by the acute angle α.

Observation 1 *Let A and B be two circles with centers C_A and C_B and radii r_A and r_B, respectively. Then A and B intersect at an acute angle if and only if $|C_AC_B|^2 \geq r_A^2 + r_B^2$. Moreover, A and B intersect orthogonally, if and only if $|C_AC_B|^2 = r_A^2 + r_B^2$.*

For an arrangement of circles we call the straight-line drawing of its intersection graph that is obtained by placing the vertices on the corresponding circle centers the *embedded intersection graph*. Figure 2 depicts such an arrangement and its embedded intersection graph. In this section we prove that the embedded intersection graph is noncrossing for nonnested acute circle arrangements. Here an arrangement is nonnested if no circle contains another circle properly.

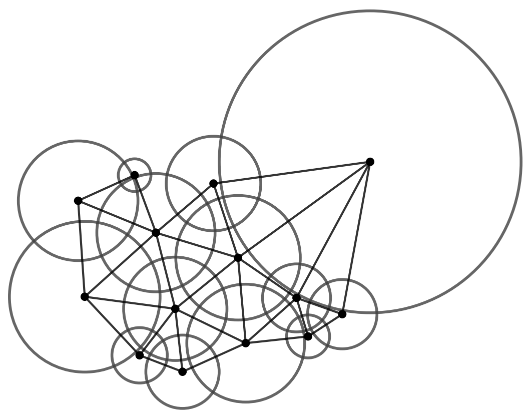

Figure 2: A nonnested circle arrangement and its embedded intersection graph.

We start with properties of arrangements of two or three nonnested orthogonal circles.

Lemma 1 *In an acute nonnested circle arrangement the center of a circle A is not contained in the closed disk determined by a circle other than A.*

Proof: Let A and B be two nonnested circles with centers C_A and C_B and radii r_A and r_B, respectively. Assume that C_A lies inside B. Obviously, A and B intersect in some angle α, since otherwise the circles are nested. It holds that $|C_AC_B|^2 = r_A^2+r_B^2-2r_Ar_B\cos(\pi-\alpha)$. Further, since C_A is in B, we have $|C_AC_B| \leq r_B$ and thus $|C_AC_B|^2 \leq r_B^2$. We get that $r_A^2+r_B^2-2r_Ar_B\cos(\pi-\alpha) \leq r_B^2$. Since $0 \leq \alpha \leq \frac{\pi}{2}$ it follows that $0 \geq \cos(\pi-\alpha) \geq -1$ and thus $r_A^2+r_B^2 \leq r_B^2$. Since $r_A, r_B > 0$ this is a contradiction. □

Lemma 2 *In an acute nonnested circle arrangement for every pair of circles A and B and every point p on A it holds that B intersects C_Ap in at most one point.*

Proof: Let A and B are two circles with centers C_A and C_B and radii r_A and r_B, respectively. Assume for a contradiction that p is a point on A such that B intersects C_Ap twice. We call these intersection points q and s and denote the midpoint between q and s with t with $0 < |C_Aq| < |C_At| < |C_As| \leq r_A$.

By Lemma 1 the center C_B of the circle B has to be outside of the circle A. So for the circle B to have any point in the inside of the circle A, the circle B has to intersect the circle A in a point u (see Figure 3). Since the circles A and B intersect in an angle $\alpha \leq \pi/2$, we have an angle

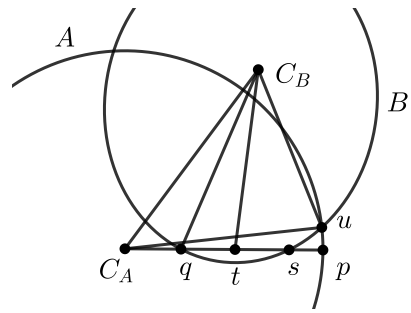

Figure 3: Schematic drawing of the construction in the proof of Lemma 2. One of the circles is drawn as an ellipse to get a better illustration.

$\pi - \alpha$ at u between $C_A u$ and $C_B u$. And since sq is a chord of the circle B, sqC_B spans an isosceles triangle with height $C_B t$. Thus, we have a right angle at t between $C_A p$ and $C_B t$. It follows that

$$|C_A C_B|^2 = r_A^2 + r_B^2 - 2r_A r_B \cos(\pi - \alpha),$$

$$|C_A C_B|^2 = |C_A t|^2 + |C_B t|^2 \qquad \text{and} \qquad r_B^2 = \left(\frac{|qs|}{2}\right)^2 + |C_B t|^2.$$

We obtain

$$\begin{aligned} & |C_A C_B|^2 = r_A^2 + r_B^2 - 2r_A r_B cos(\pi - \alpha) \\ \Leftrightarrow \quad & |C_A t|^2 + |C_B t|^2 = r_A^2 + -2r_A r_B \cos(\pi - \alpha) + \left(\frac{|qs|}{2}\right)^2 + |C_B t|^2 \\ \Leftrightarrow \quad & |C_A t|^2 = r_A^2 + \left(\frac{|qs|}{2}\right)^2 - 2r_A r_B \cos(\pi - \alpha). \end{aligned}$$

Since $0 \leq \alpha \leq \frac{\pi}{2}$ it follows that $0 \geq \cos(\pi - \alpha) \geq -1$. And thus, $r_A \leq |C_A t| < |C_A s|$. We see that s lies outside of the circle A and not on $C_A p$, this is a contradiction. So there is no circle B that intersects the line segment $C_A p$ twice. □

Lemma 3 *In an acute nonnested circle arrangement for every intersecting pair of circles A and B there is no third circle D that shares a point with the line segment between the centers C_A and C_B.*

Proof: Let A, B and D be three circles with centers C_A, C_B and C_D and radii r_A, r_B and r_D, respectively. The circles A and B intersect. Assume for a contradiction that the circle D shares a point with the line segment $C_A C_B$. There are three cases: (i) D properly intersects the line segment $C_A C_B$ once, (ii) D intersects the line segment $C_A C_B$ twice, or (iii) D touches the line segment $C_A C_B$.

If the circle D intersects the line segment between C_A and C_B only once, either C_A or C_B would be inside D; a contradiction to Lemma 1. Thus, D has to intersect the line segment $C_A C_B$ twice or touch it. We denote these intersection points q and s and the midpoint between q and s

by t with $0 < |C_A q| \leq |C_A t| \leq |C_A s| < |C_A C_B|$. Note that the case where D only touches the line segment $|C_A C_B|$ is covered by $|C_A q| = |C_A t| = |C_A s|$. Due to Lemma 2 q and s cannot lie in the same circle, so one lies in A and the other in B. Thus, D intersects both A and B. By Lemma 1 the center C_D of D has to be outside of the circles A and B. So for the circle D to have a point in the inside of the circles A and B, the circle D has to intersect the circles A (in some point u_A) and B (in some point u_B). The situation is depicted in Figure 4.

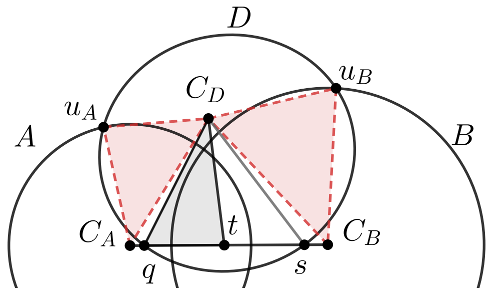

Figure 4: Schematic drawing of the construction in the proof of Lemma 3.

Since sq is a chord of the circle D, sqC_D spans an isosceles triangle with height $C_D t$. Thus, we have a right angle at t between $C_A C_B$ and $C_D t$. According to the law of cosines we obtain for $0 \leq \alpha', \alpha'' \leq \pi/2$:

$$|C_A C_D|^2 = r_A^2 + r_D^2 - 2r_A r_D \cos(\pi - \alpha'), \quad |C_A C_D|^2 = |C_A t|^2 + |C_D t|^2,$$
$$|C_B C_D|^2 = r_B^2 + r_D^2 - 2r_B r_D \cos(\pi - \alpha''), \quad |C_B C_D|^2 = |C_B t|^2 + |C_D t|^2,$$
$$r_D^2 = \left(\frac{|qs|}{2}\right)^2 + |C_D t|^2$$

Combining these equations yields

$$\begin{aligned} |C_A t|^2 &= |C_A t|^2 + |C_D t|^2 - |C_D t|^2 \\ &= |C_A C_D|^2 - |C_D t|^2 = r_A^2 + r_D^2 - 2r_A r_D \cos(\pi - \alpha') - |C_D t|^2 \\ &= r_A^2 + \left(\frac{|qs|}{2}\right)^2 - 2r_A r_D \cos(\pi - \alpha'). \end{aligned}$$

Since $0 \leq \alpha' \leq \frac{\pi}{2}$ it follows that $0 \geq \cos(\pi - \alpha') \geq -1$ and therefore $|C_A t| \geq r_A$. By a symmetric argument we see also that $|C_B t| \geq r_B$. We get $|C_A t| + |C_B t| \geq r_A + r_B$, which is a contradiction. □

We can now combine our observations to prove the following result.

Theorem 1 *The embedded intersection graph of an acute nonnested circle arrangement is non-crossing.*

Proof: Assume for contradiction that the embedded intersection graph has two edges $C_A C_B$ and $C_C C_D$ that cross in the point h. This means we have two pairs of intersecting circles A, B and

C, D, with corresponding centers C_A, C_B, C_C, C_D. Note that C_AC_B is contained in the union of the convex hulls of A and B. Hence, h has to lie in at least one of the circles A or B. By the same reasoning h also has to lie in at least one of the circles C or D. Without loss of generality we can assume that h lies in C. By Lemma 1 the circle C cannot enclose C_AC_B completely, thus it has to intersect the line segment C_AC_B. This, however, contradicts Lemma 3. □

An immediate consequence of Theorem 1 and Euler's formula is the following corollary.

Corollary 1 *The embedded intersection graph of n acute nonnested circles has at most $3n-6$ edges.*

For special arrangements we can improve the bound mentioned in Corollary 1 slightly. We begin we orthogonal circle arrangements. In the following we show that in this case the boundary face of the embedded intersection graph is at least a pentagon if we have five or more circles.

We start with a helpful observation.

Lemma 4 *In an arrangement of three pairwise orthogonal circles every point in the triangle formed by the circle centers is covered by at least one circle.*

Proof: Let A, B, C be three pairwise orthogonal circles with centers C_A, C_B, C_C and radii r_A, r_B, r_C respectively.

Assume for contradiction a point p inside the triangle $C_AC_BC_C$ such that p is not covered by any of the three circles. So its distance to any circle center is larger than the radius of that circle. Now consider the triangle C_AC_Bp, as depicted in Figure 5. We have $|C_Ap| > r_A$ and $|C_Bp| > r_B$ and $|C_AC_B|^2 = r_A^2 + r_B^2$ since A and B intersect orthogonally. Combining these equations yields

$$|C_Ap|^2 + |C_Bp|^2 = r_A^2 + r_B^2 > |C_AC_B|^2.$$

It follows that the angle between C_Ap and C_Bp in p is acute. By the same argument the angles at p between C_Ap and C_Cp, and C_Bp and C_Cp are also acute. The sum of three acute angles is less than 2π, thus we have a contradiction. □

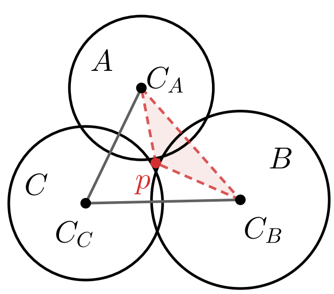

Figure 5: The point p is inside the triangle $C_AC_BC_C$ but outside the circles A, B and C.

The following lemma was proven by Chaplick et al. [4].

Lemma 5 ([4]) *No orthogonal circle intersection graph contains a K_4 or an induced C_4.*

Lemma 6 *The outer face of the embedded intersection graph of an arrangement of* $n \geq 5$ *nonnested orthogonal circles is adjacent to at least* 5 *vertices.*

Proof:

The embedded intersection graph is planar and has an outer face, whose boundary might be disconnected. In order to have less than 5 vertices on the boundary face at least one vertex v is not part of that face. All faces incident to v define a region, whose boundary $\mathcal{B}$ is either a cycle of length three or four. We can show that both cases lead to a contradiction.

Assume that $\mathcal{B}$ has three vertices. It follows that there are three pairwise orthogonal circles, whose centers coincide with the vertices of $\mathcal{B}$, as seen in Figure 6a. According to Lemma 4 three pairwise orthogonal circles cover the whole triangle between their centers. Thus, according to Lemma 1 there cannot be another circle center (in particular, v) inside that triangle. We get a contradiction.

Assume that $\mathcal{B}$ has four vertices. By Lemma 5 these vertices cannot induce a C_4 or a K_4, so the induced graph is a K_4 minus one edge as shown in Figure 6b. By Lemma 4 there can be no further circle center inside each of those triangles (see discussion in the previous paragraph). We get again a contradiction.

(a) (b)

Figure 6: (a) and (b) show intersection graphs of orthogonal circles where the outer face is adjacent to three or four vertices, respectively.

As we get a contradiction in both cases it follows that the outer face of the intersection graph of an arrangement of at least $n \geq 5$ circles is adjacent to at least 5 vertices. □

Applying Euler's formula and the fact that one face is at least of size 5 yields the following result.

Corollary 2 *The embedded intersection graph of an arrangement of* n *nonnested orthogonal circles is a plane embedding and has at most* $3n - 8$ *edges for* $n \geq 5$.

In Section 4 we show that the bound of $3n - 8$ in Corollary 2 is tight for orthogonal circle arrangements. Note that the slightly stronger statement of Corollary 2 does not hold in the acute setting, since the K_4 has a contact representation by touching disks as a consequence of the circle packing theorem. By slightly increasing the radii of that contact representation we obtain a circle arrangement, where all circles intersect at an angle of less than ϵ, for every $\epsilon > 0$. However, we can also improve the result of Corollary 1 slightly for nonorthogonal arrangements if the intersection angles between the circles is not too small. (Here, we measure the intersection angle as in the definition for the acute intersection.) To do so we need the following slightly stronger version of Lemma 4.

Lemma 7 *In an arrangement of three pairwise intersecting circles with a not necessarily common intersection angle $\pi > \alpha \geq \frac{\pi}{3}$ every point in the triangle formed by the circle centers is covered by at least one circle.*

Proof: Let A, B, C be three pairwise intersecting circles with centers C_A, C_B, C_C as in the lemma. Assume for contradiction a point p inside the triangle $C_A C_B C_C$ that is not covered by one of the three circles, as seen in Figure 5.

Let $\pi > \alpha \geq \frac{\pi}{3}$ be the angle at which the circles A and B intersect. Let d be the intersection point of A and B that lies on the same side of $C_A C_B$ as C_C. And let β bet the angle at d between $C_A d$ and $C_B d$. It holds that $\beta = \pi - \alpha$ and thus $0 < \beta \leq \frac{2\pi}{3}$. We now consider the point p which is not covered by A or B. For every point p outside the circles A, B it holds that the angle γ between $C_A p$ and $C_B p$ is smaller than the angle between $C_A d$ and $C_B d$, as seen in Figure 7. Thus $\gamma < \beta \leq \frac{2\pi}{3}$.

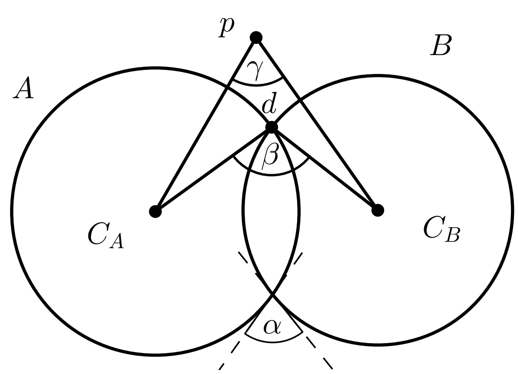

Figure 7: The point p is not covered by the circles A or B.

By the same argument the angles at p between $C_A p$ and $C_C p$, and $C_B p$ and $C_C p$ are less than $\frac{2\pi}{3}$. Thus the sum of the three angles at p is less than 2π, a contradiction. □

As promised we now improve the upper bound for circle arrangements with intersection angle of $\frac{\pi}{2} > \alpha \geq \frac{\pi}{3}$.

Lemma 8 *The embedded intersection graph of $n \geq 4$ nonnested circles where every two intersecting circles intersect at an angle of $\frac{\pi}{2} > \alpha \geq \frac{\pi}{3}$ has at most $3n - 7$ edges.*

Proof: Assume there exists a nonnested arrangement of $n \geq 4$ circles that intersect at an angle of $\frac{\pi}{2} > \alpha \geq \frac{\pi}{3}$ such that the outer face is a triangle. It follows that there are three pairwise intersecting circles, whose center lie on the vertices adjacent to the outer face. According to Lemma 7 three pairwise intersecting circles that intersect at an angle of at least $\frac{\pi}{3}$ cover the whole triangle between their centers, thus according to Lemma 1 there cannot be another circle center within that triangle, a contradiction to $n \geq 4$.

Thus, the outer face of the intersection graph of an arrangement of at least $n \geq 4$ circles is adjacent to at least 4 vertices. Applying Euler's formula yield the upper bound of at most $3n - 7$ edges in an arrangement of $n \geq 4$ nonnested circles where every two intersecting circles intersect at an angle of $\alpha \geq \frac{\pi}{3}$. □

3 Bounds for nested orthogonal arrangements

In this section we prove an upper bound of $\left(4+\frac{5}{11}\right)n$ edges for intersection graphs of arrangements of n orthogonal circles that allow nested circles. We first discuss the general approach and introduce necessary terminology before continuing with details and proofs.

For every circle C in an arrangement $\mathcal{A}$ we define its *depth* $t(C)$ as the maximum cardinality of a set of pairwise nested circles in $\mathcal{A}$ that are properly contained in C. A circle with depth 0, i.e., it contains no circles properly, is referred to as *shallow* otherwise as *deep*.

As a first step we show that in every arrangement we can find a circle with depth at most 1 that is orthogonal to at most seven deep circles (Lemma 13). We select one circle with this property and name it the *red circle*. We then look at the circles properly contained in the red circle. We call these circles *black circles*; see Figure 8. The key observation is that we can delete the set of black circles from the arrangement and by doing so we only lose few edges from the intersection graph, i.e. at most $(4+5/11)\cdot n_B$ for n_B black circles. To obtain this bound we distinguish between intersections between the black circles and intersections between any two black circle and a circle that intersects a black and the red circle (such circles are called *green circles*). To make our analysis work we have to partition the black circles further. If a black circle center lies on the boundary of the embedded intersection graph induced by the arrangement of black circles we call the corresponding circle *boundary black circle*, otherwise *inner black circle*.

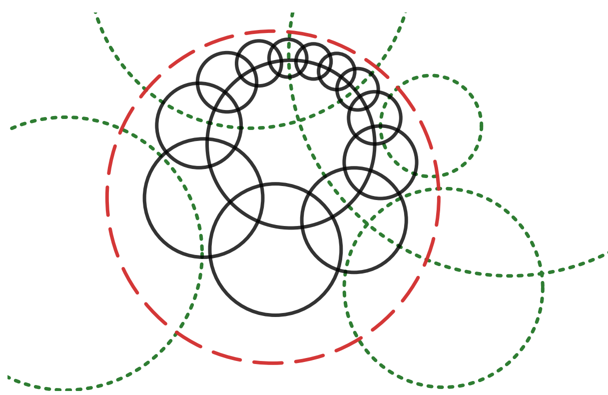

Figure 8: Illustration of the red, black and green circles. This arrangement has only one inner black circle.

We color edges in the intersection graph according to the color of the corresponding circles as follows: An intersection between a black and a green circle yields a *green edge* and an intersection between two black circles yields a *black edge*. If there are n_B black circles and b of those are boundary black circles, then we have at most $3n_B-b-3$ black edges as a consequence of Euler's formula and Theorem 1. We will prove that each black circle can be orthogonal to at most two green circles (Lemma 9). In particular, the inner black circles can only be intersected by green circles with depth at least 1 (Lemma 15). We can chose the red circle so that there are at most seven deep green circles. The intersection graph of these seven green circles has at most eight edges (Observation 2 and Lemma 17). We exploit this fact to show that only eight inner black circles can be orthogonal to two green circles (Lemma 18). As a final observation we show that if there are at most 11 black circles in the red circle, there are at most 3 inner black circles that intersect

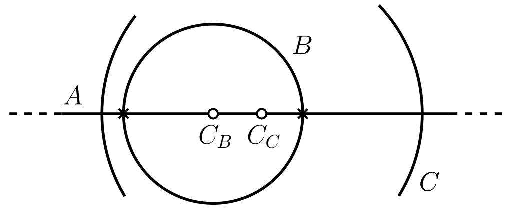

Figure 9: Situation in the proof of Lemma 10 when C contains $A \cap B$. C_B (C_C) is the center of B(C).

two of the green circles. We can then combine our findings to prove that we can always find a set of n_B black circles that intersects at most $(4 + 5/11)\, n_B$ circles.

We are now continuing with the proofs and details. We begin by stating a few properties of arrangements of orthogonal circles.

Lemma 9 *Let A and B be two nested circles. There are at most two circles that intersect both A and B orthogonally.*

Proof: Assume that there are two nested circles A and B (B lies inside A) that both intersect at least three circles D, E and F orthogonally. Consider the intersection graph of A, B, D, E and F. If the circles D, E and F are pairwise orthogonal to each other the vertices corresponding of A, D, E and F form a K_4, a contradiction due to Lemma 5. However, if two of the circles D, E and F are not orthogonal to each other their corresponding vertices together with B and A induce a C_4, which yields a contradiction due to Lemma 5.

Thus, at most two circles that intersect both A and B orthogonally. □

Lemma 10 *If a circle C intersects the circles A and B orthogonally, then one of the following two conditions holds: (i) A and B do not intersect, or (ii) A and B are orthogonal and C contains precisely one of the two intersection points of A and B.*

Proof: We prove that if (i) does not hold, then (ii) holds. So assume A and B intersect. We apply a Möbius transformation that maps A to a straight line. Note that such a transformation is conformal and thus maintains the angles; see Figure 9. The centers of B and C will then have to lie on A. Clearly, if C contains both points of $A \cap B$ then it also has to contain B, but since B intersects C, we have a contradiction. Also, if C does not contain any point of $A \cap B$, then it has to be either contained in B or is to the left or right of B along A, but since B intersects C, we have again a contradiction. □

Lemma 11 *In an arrangement of orthogonal circles let A and B be two circles that intersect. All circles that are orthogonal to A and B that contain the same intersection point of A and B are nested.*

Proof: Assume that there are two nonnested circles C and D that both contain the same intersection point u of A and B. Since C and D contain u but are not nested, they must intersect each other. Both also intersect A and B. This means the intersection graph of the four circles is a K_4. This contradicts Lemma 5. □

The following lemma is again taken from Chaplick et al. [4, Lemma 5]. The "Moreover"-part is not explicitly written down, but it is apparent from the construction given in its proof.

Lemma 12 ([4]) *Every arrangement of orthogonal circles has a circle that is orthogonal to at most seven other circles. Moreover, this circle is a shallow circle.*

We can deduce a similar lemma for deep circles.

Lemma 13 *Every arrangement of orthogonal circles with nested circles has a circle C with depth $t(C) = 1$ that is orthogonal to at most seven other circles with depth at least 1.*

Proof: Let $\mathcal{A}$ be an arrangement of orthogonal circles. By deleting all shallow circles we obtain the arrangement $\mathcal{A}'$. According to Lemma 12 we can find a shallow circle C in $\mathcal{A}'$ that is orthogonal to at most seven other circles. Since C is shallow in $\mathcal{A}'$ it has depth $t(C) = 1$ in the arrangement $\mathcal{A}$. □

In the following we select any circle that meets the requirements of Lemma 13 and refer to it as the red circle. We remind the reader that we call the circles contained in the red circle the black circles.

Lemma 14 *Let S_B be the set of black circles inside a red circle C with $|S_B \geq 2|$. The set S_B corresponds to a vertex set V_B in the intersection graph incident to no more than $4n_B + i_2 - 3$ edges, for $n_B = |S_B|$ and i_2 being the number of inner black circles in S_B, that each are orthogonal to two circles not in S_B.*

Proof:

Let C be the red circle. We count the edges incident to V_B. Edges with two endpoints in V_B are black edges, edges with one endpoint in V_B are green edges. We denote the number of boundary black circles by b. According to Theorem 1 the embedded intersection graph of the arrangement restricted to the S_B is planar. Moreover, this planar graph has b vertices on its outer face. Thus, by Euler's formula we have at most than $3n_B - b - 3$ black edges.

We now count the green edges. Every circle $D \notin S_B$ that intersects a circle in S_B has to intersect C as well. According to Lemma 9, each of the n_B black circles is orthogonal to at most two green circles. By our assumption $n_B - b - i_2$ black inner circles intersect at most one green circle. Thus, we have at most $2n_B - (n_B - b - i_2) = n_B + b + i_2$ green edges. Adding the $3n_B - b - 3$ black edges yields the upper bound of $4n_B + i_2 - 3$ as stated in the lemma. □

Lemma 15 *Every green circle intersecting an inner black circle is a deep circle.*

Proof: Let D be the red circle and S_B be the set of black circles. Assume for a contradiction that there is a shallow green circle E with center C_E that intersects an inner black circle $F \in S_B$ with center C_F. Note that E being a green circle also has to intersect D. Hence E and D are not nested and thus by Lemma 1 C_E is outside of D; see Figure 10.

Let $\mathcal{A}$ be the arrangement consisting of the circles in S_B and E. All circles in S_B and the circle E have depth 0 so the arrangement $\mathcal{A}$ is nonnested. According to Theorem 1 the embedded intersection graph $G(\mathcal{A})$ is noncrossing.

Let $\mathcal{B}$ be the arrangement consisting only of the circles in S_B. Note, that $\mathcal{B}$ is also noncrossing.

Since C_E is outside D it lies in the outer face of $G(\mathcal{B})$. On the other hand F is an inner circle, so its corresponding vertex is not on the boundary of $G(\mathcal{B})$. The straight-line edge between C_E and C_F must intersect an edge on the boundary of the outer face of $G(\mathcal{B})$. This yields a crossing and thus a contradiction. □

By Lemma 12 a red circle C intersects at most 7 deep circles. We now take a look at the possible intersections of the seven deep circles. We start with the following observation.

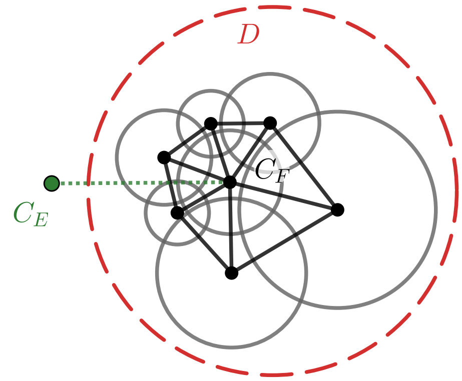

Figure 10: Illustration of the situation in the proof of Lemma 15.

Observation 2 *Let I_C be the set of deep circles that intersect a red circle. The intersection graph of I_C has*

- *no induced C_4, according to Lemma 5 and*
- *no induced C_3, since every circle in I_C is orthogonal to the red circle and according to Lemma 5 there is no K_4 in the intersection graph of the arrangement consisting of I_C and C.*

By a case distinction we can limit the graphs that fulfil the constraints listed in Observation 2 as follows.

Lemma 16 *Let G be a graph with at most seven vertices and no induced C_3 or C_4. Then one of the following holds:*

(i) G has a vertex of degree at most 1.

(ii) G is a C_7, C_6 or a C_5.

(iii) G is two C_5s glued together at a path with two edges.

Proof: If G has at most four vertices, (i) obviously holds. If G has more than four vertices, we further analyse G as follows.

If (i) does not hold, then G contains a cycle. If the shortest cycle in G has length 7, (ii) holds.

If the shortest cycle has length 6, we have two cases: If G has only 6 vertices, then (ii) holds. Or if G has 7 vertices, consider the vertex v not on that cycle of length six. If v has at least two neighbors, these neighbors must lie on the C_6 and thus yield a shortcut. This is not possible since the C_6 is the shortest cycle in G. Hence the vertex v is incident to at most one edge and (i) holds.

If the shortest cycle has length 5, we have three cases. First, if G has only 5 vertices, then (ii) holds. Second, if G has 6 vertices, then the sixth vertex is incident to at most one edge by the same argument used in the case of 7 vertices and a C_6 and (i) holds. Thirdly, if G has 7 vertices and if (i) and (ii) do not hold, consider the two vertices v, w not on the C_5. Since (i) does not hold v, w have degree at least two. If either one of them has two neighbors on the C_5, it yields a shortcut and we would have a shorter circle. This contradicts the assumption that the C_5 is the shortest cycle in G, so both have just one neighbor on the cycle. Since u and v have degree of at least 2 they are connected by an edge. Let v' and w' be the neighbors of v and w on the five-cycle, respectively. The length of the shortest path from v' to w' on the C_5 is either one or two. If it is one, then the path v', v, w, w', v' would be a C_4, a contradiction. So the length of the shortest path is two and G is two C_5 glued together at the path from v' to w'. Thus, (iii) holds. □

Lemma 17 *Every graph G with at most seven vertices without an induced C_3 or C_4 has at most 8 edges.*

Proof: We delete vertices with degree 1 until no such vertex remains. Let G' be the remaining graph. Since G' fulfils the condition (ii) or (iii) of Lemma 16 it can be made acyclic by deleting at most two edges. Hence, also G can be made acyclic by deleting at most two edges. It follows that that G has at most $(n-1)+2$ edges, if it has n vertices. Thus, $n \leq 7$ implies that G has at most 8 vertices. □

We can now bound the number of intersection points of the circles in I_C.

Lemma 18 *Let C be the red circle and let I_C be the set of deep circles intersecting C. The arrangement of circles in I_C has at most sixteen intersection points of which eight are inside of C.*

Proof: According to Lemma 17 the intersection graph of I_C has at most eight edges. Hence, there are eight pairs of intersection points in the arrangement consisting of the circles in I_C. Due to Lemma 10 for every pair exactly one intersection point is inside of C. Thus, at most eight intersection points of circles in I_C are inside the circle C. □

Lemma 19 *In the intersection graph of every arrangement of orthogonal circles we can find a nonempty subset V_B that is incident to at most $4n_B + 5$ edges, where $n_B = |V_B|$.*

Proof: Let $\mathcal{A}$ be an arrangement of orthogonal circles. According to Lemma 13 we can a find a red circle C with depth $t(C) = 1$ that is orthogonal to at most seven deep circles. We denote the black circles by S_B and set $n_B = |S_B|$. Further let V_B denote the vertex set corresponding to S_B.

We now prove that there are at most 8 inner black circles in S_B that are orthogonal to two circles not in S_B. According to Lemma 15 the inner black circles can only be intersected by deep green circles. If a black circle intersects two green circles, then the green circles have to intersect, otherwise the intersection graph of the black, the two green and the red circle would induce a C_4. According to Lemma 10 a black circle that intersects two green circles contains their intersection point. Lemma 11 states that all circles containing the same intersection point must be nested. Since the black circles are not nested, only one black circle contains a given intersection point. By Lemma 18 the seven deep green circles have at most eight intersection points inside C. Thus, at most eight inner black circles are orthogonal to two deep green circles. We now apply Lemma 14 with $i_2 \leq 8$ to obtain that V_B is incident to at most $4n_B - 3 + i_2 = 4n_B + 5$ edges. Note that when $n_B = 1$ (the case not covered by Lemma 14) then by Lemma 9 V_B is incident to at most 2 edges, which is less than $4n_B + 5 = 9$. □

Our goal is to apply the previous lemma for bounding the density of the intersection graph. If we can repeatedly take out vertex sets of size k such that the k vertices are incident to at most ck edges (for a constant c), then the density of the graph is no more than cn, for n being the number of vertices. Unfortunately, because of the additive constant Lemma 19 is too weak if the subsets are small. Hence, we analyse small sets separately to get a better bound. In the remainder of this part we analyse arrangements with a small number of inner black circles to prove the upper bound of $\left(4+\frac{5}{11}\right)\cdot n$ edges in the intersection graph of an arrangement of n circles. We start with a slightly stronger statement of Lemma 19.

Corollary 3 *In the intersection graph of every arrangement of orthogonal circles we can find a nonempty subset V_B that is incident to at most $4n_B+5$ edges where $n_B=|V_B|$. Moreover, if at most three vertices corresponding to inner black circles of V_B are incident to two green edges each, then V_B is incident to at most $4n_B$ edges.*

Proof: We can follow the proof of Lemma 14 and use the fact that for the "Moreover"-part we have $i_2 \leq 3$. □

Before proving the desired bound in Lemma 23 we provide some necessary lemmas.

Lemma 20 *Let C be the red circle in an orthogonal circle arrangement and let S_C be the set of the black circles properly nested in C. If an inner black circle of S_C is intersected by two green circles, then S_C contains at least 8 boundary black circles.*

Proof: Let D be an inner black circle that is intersected by the two green circles E and F. Since both E and F intersect D and C, they must intersect each other. Otherwise the corresponding vertices of E, D, F and C would induce a C_4, which would violate Lemma 5. Thus, E and F intersect and according to Lemma 10 D contains one of the intersection points. If D is an inner circle, then its corresponding vertex v_D in the (embedded) intersection graph G_C of S_C is not adjacent to the outer face. So v_D lies on the interior of a cycle of vertices corresponding to boundary black circles. Let $A=(a_1,...,a_x)$ be such a cycle that is inclusion-minimal.

Every green circle intersecting D must also intersect at least two circles corresponding to vertices in A. So the vertices v_E and v_F corresponding to the green circles E and F must each have at least two neighbors in A. According to Lemma 11 and since the black circles are not nested, there can only be one black circle intersecting both E and F. This circle is the circle D. Thus, the vertices v_E and v_F cannot have a common neighbor.

Let a_i, a_j be the neighbors of E and a_k, a_l be the neighbors F. If two of the neighbors are adjacent, say a_i and a_k then a_i, v_E, a_k, v_F induce a C_4, which would contradict Lemma 5. Thus, A muss contain at least 8 vertices. □

As an immediate consequence from the previous lemma we get.

Lemma 21 *Let C be the red circle in an orthogonal circle arrangement and let S_C be the set of the circles properly nested in C. If $|S_C| \leq 11$, then S_C contains at most three inner black circles that each are intersected by two green circles.*

Proof: Assume for contradiction that $|S_C| \leq 11$ and S_C contains four inner circles that are intersected by two green circles each. It follows that S_C contains at most seven boundary black circles. This is a contradiction to Lemma 20. □

Lemma 22 *In the intersection graph of an arrangement of orthogonal circles we can find a nonempty subset V_C that is incident to at most $4n+5$ edges, where $n=|V_C|$. Moreover, if $n \leq 11$, then the subset is incident to at most $4n$ edges.*

Proof: According to Corollary 3 there is a non empty subset V_C in the intersection graph of every orthogonal circle arrangement that is incident to at most $4n+5$ edges, where $n=|V_C|$. However, according to Lemma 21, if $n < 11$ then there are at most 3 inner black circles that are orthogonal to two green circles. In this case, due to Corollary 3, V_C is incident to at most $4n$ edges.

□

Lemma 23 *In the intersection graph of an arrangement of orthogonal circles we can find a subset V_C of n vertices that has at most $\left(4 + \frac{5}{11}\right) n$ edges.*

Proof: According to Lemma 22 there is a nonempty subset V_C in the intersection graph of every orthogonal circle arrangement that is incident to at most $4n + 5$ edges, where $n = |V_C|$. Moreover, if $n \leq 11$, then the subset is incident to at most $4n$ edges.

If $n \leq 11$, then V_C is incident to at most $4n < \left(4 + \frac{5}{11}\right) n$ edges.

If $n > 11$, then V_C is incident to at most $4n + 5 = \left(4 + \frac{5}{n}\right) n \leq \left(4 + \frac{5}{11}\right) n$ edges. □

Theorem 2 *The intersection graph of an arrangement of n orthogonal circles has at most $\left(4 + \frac{5}{11}\right) n$ edges.*

Proof:

Assume there exist arrangements with n orthogonal circles, whose intersection graphs have more than $\left(4 + \frac{5}{11}\right) n$ edges. Consider a smallest such arrangement $\mathcal{A}$ in terms of numbers of circles and its intersection graph $G(\mathcal{A}) = (V, E)$. By Lemma 23 there exists a subset $S \subset V$ of n' vertices that is incident to at most $\left(4 + \frac{5}{11}\right) n'$ edges. We take out S and all incident edges. The new graph has $(n - n')$ vertices and more than $\left(4 + \frac{5}{11}\right) (n - n')$ edges. This contradicts the assumption that $\mathcal{A}$ is minimal. □

We conclude this section with a short discussion whether our results might carry over to (nested) acute circle arrangements. In contrast to the nonnested case, our results cannot be easily generalized for the nested setting with acute intersections. One of the main reason for this is that both the K_4 and the C_4 have a representation by touching circles (since they are planar). Hence the application of Lemma 5 is ruled out and the proof strategy that leads to Theorem 2 cannot be reused. Moreover, if we consider arbitrary small but nonzero intersection angles between circles, then we can take any triangulation, embed its incidences as a circle packing and enlarge the radii just slightly. By this, we generate an acute circle arrangement of n circles with up to $3n - 6$ pairs of circles that properly intersect in an arbitrary small angle. This shows that the bound of Theorem 2 cannot be achieved in the acute case, even if we disallow touching circles. It would be interesting to know in greater detail how the (lower bound for) the intersection angles effects the maximum edge density in the corresponding intersection graph. We leave this as a direction for future research.

4 Lower bounds

In this section we discuss lower bound constructions. Our ideas are based on the arrangement $\mathcal{B}_{x,a}$, parametrized by two integers $a \geq 5$ and $x \geq 1$, which is constructed as follows. We start with arranging a circles with the same radius in such a way that their centers lie on a circle and two neighboring circles intersect. We call these circles the *satellite circles*. We add another circle (called *hub circle*) to this arrangement such that it intersects every satellite circle orthogonally. We name this arrangement a *wheel of circles*. An arrangement $\mathcal{B}_{x,a}$ is then constructed by "nesting" x wheels of circles with a satellite circles each inside each other such that each satellite circle of one

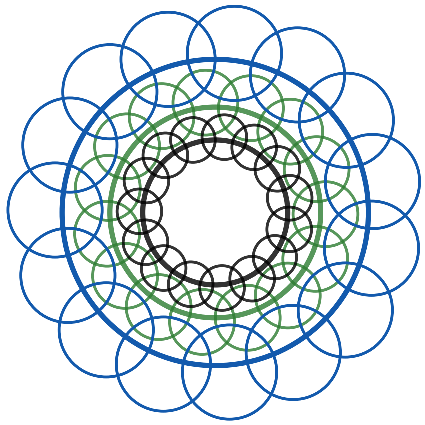

Figure 11: The arrangement $\mathcal{B}_{3,15}$. Hub circles are drawn with thick, satellite circles with thin lines. Corresponding satellite and hub circles have the same color.

wheel intersects two satellite circles of the next wheel and two satellite circles of the previous wheel. We postpone the technical details of this construction (including the proof that the arrangement is orthogonal) to the end of this section.

Lemma 24 *The intersection graph of $\mathcal{B}_{x,a}$ has $x \cdot (a+1)$ vertices and exactly $4xa - 2a$ edges.*

Proof: The arrangement consists of x wheel of circles, each having a satellite circles and one hub circles. Thus, the intersection graph has $x \cdot (a+1)$ vertices. Every vertex corresponding to a hub circle has clearly degree a. Further, every vertex corresponding to a satellite circle has degree 7, except those corresponding to a satellite circle on the inner or outermost wheel of circles, which have degree 5. So the sum of the vertex degrees is $\sum_{v \in V(G_{x,a})} \deg(v) = ax + 7a(x-2) + 5a \cdot 2 = 8xa - 4a$. This number equals twice the number of edges, and therefore the intersection graph has $4xa - 2a$ edges. □

Lemma 25 *For every n there is an arrangement of orthogonal circles, whose intersection graph has n vertices and at least $4n - O(\sqrt{n})$ edges.*

Proof: We set $x = \lfloor \sqrt{n} \rfloor$ and $a = \lceil \sqrt{n} \rceil - 2$. Note that for any positive real number t we have $\lfloor t \rfloor (\lceil t \rceil - 1) < t^2 < (\lfloor t \rfloor + 1)\lceil t \rceil$. Hence, for our choice of parameters the arrangement $\mathcal{B}_{x,a}$ has by Lemma 24

$$x \cdot (a+1) = \lfloor \sqrt{n} \rfloor \left(\lceil \sqrt{n} \rceil - 1 \right) < n$$

vertices. We introduce additional independent circles so that $B_{x,a}$ has exactly n vertices. Also by Lemma 24 we get that this arrangement has at least

$$\begin{aligned} 4xa - 2a &= 4(\lfloor \sqrt{n} \rfloor)(\lceil \sqrt{n} \rceil - 2) - 2\lceil \sqrt{n} \rceil + 4 \\ &= 4(\lfloor \sqrt{n} \rfloor + 1 - 1)\lceil \sqrt{n} \rceil - 8\lfloor \sqrt{n} \rfloor - 2\lceil \sqrt{n} \rceil + 4 \\ &= 4(\lfloor \sqrt{n} \rfloor + 1)\lceil \sqrt{n} \rceil - O(\sqrt{n}) \\ &> 4n - O(\sqrt{n}) \end{aligned}$$

many edges and the statement of the lemma follows. Note that for very small n our construction is degenerate, which is however covered by the big-O error term. $\square$

We now give a lower bound for nonnested orthogonal circles.

Lemma 26 *For every $n \geq 6$ for which $n \bmod 5 = 1$ the arrangement $\mathcal{B}_{((n-1)/5),5}$ with only the innermost hub circle is nonnested and its n-vertex intersection graph has exactly $3n - 8$ edges.*

Proof: Let G be the intersection graph of $\mathcal{B}_{((n-1)/5),5}$ in which we have deleted all but the innermost hub circle and let m its number of edges. The arrangement is nonnested by construction. Every vertex in G has degree 6, except the vertices that correspond to the inner most satellite circles, the only hub circle and the vertices adjacent to the outer face of G. The 6 vertices corresponding to inner most satellite circles and the one hub circle have degree 5. The 5 vertices on the outer face of G have degree 4. This gives us $2m = \sum_{v \in V(G)} \deg(v) = 6n - 6 - 5 \cdot 2 = 6n - 16$. Thus, the intersection graph has n vertices and $3n - 8$ edges. $\square$

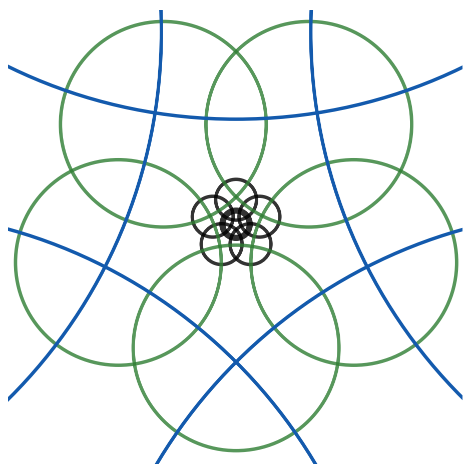

Figure 12: Detail of the arrangement used to prove the lower bound in the nonnested case in Lemma 26.

Chaplick et al. [4] investigated the maximal number of triangular cells in an orthogonal circle arrangement. They proved an upper bound of $4n$ and gave a lower bound of $2n$ triangular cells, which they later improved to $3n - 3$. This bound can be improved by taking the arrangement $\mathcal{B}_{x,a}$ and place a small (orthogonal) circle around every intersection point. This implies the following lemma.

Lemma 27 *For infinitely many values of k there is an arrangement of $n = 36k$ orthogonal circles with at least $\left(3 + \frac{5}{9}\right) n - O\left(\sqrt{n}\right)$ triangular cells.*

Proof: We construct the arrangement $\mathcal{A}$ by taking the arrangement $\mathcal{B}_{x,a}$ described above and drawing a small circle over every intersection point, such that the small circle only intersects the two circles corresponding to the intersection point. That this is always possible can be seen by applying a Möbius transformation that maps the two intersecting circles to straight lines, that intersect at right angles. Note that such a transformation is conformal and thus maintains the angles. We can now draw a circle around their intersection point that does not intersect any other circles. Reversing the inversion gives us the small circle. Each of those new circles induces four triangular cells.

According to Lemma 24 the intersection graph of $\mathcal{B}_{x,a}$ has $4xa - 2a$ edges. Thus, $\mathcal{B}_{x,a}$ has $8xa - 4a$ intersection points. The arrangement $\mathcal{A}$ has therefore $36k = x \cdot (a+1) + 8xa - 4a$ circles and at least $4 \cdot (8xa - 4a)$ triangular cells.

We set $x = 4\left(\lceil\sqrt{k}\rceil - 1\right)$ and $a = \lfloor\sqrt{k}\rfloor$. Note that for any positive real number t we have $\lfloor t\rfloor(\lceil t\rceil - 1) < t^2 < (\lfloor t\rfloor + 1)\lceil t\rceil$. Hence, for our choice of parameters the arrangement A has at most

$$\begin{aligned}
9xa - 4a + x &= 9 \cdot 4\left(\lceil\sqrt{k}\rceil - 1\right) \cdot \lfloor\sqrt{k}\rfloor - 4 \cdot \lfloor\sqrt{k}\rfloor + 4\left(\lceil\sqrt{k}\rceil - 1\right) \\
&= 36\left(\lceil\sqrt{k}\rceil - 1\right)\lfloor\sqrt{k}\rfloor - 4\left(\lfloor\sqrt{k}\rfloor - \lceil\sqrt{k}\rceil + 1\right) \\
&\leq 36\left(\lceil\sqrt{k}\rceil - 1\right)\lfloor\sqrt{k}\rfloor < 36k
\end{aligned}$$

circles. We fill $\mathcal{A}$ up such that it has exactly $36k$ circles. So this arrangement has at least

$$\begin{aligned}
32xa - 16a &= 32 \cdot 4\left(\lceil\sqrt{k}\rceil - 1\right) \cdot \lfloor\sqrt{k}\rfloor - 16\lfloor\sqrt{k}\rfloor \\
&= 128\left(\lceil\sqrt{k}\rceil - 1\right)\lfloor\sqrt{k}\rfloor - 16\lfloor\sqrt{k}\rfloor \\
&= 128\lceil\sqrt{k}\rceil\left(\lfloor\sqrt{k}\rfloor + 1 - 1\right) - 128\lfloor\sqrt{k}\rfloor - 16\lfloor\sqrt{k}\rfloor \\
&= 128\lceil\sqrt{k}\rceil\left(\lfloor\sqrt{k}\rfloor + 1\right) - 128\lceil\sqrt{k}\rceil - 128\lfloor\sqrt{k}\rfloor - 16\lfloor\sqrt{k}\rfloor \\
&> 128k - O(\sqrt{k})
\end{aligned}$$

many triangular cells. We remind the reader that the arrangement $\mathcal{A}$ has $n = 36k$ circles thus the arrangement has at least

$$128\left(\frac{n}{36}\right) - O\left(\sqrt{\frac{n}{36}}\right) = \frac{32}{9}n - O(\sqrt{n})$$

triangular cells. □

One can derive a lower bound example for acute nonnested circle arrangements out of the former construction by shrinking the circles in such a way that they still intersect. This will reduce the intersection angles. Thus, we obtain an acute circle arrangement with n circles whose intersection graph has exactly $3n - 8$ edges.

In the remainder of this section we review the formal construction of the arrangement $\mathcal{B}_{x,a}$ including a proof of its orthogonality.

We define the following constants and points in the plane for later reference.

- $\alpha = \frac{\sqrt{\cos\left(\frac{2\pi}{a}\right) - \cos\left(\frac{4\pi}{a}\right) + \sqrt{2}\cos\left(\frac{\pi}{a}\right)}}{\sqrt{2}\cos\left(\frac{2\pi}{a}\right)}$
- $d_i = \frac{\alpha^{i-1}}{\sqrt{2} \cdot \sin\left(\frac{\pi}{a}\right)}$
- $s_i = \alpha^{i-1}$

- $C_{i,j} = \begin{cases} \left(d_i, \frac{2\pi \cdot j}{a}\right), & \text{if } i \text{ is even} \\ \left(d_i, \frac{2\pi \cdot j}{a} + \frac{\pi}{a}\right), & \text{else} \end{cases}$
- $h_i = s_i \cdot \sqrt{\frac{1}{2 \cdot \sin\left(\frac{\pi}{a}\right)^2} - 1}$

The construction is guided by a set of x concentric circles O_i centered at the origin of radii d_i for $1 \leq i \leq x$; we refer to these circles as *orbits*. These circles will not be part of the final arrangement. On each of the orbit circles O_i we place the centers of a circles $S_{i,j}$ for $1 \leq j \leq a$ with radius s_i such that the centers are equidistant; we refer to these as *satellite circles*. The center of a circle $S_{i,j}$ for $1 \leq i \leq x$, $1 \leq j \leq a$ is $C_{i,j}$ As last step we add x concentric circles H_i with center on the origin and radii h_i for $1 \leq i \leq x$; we refer to these circles as *hub circles*. As before we call a hub circle together with the satellite circles it intersects a *wheel of circles*. The satellite circles and the hub circles form the arrangement $\mathcal{B}_{x,a}$.

We prove now that this arrangement is an orthogonal circle arrangement.

Lemma 28 *The arrangement $\mathcal{B}_{x,a}$ is an orthogonal circle arrangement.*

Proof: We prove the lemma by checking the following conditions.

1. **In every wheel of circles the satellite circles intersect each other orthogonally.** Let us first concentrate on the innermost wheel of circles. Consider the equilateral a-gon formed by the centers of the satellite circles $C_{1,1}, ..., C_{1,a}$. An edge length of the a-gon is (here between $C_{1,j}$ and $C_{1,j+1)}$

$$\begin{aligned} D &= \sqrt{2 \cdot d_1^2 - 2 \cdot d_1^2 \cdot \cos\left(\frac{2\pi \cdot (j+1)}{a} - \frac{2\pi \cdot j}{a}\right)} \\ &= \sqrt{2 \cdot \left(\frac{1}{\sqrt{2} \cdot \sin\left(\frac{\pi}{a}\right)}\right)^2 \left(1 - \cos\left(\frac{2\pi}{a}\right)\right)} \\ &= \sqrt{\frac{1 - \cos\left(\frac{2\pi}{a}\right)}{\sin\left(\frac{\pi}{a}\right)^2}} \\ &= \sqrt{2}. \end{aligned}$$

   Since $s_1 = 1$, the radii of the satellite circles is 1. Thus every two neighboring circles intersect orthogonally. Note that all other wheel of circles are just scaled up copies of the innermost wheel of circles. Hence, also here the intersections are orthogonally.

2. **In a wheel of circles the hub circle intersects every satellite circle orthogonally.** Due to rotational symmetry it suffices to prove the statement for one satellite circle. We consider now the intersection point of a satellite circle with a hub circle. The radius of the hub circle is h_i and the radius of the satellite circles is s_i. It holds that

$$h_i^2 + s_i^2 = \left(s_i \cdot \sqrt{\frac{1}{2 \cdot \sin\left(\frac{\pi}{a}\right)^2} - 1}\right)^2 + s_i^2 = \left(\frac{\alpha^{i-1}}{\sqrt{2} \cdot \sin\left(\frac{\pi}{a}\right)}\right)^2 = d_i^2.$$

   Since d_i is the distance between between the center of the satellite circle and the center of the hub circle (origin) we get by Observation 1 that both circles intersect orthogonally.

3. **The satellite circles of two adjacent wheels of circles intersect orthogonally.** Due to symmetry it suffices to check the condition between the circles with center $C_{1,1}$ and $C_{2,1}$. It holds that

$$\|C_{1,1}, C_{2,1}\|_2^2 = {d_2}^2 - 2\cos\left(\frac{\pi}{a}\right) d_1 \cdot d_2 + {d_1}^2 = \frac{\alpha^2 - 2\cos\left(\frac{\pi}{a}\right)\alpha + 1}{2\sin\left(\frac{\pi}{a}\right)^2}.$$

If we can show that $\|C_{1,1}, C_{2,1}\|_2^2 = {s_1}^2 + {s_2}^2 = 1 + \alpha^2$, then by Observation 1 the orthogonality is proven. With computer algebra software we have checked that indeed

$$\alpha^2 + 1 = \frac{\alpha^2 - 2\cos\left(\frac{\pi}{a}\right)\alpha + 1}{2\sin\left(\frac{\pi}{a}\right)^2}.$$

4. **No hub circle intersects any circle that is not part of its wheel of circles.** Let H be a hub circle and let W_1 and W_2 be two neighboring circles from the same wheel of circles. Further let C be a circle from an adjacent wheel of circles that intersect W_1 and W_2. We invert all four circles in a circle with center on one of the intersection points of W_1 and W_2. This turns W_1 and W_2 into straight lines that intersect orthogonally and the circles H and C into circles that intersect these straight lines orthogonally . The inversions of H and C are nested and thus do not intersect. Thus, also H and C do not intersect.

Since these four conditions hold, all intersection points in the arrangement belong to circles that intersect orthogonally. □

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