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Maximal origami flip graphs of flat-foldable vertices: properties and algorithms

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Abstract. *Flat origami* studies straight line, planar graphs $C = (V, E)$ drawn on a region $R \subset \mathbb{R}^2$ that can act as crease patterns to map, or fold, R into $\mathbb{R}^2$ in a way that is continuous and a piecewise isometry exactly on the faces of C. Associated with such crease pattern graphs are *valid mountain-valley (MV) assignments* $\mu : E \to \{-1, 1\}$, indicating which creases can be mountains (convex) or valleys (concave) to allow R to physically fold flat without self-intersecting. In this paper, we initiate the first study of how valid MV assignments of single-vertex crease patterns are related to one another via *face-flips*, a concept that emerged from applications of origami in engineering and physics, where flipping a face F means switching the MV parity of all creases of C that border F. Specifically, we study the origami flip graph $\mathrm{OFG}(C)$, whose vertices are all valid MV assignments of C and edges connect assignments that differ by only one face flip. We prove that, for the single-vertex crease pattern A_{2n} whose $2n$ sector angles around the vertex are all equal, $\mathrm{OFG}(A_{2n})$ contains as subgraphs all other origami flip graphs of degree-$2n$ flat origami vertex crease patterns. We also prove that $\mathrm{OFG}(A_{2n})$ is connected and has diameter n by providing two $O(n^2)$ algorithms to traverse between vertices in the graph, and we enumerate the vertices, edges, and degree sequence of $\mathrm{OFG}(A_{2n})$. We conclude with open questions on the surprising complexity found in origami flip graphs of this type.

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1 Introduction

When folding a piece of paper into a flat object, the creases that are made will be straight lines. This describes *flat origami* [7], which we formally model with a pair (C, P), called the *crease pattern*, where P is a closed region of the plane (our paper), and the set of creases $C = (V(C), E(C))$ is a plane graph on P with straight line segments for the edges. (When the exact shape of the paper P is not important, we will refer to the crease pattern merely as C.) If there exists a mapping $f : P \to \mathbb{R}^2$ that is continuous, non-differentiable along the edges of C, and an isometry on each face of C, then we say that (C, P) is *locally flat-foldable*. Also, folded creases come in two types when viewing a fixed side of the paper: *mountain* creases, which fold away in a convex manner, and *valley* creases, which are concave. We model this with a function $\mu : E(C) \to \{-1, 1\}$, called a *mountain-valley (MV) assignment* for the crease pattern C, where 1 (respectively -1) represents a mountain (respectively valley) crease. A MV assignment μ is called *valid* if μ can be used to fold C into a flat object without the paper intersecting itself.

Capturing mathematically how paper self-intersection works, and how it can be avoided, to achieve *global* flat-foldability is difficult. In fact, determining if a crease pattern (C, P) is globally flat-foldable is NP-hard [1, 4], even if a specific MV assignment is already given. In the special case where the crease pattern has only one vertex in the interior of P, called a *single-vertex crease pattern* or a *flat vertex fold*, determining if a MV assignment is valid is not straight-forward [7, 13] but can be determined in linear time [5]. Indeed, there are many open questions that remain about enumerating valid MV assignments [8] and understanding their structure [12], even for very simple crease patterns.

Flat-foldability and valid MV assignments have been of interest to scientists in the study of origami mechanics and their application in constructing metamaterials [14], even in the case of single-vertex crease patterns [11]. A concept that has emerged from such applications is that of a *face flip*, where a valid MV assignment μ is altered by switching only the mountains and valleys that surround a chosen face F, denoting the new MV assignment (which may or may not be valid) by μ_F. Face flips were first introduced in the literature by VanderWerf [16] and have been utilized in applications ranging from tuning metamaterials [14] to analyzing the statistical mechanics of origami tilings [3].

In this paper, we explore the relationships between valid MV assignments of a given crease pattern C using a tool called the *origami flip graph*, denoted $\mathrm{OFG}(C)$, which is a graph whose vertices are all valid MV assignments of C and where two vertices μ and ν are connected by an edge if and only if there exists a face F of C such that flipping F changes μ to ν (and vice-versa, i.e., $\nu = \mu_F$). We may view face flips and the paths they generate on $\mathrm{OFG}(C)$ as a reconfiguration setting for some problems in computational origami.

Origami flip graphs were introduced in [2], but only in the context of origami tessellations (crease patterns that form a tiling of the plane). In the present work, we focus on flat-foldable crease patterns that have a single vertex in the paper's interior, called *flat vertex folds*, with the additional requirement that the sector angles between the creases are all equal. We denote such a crease pattern by A_{2n} where $2n$ is the degree of the vertex. In Section 2, we provide background on flat origami and show that $\mathrm{OFG}(A_{2n})$ serves as a maximal "superset" graph for flat vertex folds–if C is any other flat vertex fold of degree $2n$, then $\mathrm{OFG}(C)$ is a subgraph of $\mathrm{OFG}(A_{2n})$. In Sections 3 and 4, we prove that $\mathrm{OFG}(A_{2n})$ is connected using two different algorithms for finding paths in this graph, one of which further proves that the diameter of $\mathrm{OFG}(A_{2n})$ is n. In Section 5 we describe an algorithm for computing the size of $\mathrm{OFG}(A_{2n})$, generating a sequence that was not originally in the Online Encyclopedia of Integer Sequences, and find a formula for this as well as

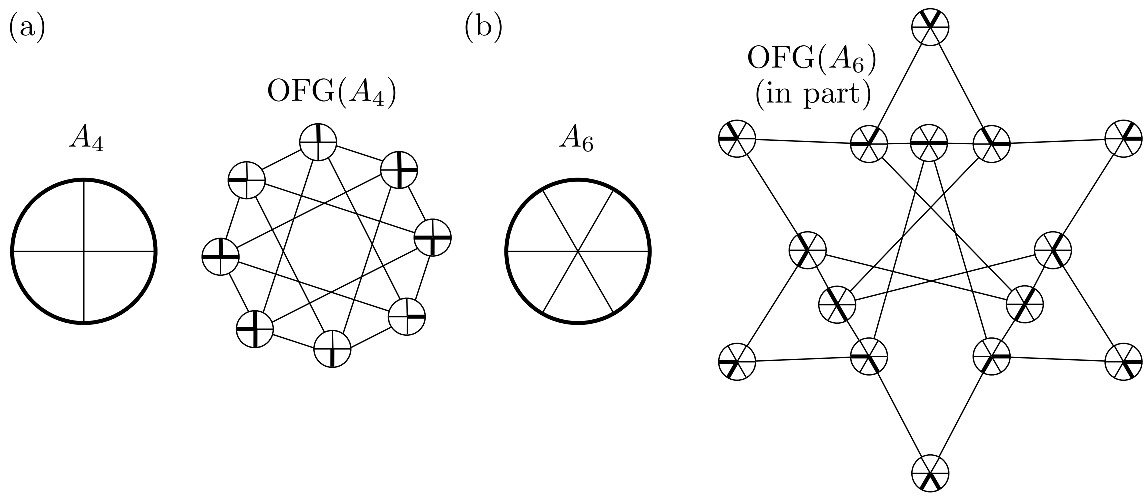

Figure 1: The crease patterns (a) A_4 and (b) A_6 along with their origami flip graphs (for OFG(A_6) only the vertices with $M-V=-2$ are shown). Each vertex is labeled with the valid MV assignment to which it corresponds (bold/non-bold means mountain/valley, respectively).

for the degree sequence of OFG(A_{2n}). We conclude with open questions and a discussion of future work.

2 Background and maximality of OFG(A_{2n})

Let (A_{2n}, P) denote the crease pattern that contains only one vertex v in the interior of P, where v has degree $2n$ and the angles between consecutive creases around v are all equal (to π/n). We normally let P be a disc with v at the center. Let $e_1, \ldots, e_{2n}$ denote the creases in A_{2n} and α_i denote the face between e_i and e_{i+1} (with the indices taken cyclically, so α_{2n} is between e_{2n} and e_1).

A basic result from flat origami theory is *Maekawa's Theorem*, which states that, if v is a vertex in a flat-foldable crease pattern with valid MV assignment μ, then the difference between the number of mountain and valley creases at v under μ must be two, often denoted by $M-V=\pm 2$ [7]. However, in the case of the crease pattern A_{2n} Maekawa's Theorem is stronger:

Theorem 1 (Maekawa for A_{2n}) *A MV assignment μ on A_{2n} is valid if and only if*

$$\sum_{i=1}^{2n} \mu(e_i) = \pm 2.$$

Theorem 1 is proved in [2, 9], but a summary of the sufficient direction is: Find a pair of consecutive creases e_i, e_{i+1} in A_{2n} with $\mu(e_i) \neq \mu(e_{i+1})$ and fold them (making a "crimp") to turn the paper into a cone, on which we now have the crease pattern $A_{2(n-1)}$ and a MV assignment that still has $\sum \mu(e) = \pm 2$. We have two options when forming this cone; it could be convex or concave, which will be preserved as we repeat this process until there are only two creases left,

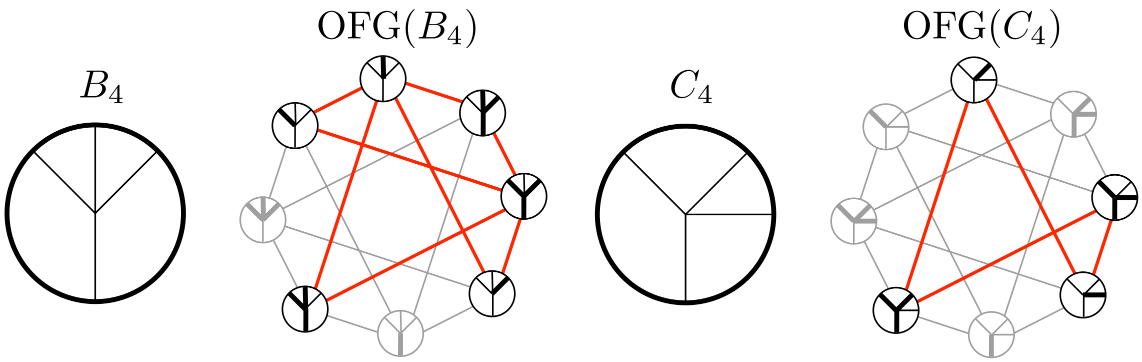

Figure 2: Other flat vertex folds B_4 and C_4 of degree 4 and their origami flip graphs, viewed as subgraphs of $\mathrm{OFG}(A_4)$.

which must both be mountains (if the cone is convex) or both be valleys (if concave). This gives us a flat folding of the original vertex A_{2n}.

Examples of the origami flip graphs $\mathrm{OFG}(A_4)$ and $\mathrm{OFG}(A_6)$ are shown in Figure 1, although in the latter case only half of the vertices (those whose MV assignment satisfies $\sum \mu(e) = -2$) are shown. The vertices in these graphs are labeled with their corresponding valid MV assignment, where bold creases are mountains and non-bold means valley, a convention we will use throughout this paper. In [10], it is proved that $\mathrm{OFG}(C)$ is bipartite whenever C is a flat-foldable, single-vertex crease pattern, although we will not be making particular use of that here.

Theorem 1 tells us that any MV assignment of A_{2n} that satisfies $M - V = \pm 2$ will be valid. Therefore, there are $2\binom{2n}{n-1}$ vertices in $\mathrm{OFG}(A_{2n})$.

We will now show that the origami flip graph of A_{2n} has maximal size over all origami flip graphs of flat vertex folds of degree $2n$, and further that such origami flip graphs are all subgraphs of $\mathrm{OFG}(A_{2n})$. The idea is that when all the sector angles of a flat vertex fold are equal, the only requirement for a MV assignment to be valid is that it satisfies Maekawa's Theorem. If, on the other hand, the sector angles are not all equal, then other restrictions will apply. For example, if a flat-foldable, single-vertex crease pattern C has consecutive sector angles $\alpha_{i-1}, \alpha_i, \alpha_{i+1}$ where α_i is strictly smaller than both α_{i-1} and α_{i+1}, then the creases e_i and e_{i+1} bordering α_i must have different MV parity, so $\mu(e_i) \neq \mu(e_{i+1})$ must hold in any valid MV assignment μ of C. (This is known as the Big-Little-Big Lemma; see [7].) This implies that the faces α_{i-1} and α_{i+1} can *never* be individually flipped under a valid MV assignment μ, since doing so would make $\mu(e_i) = \mu(e_{i+1})$. Other restrictions on when faces in a single-vertex crease pattern can be flipped are detailed in [10], but since A_{2n} does not have such restrictions, its origami flip graph will have the most edges possible. Examples of this when $2n = 4$ are shown in Figure 2. We formalize and slightly expand this in the following Theorem.

Theorem 2 *Let C be a flat-foldable, single-vertex crease pattern of degree $2n$ that is not A_{2n}. Then $\mathrm{OFG}(C)$ is isomorphic to at least $2n$ distinct, labeled subgraphs of $\mathrm{OFG}(A_{2n})$.*

Proof: Suppose we have an arbitrary flat vertex fold C with degree $2n$, creases $c_1, \ldots, c_{2n}$, and angles β_i. Let ν be a valid MV assignment of C. Then, ν also represents a valid MV assignment for A_{2n}. Specifically, if $e_1, \ldots, e_{2n}$ are the creases in A_{2n} and we define μ by $\mu(e_i) = \nu(c_i)$, then μ will be a valid MV assignment on A_{2n} by Theorem 1 (since ν must satisfy Maekawa's Theorem).

Thus, we have a mapping f between all MV assignments ν of C and some MV assignments μ of A_{2n} ($f(\nu) = \mu$). If $\{\nu, \nu_{\beta_i}\}$ is an edge of OFG(C) (where β_i is flipped to make this edge), then $\{f(\nu), f(\nu_{\beta_i})\}$ is an edge of OFG(A_{2n}). That is, $\nu(c_i) = -\nu_{\beta_i}(c_i)$ and $\nu(c_{i+1}) = -\nu_{\beta_i}(c_{i+1})$ and $\nu(c) = \nu_{\beta_i}(c)$ for all other creases c of C. The same relationship holds true between $f(\nu)$ and $f(\nu_{\beta_i})$. That is, flipping the corresponding face α_i (between e_i and e_{i+1} in A_{2n}) in $f(\nu)$ will result in $f(\nu_{\beta_i})$. This can be written as $f(\nu)_{\alpha_i} = f(\nu_{\beta_i})$, which implies that $\{f(\nu), f(\nu_{\beta_i})\}$ is an edge of OFG(A_{2n}). Therefore, OFG(C) is isomorphic to a subgraph of OFG(A_{2n}).

Furthermore, our labeling of the creases e_i in A_{2n} was arbitrary, and by the rotational symmetry of A_{2n} we had $2n$ different ways we could have done this, resulting in at least $2n$ distinctly-labeled copies of OFG(C) (since $C \neq A_{2n}$) that may be found in OFG(A_{2n}). □

If μ is a valid MV assignment for a crease pattern C, then we say that a face F of C is *flippable under* μ if μ_F is also a valid MV assignment for C. In what follows, we will make extensive use of the following Lemma.

Lemma 1 *Let μ be a valid MV assignment of A_{2n}. Then a face α_k is not flippable under μ if and only if $\mu(e_k) = \mu(e_{k+1}) \neq (\sum \mu(e_i))/2$.*

Proof: By Theorem 1, μ_{α_k} will be an invalid MV assignment if and only if $\sum \mu_{\alpha_k}(e_i) \neq \pm 2$. This will only happen if $\mu(e_k) = \mu(e_{k+1})$ (i.e., the creases that border α_k have the same MV assignment under μ) and this value, $\mu(e_k)$, is different from the majority of the creases in μ. For example, if $\sum \mu(e_i) = 2$ and $\mu(e_k) = \mu(e_{k+1}) = -1$, then $\sum \mu_{\alpha_k}(e_i) = 6$, meaning that μ_{α_k} violates Theorem 1 and thus is invalid. All other possibilities for $\mu(e_k)$ and $\mu(e_{k+1})$ preserve the MV summation invariant and thus allow α_k to be flippable under μ. □

We will utilize the following definition in Section 4: given two MV assignments μ and ν of A_{2n}, let $S(\mu, \nu)$ denote the set of creases $e_1, \ldots, e_{2n}$ with $\mu(e_i) \neq \nu(e_i)$. This set is useful because it provides us with a quantity that is face-flip invariant.

Lemma 2 *The parity of $|S(\mu, \nu)|$ (the size of $S(\mu, \nu)$) is invariant under face flips. That is, if μ and ν are valid MV assignment of A_{2n}, then the $|S(\mu, \nu)|$ will have the same even/odd parity as $|S(\mu_{\alpha_i}, \nu)|$ for any face α_i of A_{2n}.*

Proof: Suppose we flip a face α_i of A_{2n} under μ. Then we are changing the MV assignments of two creases. This will change the size of $S(\mu, \nu)$ by either 0 (if exactly one of e_i and e_{i+1} is different between μ and ν) or 2 (if e_i and e_{i+1} are both the same or both different between μ and ν). Therefore, the parity of $|S(\mu, \nu)|$ is invariant under face flips. □

3 Connectivity of OFG(A_{2n})

In this section, we present an algorithm for face-flipping between any two valid MV assignments μ and ν of A_{2n}. This will prove that OFG(A_{2n}) is connected. In contrast, if C is an arbitrary flat vertex fold, then OFG(C) is not always connected. We invite the reader to verify that the degree-6 flat vertex fold with sector angles $(45°, 15°, 60°, 85°, 75°, 80°)$ has two disconnected 4-cycles for its origami flip graph. (Determining the connectivity of OFG(C) for general flat vertex folds C is quite convoluted and beyond the scope of this paper; see [10] for details.) Note that we will provide a different algorithm in Section 4 that also proves the connectedness of OFG(A_{2n}), but the one given here, which we will call the FEA-Shwoop algorithm, is useful for other reasons, such as in an elegant proof of Lemma 3 below and as a warm-up for what follows.

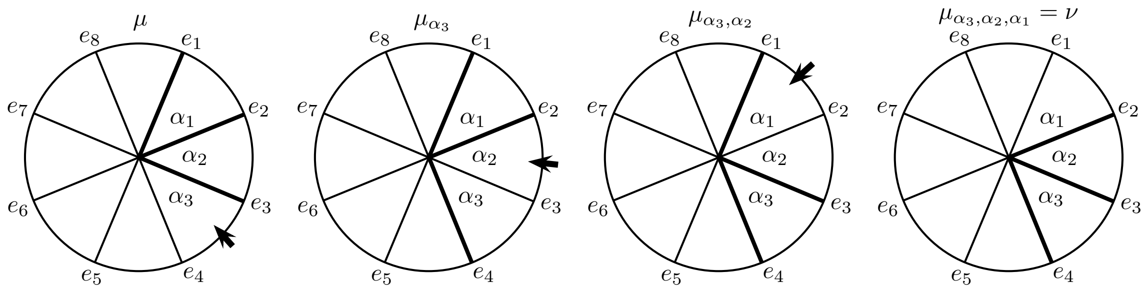

Figure 3: An example of a shwoop sequence of face flips that converts μ to ν.

In the algorithm, we start with crease e_1. If $\mu(e_1) = \nu(e_1)$, then we move on to crease e_2. If $\mu(e_1) \neq \nu(e_1)$, then we would like to flip the face α_1, since $\mu_{\alpha_1}(e_1) = \nu(e_1)$, and then continue the algorithm on crease e_2 comparing μ_{α_1} with ν.

However, if e_1 and e_2 have the same MV assignment under μ, then α_1 might not be flippable under μ if it falls under Lemma 1; such a μ and α_1 are shown in Figure 3. Since α_1 is not flippable, we move to α_2 and check to see if it is flippable under μ. If so, then we flip it, and doing so will make α_1 flippable (since it will no longer satisfy Lemma 1). Then we have $\mu_{\alpha_2,\alpha_1}(e_1) = \nu(e_1)$, and we may proceed with crease e_2 comparing μ_{α_2,α_1} and ν. If α_2 is not flippable, then we try to flip the next face, α_3. Eventually we will find some face α_i that can be flipped (otherwise μ would be all mountain or all valley creases and violate Maekawa's Theorem) and then we can flip the sequence of faces $\alpha_i, \alpha_{i-1}, \alpha_{i-2}, \ldots, \alpha_1$. We call this sequence of flipping faces in order to make $\mu_{\alpha_i,\ldots,\alpha_1}(e_1) = \nu(e_1)$ a *shwoop*, and an example of such a shwoop is shown in Figure 3.

Thus, our algorithm is to start by comparing $\mu(e_1)$ and $\nu(e_1)$, flipping α_1 or performing a shwoop to make them agree on e_1 if needed, and then moving on to e_2, and so on. We call this algorithm FEA-SHWOOP(A_{2n}, μ, ν), and pseudocode for it is shown in Algorithm 1. (FEA stands for Flipping Equal Angles.)

Theorem 3 *The* FEA-SHWOOP(A_{2n}, μ, ν) *algorithm inputs two valid MV assignments for* A_{2n} *and outputs a sequence of faces that, when flipped in order, will provide a sequence of valid MV assignments that start with* μ *and end with* ν*.*

Proof: As previously described, the algorithm uses single face-flips and shwoops to generate a sequence of valid MV assignments of A_{2n} that, starting with μ, make the MV parity of creases $e_1, e_2, e_3, \ldots$, in order, agree with that of ν. We need to prove that (1) finding faces to perform a shwoop is always possible and (2) that when the algorithm terminates after $i = 2n - 1$, the resulting MV assignment will be ν.

Suppose we are at stage $i = k$ in the algorithm where we have valid MV assignments μ_F and ν for A_{2n} where F is the sequence of faces we've already flipped, $\mu_F(e_i) = \nu(e_i)$ for $i = 1, \ldots, k-1$, and $\mu_F(e_k) \neq \nu(e_k)$.

Then, if α_k is flippable under μ_F, we flip it so that $\mu_{F\cup\{\alpha_k\}}(e_k) = \nu(e_k)$ and move on to $i = k + 1$.

If we cannot flip α_k under μ_F, then that means, for example, that μ_F is majority-mountain and e_k and e_{k+1} are both valleys under μ_F. So we look to see if we can flip face α_{k+1} under μ_F. If that's not possible, then we look at face α_{k+2}, and continue in search of a flippable face α_{k+j} under μ_F for some j.

Algorithm 1: The FEA (Flipping-Equal-Angles) Shwoop algorithm.

FEA-Shwoop(A_{2n}, μ, ν)

```
Let S = {}, η = μ
for i = 1 to 2n − 1 do
    Let m = 0
    if η(e_i) ≠ ν(e_i) then
        if face α_i of A_2n is flippable under η then
            Replace η with η_{α_i}
            Append α_i to S
        else
            while face α_i of A_2n is not flippable under η do
                Let m = m + 1, i = i + 1
            Replace η with η_{α_i}
            Append α_i to S
            for j = m to 1 do
                Let i = i − 1 // This is the shwoop.
                Replace η with η_{α_i}
                Append α_i to S
Output S
```

Suppose we get all the way to α_{2n-1} without finding a flippable face under μ_F. That means that $\mu_F = \nu$ on creases $e_1, \ldots, e_{k-1}$ and, assuming μ_F is majority-mountain, that $\mu_F = -1$ (valley creases) on $e_k, \ldots, e_{2n}$ (since face α_{2n-1} borders the creases e_{2n-1} and e_{2n}). Since μ_F is a valid MV assignment, this means that ν must also be all valleys on $e_k, \ldots, e_{2n}$, for if it were anything else, then ν would have fewer valley creases than μ_F and thus violate Maekawa's Theorem. This contradicts our assumption that μ_F and ν disagreed on crease e_k, and so our supposition is false.

Thus, we will find a face α_{k+j} that is flippable under μ_F where $k + j$ is no more than $2n - 1$. We then flip α_{k+j} and perform a shwoop to be able to make a new MV assignment $\mu_{F \cup \{\alpha_{k+j}, \alpha_{k+j-1}, \ldots, \alpha_k\}}$ that will agree with ν on crease e_k.

We now examine how the algorithm terminates. The last face that could be flipped in this algorithm is α_{2n-1}. Let μ_x be the last MV assignment produced up to this point (so, after step $i = 2n - 2$ in the algorithm). For step $i = 2n - 1$, suppose that $\mu_x(e_{2n-1}) = \nu(e_{2n-1})$. This means μ_x and ν agree on all the creases $e_1, \ldots, e_{2n-1}$, which implies that they must also agree on e_{2n}, for otherwise one of μ_x or ν would not satisfy Maekawa's Theorem despite both being valid. Thus $\mu_x = \nu$ and the algorithm completes successfully.

Similarly, if $\mu_x(e_{2n-1}) \neq \nu(e_{2n-1})$ then we must also have that $\mu_x(e_{2n}) \neq \nu(e_{2n})$. Then flipping face α_{2n-1} will make $\mu_{x,\alpha_{2n-1}} = \nu$, and this face-flip must be possible because ν is a valid MV assignment for A_{2n}. Thus the algorithm completes successfully in this case as well. □

Corollary 1 *The flip graph* OFG(A_{2n}) *is connected.*

The FEA-Shwoop(A_{2n}, μ, ν) algorithm uses a nested loop, each of which are $O(n)$, and therefore the running time of the whole algorithm is $O(n^2)$. Note that this algorithm does *not* necessarily produce the shortest path in OFG(A_{2n}) between μ and ν, since the choice of the first crease e_1 to consider might not be the most efficient, for instance.

4 Diameter of $\mathrm{OFG}(A_{2n})$

There is a different algorithm that we could use to flip between any two valid MV assignments μ and ν of A_{2n}, one that also proves that the diameter of $\mathrm{OFG}(A_{2n})$ is n. We call this algorithm FEA-HALVES(A_{2n}, μ, ν).

Recall from Section 2 that, if μ and ν are two valid MV assignments of A_{2n}, then $S(\mu, \nu)$ is the set of creases e_i with $\mu(e_i) \neq \nu(e_i)$.

Lemma 3 *If μ and ν are two valid MV assignments of A_{2n}, then $|S(\mu, \nu)|$ is even.*

Proof: This can be proven using only Maekawa's Theorem by considering the sums $\sum \mu(e_i)$ and $\sum \nu(e_i)$ mod 4. That is, these two sums are equivalent mod 4, and if we remove the creases e_i with $\mu(e_i) = \nu(e_i)$ these sums will still be equivalent mod 4, meaning $\sum_{e \in S(\mu,\nu)} \mu(e) \equiv \sum_{e \in S(\mu,\nu)} \nu(e)$ mod 4. But we also have $\sum_{e \in S(\mu,\nu)} \mu(e) = -\sum_{e \in S(\mu,\nu)} \nu(e)$, which implies the result.

A more elegant proof, however, uses Lemma 2 and Corollary 1. That is, $|S(\mu, \mu)| = 0$, and if we already know that $\mathrm{OFG}(A_{2n})$ is connected, then since the parity of $|S(\mu, \nu)|$ is invariant under face flips, all values of $|S(\mu, \nu)|$ must be even. □

In lieu of Lemma 3, let us denote $S(\mu, \nu) = \{e_{i_1}, \dots, e_{i_{2k}}\}$, where $i_1 < \dots < i_{2k}$. For $i < j$ let us denote $B(e_i, e_j) = \{\alpha_i, \alpha_{i+1}, \dots, \alpha_{j-1}\}$, which is the set of all faces of A_{2n} between creases e_i and e_j. Define

$$B(\mu, \nu) = B(e_{i_1}, e_{i_2}) \cup B(e_{i_3}, e_{i_4}) \cup \dots \cup B(e_{i_{2k-1}}, e_{i_{2k}}) = \bigcup_{j=1}^{k} B(e_{i_{2j-1}}, e_{i_{2j}}).$$

That is, $B(\mu, \nu)$ is a set of faces of A_{2n} between pairs of creases that have different MV parity under μ and ν. The complement set $\overline{B(\mu, \nu)}$ among the faces in A_{2n} will be a similar set, and thus the sets $B(\mu, \nu)$ and $\overline{B(\mu, \nu)}$ divide the faces of A_{2n} into (probably not equal-sized) "halves."

We may now summarize the FEA-HALVES algorithm: Find a flippable face $\alpha_{j_1} \in B(\mu, \nu)$. We then claim that $B(\mu_{\alpha_{j_1}}, \nu)$ will equal $B(\mu, \nu) \setminus \{\alpha_{j_1}\}$, and so we repeat, finding a flippable face $\alpha_{j_2} \in B(\mu_{\alpha_{j_1}}, \nu)$, and so on, producing an ordering $\alpha_{j_1}, \alpha_{j_2}, \dots$ of all the faces in $B(\mu, \nu)$ that, when flipped in order, will convert μ to ν.

Lemma 4 *For valid MV assignments μ and ν of A_{2n}, there exists a flippable face $\alpha_j \in B(\mu, \nu)$ such that $B(\mu_{\alpha_j}, \nu) = B(\mu, \nu) \setminus \{\alpha_j\}$.*

Proof: For a set C of creases, let $M(C, \mu) =$ the number of mountain creases in C under a MV assignment μ and similarly define $V(C, \mu)$ for valleys. Assume without loss of generality that μ is majority-valley on A_{2n}. Then, if $\overline{S(\mu, \nu)}$ denotes the compliment of $S(\mu, \nu)$ among the creases in A_{2n}, we have, by Maekawa's Theorem applied to μ,

$$M(S(\mu, \nu), \mu) + M(\overline{S(\mu, \nu)}, \mu) - V(S(\mu, \nu), \mu) - V(\overline{S(\mu, \nu)}, \mu) = -2 \tag{1}$$

Also, since ν is valid we have

$$M(S(\mu, \nu), \nu) + M(\overline{S(\mu, \nu)}, \nu) - V(S(\mu, \nu), \nu) - V(\overline{S(\mu, \nu)}, \nu) = \pm 2. \tag{2}$$

However, by definition of $S(\mu, \nu)$, we know that $M(S(\mu, \nu), \mu) = V(S(\mu, \nu), \nu)$ and $V(S(\mu, \nu), \mu) = M(S(\mu, \nu), \nu)$. Also, $M(\overline{S(\mu, \nu)}, \mu) = M(\overline{S(\mu, \nu)}, \nu)$ and $V(\overline{S(\mu, \nu)}, \mu) = V(\overline{S(\mu, \nu)}, \nu)$. Thus Equation (2) becomes

$$V(S(\mu, \nu), \mu) + M(\overline{S(\mu, \nu)}, \mu) - M(S(\mu, \nu), \mu) - V(\overline{S(\mu, \nu)}, \mu) = \pm 2. \tag{3}$$

Case 1: ν is majority-valley. Then Equation (3) will have -2 on its right-hand side, and subtracting this from Equation (1) gives

$$M(S(\mu,\nu),\mu) - V(S(\mu,\nu),\mu) = 0. \tag{4}$$

Suppose that there is a face $\alpha_j \in B(\mu,\nu)$ whose creases e_j and e_{j+1} have different MV parity under μ, and therefore α_j is a flippable face under μ. If e_j or e_{j+1} are in $S(\mu,\nu)$, then $S(\mu_{\alpha_j},\nu)$ will be either $S(\mu,\nu)\setminus\{e_j,e_{j+1}\}$ or $(S(\mu,\nu)\setminus\{e_j\})\cup\{e_{j+1}\}$ or $(S(\mu,\nu)\setminus\{e_{j+1}\})\cup\{e_j\}$, and so $B(\mu_{\alpha_j},\nu)$ will equal $B(\mu,\nu)$ but with the face α_j removed, as desired. If neither e_j nor e_{j+1} are in $S(\mu,\nu)$, then they will be elements of $S(\mu_{\alpha_j},\nu)$, but, by definition of $B(\mu,\nu)$, this means that α_j will not be an element of $B(\mu_{\alpha_j},\nu)$, and so $B(\mu_{\alpha_j},\nu) = B(\mu,\nu)\setminus\{\alpha_j\}$.

On the other hand, if there is no face $\alpha_j \in B(\mu,\nu)$ with $\mu(e_j) \neq \mu(e_{j+1})$, then by Equation (4) there must be a face $\alpha_j \in B(\mu,\nu)$ with $\mu(e_j) = \mu(e_{j+1}) = -1$ (both valleys, since they can't all be mountains), in which case, α_j is flippable by Lemma 1. Then, α_j must be in some component $B(e_{i_k}, e_{i_{k+1}})$ in $B(\mu,\nu)$ that has only valley creases under μ, whereby $B(\mu_{\alpha_k},\nu) = B(\mu,\nu)\setminus\{\alpha_j\}$.

Case 2: ν is majority-mountain. Then, Equation (3) will have $+2$ on its right-hand-side, and subtracting from Equation (1) gives

$$M(S(\mu,\nu),\mu) - V(S(\mu,\nu),\mu) = -2.$$

This means that we have at least two valley creases in $S(\mu,\nu)$ under μ. Let $e_{i_m} \in S(\mu,\nu)$ be a valley crease under μ, and let α_j be the face in $B(\mu,\nu)$ that borders e_{i_m}. We claim that α_j is a flippable face under μ: If the other crease bordering α_j is also a valley under μ, then since μ is majority-valley, μ_{α_j} will be majority-mountain and still satisfy Maekawa's Theorem. If the other crease bordering α_j is a mountain under μ, then μ_{α_j} is still majority-valley and satisfies Maekawa because μ did. In both cases we have that μ_{α_j} is a valid MV assignment. Then $B(\mu_{\alpha_j},\nu)$ will have one fewer face than $B(\mu,\nu)$, the missing face being α_j, and the Lemma is proved. □

Algorithm 2: The FEA (Flipping-Equal-Angles) Halves algorithm.

FEA-Halves(A_{2n},μ,ν)

Let $L = B(\mu,\nu)$, $S = \{\}$, $\eta = \mu$
if Length(L) $> n$ **then**
  Let L = the complement of $B(\mu,\nu)$ in A_{2n}
Let m =Length(L)
for $i = 1$ **to** m **do**
  Find $\alpha \in L$ such that L is flippable under η
  Append α to S
  Replace η with η_α and L with $L \setminus \{\alpha\}$
Output S

Therefore, the FEA-Halves algorithm (see Algorithm 2) will input two valid MV assignments, μ and ν for A_{2n} and compute the set of faces $B(\mu,\nu) = \bigcup_{j=1}^{k} B(e_{i_{2j-1}}, e_{i_{2j}})$ as well as the complement set of faces (in A_{2n}) $\overline{B(\mu,\nu)} = B(e_{i_{2k}}, e_{i_1}) \cup \bigcup_{j=1}^{k-1} B(e_{i_{2j}}, e_{i_{2j+1}})$. Since these form a disjoint union of all the faces in A_{2n}, one of $B(\mu,\nu)$ and $\overline{B(\mu,\nu)}$ will have size $\leq n$. Pick that set, say it's $B(\mu,\nu)$, and apply Lemma 4 repeatedly to generate a sequence of at most n face flips that will transform μ into ν. This proves most of the following theorem.

Theorem 4 *The flip graph* $\mathrm{OFG}(A_{2n})$ *is connected and has diameter* n.

Proof: To see that the diameter of $\mathrm{OFG}(A_{2n})$ equals n, let μ be any valid MV assignment of A_{2n} and consider the complement MV assignment $\overline{\mu}$ which is μ but with all the mountains and valleys reversed. To transform μ to $\overline{\mu}$, every crease needs to be flipped, and (since there are $2n$ creases and each face flip switches two creases) doing this this requires at least n face flips. Since every crease is flipped, this gives us a lower bound on the required number of face flips between nodes of $\mathrm{OFG}(A_{2n})$. The FEA-Halves algorithm guarantees at most n face flips to flip from μ to $\overline{\mu}$, so the diameter of $\mathrm{OFG}(A_{2n})$ is n. Examples that require n face flips can be readily found (for example, let μ have $\mu(e_i) = 1$ for $i = 1, 3, 5, \ldots, 2n-3$ and $\mu(e_i) = -1$ for $i = 2, 4, 6, \ldots, 2n$ and $i = 2n-1$). □

Like FEA-Shwoop, the FEA-Halves(A_{2n}, μ, ν) algorithm runs in $O(n^2)$ time since each pass through $B(\mu, \nu)$ to search for a flippable face takes $O(n)$ steps and $\mathrm{Length}(B(\mu, \nu))$ is $O(n)$.

5 Counting edges of $\mathrm{OFG}(A_{2n})$

We saw in Section 2 that $\mathrm{OFG}(A_{2n})$ has $2\binom{2n}{n-1}$ vertices. Counting the edges in $\mathrm{OFG}(A_{2n})$ is not as straight-forward. We first perform this enumeration using the method shown in Algorithm 3, which we call Edge-Count(n). This takes each valid MV assignment μ of A_{2n} and uses Lemma 1 to compute the degree of μ in $\mathrm{OFG}(A_{2n})$: each vertex μ will have degree $2n$ unless there are non-flippable faces (bordered by "VV" if μ is majority-mountain or by "MM" if μ is majority-valley) which must then be subtracted from $2n$. We then take the sum of the vertex degrees and divide by two to find the number of edges.

Algorithm 3: Counting the edges in $\mathrm{OFG}(A_{2n})$.

Edge-Count(n)

```
Let L = 2(2n choose n−1), MVAssigns = all L valid MV assignments of A_2n
for i = 1 to L do
    if MVAssigns[i] is majority mountain then
        Let Deg[i] = 2n−(number of "VV" in MVAssigns[i])
    if MVAssigns[i] is majority valley then
        Let Deg[i] = 2n−(number of "MM" in MVAssigns[i])
Output (∑ Deg[i])/2
```

The output of Edge-Count(n) for $n = 1$ to $n = 13$ is

$$2, 16, 84, 400, 1820, 8064, 35112, 151008, 643500, 2722720, 11454872, 47969376, 200107544.$$

When we first encountered it, this sequence Edge-Count(n) did not appear in the Online Encyclopedia of Integer Sequences. It has now been added by one of the authors and is sequence A352626 [15].

The running time of this algorithm is super-exponential in n, since it needs to calculate and check every valid MV assignment of A_{2n}. Fortunately, we can do better.

Theorem 5 *The number of edges in* $\text{OFG}(A_{2n})$ *is* $\frac{(n+1)(3n-2)}{2n-1}\binom{2n}{n-1}$.

Note that the formula in Theorem 5 matches the output of Edge-Count(n). We prove this formula using a probabalistic approach.

Proof: In any uniformly chosen at random MV assignment of A_{2n}, some faces will be flippable and some will not be flippable. Define random variables G = the number of flippable faces in a MV assignment of A_{2n} (or "good" faces) and B = the number of unflippable faces (or "bad" faces). Also let $\mathbf{1}_{\alpha_i}$ denote the indicator random variable for α_i being a bad face. That is, $B = \mathbf{1}_{\alpha_1} + \mathbf{1}_{\alpha_2} + \cdots + \mathbf{1}_{\alpha_{2n}}$. Then linearity of expectation gives us

$$\mathbb{E}[G] = \mathbb{E}[2n - B] = \mathbb{E}[2n - \Sigma \mathbf{1}_{\alpha_i}] = 2n - \sum \mathbb{E}[\mathbf{1}_{\alpha_i}] = 2n - 2nP[\alpha_i \text{ is bad}].$$

Now, by Lemma 1, $P[\alpha_i \text{ is bad}] = P[e_i \text{ and } e_{i+1} \text{ are minority}] =$

$$P[((e_i \text{ and } e_{i+1} \text{ are V}) \text{ and } (\mu \text{ is majority M})) \text{ or } ((e_i \text{ and } e_{i+1} \text{ are M}) \text{ and } (\mu \text{ is majority V}))]$$

$$= 2P[e_i \text{ and } e_{i+1} \text{ are V and } \mu \text{ is majority M}] =$$

$$2P[\mu \text{ is majority M}]P[e_i \text{ and } e_{i+1} \text{ are V} | \mu \text{ is majority M}]$$

$$= 2(1/2)P[e_i \text{ and } e_{i+1} \text{ are V} | \mu \text{ is majority M}]$$

$$= \frac{\binom{2n-2}{n-3}}{\binom{2n}{n-1}} = \frac{(n-1)(n-2)}{2n(2n-1)}.$$

Therefore $\mathbb{E}[G] = 2n(1 - \frac{(n-1)(n-2)}{2n(2n-1)})$. However, since MV assignments μ of A_{2n} form the vertices of $\text{OFG}(A_{2n})$, we have that $\mathbb{E}[G] = \mathbb{E}[\deg(\mu) \text{ in OFG}(A_{2n})]$, and

$$\mathbb{E}[\deg(\mu)] = \frac{1}{|V|}\sum_{\mu \in V} \deg(\mu) = \frac{2|E|}{|V|}$$

where V and E are the vertices and edges in $\text{OFG}(A_{2n})$, respectively. Thus we have

$$|E| = \frac{|V|}{2} 2n \left(1 - \frac{(n-1)(n-2)}{2n(2n-1)}\right) = \frac{(n+1)(3n-2)}{2n-1}\binom{2n}{n-1},$$

as desired. □

The Edge-Count(n) algorithm can be used to generate the degree sequence for $\text{OFG}(A_{2n})$. Let $f_k(2n)$ denote the number of vertices of degree k in $\text{OFG}(A_{2n})$, so that Edge-Count(n) = $(1/2)\sum_{k=n-2}^{2n} k f_k(2n)$. The values for $f_k(2n)$ for $2 \le n \le 6$ and the possible degrees k are shown in Table 1, and studying these led to the following formula.

Theorem 6 *The number of vertices of degree* k *in* $\text{OFG}(A_{2n})$ *is*

$$f_k(2n) = \frac{4n}{n+1}\binom{n+1}{k-n-1}\binom{n-2}{k-n-2},$$

for $n+2 \le k \le 2n$.

$2n\backslash k$	4	5	6	7	8	9	10	11	12
4	8								
6		12	18						
8			16	64	32				
10				20	150	200	50		
12					24	288	720	480	72

Table 1: Values for $f_k(2n)$ generated by running EDGE-COUNT(n).

We provide a combinatorial proof of this result developed by Jonah Ostroff.

Proof: We will enumerate the number of valid MV assignments μ of A_{2n} that are majority-mountain with b non-flippable faces; such a vertex in OFG(A_{2n}) will have degree $k = 2n - b$, and this enumeration will equal $f_k(2n)/2$. In this situation we will have $n+1$ mountains, $n-1$ valleys, and by Lemma 1 there should be exactly b pairs of consecutive creases around A_{2n} that are "VV" under μ. That means there are exactly $n-b-1$ valley creases that are *not* followed by a valley (say, going clockwise around the vertex). Therefore we are counting the number of ways to arrange mountains and valleys so that there are exactly $n-b-1$ runs of consecutive valleys.

We can construct such MV assignments as follows:

- First we place the $n+1$ mountains around a circle and mark one of them as the "start" point.
- Then we place boxes in $n-b-1$ of the $n+1$ spaces between the mountains.
- Place one valley in each of the $n-b-1$ boxes. Then place the remaining b valleys in any of the $n-b-1$ boxes; by a "stars and bars" counting argument there are $\binom{n-b-1+b-1}{b} = \binom{n-2}{b}$ ways to do this.

This gives us a MV assignment with the required conditions, but we've only counted ones that "start" with a mountain crease. Call the set of these MV assignments A. We rotate each member of A around the A_{2n} crease pattern to get a bigger set of MV assignments, B, with $2n|A|$ elements. We claim that each MV assignment we are looking for (valid, majority-mountain with exactly b non-flippable faces) appears in B exactly $(n+1)$ times. To see this, let μ meet our required conditions and suppose μ has no rotational symmetry (meaning that each rotation of μ in A_{2n} is a MV assignment distinct from μ). Then a rotated version of μ will appear in A exactly $(n+1)$ times, since there are $(n+1)$ mountains in μ. These rotations of μ in A will result in exactly $(n+1)$ copies of μ appearing in B.

On the other hand, suppose μ has rotational symmetry, say $r^j(\mu) = \mu$ for some j that divides $2n$, where $r(\mu)$ is μ rotated by π/n in A_{2n}. Let $2n = qj$. Then a rotated copy of μ will appear in A exactly $(n+1)/q$ times (that is, it would be $(n+1)$ times, one for each mountain in μ, but every qth one is a duplicate because of the rotational symmetry). Each of these rotated copies of μ are rotated a full $2n$ times in B, each giving us q copies of μ in B. That's a total of $q(n+1)/q = (n+1)$ copies of μ in B.

Therefore, the number of valid MV assignments of A_{2n} that are majority-mountain and have exactly b non-flippable faces is

$$\frac{2n}{n+1}\binom{n+1}{n-b-1}\binom{n-2}{b}.$$

To include the majority-valley cases, we multiply by two. Substituting $b = 2n - k$ and simplifying gives the desired result. □

6 Conclusion

We have seen how the origami flip graph of A_{2n} has the largest size among the flip graphs of flat vertex folds of degree $2n$, that it contains all such origami flip graphs as subgraphs, and that it is a connected graph with diameter n. Furthermore, the algorithms used to prove these facts could be useful in further studies of origami flip graphs. For example, the FEA-SHWOOP algorithm has the interesting property that it provides a way to flip between any two valid MV assignments of A_{2n} without flipping the face α_{2n}. Since the labeling of the faces was arbitrary, this means that we can always avoid flipping a chosen face and still traverse the origami flip graph. This feature is used in the forthcoming paper [10] to help classify when $\mathrm{OFG}(C)$ will be connected for arbitrary flat vertex folds C. Indeed, [10] also explores when the FEA-SHWOOP algorithm can be used in other situations besides the crease pattern A_{2n}.

Despite A_{2n} being, in a sense, the most simple case of all degree-$2n$ flat vertex folds, as it requires only Maekawa's Theorem to determine if a MV assignment will be valid, its origami flip graph nonetheless exhibits surprising complexity. Further details on the structure of $\mathrm{OFG}(A_{2n})$ remains unexplored. For instance, Theorem 2 does not tell the whole story about the number of copies of $\mathrm{OFG}(C)$ that can be found in $\mathrm{OFG}(A_{2n})$.

Open Problem 1 *If C is a flat vertex fold of degree $2n$, how do we determine the exact number of distinct labeled subgraphs of $\mathrm{OFG}(A_{2n})$ that are isomorphic to $\mathrm{OFG}(C)$?*

There are various computational origami reconfiguration problems other than those already considered in this paper that have yet to be explored. Perhaps the next most basic one to consider is the following:

Open Problem 2 *What is the computational complexity of finding a shortest path between two vertices μ and ν in $\mathrm{OFG}(A_{2n})$?*

Readers familiar with closed meanders might suspect a connection between them and flat foldings of A_{2n}. A *closed meander* of order n is a closed curve in $\mathbb{R}^2$ that crosses a given directed line $2n$ times [6]. The cross-section of any flat folding of A_{2n} can be viewed as a closed meander if we draw a directed line through (and perpendicular to) the cross-section. However, the number of homeomorphically-distinct closed meanders of order n, denoted M_n, is not equal to the number of vertices in $\mathrm{OFG}(A_{2n})$, since different layer orders of the paper give us different meanders but not different MV assignments, and the latter is all we care about in origami flip graphs. Still, there could be a way to extend our work in this paper to include layer orderings.

Open Problem 3 *Is there a way to create a flip graph for the space of closed meanders of order n where the number of vertices is M_n? Or, equivalently, an origami flip graph for A_{2n} that also considers different paper layer orders as distinct foldings?*

Also, we have seen that determining the degree sequence of $\mathrm{OFG}(A_{2n})$ involves the different ways to separate the valleys (assuming we're majority-mountain) into runs of consecutive valleys. In other words, if we have $n-1$ valleys we are considering the integer partitions of $n-1$. The different integer partitions affect $f_k(2n)$ for different k, so their influence is lost in Theorem 6. However, perhaps another connection is possible.

Open Problem 4 *Can we clarify the role that integer partitions of $n-1$ play in* $\mathrm{OFG}(A_{2n})$*?*

This is further evidence, also seen in [7], that the single-vertex case of flat origami continues to possess more combinatorial richness than one would originally expect.

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