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On the Maximum Number of Crossings in Star-Simple Drawings of K_n with No Empty Lens

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Abstract. A *star-simple* drawing of a graph is a drawing in which adjacent edges do not cross. In contrast, there is no restriction on the number of crossings between two independent edges. We forbid empty lenses, i.e., every lens is required to enclose a vertex, and show that with this restriction $3 \cdot (n-4)!$ is an upper bound on the number of crossings between two edges of a star-simple drawing of K_n. It follows that $n!$ bounds the total number of crossings in the drawing. This is the first finite upper bound on the number of crossings in star-simple drawings of the complete graph K_n with no empty lens. For a lower bound we construct a star-simple drawing of K_n with no empty lens in which a pair of edges contributes $5^{n/2-2}$ crossings.

1 Introduction

A *drawing* of a graph G is a representation of G in the plane where vertices are represented by pairwise distinct points, and edges are represented by Jordan arcs whose endpoints correspond to

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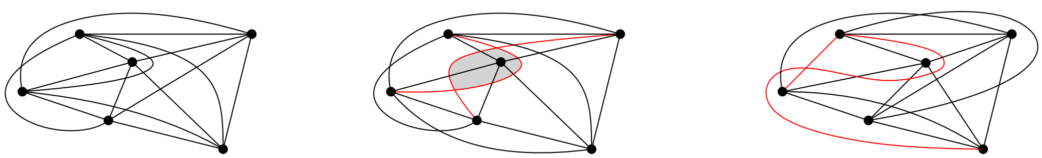
the vertices of the edge. Additionally, edges contain no other vertices, every common point of two edges is either a proper (transversal) crossing or a common endpoint, the number of crossings is finite, and no three edges cross at a single point. A *simple drawing* is a drawing in which adjacent edges do not cross, and independent edges cross at most once.

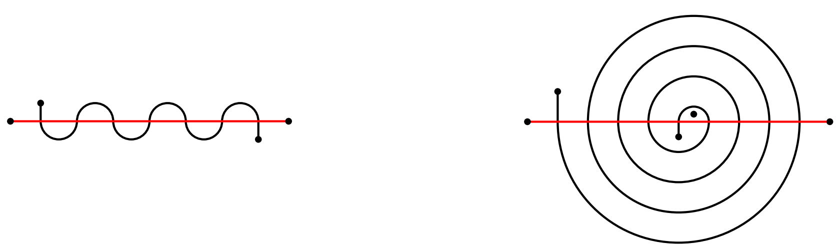
We study a broader class of drawings, which we call *star-simple* drawings, where adjacent edges do not cross, but independent edges may cross any number of times; see Figure 1 for illustration. In such drawings, for every vertex v the induced substar centered at v is simple, that is, the drawing restricted to the edges incident to v forms a plane drawing. In the literature, these drawings also appear under the name *semi-simple* [1, 2] (with and without dash), and the condition that adjacent edges do not cross appears as *Rule +* [12] or *star condition* [7]. We advocate using the term star-simple for these drawings because it is more descriptive.



(a) simple (b) star-simple but not simple (c) not star-simple

Figure 1: Three types of drawings of K_6. A nonempty lens is shaded in (b).

We are interested in bounding the number of crossings in star-simple drawings. In contrast to simple drawings, star-simple drawings can have regions that are 2-gons, bounded by (continuous parts of) two edges. We call such a region a *lens*; see Figure 1b. In the literature, lenses are also called *bigons* or *digons*. A lens is *empty* if it has no vertex in its interior. If empty lenses are allowed, the number of crossings in star-simple drawings of graphs with at least two independent edges is unbounded because two edges can be "twisted" arbitrarily, as illustrated in Figure 2a. Therefore we restrict our attention to star-simple drawings with no empty lens. Still, this restriction does not suffice to guarantee a bounded number of crossings because an edge can "spiral" through another edge, as illustrated in Figure 2b. However, we will show that star-simple drawings *of the complete graph K_n* with no empty lens have a bounded number of crossings.



(a) twisting (b) spiraling

Figure 2: Edge pairs with an unbounded number of crossings.

It is well known that for every graph G, any drawing of G that minimizes the number of crossings is simple: if two edges form a lens, they can be locally redrawn to decrease the total number of crossings. This redrawing can change the homotopy type of the affected edge, with respect to the set of vertices. However, if a lens formed by the two edges is empty, the redrawing keeps the homotopy types of the edges unchanged. This motivates the following definition.

A drawing D of a graph is called *reduced* if the number of crossings in D is minimum in its homotopy class; that is, it is not possible to decrease the number of crossings by a continuous deformation (isotopy) of the edges while avoiding passing over vertices. By the previous argument, a reduced drawing has no empty lens. The converse is also true: a drawing with no empty lens is reduced [5, Lemma 3.1].

Results and related work. Empty lenses play a crucial role in the context of crossing lemmas for multigraphs see e.g. [6, 13]. This is because a group of parallel edges can be drawn without a single crossing. Hence, for general multigraphs there is no hope to get a lower bound on the number of crossings as a function of the number of edges. However, if empty lenses are forbidden, we cannot draw arbitrarily many parallel edges.

Star-simple drawings have also been considered in the context of crossing minimization. Balko et al. [2] study the monotone star-simple odd crossing number (denoted by mon-ocr$_+$ in Schaefer [16]). This is the smallest number of pairs of edges that cross an odd number of times in a monotone star-simple drawing of G. They show that in the case of K_n this variant of the crossing number equals the Hill number $H(n)$, that is, the conjectured minimum number of crossings in any drawing of K_n.

Kynčl [8, Section 5 “Picture hanging without crossings”] proposed a construction of two edges in a graph on n vertices with an exponential number (2^{n-4}) of crossings and no empty lens; see Figure 3. This configuration can be completed to a star-simple drawing of K_n, cf. [14]. For $n = 6$ it is possible to have one more crossing while maintaining the property that the drawing can be completed to a star-simple drawing of K_6; see Figure 4. Repeated application of the doubling construction of Figure 3 leads to two edges with $2^{n-4} + 2^{n-6}$ crossings in a graph on n vertices. This configuration can be completed to a star-simple drawing of K_n. In Section 4 we introduce generalizations of the doubling technique. Using this technique we can construct a pair of edges with $5^{n/2-2}$ crossings such that $n - 4$ additional vertices are enough to hit all the lenses formed by the two edges, i.e., to make it a drawing with no empty lens. In addition we show that this drawing of two edges can be extended to a star-simple drawing of K_n.

Whereas results on the lower bound were known, the question if there exists an upper bound on the maximum crossing number in these drawings has still been open. In this work, we will give an upper bound of $3(n-4)!$ for the maximum crossing number of star-simple drawings of K_n with no empty lens and therefore answer this question positively.

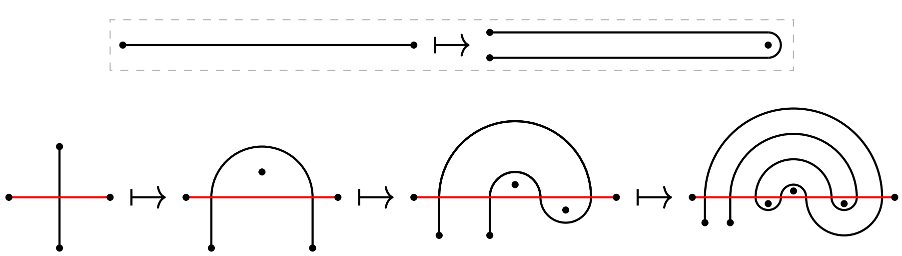
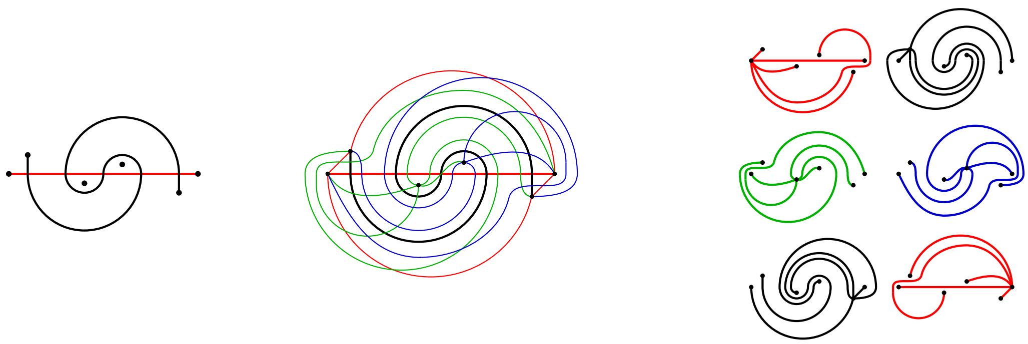


Figure 3: The doubling construction yields a number of crossings which is exponential in the number of vertices which hit all lenses.



(a) 5 crossings (b) star-simple completion (c) the stars of the drawing

Figure 4: A star-simple drawing of K_6 with a pair of edges crossing 5 times.

2 Preliminaries

In this section we first introduce some notation and then discuss two types of configurations (deadlocks and spirals) that cannot occur in reduced star-simple drawings of K_n.

Curves and edges. We use the term *curve* as a synonym for Jordan arc, that is, the image $\gamma = \mathrm{Im}(f_\gamma)$ of the closed unit interval under an injective continuous map $f_\gamma : [0,1] \to \mathbb{R}^2$. The points $f_\gamma(0)$ and $f_\gamma(1)$ are the *endpoints* of γ. For a *closed* curve the two endpoints coincide and the corresponding map is injective on $[0,1)$ only. A curve γ_1 is a *subcurve* of a curve γ_2 if $\gamma_1 \subseteq \gamma_2$. Two curves *overlap* if they have a common subcurve; otherwise they are *nonoverlapping*. Sometimes we consider curves as directed objects and refer to them as a curve *from* one endpoint *to* the other.

An edge in a drawing is represented by a curve. In a slight abuse of notation we use the term *edge* to refer to both the edge of an abstract graph and the curve that represents this edge in the drawing under consideration. In that sense, an edge *is* a curve. Whenever we use the term curve explicitly, these objects may or may not be part of the graph or drawing under consideration.

Arrangements and lenses. Let Γ be a set of curves such that any pair of curves of Γ has a finite set of intersection points. The *arrangement* $\mathcal{A}(\Gamma)$ induced by Γ is the partition of $\mathbb{R}^2$ into *vertices* (endpoints, crossings, and touchings of curves), *edges* (maximal vertex-free components of curves), and *faces* (maximal connected subsets of the complement $\mathbb{R}^2 \setminus \bigcup_{\gamma\in\Gamma} \gamma$). Two arrangements are *isomorphic* if there is an orientation-preserving homeomorphism of the plane transforming the curves of one arrangement into the curves of the other. Such a homeomorphism bijectively maps vertices to vertices, edges to edges, and faces to faces, so that incidences and the circular order (of edges around vertices and faces) are preserved.

A *lens* in an arrangement $\mathcal{A}(\Gamma)$ is an open region bounded by a closed curve that is the union of two subcurves $\delta_1 \subseteq \gamma_1 \in \Gamma$ and $\delta_2 \subseteq \gamma_2 \in \Gamma$ such that δ_1 and δ_2 are internally disjoint and have the same endpoints. A lens L is *minimal* if no lens induced by the same pair of curves is strictly contained in L.

Let Γ be a set of curves corresponding to the set of edges of a drawing of a graph G. Every lens of the induced arrangement contains a minimal lens that is induced by the same pair of curves. A lens is *empty* if it does not contain a vertex of G.

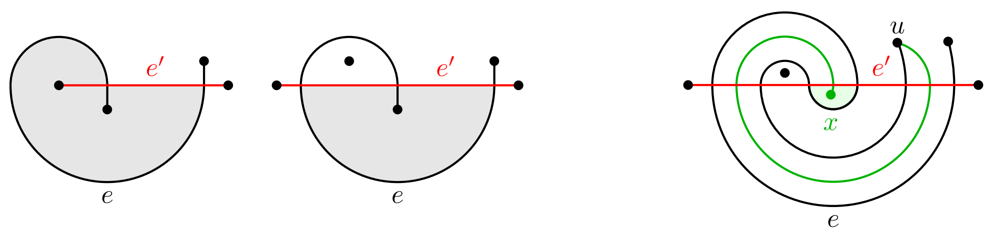
Deadlocks. Let us study the arrangement $\mathcal{A}(e, e')$ induced by two curves e and e' that are independent (that is, they do not share an endpoint). In any such arrangement each endpoint of e and e' lies on the boundary of exactly one face, so we say that it *belongs* to this face. We start by observing that for two independent edges in a star-simple drawing of the complete graph, all endpoints of the edges belong to the same face of the induced arrangement. This fact was used earlier by Aichholzer et al. [1] and by Kynčl [9, p. 18].

Lemma 1 ([1, 9]) *Let e and e' be independent edges in a star-simple drawing of the complete graph. Then the four vertices of e and e' belong to the same face of $\mathcal{A}(e, e')$.*

Proof: Let u and v be endpoints of the two edges belonging to different faces. If $u \in e$ and $v \in e'$, then the edge u, v of K_n has to violate the star-simplicity, see Figure 5a. If u and v belong to e, then fix $w \in e'$. Now, either u, w or v, w is a pair belonging to different faces, whence again we are in the first case. □

A *deadlock* is a pair e, e' of independent curves for which not all vertices belong to the same face of the arrangement $\mathcal{A}(e, e')$ induced by e and e', Figure 5a shows two deadlocks. From Lemma 1 we know that deadlock configurations do not occur in star-simple drawings of complete graphs.

Spirals. Now suppose that D is a star-simple drawing of a complete graph with no empty lens. In this case we can argue that e and e' do not form a configuration as the black edge e and the red edge e' in Figure 5b. Indeed, this configuration has an inner lens L (shaded green in the figure) and by assumption this lens is non-empty, that is, L contains a vertex x. Let u be a vertex of e. The edge xu (the green edge in the figure) must not cross e, hence it follows the "tunnel" formed by e. This yields a deadlock configuration of the edges xu and e'. Also note that if in Figure 5b we connect x to a vertex of e' with a curve γ that does not cross e', then γ necessarily forms a deadlock with e.



(a) deadlocks (b) spiral

Figure 5: Configurations that do not appear in star-simple drawings of complete graphs with no empty lens.

We use this intuition to formally define spirals. A curve γ_1 *forms a spiral with* a curve γ_2 if the curves γ_1 and γ_2 form a lens L such that every curve that connects some point in the interior of L to some endpoint of γ_1 without crossing γ_1 forms a deadlock with γ_2. Two curves γ_1 and γ_2 *form a spiral* if γ_1 forms a spiral with γ_2 or γ_2 forms a spiral with γ_1 (or both).

Lemma 2 *In a star-simple drawing of a complete graph with no empty lens no pair of edges forms a spiral.*

Proof: Consider a pair e, e' of edges in a star-simple drawing D of a complete graph with no empty lens, and suppose for the sake of a contradiction that e and e' form a spiral. Without loss of generality, suppose that e forms a spiral with e'. Then by definition e and e' form a lens L such that every curve that connects some point in the interior of L to some endpoint of e without crossing e forms a deadlock with e'. Let u be an endpoint of e. As D has no empty lens, there is a vertex v of D inside L. The edge uv in D connects $v \in L$ to u and, therefore, forms a deadlock with e'. This contradicts Lemma 1, which states that the edges uv and e' of D do not form a deadlock. □

Unlocked pairs and loose meanders. A pair of curves that does not form a deadlock is called an *unlocked pair* of curves. In enumerative combinatorics unlocked pairs are also studied as *open meanders*, see sequence A005316 in The On-Line Encyclopedia of Integer Sequences, and it is a major open problem to find a precise formula or tight asymptotics [3, 10, 11, 15]. An unlocked pair is a *loose meander* if it does not form a spiral. It is convenient to think of an unlocked pair as given in a *standard representation*: All the vertices of the two curves e and e' are on the outer face, the curve e' is horizontal, and the other curve e forms a wiggling "meander" curve such that all crossings of e with the line supporting e' lie within e'.

By Lemma 1 and Lemma 2 every pair of edges in a reduced star-simple drawing of K_n, i.e., in a star-simple drawing with no empty lens, forms a loose meander. It is an open question whether there are loose meanders that cannot be completed to a star-simple drawing of K_n.

Exit-curves. Consider an unlocked pair e, e' of curves in standard representation. Let L be a minimal lens in $\mathcal{A}(e, e')$, and let p be a point in the interior of L. An *exit-curve* for p along e through e' is a curve γ between p and some point q in the outer face of $\mathcal{A}(e, e')$ such that γ is disjoint from e and γ has the minimum number of crossings with e' (among all curves between p and q that are disjoint from e), see Figure 6.

We use exit-curves to model loose meanders that arise from a star-simple drawing of K_n with no empty lens. The point p models a vertex in L (which must exist), and the exit-curve for p along e models an edge between p and an endpoint of e (which must be drawn without crossing e). The following property of exit-curves, which we prove later in Section 3, turns out to be very useful.

Lemma 3 *Let e, e' be a loose meander, and let p be a point in the interior of a minimal lens in $\mathcal{A}(e, e')$. Then every exit-curve for p along e through e' forms a loose meander with e'.*

3 Crossings of pairs of edges

In this section we derive an upper bound on the number of crossings of two edges in a star-simple drawing of K_n with no empty lens. Actually, we prove a more general statement that bounds the number of crossings of a loose meander in such a drawing.

Theorem 4 *Let $C(k)$ denote the maximum number of crossings in a loose meander where all lenses can be hit by k points. Then $C(k) \leq \mathrm{e} \cdot k!$, where $\mathrm{e} \approx 2.718$ is Euler's number.*

Proof: Let e, e' be a loose meander in standard representation, and let all lenses of $\mathcal{A}(e, e')$ be hit by the points $p_1, \ldots, p_k$. For each $i = 1, \ldots, k$, select an exit-curve e_i for p_i along e through e'. See Figure 6 for an example of an exit-curve. By Lemma 3 we know that e_i, e' is a loose meander, for every $i \in \{1, \ldots, k\}$. In addition, we claim the following two properties:

(P1) All lenses of $\mathcal{A}(e_i, e')$ are hit by the $k-1$ points $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_k$, for each $i \in \{1, \dots, k\}$.

(P2) Between any two crossings of e and e' from left to right, that is, in the order along e', there is at least one crossing of e' with one of the exit-curves e_i.

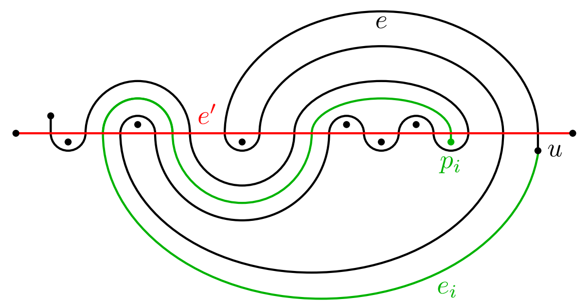


Figure 6: A loose meander e, e' with a point p_i and an exit-curve e_i for p_i.

Before proving these two properties, we show that they imply the statement of the theorem. More precisely, we prove by induction on k that

$$C(k) \le k! \cdot \sum_{s=0}^{k} \frac{1}{s!}.$$

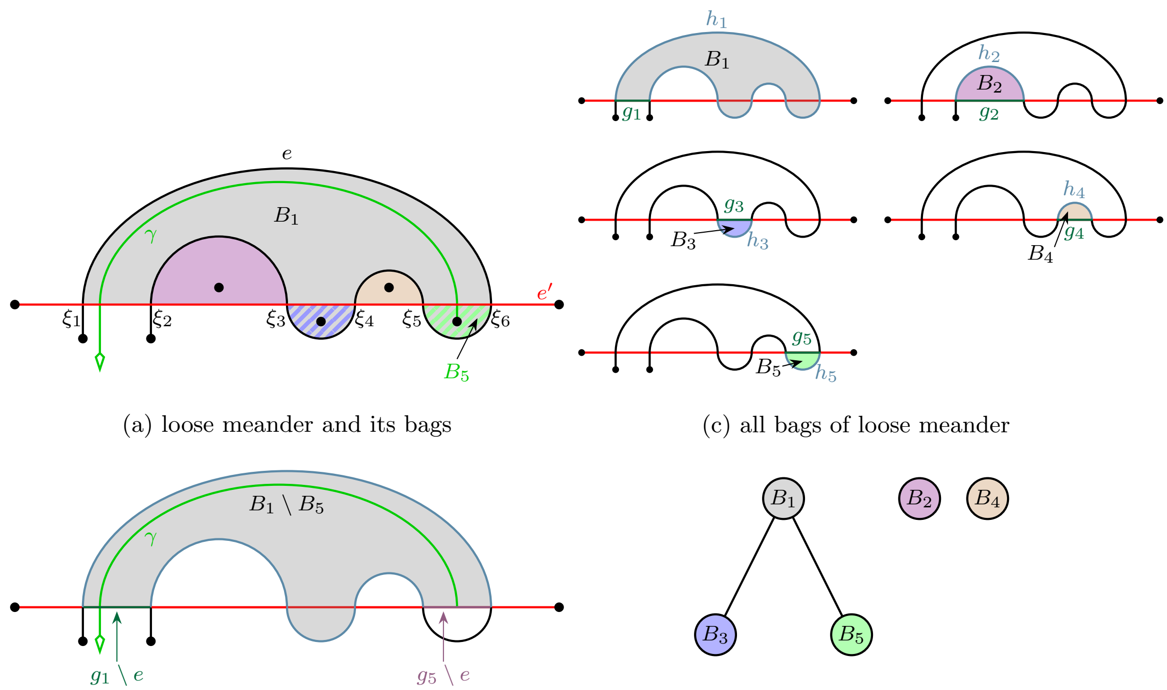
In the base case $k = 0$ we have $C(0) = 1 = 0!$. For general $k > 0$, from (P1) we see that the number X_i of crossings between e_i and e' is upper bounded by $C(k-1)$. From (P2) we obtain that $C(k) \le 1 + \sum_i X_i$. Combining these and using the inductive hypothesis we get

$$C(k) \le k \cdot C(k-1) + 1 \le k \cdot (k-1)! \cdot \sum_{s=0}^{k-1} \frac{1}{s!} + 1 = k! \cdot \sum_{s=0}^{k} \frac{1}{s!} < k! \cdot \mathrm{e}. \qquad \square$$

Bags and gaps. For the proof of the two claims we need some notation. Let $\xi_1, \xi_2, \dots, \xi_N$ be the crossings of e and e' indexed according to the left to right order along the horizontal curve e'. Let g_i and h_i be the subcurves of e' and e, respectively, between crossings ξ_i and ξ_{i+1}. Observe that $g_i \cup h_i$ forms a closed Jordan curve. The bounded region enclosed by $g_i \cup h_i$ is the *bag* B_i and g_i is the *gap* of the bag B_i, see Figure 7. A bag B_i where h_i does not cross e' corresponds to a minimal lens in $\mathcal{A}(e, e')$ and vice versa. The following observation is crucial.

Lemma 5 *For two bags B_i and B_j the open interiors are either disjoint or one is contained in the other.*

Proof: If $i = j$, then the statement holds trivially. Hence suppose that $i \neq j$. Assume that there exists a point p in the open interior of $B_i \cap B_j$. We have to show that $B_i \subseteq B_j$ or $B_j \subseteq B_i$. Let γ be a curve from p to some point in the exterior face of the arrangement $\mathcal{A}(e, e')$ with $\gamma \cap e = \emptyset$. (Such a curve exists because e is a curve, which has no self-intersections by definition, and so $\mathbb{R}^2 \setminus e$ is connected.) Moreover, we may take γ to be a combinatorially shortest such curve, in the sense that it intersects every edge of $\mathcal{A}(e, e')$ in at most one point, see Figure 7a for an example.



(a) loose meander and its bags

(c) all bags of loose meander

(b) allowed route to the outer face after leaving B_5

(d) containment order of loose meander

Figure 7: A loose meander and its bags

Every bag B_x is bounded by a closed Jordan curve $g_x \cup h_x$, where $h_x \subseteq e$. Therefore, in order to leave B_i and B_j, the curve γ must cross the gaps g_i and g_j, respectively. Moreover, by the shortness assumption on γ, once the curve exits a bag it cannot re-enter it. As $(g_i \setminus e) \cap (g_j \setminus e) = \emptyset$, the curve γ leaves one bag strictly before the other. Without loss of generality assume that γ leaves first B_i and then B_j. The open interior of B_i is disjoint from e, hence h_j is outside of it. Since γ leaves B_i only once, by crossing g_i we also have that g_j is outside the open interior of B_i, see Figure 7b. Hence, the closed curve $g_j \cup h_j$ and the open interior of B_i are disjoint. Since both contain p in the interior we have $B_i \subset B_j$. □

Lemma 5 implies that the containment order on the bags is a downwards branching forest. The minimal elements in the containment order are the minimal lenses in $\mathcal{A}(e, e')$. Now we are ready to prove Lemma 3, which was stated earlier in Section 2.

Lemma 3 *Let e, e' be a loose meander, and let p be a point in the interior of a minimal lens in $\mathcal{A}(e, e')$. Then every exit-curve for p along e through e' forms a loose meander with e'.*

Proof: Let L° be a minimal lens in $\mathcal{A}(e, e')$ and let p be a point in L°. Let γ be an exit-curve for p along e through e'. The second endpoint of γ is a point q in the outer face of $\mathcal{A}(e, e')$. Let $L^\circ = B_{i_1} \subset B_{i_2} \subset \ldots \subset B_{i_t}$ be the maximal chain of bags with minimal element L°.

In order to get from p to q, the curve γ must leave each of the bags $B_{i_1}, \ldots, B_{i_t}$. As γ is disjoint from e by definition, it can only leave a bag through its gap. Therefore, the curve γ crosses e' at the gaps $g_{i_1}, \ldots, g_{i_t}$, in this order. Moreover, as γ minimizes the number of crossings with e', it crosses each of these gaps exactly once. In other words, for every exit-curve the sequence of faces in the arrangement $\mathcal{A}(e, e')$ that it traverses is uniquely determined. As a consequence also the arrangement $\mathcal{A}(\gamma, e')$ is uniquely determined up to isomorphism.

We have to show that γ, e' is a loose meander. First we argue that γ, e' is an unlocked pair. Since e, e' is a loose meander e does not form a spiral with e'. Hence, there exists some curve γ' from a point $p' \in L^\circ$ to an endpoint of e such that γ' does not cross e and γ', e' is an unlocked pair. Among all eligible curves we select γ' to have a minimum number of crossings with e'. As γ' does not cross e, it also (like γ) crosses the gaps $g_{i_1}, \ldots, g_{i_t}$ in this order. However, it is not clear a priori that the curve γ' crosses each of these gaps only once and that it does not cross any other gaps. But we claim that this is the case, and so γ' has the same number, order, and type of crossings with e' as γ.

In order to prove the claim suppose for the sake of a contradiction that γ' has more crossings with e' than γ. As a consequence of the forest structure of the bag containment order, the curve γ' crosses every gap in the sequence $g_{i_1}, \ldots, g_{i_t}$ an odd number of times and every other gap an even number of times. Since γ' has more crossings with e' than γ there is some gap g which is crossed at least two times by γ'. In the next three paragraphs we argue in detail that there is a detour of γ' with ends at consecutive crossings on g. Such a detour can be shortcut to obtain a curve γ'' which can replace γ' and has fewer crossings with e'. This contradicts the choice of γ'.

Consider first the case that γ' crosses a gap g_{i_k}, for some $k \in \{1, \ldots, t\}$, for an odd number of times that is at least three. Order the crossings $c_1, \ldots, c_x$ of γ' with g_{i_k} as they appear along γ', when tracing the curve starting from p'. A crossing c_j, for j odd, corresponds to a point where γ' leaves the bag B_{i_k}. Conversely, for j even, a crossing c_j corresponds to a point where γ' enters the bag B_{i_k}. So the subcurve of γ' between c_{2i} and c_{2i+1} together with the subcurve of g_{i_k} between c_{2i} and c_{2i+1} forms a closed Jordan curve whose closed bounded region B'_i is contained in B_{i_k}. Given that γ' is a curve (and therefore does not cross itself), any two such regions B'_i and B'_j are either disjoint or one contains the other. As γ', e' form an unlocked pair, no endpoint of γ' lies in any of the regions B'_i since they form lenses. Hence, the first crossing c_1, that leaves B_{i_k} by definition, cannot lie in any of the regions B'_i. It follows that there exists a region B'_i so that c_{2i} and c_{2i+1} are consecutive crossings of γ' and g_{i_k} not only along γ' but also along g_{i_k}.

Let f be a point on γ' just (sufficiently close) before c_{2i}, and let ℓ be a point on γ' just after c_{2i+1}. We modify γ' by replacing the subcurve $\gamma'_{f\ell}$ from f to ℓ by a new curve δ from f to ℓ that (sufficiently closely) follows g_{i_k} so that δ is disjoint from e' and from $\gamma' \setminus \gamma'_{f\ell}$. We claim that the resulting curve $\gamma'' = (\gamma' \setminus \gamma'_{f\ell}) \cup \delta$ forms an unlocked pair with e'. To see this, observe that the arrangements $\mathcal{A}(\gamma', e')$ and $\mathcal{A}(\gamma'', e')$ differ only by the closed Jordan curve $\gamma_{f\ell} \cup \delta$, and the bounded side of this curve does not contain any endpoint of the edges e' and γ' (which has the same endpoints as γ''). Therefore, given that γ', e' is an unlocked pair, the same holds for γ'', e'. However, the curve γ'' has fewer crossings with e', in contradiction to the minimality in the definition of γ'.

The case where γ' crosses a gap that is not among $g_{i_1}, \ldots, g_{i_t}$ an even number of times can be handled analogously. The only difference is in the parity of the crossings c_i between γ' and g_{i_k} and that the bag B_{i_k} does not contain an endpoint of γ'. In conclusion we have shown that the arrangements $\mathcal{A}(\gamma', e')$ and $\mathcal{A}(\gamma, e')$ are isomorphic. In particular, the two curves γ and e' form an unlocked pair, as claimed.

To show that γ, e' is a loose meander it remains to show that γ does not form a spiral with e'. So consider a minimal lens L in $\mathcal{A}(\gamma, e')$. It suffices to show that there exists some curve δ from some point $d \in L$ to an endpoint of γ so that δ does not cross γ and δ, e' is an unlocked pair. To show this we argue as follows.

First, we claim that there exists a minimal lens L' formed by e and e' with $L' \subset L$. Assuming this holds, we select $d \in L'$ arbitrarily and let δ be an exit-curve for d along e through e'. See Figure 8 for illustration. By the first part of this proof we know that δ, e' is an unlocked pair. Moreover, as both δ and γ are exit-curves along e through e', we can select δ so that it does not cross γ. (Due to the forest structure of the bag containment order, if δ should reach a bag visited by γ, then it can simply follow γ from this point onward.) So we have found a curve δ to certify that γ does not form a spiral with e'.

In order to complete the proof, it remains to establish the claim about the existence of a lens $L' \subset L$ formed by e and e'. First note that $e \cap L \neq \emptyset$ because γ aims to minimize the number of crossings with e' by definition. If e was disjoint from L, then we could remove the two crossings between γ and e' that form L and route γ along e' instead. Summing up the arguments and definitions given before: (1) e does not cross γ, (2) no endpoint of e is inside L, and (3) e does not cross itself. It follows that there is a lens formed by e and e' inside L, and this lens in turn contains a minimal lens L' formed by e and e'. □

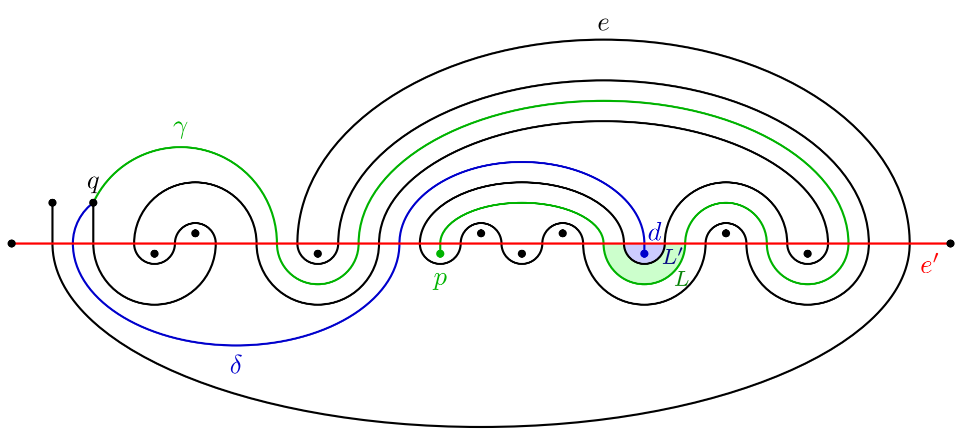


Figure 8: An exit-curve γ (green) for p along e through e' that forms a lens L with e' and an exit-curve δ (blue) for d along e through e' that forms a lens $L' \subset L$ with e'.

As a next step, we reformulate and prove the second claim (P2).

Lemma 6 *For each pair ξ_i, ξ_{i+1} of consecutive crossings along e' there is a lens L such that the exit path e_j of a point $p_j \in L$ crosses e' between ξ_i and ξ_{i+1}.*

Proof: The subcurve of e' between ξ_i and ξ_{i+1} forms the gap g_i of the bag B_i. Let L be any of the minimal elements below B_i in the containment order of bags, that is, $L \subseteq B_i$ is a minimal lens formed by e and e'. By assumption, there exists a point $p_j \in L \subseteq B_i$. It follows that the exit-curve e_j of p_j crosses g_i, that is, the curve e_j crosses e' between ξ_i and ξ_{i+1}. □

It remains to prove (P1). Recall that (P1) states that all lenses formed by e_i and e' are hit by the $k-1$ points $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_k$.

Proof of (P1): We know by Lemma 3 that e_i, e' is an unlocked pair, that is, the endpoints of e_i and e' belong to the same face of the arrangement $\mathcal{A}(e_i, e')$, which is the outer face. In particular, the endpoint p_i of e_i belongs to the outer face (and is not in any lens) of $\mathcal{A}(e_i, e')$. Since by the proof of Lemma 3 every lens of $\mathcal{A}(e_i, e')$ contains a lens of $\mathcal{A}(e, e')$, it also contains one of the points $p_1, \ldots, p_k$ that hit all lenses of $\mathcal{A}(e, e')$ by assumption. Altogether, all lenses of $\mathcal{A}(e_i, e')$ are hit by the $k-1$ points $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_k$. □

4 Lower Bound

In this section we discuss constructions for loose meanders with many crossings. In the introduction we explained the doubling construction which when initialized with a simple crossing yields a loose meander with k minimal lenses and 2^k crossings with the horizontal edge, see Figure 3. The doubling technique initialized with the loose meander from Figure 4a yields a loose meander with k minimal lenses and $2^k + 2^{k-2}$ crossings.

We now define a *product* operation for loose meanders. Let M_1, M_2 be loose meanders such that M_i has k_i minimal lenses and c_i crossings. Then the product $M_1 \otimes M_2$ is a loose meander with $k_1 + k_2$ minimal lenses and $c_1 \cdot c_2$ crossings. We note in passing that the product can be defined in the more general context of open meanders.

Let M_1 be drawn such that all the crossings are in the interval $(0, 1)$ of the x-axis. Draw M_2 with a pen of thickness 1, the meander edge will appear as a tube of width 1 in the drawing. Now cut the meander-edge M_1 open along the x-axis. We call the resulting drawings H_a and H_b where H_a denotes the part of M_1 above the x-axis and H_b the one below. We place H_a before the first crossing of M_2 with the x-axis and H_b after the last crossing. It may be necessary to rotate one of H_a or H_b by 180° to make their left sides be on the left side of the tube formed by M_2. Then we draw a bundle of c_1 parallel curves in the M_2-tube such that each curve connects the two copies of a crossing point of M_1. Figure 9 illustrates the construction. The orientation of $M_1 \otimes M_2$ is inherited from M_1, that is, the starting point of the oriented meander-edge of M_1 is the starting point for the meander-edge of the product.

From the construction it is obvious that the product $M_1 \otimes M_2$ has $k_1 + k_2$ minimal lenses and $c_1 \cdot c_2$ crossings. Less obvious is that the product is indeed a loose meander, that is, it has no deadlock and no spiral.

Proposition 7 *The product $M_1 \otimes M_2$ of two loose meanders has no deadlock and forms no spiral, that is, it is a loose meander.*

Proof: Let e_i be the meander-edge of M_i. Since M_1 and M_2 form no deadlock the end-points of e_1 and e_2 are accessible from the outside in M_1 and M_2. This accessibility is inherited by the endpoints of the meander-edge $e_1 \otimes e_2$ of the product.

Let e, e' be a loose meander and v be a vertex in one of its minimal lenses. Recall that the exit-path of v is a curve γ_v connecting v to the outside such that γ_v avoids e and has as few crossings with e' as possible. Edge e has no spiral around v if and only if the exit-path γ_v has no deadlock with e'.

Now consider a lens-vertex v of the product $M_1 \otimes M_2$. The lens of v can clearly be assigned to M_1 or M_2. Let v belong to a lens of M_1 and note that v belongs to one of H_a or H_b. Let γ_v be an exit-curve of v in M_1. The corresponding exit-curve γ'_v of v in the product is the stretched and bent version of γ_v. For $s \in \{a, b\}$, if $v \in H_s$, then γ_v has no arc in H_s enclosing v, as such an arc would create a deadlock. In H_s the arcs of γ_v and γ'_v are identical, hence γ'_v has no arc in H_s enclosing v, that is, γ'_v has no deadlock and $M_1 \otimes M_2$ has no spiral around v.

Now let v belong to a lens of M_2. Consider the exit-path γ_v in the drawing of M_2 where e_2 is drawn with the thickness of the tube. After placing the stretched M_1 in the tube the path γ_v remains an exit-path for v in the product. The path γ_v still has no deadlock with e' whence $M_1 \otimes M_2$ has no spiral around v. □

We remark that the set of meanders forms a monoid with the product operation, that is, the product is associative and has a neutral element, the simple crossing. The product fails to be commutative.

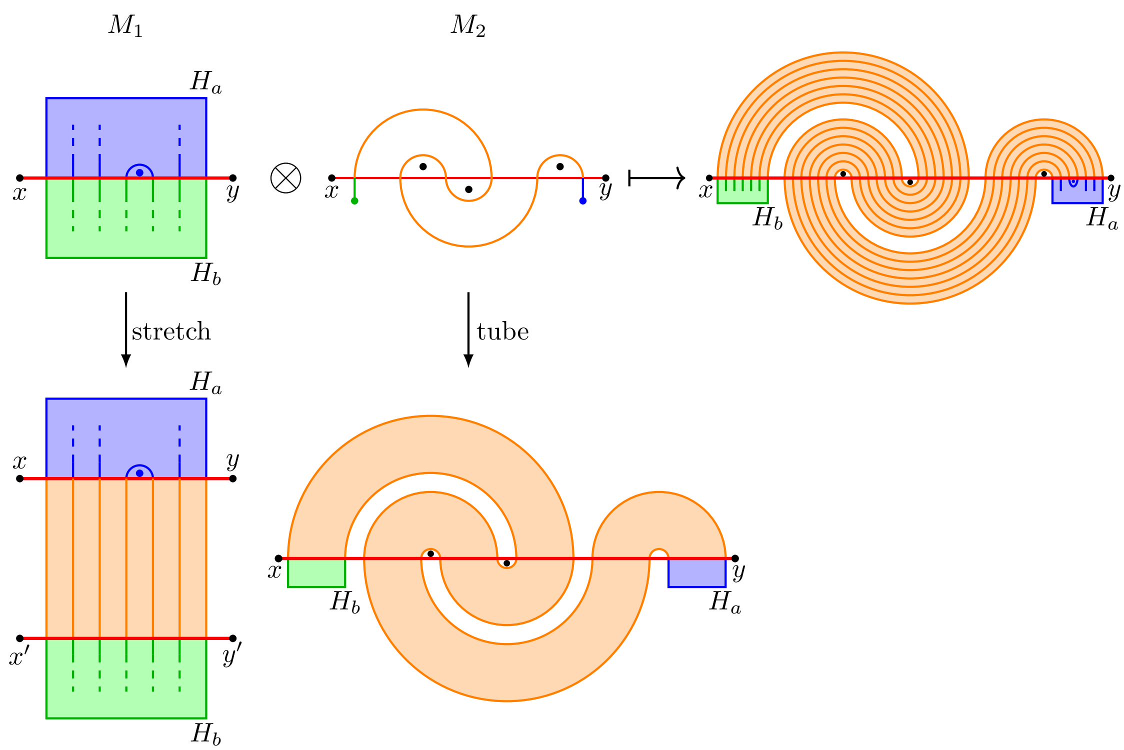


Figure 9: The product $M_1 \otimes M_2$ of two loose meanders.

Let M be the loose meander with 5 crossings and 2 lenses depicted in Figure 4a. Repeatedly taking products with M we obtain loose meanders $M^{\otimes k}$ with 5^k crossings and $2k$ lenses as illustrated in Figure 10. We summarize:

Corollary 8 *There are loose meanders with $2k$ minimal lenses and 5^k crossings.*

It has been shown in fifth author's PhD thesis [14] that the edge pair e, e' of the loose meander obtained by iterated doubling initialized with a simple crossing can be completed to a star-simple drawing of K_n. An obvious question is whether loose meanders with $2k$ lenses and 5^k crossings can also be integrated in a star-simple drawing of K_{2k+4}. It turns out that the product construction is compatible with completability.

Let M be a loose meander formed by the edge pair (e, e'). A *completion of* M is a star-simple drawing of K_n with no empty lens that contains M, i.e. the two edges e and e', and moreover, every crossing of an edge e'' of the drawing with the horizontal line supporting e' is a crossing of e'' and e'. A loose meander (e, e') is *completable* if a completion of (e, e') exists.

Theorem 9 *Let M_1 and M_2 be completable loose meanders. Then also $M_1 \otimes M_2$ is a completable loose meander.*

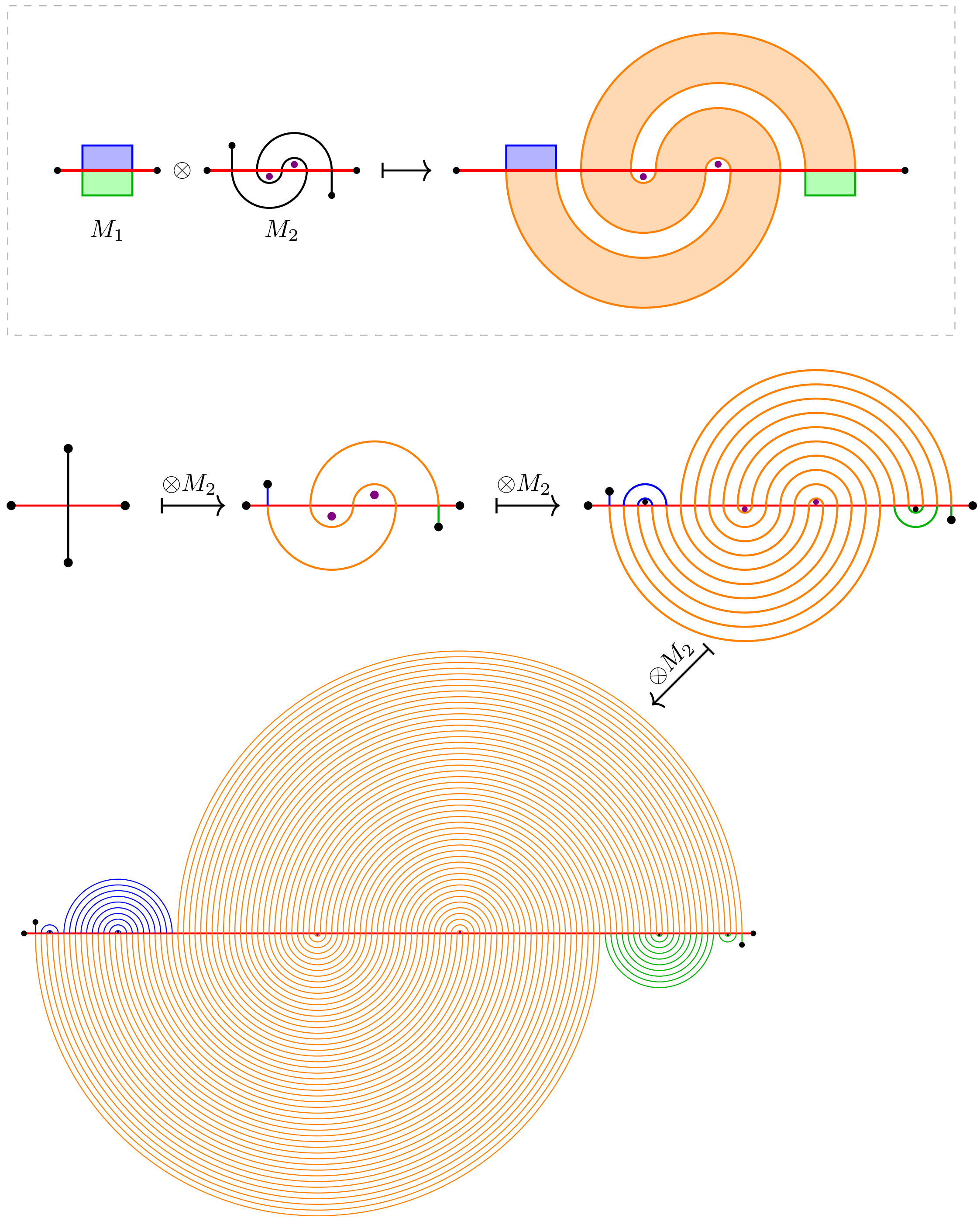


Figure 10: The exponentiation of the drawing with five crossings and two lens vertices yields an exponential number of crossings with base $\sqrt{5}$.

Proof: We will produce a drawing of the complete graph containing $M_1 \otimes M_2$. The drawing is constructed in three steps. Note that we will use a drawing of K_8 deferring from the one given in Figure 4. The chosen complete drawing has the advantage that the end vertices of the straight-line edge are embedded in the outer face.

Let $M_1 = (e_1, e')$ have k minimal lenses and let u_0, u_1 and w_0, w_1 be the vertices of e_1 and e', respectively. Let $\mathcal{D}_1$ be a completion of M_1. The vertex set of $\mathcal{D}_1$ shall be $V_1 = \{v_1, \ldots, v_k, u_0, u_1, w_0, w_1\}$. Now delete the edge e' from $\mathcal{D}_1$ and for $i = 0, 1$ split w_i into w_i' and w_i'' so that edges approaching w_i from the upper halfspace end at w_i' and edges approaching w_i from the lower halfspace end at w_i''. Now move w_0' and w_1' into the upper halfspace and w_0'', w_1'' into the lower halfspace, respectively. This yields a drawing $\mathcal{D}_1^*$. We now consider the 'product' of $\mathcal{D}_1^*$ with $M_2 = (e_2, e')$: Cut $\mathcal{D}_1^*$ along the segment which used to be e'. Let $\mathcal{D}_a$ and $\mathcal{D}_b$ be the parts above and below the segment, respectively. For later use we let V_a and V_b be the vertices that belong to $\mathcal{D}_a$ and $\mathcal{D}_b$, respectively. Place $\mathcal{D}_a$ and $\mathcal{D}_b$ with appropriate orientation at the beginning and the ending of the M_2-tube obtained by thickening e_2 and draw a bundle of parallel curves in the M_2-tube such that each curve connects the two copies of a crossing point of $\mathcal{D}_1^*$. Let $\mathcal{D}_1^2$ denote the subdrawing of $\mathcal{D}_1^*$ obtained by deleting w_0', w_1', w_0'', w_1'' and the incident edges. Now $\mathcal{D}_1^2$ is a drawing of K_{k+2}, see Figure 11. Before continuing we note that the drawing $\mathcal{D}_1^2$ is isomorphic to a subdrawing of $\mathcal{D}_1^*$ and, therefore, star-simple. For each vertex v of $\mathcal{D}_1^2$ we now define a *hose*. If (v, w_0') is an edge of $\mathcal{D}_1^*$ the hose h_v of v is a narrow corridor that follows the edge (v, w_0') to connect v with the outer face of $\mathcal{D}_1^2$, otherwise the hose h_v follows (v, w_0''). Since in $\mathcal{D}_1$ the star of w_0 has no crossings we can adapt the widths of the hoses so that they are pairwise disjoint.

Now consider a completion $\mathcal{D}_2$ of M_2. Let M_2 have ℓ minimal lenses and let x_0, x_1 and w_0, w_1 be the vertices of the meander edges e_2 and e', respectively (since w_0, w_1 do not appear in $\mathcal{D}_1^2$ we can use them again). The vertex set of $\mathcal{D}_2$ shall be $V_2 = \{y_1, \ldots, y_\ell, x_0, x_1, w_0, w_1\}$. Consider the subdrawing $\mathcal{D}_2^*$ of $\mathcal{D}_2$ induced by the vertices $V_2 \setminus \{x_0, x_1, w_0, w_1\}$. In this drawing we can integrate a copy of $\mathcal{D}_1^2$ along the curve that used to represent e_2, like the gray tube in Figure 13. We assume that $\mathcal{D}_a$ is at the end of the e-tube which corresponds to x_0 while $\mathcal{D}_b$ is at the end corresponding to x_1. The next task is to introduce edges connecting vertices of $\mathcal{D}_2^*$ with vertices of $\mathcal{D}_1^2$. We assume that there is a box B_a such that the hoses of vertices in V_a all end on one side of B_a. For each vertex y_i of $\mathcal{D}_2^*$ we draw a bundle of $|V_a|$ essentially parallel curves from y_i to the other side of the box B_a. By drawing the bundle of y_i along the curve of the edge (y_i, x_0) of $\mathcal{D}_2$ we can make sure that the bundles inherit the disjointness from the star of x_0 in $\mathcal{D}_2$. For each v of $\mathcal{D}_a$ route as many internally disjoint curves starting at v in the hose h_v to B_a as there are bundles attached to B_a. Finally, complete the partially drawn edges which end on the two sides of B_a by connecting them in a crossing minimal way so that each $v \in V_a$ gets an edge to each y_i. Figure 12 shows an example of the connections within a box. Note that a crossing pair of edges corresponds to distinct bundles on both sides of the box so that we only introduce crossings which are unproblematic for a star-simple drawing.

Vertices of $\mathcal{D}_b$ are connected to the y_i in a similar way. This time we route a bundle of size $|V_b|$ from y_i along the curve of the former edge (y_i, x_1) to a box B_b. Let $\mathcal{D}'$ be the drawing obtained in this step. Figure 13 shows an example of a combined drawing $\mathcal{D}'$.

It remains to add the horizontal edge $e' = (w_0, w_1)$ to the drawing and connect the two vertices with all the vertices of $\mathcal{D}'$. By looking at the edge (y_i, w_0) in $\mathcal{D}_2$ we see that from y_i we can draw a curve to the outer face that avoids crossings with edges from the star of y_i. For a vertex $v \in V_a \cup V_b$ we also find such a curve. For $v \in V_a$, draw the curve leftmost in the hose of v and continue trough B_a so that it ends left of all the existing bundles. This can be done for all the vertices of V_a such that their curves are disjoint and form a new bundle on the other side of B_a. From there the

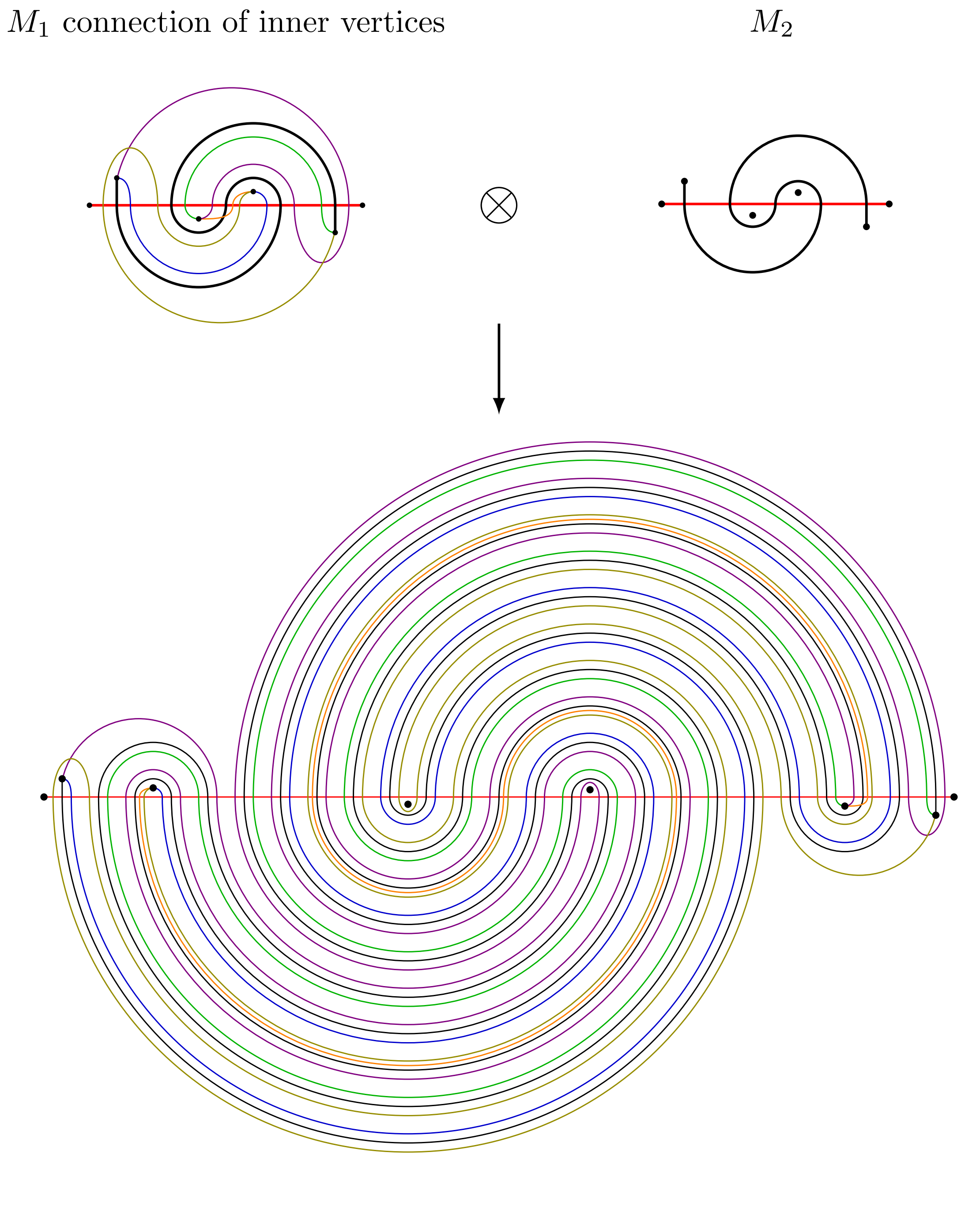


Figure 11: Illustration of completion of the inner meanders derived by M_1

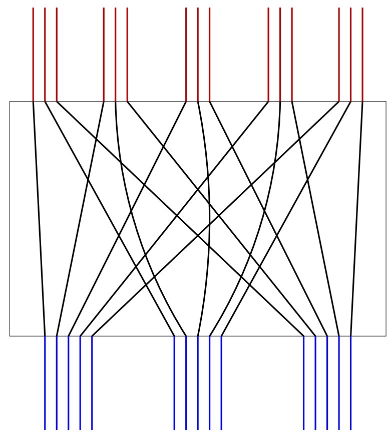


Figure 12: Connecting the partial edges in a box B_a.

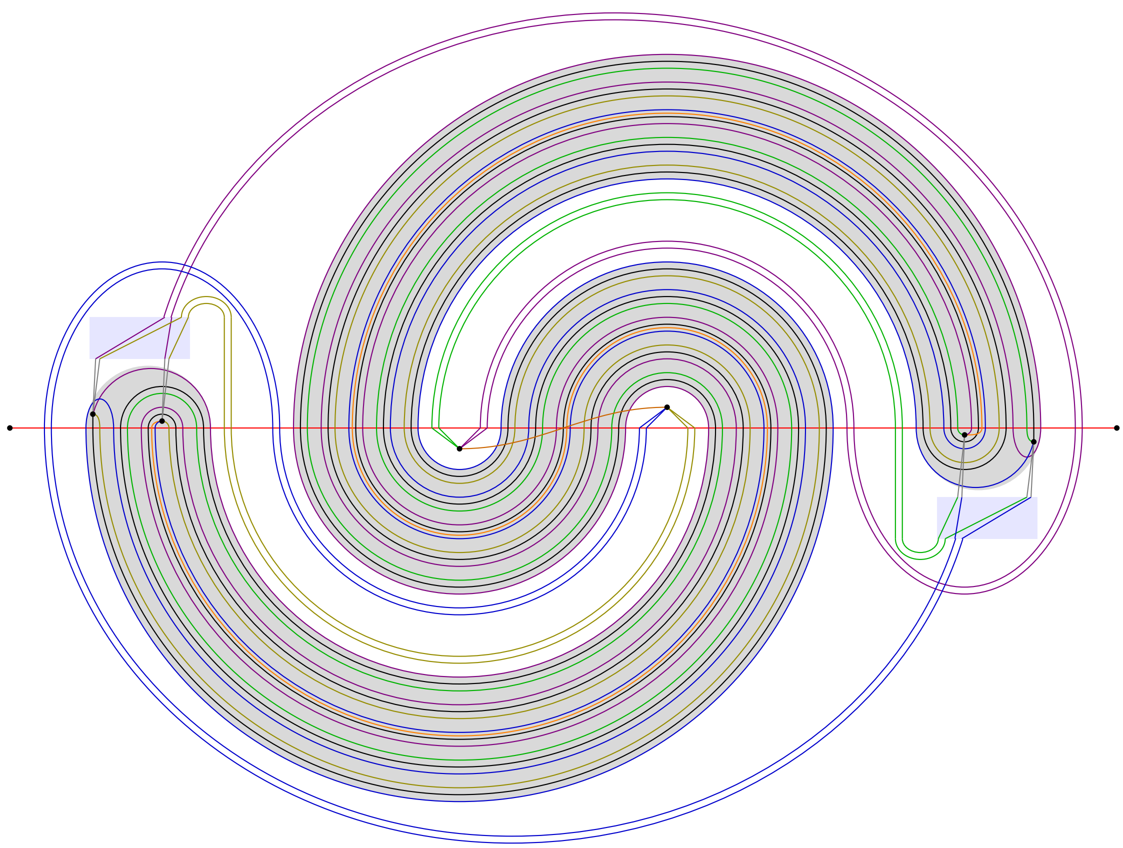


Figure 13: Connection of the M_1 vertices to the M_2 lens vertices

bundle can follow the curve that represents the edge (x_0, w_0) of $\mathcal{D}_2$ to the outer face.

Each of the curves that lead from vertices of $\mathcal{D}'$ to the outer face can be used to obtain two edges, one that connects to w_0 and one that connects to w_1. All these edges can be realized so that the stars of w_0 and w_1 are simple. Figure 14 shows an example. This completes the proof of Theorem 9. □

5 Crossings in complete drawings and open questions

In Section 3 we gave an upper bound on the number of crossings of a loose meander. Accounting for the four endpoints of the two edges of the loose meander we have $k \leq n - 4$ in Theorem 4. Therefore, we obtain that $3(n-4)!$ is an upper bound on the number of crossings of a pair of edges in a star-simple drawing of K_n without empty lens. This directly implies that the drawing of K_n has at most $n!$ crossings.

In Section 4 we introduced the product of loose meanders. Based on the product we could construct star-simple drawings with exponentially many crossings between a pair of edges with the basis of the exponential function being $\sqrt{5}$.

Figure 15 shows a loose meander with 4 minimal lenses and 27 crossings. It was obtained using a corrected version of the algorithm described in [1, 4]. Using products we get a family of loose meanders with $4k$ minimal lenses and $27^{k/4}$ crossings. We have convinced ourselves that the example and therefore the corresponding family can be completed. By Theorem 9, this raises the basis of the exponential function from $5^{1/2} \sim 2.236$ to $27^{1/4} \sim 2.28$.
We leave the following problems:

- How many crossings can a star-simple drawing of K_n have? In particular is the growth singly exponential or larger?
- How many crossings can a loose meander with k lenses have? If the growth is singly exponential, what is the basis?
- Is every loose meander completable to a star-simple drawing of a complete graph without empty lens?

Acknowledgments

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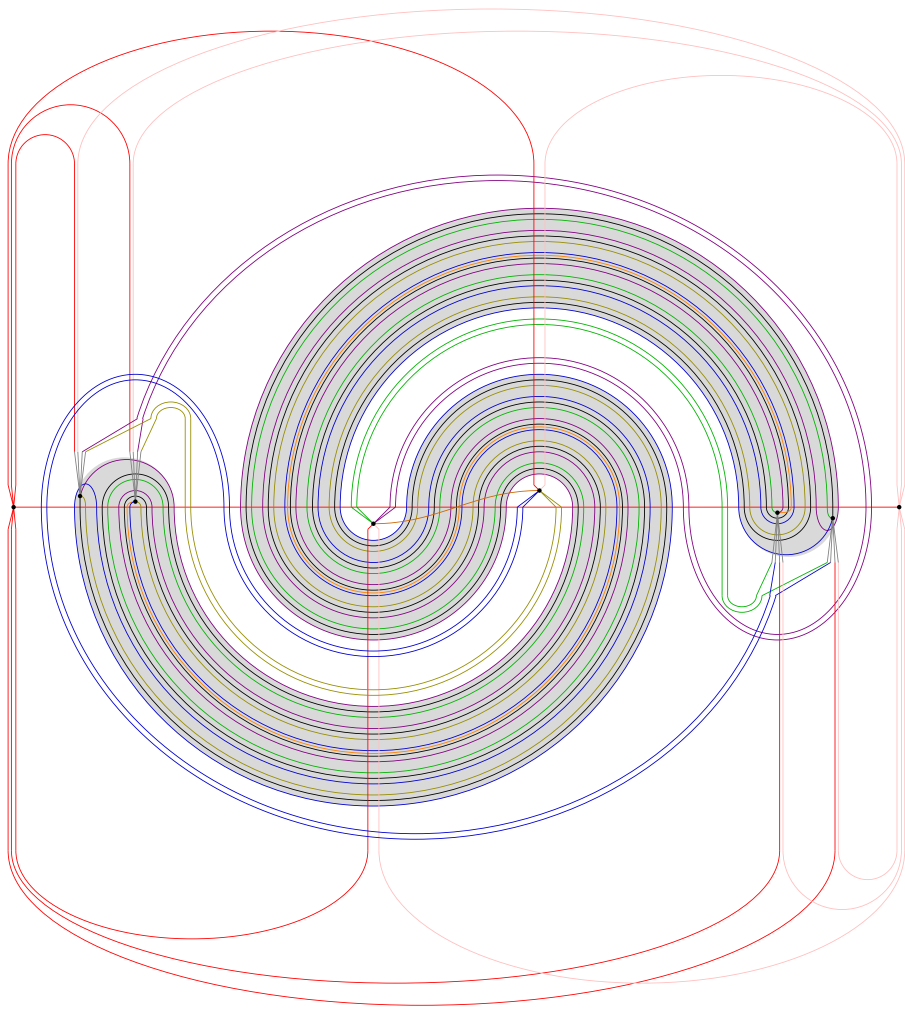

Figure 14: Connection of the vertices of e' to the vertices of M_1 and M_2.

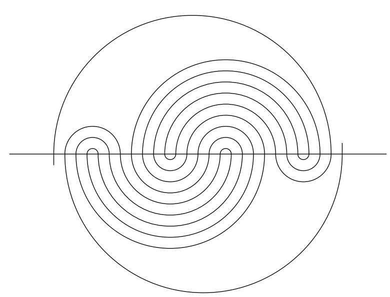

Figure 15: Meander with 4 lenses and 27 crossings.

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