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The Complexity of Drawing a Graph in a Polygonal Region

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Abstract. We prove that the following problem is complete for the existential theory of the reals: Given a planar graph and a polygonal region, with some vertices of the graph assigned to points on the boundary of the region, place the remaining vertices to create a planar straight-line drawing of the graph inside the region. This establishes a wider context for the NP-hardness result by Patrignani on extending partial planar graph drawings. Our result is one of the first showing that a problem of drawing planar graphs with straight-line edges is hard for the existential theory of the reals. The complexity of the problem is open in the case of a simply connected region.

We also show that, even for integer input coordinates, it is possible that drawing a graph in a polygonal region requires some vertices to be placed at irrational coordinates. By contrast, the coordinates are known to have bounded bit complexity for the special case of a convex region, or for drawing a path in any polygonal region.

In addition, we prove a Mnëv-type universality result—loosely speaking, that the solution spaces of instances of our graph drawing problem are equivalent, in a topological and algebraic sense, to bounded algebraic varieties.

1 Introduction

There are many examples of structural results on graphs leading to beautiful and efficient geometric representations. Two highlights are: Tutte's polynomial-time algorithm [59] that draws any 3-connected planar graph with convex faces inside any fixed convex drawing of its outer face; and

Some preliminary results appeared in the Proceedings of the 26th International Symposium on Graph Drawing and Network Visualization (GD 2018) [37]. A short video explaining the paper's content can be found here: https://youtu.be/JbmWLnY1hGk.

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Schnyder's tree realizer result [55] that provides a straight-line drawing of any n-vertex planar graph on an $n \times n$ grid.

On the other hand, there are geometric representations that are intractable, either in terms of the required coordinates or in terms of computational complexity. As an example of the former, a representation of a planar graph as touching disks (Koebe's theorem) is not always possible with rational numbers, nor even with roots of low-degree polynomials [11]. As an example of the latter, Patrignani considered a generalization of Tutte's theorem and proved that it is NP-hard to decide whether a graph has a straight-line planar drawing when part of the drawing is fixed [47]. He was unable to show that the problem lies in NP because of coordinate issues.

This, and many other geometric problems are not known to lie in NP, but lie in a larger class, $\exists\mathbb{R}$, defined by existentially quantified real (rather than Boolean) variables. Showing that a geometric representation problem is complete for $\exists\mathbb{R}$ is a stronger intractability result, often implying lower bounds on coordinate sizes. For example, Kang and Müller [33] showed that deciding if a graph can be represented as intersecting disks is $\exists\mathbb{R}$-complete. The relaxation from touching disks (Koebe's theorem) to intersecting disks implies that disk centers and radii can be restricted to integers, but Kang and Müller show that an exponential number of bits may be required (see also [39]).

In this paper we prove that an extension of Tutte's problem is $\exists\mathbb{R}$-complete. We call it the "Graph in Polygon" problem. See Figure 1. The input is a graph G and a polygonal region R that may be unbounded and may have holes, i.e., is not necessarily simply connected. Some vertices of G are assigned fixed positions on the boundary of R. The question is whether G has a straight-line planar drawing inside R respecting the fixed vertices. Boundary points of R may be used in the drawing. A straight-line planar drawing (see Figure 2(a,b)) means that vertices are represented as distinct points, and every edge is represented as a straight-line segment joining its endpoints, and no two of the closed line segments intersect except at a common vertex. (In particular, no vertex point may lie inside an edge segment, and no two segments may cross.)

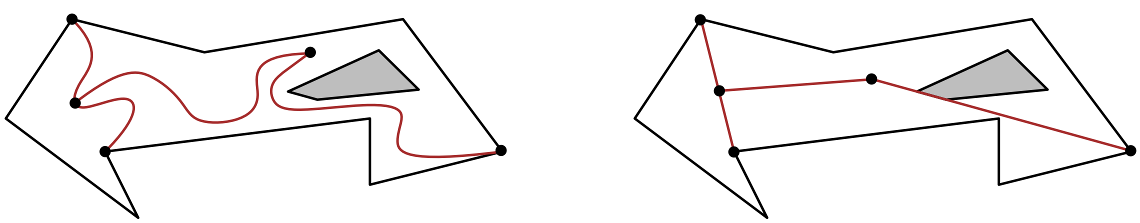

Figure 1: The Graph in Polygon problem. Left: a polygonal region with one hole and a graph to be embedded inside the region. The three vertices on the boundary are fixed; the others are free. Right: a straight-line embedding of the graph in the region. Note that we allow an edge of the drawing (in red) to include points of the region boundary.

Furthermore, we give a simple instance of Graph in Polygon with integer coordinates where a vertex of G may need irrational coordinates in any solution, thus defeating the naive approach to placing the problem in NP.

More generally, we will prove a "universality" theorem stating that the solution spaces of instances of Graph in Polygon are complicated, in particular, as complicated as bounded algebraic varieties. This "Mnëv-type" result is described in more detail below.

The Graph in Polygon problem is a very natural one that arises in many practical applications including dynamic and incremental graph drawing. Questions of the coordinates (or grid size) required for straight-line planar drawings of graphs are fundamental and well-studied [61]. It

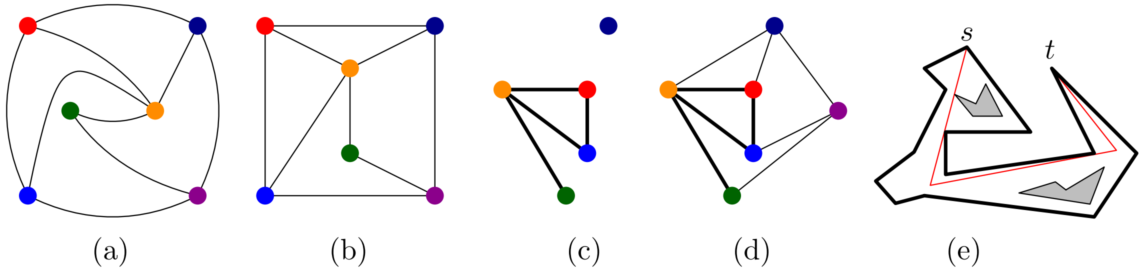

Figure 2: (a) A planar graph G. (b) A straight-line drawing of G. (c) A partial drawing Γ of G. (d) Extension of Γ to a straight-line drawing of G. (e) A minimum-link s-t path in a polygonal region.

is surprising that a problem as simple and natural as Graph in Polygon is so hard and requires irrational coordinates.

We state our results in Section 1.1 below, but first we give some background on the existential theory of the reals, Mnëv's universality result, and relevant graph drawing results. In particular, we explain that our problem is a generalization of the problem of extending a partial drawing of a planar graph to a straight-line drawing of the whole graph, called Partial Drawing Extensibility. See Figure 2(c,d).

Existential Theory of the Reals. In the study of geometric problems, the complexity class $\exists\mathbb{R}$ plays a crucial role, connecting purely geometric problems and real algebra. Whereas NP is defined in terms of existentially quantified Boolean variables, $\exists\mathbb{R}$ deals with existentially quantified real variables.

Consider a first-order formula over the reals that contains only existential quantifiers,

$$\exists x_1, x_2, \ldots, x_n : \Phi(x_1, x_2, \ldots, x_n),$$

where $x_1, x_2, \ldots, x_n$ are real-valued variables and Φ is a boolean quantifier-free formula with polynomial equalities and inequalities as atoms. We assume that the polynomials have integer coefficients. The Existential Theory of the Reals (ETR) problem takes such a formula as an input and asks whether it is satisfiable. The complexity class $\exists\mathbb{R}$ consists of all problems that reduce in polynomial time to ETR. Note that ETR lies in PSPACE and is therefore decidable [17]. This gives the best known algorithms for Graph in Polygon under worst-case considerations, and due to $\exists\mathbb{R}$-completeness we cannot hope for better algorithms, unless $\exists\mathbb{R} = \text{NP}$.

Many problems in combinatorial geometry and geometric graph representation naturally lie in $\exists\mathbb{R}$ and furthermore, many have been shown to be $\exists\mathbb{R}$-complete. Some of the earlier examples are: stretchability of pseudoline arrangements [38, 43, 54]; recognition of segment intersection graphs [35], (recently proved even for bounded vertex degrees [53]); and recognition of disk intersection graphs [39].

For straight-line drawings that allow crossings, there are $\exists\mathbb{R}$-completeness proofs for: drawing graphs with specified edge lengths (even for unit lengths) [50]; drawing a graph so a specified subset of edges have at most one crossing each [51]; finding the rectilinear crossing number of a graph [15], or the local rectilinear crossing number (the number of crossings per edge) [52]; and simultaneous geometric embedding [19, 36], even for a fixed number of graphs [53].

For straight line graph drawings without crossings, there are ∃ℝ-completeness proofs for the problems of slope number and segment number [22,29,45], and for the problem of drawing "matchstick" graphs with unit length edges [1]. Recently, Bieker [14] used Theorem 2 to show that finding a straight-line non-crossing drawing with vertices on specified line segments is ∃ℝ-complete.

For surveys on ∃ℝ, see [18, 38, 49]. The recent thesis of Bieker [14] summarizes ∃ℝ-complete graph drawing problems.

The recent proof that the Art Gallery Problem is ∃ℝ-complete [4] provides the framework we follow in our proof. This framework has also been used to establish ∃ℝ-completeness of the geometric packing problem and training neural networks [6, 8].

Finally, we mention a new characterization of ∃ℝ by Erickson, van der Hoog, and Miltzow [25]. It is based on the equivalent definition of NP in terms of a witness and a verification algorithm: decision problem X belongs to NP if there is a verification algorithm A such that for any instance I of X, I is a yes-instance if and only if there is a witness $w \in \{0,1\}^*$ for which $A(w, I)$ returns true. Here the verification algorithm is required to run in polynomial time on the word RAM model of computation. For the new characterization of ∃ℝ in terms of a *real* verification algorithm the witness is allowed to contain real numbers and the verification algorithm runs in polynomial time on the real RAM model of computation.

Mnëv's Universality Theorem. The complexity class ∃ℝ is closely linked to Mnëv's universality theorem, which provides a deep mathematical connection between simple geometric problems and semi-algebraic sets [43, 48]. At a very high level, a universality theorem states that we can represent *any* object of type A as an object of type B preserving property C. Mnëv's original universality theorem included all semi-algebraic sets as objects of type A, but the notion of equivalence was weaker and much more involved. Mnëv's objects of type B were order types, also known as oriented matroids, pseudoline arrangements, or chirotopes. They capture basic combinatorial properties of points in the Euclidean plane.

Although universality results for semi-algebraic sets seemed to be specific to ∃ℝ-complete problems, such a universality result was recently proved for an NP-complete problem, specifically, the version of the Art Gallery Problem where the guards only need to see the polygon vertices [12,56]. The underlying idea is that semi-algebraic sets can be triangulated [28], and thus there are also simplicial complexes with the identical topology. This weakens the apparent link between ∃ℝ and topological universality.

Planar Graph Drawing. The field of Graph Drawing investigates ways of representing graphs geometrically [44], but we focus on the most basic representation of planar graphs, with points for vertices and straight-line segments for edges, such that segments intersect only at a common endpoint. By Fáry's theorem [26], every planar graph admits such a straight-line planar drawing.

Tutte, in his famous paper, "How to Draw a Graph" [59], gave a polynomial-time algorithm for finding a straight-line planar drawing of a graph. More generally, given a combinatorial planar embedding (a specification of the faces) and given a convex polygon drawing of the outer face of the graph, his algorithm produces a planar straight-line drawing respecting both. The method is to augment to a 3-connected planar graph and then reduce the problem to solving a linear system involving barycentric coordinates for each internal vertex. Tutte proved that the linear system has a unique solution and that the solution yields a drawing with convex faces (for a 3-connected graph). The linear system can be solved in polynomial time. For a discussion of coordinate bit complexity see Section 6.

There is a rich literature on implications and variations of Tutte's result. We concentrate on the aspects of drawing a planar graph in a constrained region, or when part of the drawing is fixed. (We leave aside, for example, the issue of drawing graphs with convex faces, which also has an extensive literature.)

Our focus will be on straight-line planar graph drawings, but it is worth mentioning that for general planar drawings (with curved edges), the problem of drawing a graph in a constrained region is equivalent to the problem of extending a partial planar drawing, and there is a polynomial-time algorithm for the decision version of the problem [9] as well as a characterization via forbidden substructures [31]. Furthermore, there is an algorithm to construct such a drawing in which each edge is represented by a path with linearly many segments [21].

For the remainder of this paper we assume straight-line planar drawings, which makes the problems harder. The problem of drawing a graph in a constrained region is formalized as Graph in Polygon, defined above, and more precisely in Subsection 1.1. The problem of finding a planar straight-line drawing of a graph after part of the drawing has been fixed, is called Partial Drawing Extensibility in the graph drawing literature, and the complexity of the problem was formulated as an open question in [16].

Graph in Polygon is more general than Partial Drawing Extensibility, as we now argue. Given an instance of Partial Drawing Extensibility for graph G with fixed subgraph H, we construct an instance of Graph in Polygon by making a point hole for each vertex of H and assigning the vertex to the point. Then an edge of H can only be drawn as a line segment joining its endpoints, so we have effectively fixed H. The region R is then the (unbounded) punctured plane. We now have an instance of Graph in Polygon of polynomial size that has a solution if and only if G has a planar straight-line drawing that extends the drawing of H.

There is no easy reduction in the other direction because Graph in Polygon allows an edge to be drawn as a segment that touches, or lies on, the boundary of the region, which prevents the natural idea of modelling boundary edges as fixed edges of the graph in the problem Partial Drawing Extensibility. However, the version of Graph in Polygon where interiors of edges must lie in the interior of the region is equivalent to Partial Drawing Extensibility.

We now summarize results on Partial Drawing Extensibility, beginning with positive results. Besides Tutte's result that a convex drawing of the outer face can always be extended, there is a similar result for a star-shaped drawing of the outer face [30], and a polynomial-time algorithm to decide the case when a convex drawing of one cycle in the graph is fixed [40]. Also Gortler et al. [27] gave an algorithm, extending Tutte's algorithm, that succeeds in some (not well-characterized) cases for a simple non-convex drawing of the outer face. In another direction, there is a characterization of (graph, outer-cycle) pairs that can be be realized for any simple drawing of the outer cycle [46].

The Partial Drawing Extensibility problem was shown to be NP-hard by Patrignani [47]. This implies that Graph in Polygon is NP-hard. However, there are two natural questions about partial drawing extensibility that remain open: (a) does the problem belong to the class NP (discussed in detail by Patrignani [47]), and (b) does the problem remain NP-hard when a combinatorial embedding of the graph is given and must be respected in the drawing. Our results shed light on these questions for the more general Graph in Polygon problem: the problem cannot be shown to lie in NP by means of giving the vertex coordinates, and the problem is $\exists\mathbb{R}$-complete even when a combinatorial embedding of the graph is given.

In addition to Tutte's result, there is another special case of Graph in Polygon that is well-solved, namely when the graph is just a path with its two endpoints s and t fixed on the boundary of the region. See Figure 2(e). In this special case, the problem is equivalent to the *Minimum Link*

Path problem, since a path of k edges can be drawn inside the region if and only if the minimum link distance between s and t is less than or equal to k. Minimum link paths in a polygonal region can be found in polynomial time [42], and in linear time for a simple polygon [57]. The complexity of the coordinates is well-understood (see Section 6).

1.1 Our Contributions

Our problem is defined as follows.

> **Graph in Polygon**
> **Input:** A planar graph G and a polygonal region R with some vertices of G assigned to fixed positions on the boundary of R.
> **Question:** Does G admit a planar straight-line drawing inside R respecting the fixed vertices?

The graph may be given abstractly, or via a *combinatorial embedding* which specifies the cyclic order of edges around each vertex, thus determining the faces of the embedding. When a combinatorial embedding is specified, the final drawing must respect that embedding. Note that we allow points on the boundary of R to be used in the drawing of G. In particular, an edge of G may be drawn as a segment that touches, or lies on, the boundary of R. See Figure 1. Note that we still require the drawing of G to be "simple" in the conventional sense that no two edge segments may intersect except at a common endpoint.

Our first result, proved in Section 2, is that solutions to Graph in Polygon may involve irrational points. This will in fact follow from the proof of our main hardness result, but it is worth seeing a simple example.

Theorem 1 *There is an instance of* Graph in Polygon *with all coordinates given by integers, in which some vertices need irrational coordinates.*

Note that the theorem does not rule out membership of the problem in NP, since it may be possible to demonstrate that a graph can be drawn in a region without giving explicit vertex coordinates. We prove Theorem 1 by adopting an example from Abrahamsen, Adamaszek and Miltzow [3] that proves a similar irrationality result for the Art Gallery problem. Further discussion of the bit complexity of vertex coordinates for special cases of the Graph in Polygon problem can be found in Section 6.

Our main result, proved in Section 4, is:

Theorem 2 Graph in Polygon *is* $\exists\mathbb{R}$*-complete.*

Our proof holds whether the graph is given abstractly or via a combinatorial embedding.

In the remainder of this section, we first discuss the proof techniques used for the main theorem, and then discuss an extension of the main theorem that provides our Mnëv-type universality result.

Proof technique. The idea for proving our main theorem is to use a reduction from a problem called ETR-INV which was introduced and proved $\exists\mathbb{R}$-complete by Abrahamsen, Adamaszek and Miltzow [4]. Note that this work was generalized to a host of continuous constraint satisfaction problems [41].

Definition 3 (ETR-INV) *In the problem ETR-INV, we are given a set of real variables* $\{x_1, \ldots, x_n\}$*, and a set of equations of the form*

$$x = 1, \quad x + y = z, \quad x \cdot y = 1,$$

for $x, y, z \in \{x_1, \ldots, x_n\}$*. The goal is to decide whether the system of equations has a solution when each variable is restricted to the range* $[1/2, 2]$.

Reducing from ETR-INV, rather than from ETR, has several crucial advantages. First, we can assume that all variables are in the range $[1/2, 2]$. Second, we do not have to implement a gadget that simulates multiplication, but only inversion, i.e., $x \cdot y = 1$. Note that $x \cdot x = 1$ together with the range constraints, enforces $x = 1$, and thus the first type of constraint is not actually needed. For our purpose of reducing to Graph in Polygon, we will find it useful to further modify ETR-INV to avoid equality and to ensure planarity of the variable-constraint incidence graph, as follows:

Definition 4 (Planar-ETR-INV*) *In the problem Planar-ETR-INV*, we are given a set of real variables* $\{x_1, \ldots, x_n\}$*, and a set of equations and inequalities of the form*

$$x = 1,\ x + y \leq z,\ x + y \geq z,\ x \cdot y \leq 1,\ x \cdot y \geq 1,\ \text{ for } x, y, z \in \{x_1, \ldots, x_n\}.$$

Furthermore, we require planarity of the variable-constraint incidence graph*, which is the bipartite graph that has a vertex for every variable and every constraint and an edge when a variable appears in a constraint. The goal is to decide whether the system of equations has a solution when each variable is restricted to lie in* $[1/2, 4]$.

As a technical contribution, we prove the following.

Theorem 5 *Planar-ETR-INV* is* $\exists\mathbb{R}$*-complete.*

Universality. Our final contribution is a universality result about the complexity of solution spaces to instances of Graph in Polygon. As mentioned above, at a very high level, a universality theorem states that we can represent *any* object of type A as an object of type B preserving property C. In particular, for our universality result, the objects of type A are bounded algebraic varieties (described below) and the objects of type B are solution spaces of instances of the Graph in Polygon problem. Finally, the properties we want to preserve are algebraic and topological properties.

We begin by defining the notion of equivalence that we will use.

Let F be a finite set of polynomials $F = \{f_1, \ldots, f_k\} \subset \mathbb{Z}[x_1, \ldots, x_n]$. (Recall that $\mathbb{Z}[x_1, \ldots, x_n]$ denotes the polynomial ring over the variables $x_1, \ldots, x_n$ with integer coefficients.) The *solution space* of F is

$$V(F) = \{x \in \mathbb{R}^n : f(x) = 0,\ \forall f \in F\}.$$

We say $V(F)$ is *bounded*, if there is a ball B such that $V(F) \subseteq B$. In this case, $V(F)$ is called a 'bounded algebraic variety'.

For all the algorithmic problems that we consider, there will be a natural way to embed the solutions into $\mathbb{R}^n$, and for an instance I we will again denote the solution space by $V(I) \subseteq \mathbb{R}^n$.

Two sets $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ are *rationally equivalent* if there exists a homeomorphism $f : V \to W$ such that both f and f^{-1} are given by rational functions. A function $f : V \to W$ is a *homeomorphism*, if it is bijective, continuous, invertible and its inverse is continuous as well. The function f is *rational*, if it can be component-wise described as the ratio of polynomials. We denote rational equivalence by $V \simeq W$. Note that the composition of two homeomorphisms is a homeomorphism. Similarly, the composition of two rational functions is rational.

Our universality result is the following.

Theorem 6 (Universality of Graph in Polygon) *Let F be a finite set of polynomials with integer coefficients such that $V(F)$ is bounded. Then there exists an instance I of* Graph in Polygon *with all coordinates given by integers such that*

$$V(I) \simeq V(F).$$

To prove Theorem 6 we use two results: that Planar-ETR-INV* is universal; and that our reduction from Planar-ETR-INV* to Graph in Polygon preserves rational equivalence. The first result is in Section 3, and the second result and the final proof are in Section 5.

To illustrate the implications of preserving algebraic and topological properties we point out two consequences of Theorem 6.

Corollary 7 *Let $\mathbb{Q} \subseteq F_1 \subset F_2 \subset \mathbb{R}$ be two algebraic field extensions of $\mathbb{Q}$. Then there exists an instance of* Graph in Polygon *with all coordinates given by integers that has a solution with vertex coordinates in F_2, but has no solution with vertex coordinates in F_1.*

This means that we can essentially enforce any irrational number to be required for a solution of Graph in Polygon.

Proof. Let $p \in \mathbb{Z}[x]$ be a univariate polynomial with a zero in F_2 but not in F_1. Then according to Theorem 6 there is an instance I of Graph in Polygon whose solution space is rationally equivalent to that of p, i.e., $V(I) \simeq V(p)$. We denote by f the mapping that defines the rational equivalence.

First, we show that I has a solution with vertex coordinates in F_2. Let $a \in F_2$ such that $p(a) = 0$. By definition $f(a)$ describes a solution to I. Furthermore all coordinates of $f(a)$ are in F_2.

For the reverse direction, let $b \in V(I)$ and suppose, for the purpose of contradiction, that all coordinates of b are in F_1. Then it holds that $c = f^{-1}(b) \in F_1$ and $p(c) = 0$, which contradicts the definition of p. □

Example 8 (Topological Consequences) *Let T be a torus. Then there is an instance I of* Graph in Polygon *with all coordinates given by integers such that the solution space is homeomorphic to T.*

Note that the polynomial equation

$$p(x, y, z) = (x^2 + y^2 + z^2 + 3)^2 - 16(x^2 + y^2) = 0$$

describes a torus.

To verify the last example, we simply apply Theorem 6 to p.

The polynomial equation $x^2 - 1 = 0$ leads to an example of Graph in Polygon with a disconnected solution space.

2 Irrational Coordinates

The purpose of this section is to prove Theorem 1.

Theorem 1 *There is an instance of* Graph in Polygon *with all coordinates given by integers, in which some vertices need irrational coordinates.*

In fact, the result follows from our proof of Theorem 2, but it is interesting to have a simple explicit example. This section heavily relies on a paper by Abrahamsen et al. [3]. We repeat the key ideas of their paper and show how to adapt it for our purpose. In their paper, they studied the Art Gallery Problem. In the Art Gallery Problem, we are given a polygon P and a number k, and we want to find a set of at most k guards (points) that together see the entire polygon. We say a guard g sees a point p if the entire line-segment gp is contained inside the polygon P. Abrahamsen et al. gave a simple polygon with integer coordinates such that there exists only one way to guard it optimally, with three guards. Those guards have irrational coordinates. See Figure 3, for a sketch of their polygon.

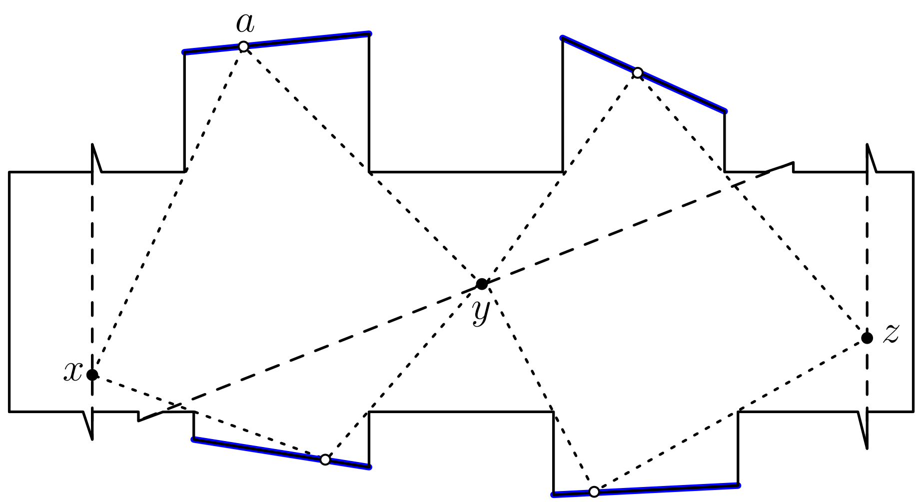

Figure 3: A sketch of the polygon from Abrahamsen et al. The three guards are indicated by black dots.

The key ingredients of their proof are as follows. First observe that the notches in the polygon boundary force there to be a guard on each of the three so-called *guard segments*, indicated by the dashed lines. Then the left guard and the middle guard together must see the top left pocket edge and bottom left pocket edge (shown in thick blue). Similarly the middle guard and the right guard together must see the top right pocket edge and bottom right pocket edge. Abrahamsen et al. specify precise coordinates for the polygon that force unique positions for the guards, and such that those positions have irrational coordinates. As shown in Figure 3, the unique guard positions result in a single point on each pocket edge that is seen by two guards. For example, point a is the only point on the top left pocket edge that is seen by x and y. In particular, the line segments xa and ya pass through reflex corners of the polygon.

We adopt this example as follows, see Figure 4. Instead of guard segments we use *variable segments* (shown in thick green), and instead of guards we use vertices. We describe variable segments in detail in Section 4, see also Figure 6. By placing notches in the polygon boundary with fixed vertices of the graph in the notches, we can force a vertex on each variable segment. The pocket edges of the previous example become variable segments. The middle variable segment, the one that contains vertex y' in the figure, is determined by a hole in the region. We used this hole in order to keep our graph drawing planar. Note that, besides the fixed vertices lying on the boundary of the region, our graph now has 7 vertices, which are forced to lie on 7 distinct variable segments. To complete the construction of our graph, we add edges between the 7 vertices to create two cycles: one containing the leftmost four vertices and the other containing the rightmost four vertices, as illustrated by the dotted lines in Figure 4.

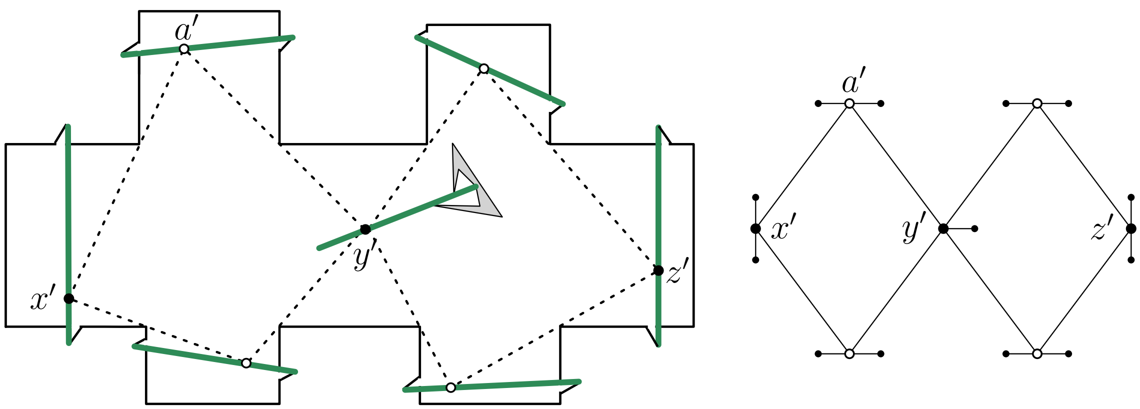

Figure 4: Left: An instance of Graph in Polygon based on Figure 3 that requires vertices at irrational coordinates. Right: The graph, with small dots indicating the fixed vertices.

Now the constraints on the three vertices x', y' and z', shown with black dots in Figure 4, are exactly the same as for the guards x, y, and z in Figure 3. All that changes is how the constraints are described. Let us give an explicit example. The guards x and y together need to see the top left pocket edge. In our new polygon, the vertices x' and y' must both be adjacent to the same vertex a', as indicated in Figure 4. This imposes the same constraints on the vertices x', y' as was imposed on the guards x, y. This translation of conditions happens in the same way for all the other pockets.

As there exists only one position to guard Abrahamsen et al.'s polygon with three guards, there exists also only one way to place the vertices in the polygon of Figure 4, and those positions have irrational coordinates.

3 Planar-ETR-INV* is $\exists\mathbb{R}$-complete and universal

The purpose of this section is to prove that Planar-ETR-INV* is $\exists\mathbb{R}$-complete (Theorem 5) and to prove that it is universal (Lemma 9). Later on we will use these results to prove that Graph in Polygon is $\exists\mathbb{R}$-complete and universal.

Theorem 5 *Planar-ETR-INV* is $\exists\mathbb{R}$-complete.*

Lemma 9 (Universality of Planar-ETR-INV*) *Let F be a finite set of polynomials $F = \{f_1, \ldots, f_k\} \subset \mathbb{Z}[x_1, \ldots, x_n]$, such that $V(F)$ is bounded. There is an instance I of the algorithmic* *problem Planar-ETR-INV* such that*

$$V(I) \simeq V(F).$$

We prove these two results about Planar-ETR-INV* based on the analogous results for ETR-INV. ETR-INV was introduced and proved $\exists\mathbb{R}$-complete by Abrahamsen, Adamaszek and Miltzow [4]. Universality was already implicit in their reduction, and was shown rigorously by Abrahamsen et al. [8]. The preprint of Abrahamsen and Miltzow [7] gives a self-contained exposition of the $\exists\mathbb{R}$-completeness of ETR-INV. To summarize, we use the following Theorem.

Theorem A ([4, 7, 8]) *ETR-INV is $\exists\mathbb{R}$-complete. Furthermore, for every instance Φ of ETR where $V(\Phi)$ is compact, there is an instance Ψ of ETR-INV such that $V(\Phi) \simeq V(\Psi)$.*

Our reduction from ETR-INV to Planar-ETR-INV* builds on the work of Dobbins, Kleist, Miltzow and Rzążewski [24]. They showed that ETR-INV is $\exists\mathbb{R}$-complete even when the variable-constraint incidence graph is planar. We cannot simply start from their result, because we still need to eliminate equalities, and the obvious idea of replacing $x + y = z$ (for example) by two constraints of the form $x + y \leq z$ and $x + y \geq z$ will destroy planarity of the variable-constraint incidence graph.

Proof of Theorem 5. First note that Planar-ETR-INV* is in $\exists\mathbb{R}$ since it is expressible as ETR.

To prove $\exists\mathbb{R}$-hardness, we reduce from ETR-INV. For this purpose let I be an instance of ETR-INV. As a first step we replace every equality constraint by the two corresponding inequality constraints. For the next step, let G_I be the variable-constraint graph of I. Let D be a drawing of G_I in the plane with edges drawn as straight line segments and vertices represented by points. This drawing may have crossing edges, but we assume that no three segments cross in a common point. We will add constraints and variables to I to obtain a new instance J, which is equivalent to I and such that the corresponding graph G_J is planar. To compute J, we replace each crossing in D by a 'crossing gadget'. Since D has at most a quadratic number of crossings, this construction takes polynomial time.

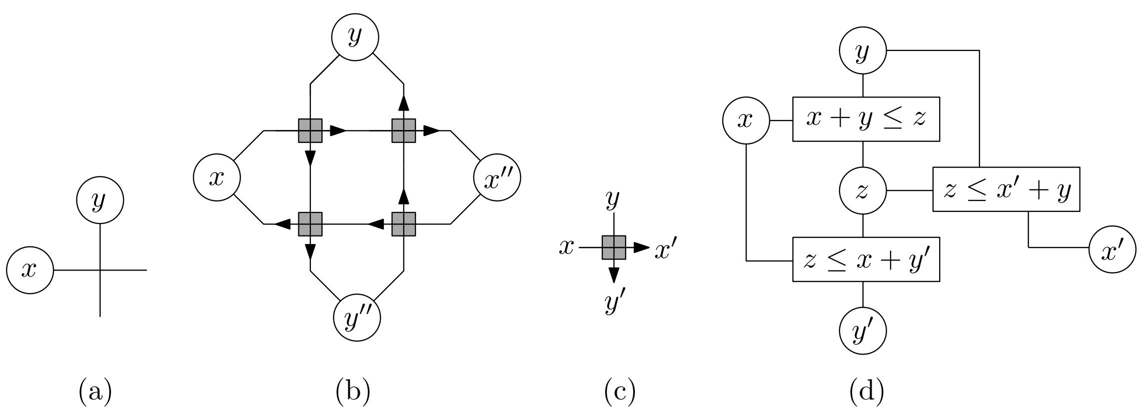

Figure 5: A crossing gadget: (a) a crossing; (b) a crossing gadget composed of 4 half-crossing gadgets that enforces $x = x''$ and $y = y''$; (c) schematic of a half-crossing gadget; (d) detail of a half-crossing gadget that ensures $x \leq x'$ and $y \leq y'$.

We first introduce a *half-crossing gadget* that almost does the job, by using an idea of Dobbins et al. [24]. Figure 5(d) illustrates this gadget, and Figure 5(c) illustrates its schematic representation.

Note that the inequalities $x + y \leq z$ and $z \leq x' + y$ ensure that $x \leq x'$. Similarly, we can observe that $y \leq y'$.

To enforce $x = x''$ and $y = y''$ we use four copies of the half-crossing gadget to build a *crossing gadget*. Figure 5(b) illustrates a crossing gadget corresponding to the crossing of Figure 5(a). The top two half-crossing gadgets (i.e., the pair of gadgets lying on the x-monotone path from x to x'') ensure $x \leq x''$. The bottom two half-crossing gadgets ensure $x \geq x''$. Together they ensure $x = x''$. The same works with y and y''.

Since the variables of ETR-INV are restricted to the range $[1/2, 2]$, the variable z will lie in the interval $[1, 4]$. This is why our definition of Planar-ETR-INV* involves a larger range for variables. Accordingly, our final step is to loosen the range restriction of all variables to $[1/2, 4]$. For any new variable z introduced in a half-crossing gadget, the larger range does no harm. For any original variable x, we will enforce the added restriction that $x \leq 2$ by adding further constraints. In particular, add two new variables a_x and b_x and add the constraints $a_x = 1$, $b_x = 1$, $a_x + b_x \geq x$. Note that the variable-constraint incidence graph remains planar, since the new constraints and variables only connect to x in the graph.

The final constructed instance J is equivalent to the original, has a planar variable-constraint incidence graph, and restricts all variables to the required range. □

Finally, we prove Lemma 9 by showing that the reduction above preserves rational equivalence.

Proof of Lemma 9. We start from the fact that ETR-INV is universal; see Theorem A. Note that $V(F)$ is a compact semi-algebraic set. Thus it is sufficient to show the following claim:

Let I be an instance of ETR-INV and J be the instance of Planar-ETR-INV* constructed as in the proof of Theorem 5. Then $V(I) \simeq V(J)$.

To see this, consider a solution $(x_1, \ldots, x_n)$ of I. Now, in the planarity gadget as described above, some new variables are introduced. By construction, every new variable y, is determined by an old variable either by the equation:

$$y = x_i + x_j, \text{ or } y = x_k, \text{ or } y = 1$$

for some i, j, k. Thus the mapping f, which is the identity on $x_1, \ldots, x_n$ and assigns each new variable y to the corresponding value as described above, is a rational and continuous function. To see that f is surjective consider a solution to J, say $(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{n+m}$, and note that all the original variables are a valid solution to I. Thus f^{-1} is described by

$$(x_1, \ldots, x_n, y_1, \ldots, y_m) \mapsto (x_1, \ldots, x_n).$$

Thus the inverse exists and is also rational and continuous. □

4 $\exists\mathbb{R}$-completeness

The purpose of this section is to prove our main result:

Theorem 2 Graph in Polygon *is* $\exists\mathbb{R}$*-complete.*

Proof. (Use two variables to express the x- and y-coordinates of each vertex. After triangulating the region, containment of a vertex in the region can be expressed as a disjunction over containment in the triangles. That graph edges are contained in the region and do not cross can be enforced using determinants to test intersections of line segments. For more details, see how Matousek [38] handles similar issues.)

To prove that the problem is ∃ℝ-hard we give a reduction from Planar-ETR-INV*. Let I be an instance of Planar-ETR-INV*. We build an instance J of Graph in Polygon such that J admits an affirmative answer if and only if I is satisfiable. The idea is to take the variable-constraint incidence graph of the instance I and represent it as a planar orthogonal graph drawing. We then replace the vertices and edges of this drawing by gadgets. In particular, for the vertices we construct gadgets to represent variables, and gadgets to enforce the addition and inversion inequalities, $x + y \leq z, x + y \geq z, x \cdot y \leq 1, x \cdot y \geq 1$. For the edges, we construct gadgets to copy and duplicate variables. Finally, we show how to combine all these gadgets to obtain an instance J of Planar-ETR-INV*. Look ahead to Figure 13 for the overall plan.

Describing Variables. We will encode the value of a variable in $[1/2, 4]$ as the position of a vertex that is constrained to lie on a line segment of length 3.5, which we call a *variable-segment*. One end of a variable-segment encodes the value $\frac{1}{2}$, the other end encodes the value 4, and linear interpolation fills in the values between. Figure 6 shows one side of the construction that forces a vertex to lie on a variable-segment. The other side is similar.

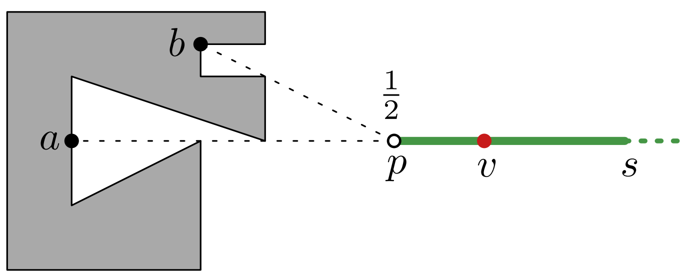

Figure 6: Variable v is represented as a point on variable-segment s (shown in green). The construction of one end of s is illustrated. In the graph, vertex v is adjacent to fixed vertices a and b on the boundary of a hole of the region (shaded). Adjacency with a forces v to lie on the line of s. Adjacency with b forces v to lie at, or to the right of, point p which is associated with the value $1/2$.

By slight abuse of notation, we will identify a variable and the vertex representing it, if there is no ambiguity. For the description of the remaining gadgets, our figures will show variable-segments (as thick green lines) without showing the polygonal holes that determine them. Our next gadget shows how to "transmit" a value by copying variables.

Copying Variables. Given a variable-segment for a variable x, we will need to "transmit" its value to other locations in the plane. To do this, we will construct a second variable-segment for a new variable y and enforce $x = y$. We first show how to construct a gadget that ensures $x \leq x'$ for a new variable x'. Then we combine four such gadgets, enforcing

$$x \leq z_1, \quad z_1 \leq y, \quad x \geq z_2, \quad z_2 \geq y.$$

This implies $x = y$.

The gadget enforcing $x \leq x'$ is depicted at the left of Figure 7. It consists of two parallel variable-segments. In general, these two segments need not be vertically aligned. In the graph

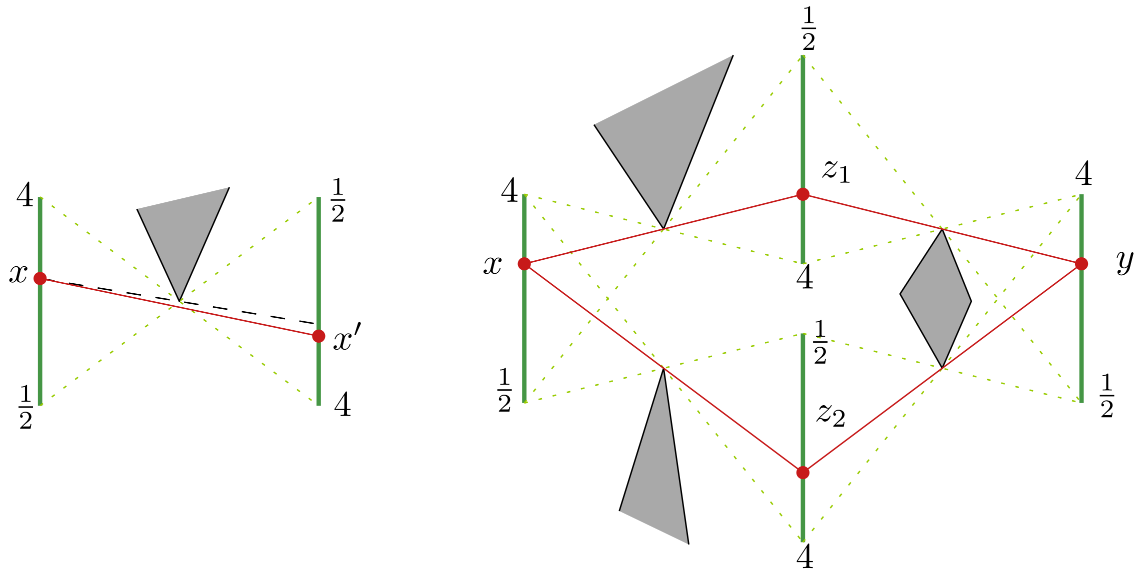

Figure 7: Copying. Left: a gadget to enforce $x \leq x'$. Right: The full gadget enforcing $x = y$.

we connect the corresponding vertices by an edge. The left and the right variables are encoded in opposite ways, i.e., x increases as the vertex moves up and x' increases as the corresponding vertex goes down. We place a hole of the polygonal region (shaded in the figure) with a vertex at the intersection point of the lines joining the top of one variable-segment to the bottom of the other. The hole must be large enough that the edge from x to x' can only be drawn to one side of the hole. An argument about similar triangles, or the "intercept theorem", also known as Thales' theorem, implies $x \leq x'$.

We combine four of these gadgets to construct our copy gadget, as illustrated on the right of Figure 7. We can think of the gadget as a way to transmit a variable along a "wire".

Duplicate Variables. Since a single variable may appear in several constraints, we need a way to split a wire into two wires, each holding the correct value of the same variable. Figure 8 shows a gadget to duplicate the variable x to variables y_1 and y_2. The gadget consists of two copy gadgets sharing the variable-segment for x. We can construct the two copy gadgets to avoid any intersections between them.

Turning. We need to encode a variable both as a vertical and as a horizontal variable-segment. To transform one into the other we use a turn gadget, as described below.

The key idea is to construct two diagonal variable-segments for variables z_1 and z_2, and then transfer the value of the vertical variable-segment to the horizontal variable-segment using z_1, z_2. This is in fact very similar to the copy gadget, except that the intermediate variable-segments are placed on a line of slope 1. We do not know if it is possible to enforce the constraint $x \leq z$ directly. However, it is sufficient to enforce $x \leq f(z)$ for some function f. See the left side of Figure 9. Interestingly, it is not necessary to know the function f. However, we do know that f is monotone and we can construct another gadget enforcing $y \geq f(z)$, for the same function f, by

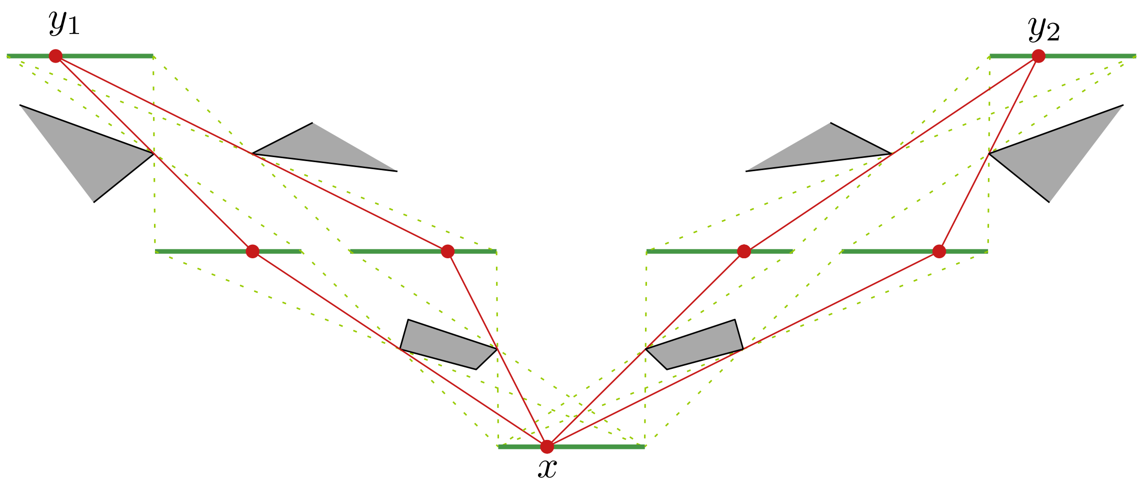

Figure 8: Duplication. The variables y_1, y_2 have both the same value as x.

making another copy of the first gadget reflected through the line of the variable-segment for z.

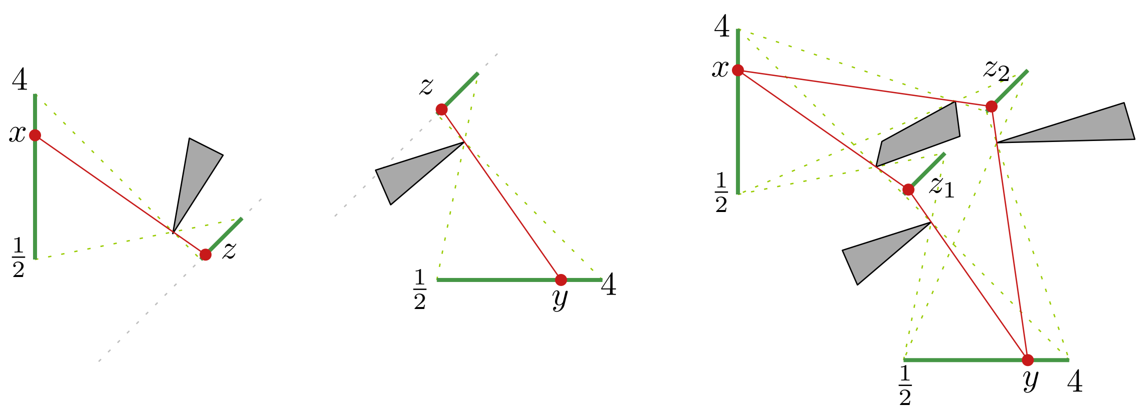

Figure 9: Turning. Left: Gadget to encode $x \leq f(z)$. Middle: Symmetric gadget to encode $y \geq f(z)$. Right: Four gadgets of the previous type combined to enforce $x = y$, for x and y on a vertical and horizontal variable-segment, respectively.

Combining four such gadgets, as on the right of Figure 9, yields the following inequalities.

$$x \leq f_1(z_1), \quad f_1(z_1) \leq y, \quad y \leq f_2(z_2), \quad f_2(z_2) \leq x$$

This implies $x = y$.

Addition. The gadget to encode $x + y \geq z$ is depicted in Figure 10. Important for correctness is that the gaps between the dotted auxiliary lines have equal lengths. This is essentially the same gadget that was used by Abrahamsen et al. [4, Lemma 31]. To keep this work self-contained we offer a sketch of an alternative correctness proof.

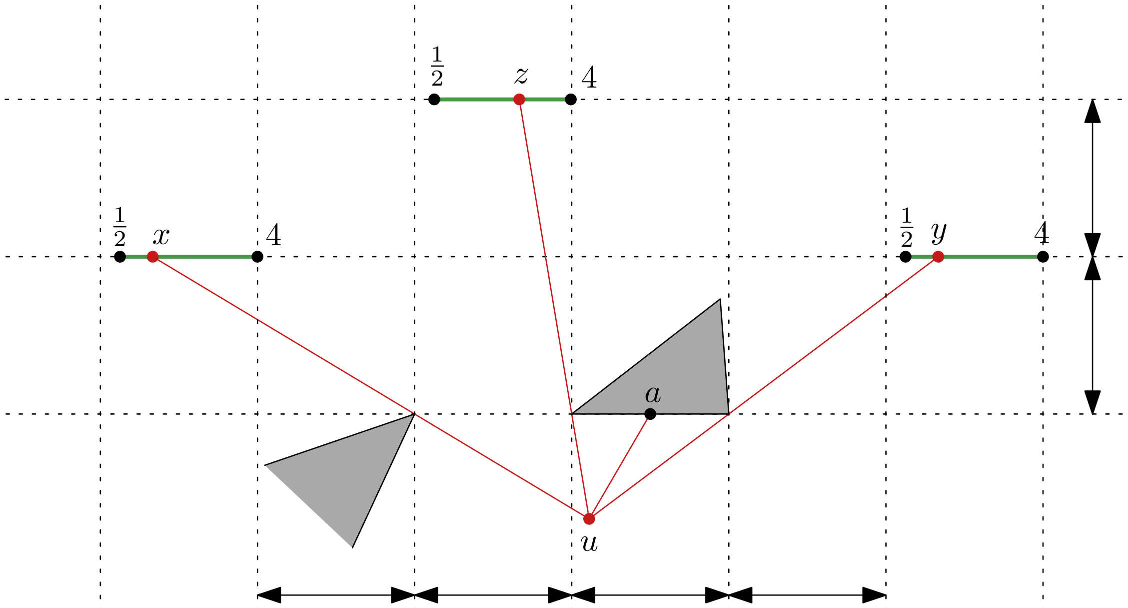

Figure 10: Addition. The three vertices x, y, z can only be connected to u if $x + y \geq z$ holds.

Lemma 10 ([4]) *The gadget in Figure 10 enforces $x + y \geq z$.*

Proof of Lemma 10. This proof is inspired by the following thought experiment. The construction is made such that $x, y = 1$ and $z = 2$ is a valid solution. Now, assume that we choose z always to be the maximum possible value. Furthermore we assume that while we fix the position of y, we move x some distance d to the left. What we would expect is that z moves by the same distance to the left. Actually, showing the last statement also proves the lemma, due to symmetry of x and y. We denote by ℓ the line that contains the variable segments of x and y. We denote by t half the distance that z moves. Note that t has a geometric interpretation as indicated in Figure 11. We need to show $d = 2t$. The lengths A, A', B, B' are defined, by Figure 11. Note that $B' = 2B$, because $\|a - b\| = \|b - c\|$. Similarly, $A' = 2A$. The lemma follows from

$$d = B' - A' = 2(B - A) = 2t.$$

□

The gadget that enforces $x + y \leq z$ is just a mirror image of the previous gadget.

Inversion. The inversion gadgets to enforce $x \cdot y \leq 1$ and $x \cdot y \geq 1$ are depicted in Figure 12. We use a horizontal variable-segment for x and a vertical variable-segment for y and align them as shown in the figure, 1.5 units apart both horizontally and vertically. We make a triangular hole with its apex at point p as shown in the figure. The graph has an edge between x and y.

For correctness, observe that if x and y are positioned so that the line segment joining them goes through point p, then, because triangles Δ_1 and Δ_2 (as shown in the figure) are similar, we have $\frac{x}{1} = \frac{1}{y}$, i.e. $x \cdot y = 1$. If the line segment goes above point p (as in the left hand side of Figure 12) then $x \cdot y \geq 1$, and if the line segment goes below then $x \cdot y \leq 1$.

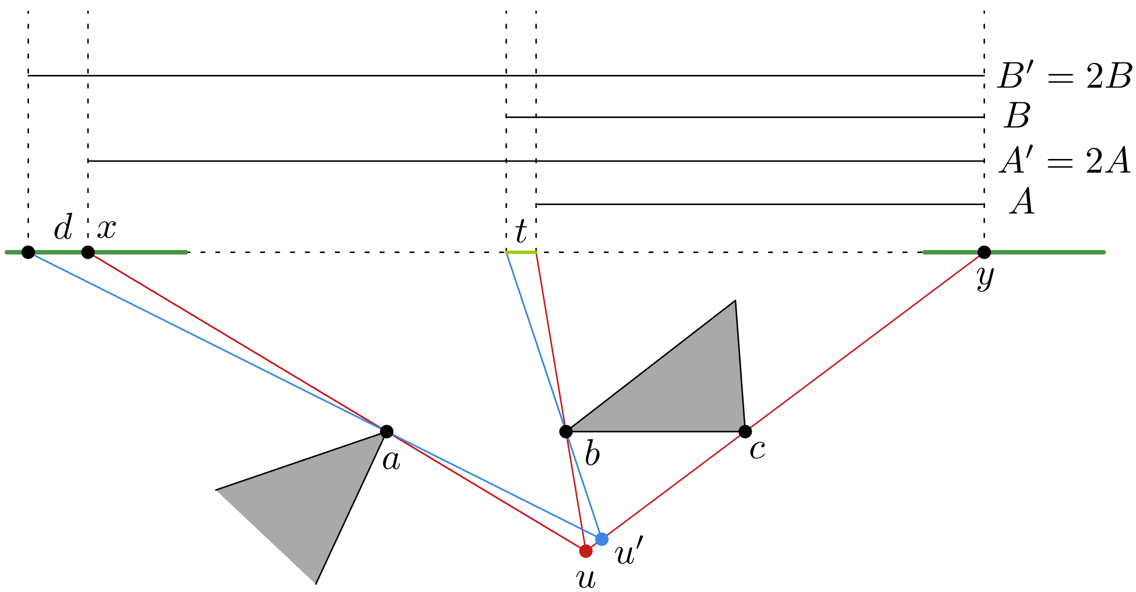

Figure 11: An illustration of the correctness of the addition gadget.

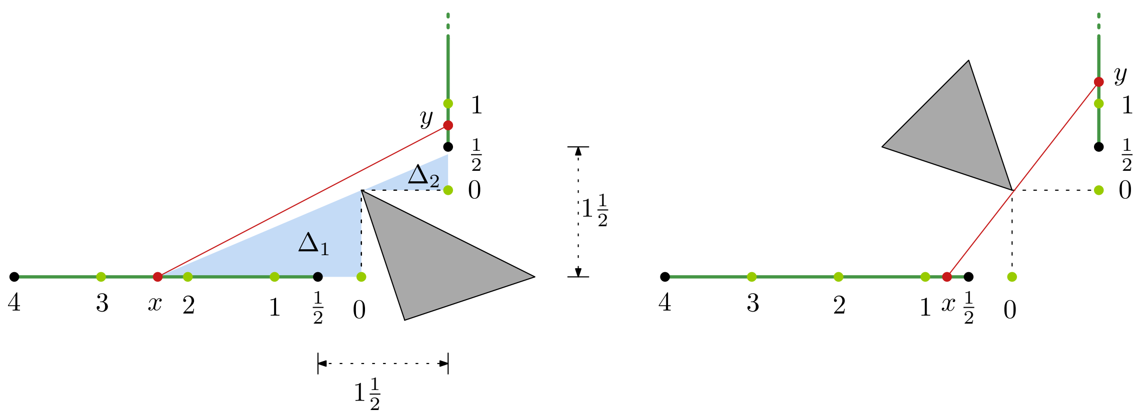

Figure 12: Inversion. Left: Gadget enforcing $x \cdot y \geq 1$. Right: Gadget enforcing $x \cdot y \leq 1$.

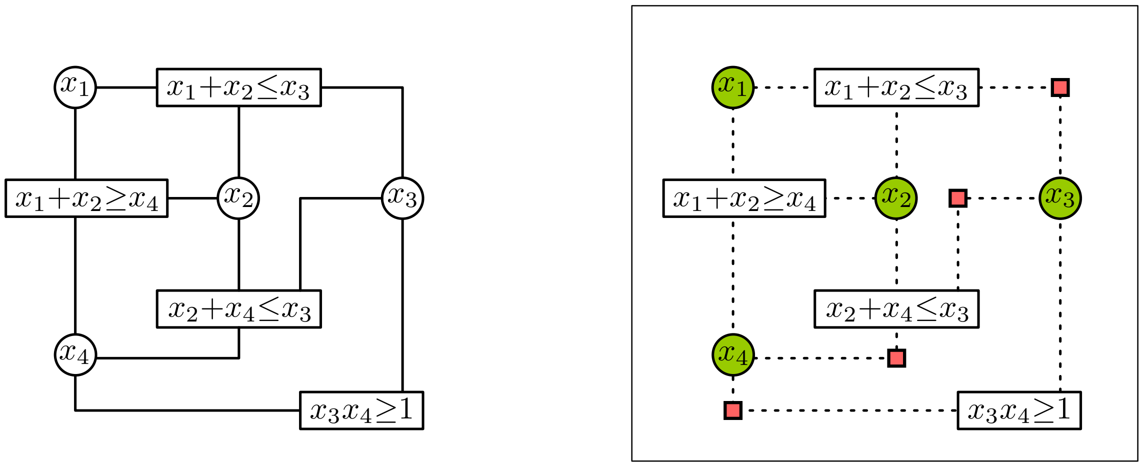

Figure 13: The overall reduction from Planar-ETR-INV* to Graph in Polygon. Left: An orthogonal drawing D of the variable-constraint graph of I. In this example every variable vertex has degree at most 3. Right: A schematic representation of the resulting instance. The variable-segments are placed inside the variable vertices (green). Every edge is replaced by a sequence of copy gadgets (dotted lines) and turn gadgets (red squares). The constraint vertices are replaced by the corresponding constraint gadgets. Note that it might be necessary to have one or several turn and copy gadgets as part of the addition and inversion gadgets.

Putting it all together. It remains to show how to obtain an instance of Graph in Polygon in polynomial time from an instance of Planar-ETR-INV*.

Let I be an instance of Planar-ETR-INV*. As a first step we split every variable, until all variables have degree at most 3. Then we compute an orthogonal planar drawing D of the planar variable-constraint incidence graph, which can be done in polynomial time using orthogonal planar graph drawing algorithms [44]. The edges of D act as wires and we replace each horizontal and vertical segment by a copy gadget. As a next step, we replace every 90 degree corner, by a turn gadget. Every constraint will be replaced by the corresponding gadget. Note that the inversion gadgets may also need a turn gadget. We add a big square to the outside, to ensure that everything is inside one polygon. See Figure 13.

It is easy to see that this can be done in polynomial time, as every gadget has a constant size description and can be easily described with rational numbers, although, we did not do it explicitly. In order to see that we can also use integers, note that we can scale everything with the least common multiple of all the denominators of all numbers appearing. This can also be done in polynomial time. □

5 Universality

This section is devoted to proving that Graph in Polygon is universal:

Theorem 6 (Universality of Graph in Polygon) *Let F be a finite set of polynomials with integer coefficients such that $V(F)$ is bounded. Then there exists an instance I of* Graph in Polygon *with all coordinates given by integers such that*

$$V(I) \simeq V(F).$$

By Lemma 9 we know that Planar-ETR-INV* is universal. Thus it is sufficient to show the following claim:

Let I be an instance of Planar-ETR-INV* and J be the instance of GRAPH IN POLYGON constructed as in the proof of Theorem 2. Then $V(I) \simeq V(J)$.

To see this, consider a solution $(x_1, \ldots, x_n)$ of I. By construction, every variable of I is explicitly represented in J by a vertex forced to lie in a line segment, or, more precisely, by many vertices constrained by our gadgets to represent the same value. For those variables, we only need a linear transformation to read off the exact value. There are two types of vertices, whose coordinates cannot be derived in this way from the old variables. The first type are the vertices z_1 and z_2 in the turn gadget as shown in Figure 9. The second type are the vertices u in the addition gadget as shown in Figure 10.

In both cases, we need to argue that the new vertices can be described by rational functions of the variables $(x_1, \ldots, x_n)$. As a preparation, we study specific mappings that are constructed in a geometric way.

Let ℓ_1 and ℓ_2 be two lines and $p \notin \ell_1 \cup \ell_2$ be a point. Furthermore we denote by ab some segment with endpoints $a \in \ell_1$, $b \in \ell_2$ and $p \in ab$. Then the position of b is uniquely determined by the position of a. In other words, $b = h(a)$, for some function h.

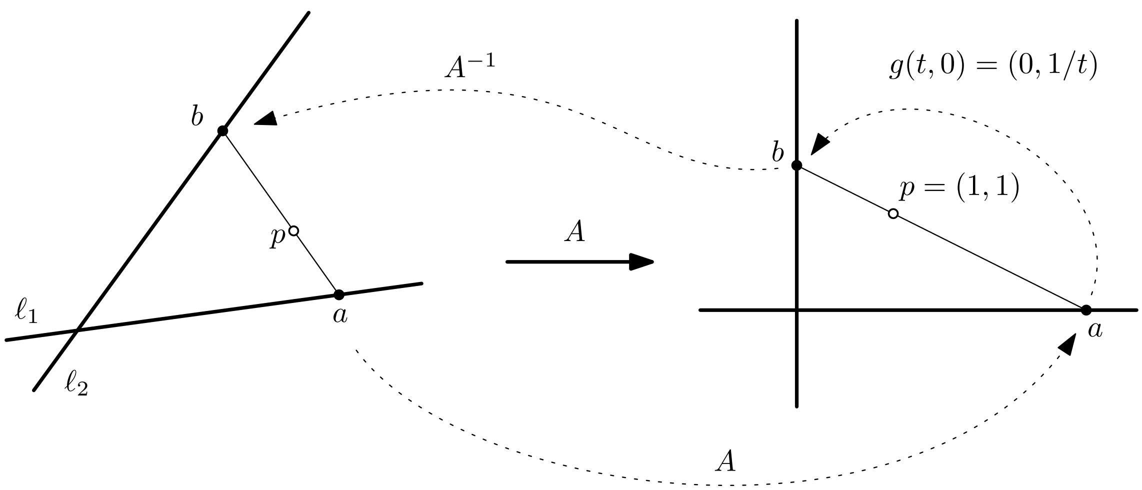

Figure 14: The bijective linear map A transforms the lines to the coordinate axis.

Lemma 11 (Rational Functions) *The function h can be described as the fractions of two linear functions.*

.See Figure 14 for an illustration. This is easy to see in case that $p = (1, 1)$ and ℓ_1 and ℓ_2 are the x-axis and y-axis respectively. (To be explicit, the point $a = (t, 0)$ is mapped to the point $b = (0, 1/t)$.)

Otherwise, consider a linear bijection A, which maps ℓ_1 to the x-axis and ℓ_2 to the y-axis. This gives us a function, h' that maps a' from the x-axis to b' on the y-axis. As noted above, h' can be described as the fraction of two linear functions. The function h is given by $A^{-1} \circ h' \circ A$, which is a rational function, as claimed. □

Now we are ready to show that z_1 and z_2 can be described as rational functions of the original variables. As both cases are the same, we do the proof only for $z = z_1$. Note that z is uniquely defined by the position of the vertex x, see Figure 9. And the coordinates of the vertex x are defined linearly in terms of the original variables of I. By Lemma 11, the coordinates of z can be described by a rational function of the original variables. This is because x and z both lie on a line and the connecting segment contains a fixed vertex.

Let us now turn our attention to the vertex u, as in Figure 10. Note that we know that u is already determined by x and y, as the position of z is determined by x and y globally. Denote by g the function that determines the position of u depending on x and y. Note that g restricted to x is rational by Lemma 11. Similarly, g restricted to y is rational. Thus g must be a rational function globally.

We are now ready to describe the mapping

$$f : V(I) \to V(J).$$

Note that the coordinates of each vertex of J is either described in a linear way, or it is a vertex in the turn or addition gadgets as discussed above. Those vertices can be described by rational functions as shown above. Thus f is a rational, continuous and injective function.

Now, we want to show that f is also surjective. Assume we have a valid drawing of J. Then we can read off a solution to I, from the position of certain vertices. We can forget about the positions of all other vertices. This gives an implicit description of f^{-1} as a linear continuous function. This finishes the proof.

6 Vertex Coordinates

Since we have shown that Graph in Polygon may require irrational coordinates for vertices in general, it is interesting to examine bounds on coordinates for special cases. In this section we discuss the bit complexity of vertex coordinates needed for two well-solved special cases of Graph in Polygon.

Tutte's algorithm [58] finds a straight-line planar drawing of a graph inside a fixed convex drawing of its outer face. Suppose the graph has n vertices and each coordinate of the convex polygon uses t bits. Tutte's algorithm runs in polynomial time, but the number of bits used to express the vertex coordinates is a polynomial function of t and n. The dependence on n means that the drawing uses "exponential area." Chambers et al. [20] gave a different algorithm that uses polynomial area—the number of bits for the vertex coordinates is bounded by a polynomial in t and $\log n$.

The other well-solved case of Graph in Polygon is the minimum link path problem, as discussed in Section 1. In this case we have a general polygonal region with holes, but the graph is restricted to be a path with endpoints s and t fixed on the boundary of the polygonal region. Based on a lower bound result of Kahan and Snoeyink [32], Kostitsyna et al. [34] proved a tight bound of $\Theta(n \log n)$ bits for the coordinates of the bends on a minimum link path. Note that the dependence on n means that this bound is exponentially larger than the bound for drawing a graph inside a convex polygon. Problem 3 below asks about the complexity of drawing a tree in a polygonal region.

7 Conclusions and Open Questions

In this paper, we studied the problem of finding a planar straight-line drawing of a graph inside a polygonal region, and showed that it is $\exists\mathbb{R}$-complete. Previous $\exists\mathbb{R}$-hardness results for graph drawing involved other representations, such as disk or segment intersection [38, 39], or involved straight-line drawings of non-planar graphs [51]. Our result is one of the first $\exists\mathbb{R}$-hardness results about drawing planar graphs with straight-line edges—along with recent results about drawings with extra conditions: prescribed face areas [24]; and drawings on few lines [22].

We conclude with some open questions:

1. Our proofs of Theorems 1 and 2 used the fact that the polygonal region may have holes and may have collinear vertices. Is Graph in Polygon polynomial-time solvable for a simple polygon (a polygonal region without holes) whose vertices lie in general position (without collinearities)?

2. Our proofs also used the assumption that points on the boundary of the region may be used in the graph drawing. If we disallow that, then Graph in Polygon is equivalent to Partial Drawing Extensibility. Is this problem $\exists\mathbb{R}$-hard? There are two versions, depending whether the graph is given abstractly or via a combinatorial embedding. In the first case the problem is known to be NP-hard [47], but in the second case even that is not known.

3. What is the complexity of Graph in Polygon when the graph is a tree? Can vertex coordinates still be bounded as for the minimum link path problem? When the tree is a caterpillar, the problem might be related to the minimum link watchman tour problem, which is known to be NP-hard [10].

4. Recently, two $\exists\mathbb{R}$-complete problems were shown to be contained in NP "under the lens of smoothed analysis" [23, 60]. It would be interesting to see if the same is true for the Graph in Polygon problem. In particular, can we build an Integer Program in an efficient way that decides with high probability if a solution exists, under a reasonable model of perturbation?

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