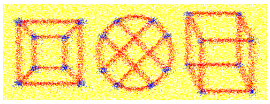
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Parameterized Algorithms for Queue Layouts

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Abstract. An *h-queue layout* of a graph G consists of a *linear order* of its vertices and a partition of its edges into h sets, called *queues*, such that no two independent edges of the same queue nest. The minimum h such that G admits an h-queue layout is the *queue number* of G. We present two fixed-parameter tractable algorithms that exploit structural properties of graphs to compute optimal queue layouts. As our first result, we show that deciding whether a graph G has queue number 1 and computing a corresponding layout is fixed-parameter tractable when parameterized by the treedepth of G. Our second result then uses a more restrictive parameter, the vertex cover number, to solve the problem for arbitrary h.

1 Introduction

An *h-queue layout* of a graph G is a linear layout of G consisting of a *linear order* of its vertices and a partition of its edges into h sets, called *queues*, such that no two independent edges of the same queue nest [32]; see Fig. 1 for an illustration. The *queue number* $\mathrm{qn}(G)$ of a graph G is the minimum number of queues in any queue layout of G. While such linear layouts represent an abstraction of various problems such as, for instance, sorting and scheduling [6,40], they also play a central role in three-dimensional graph drawing. It is known that a graph class has bounded queue

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number if and only if every graph in this class has a three-dimensional crossing-free straight-line grid drawing in linear volume [15, 22]. We refer the reader to [24, 37] for further references and applications. Moreover, it is worth recalling that *stack layouts* [36, 42] (or *book embeddings*), which allow nesting edges but forbid edge crossings, form the "dual" concept of queue layouts.

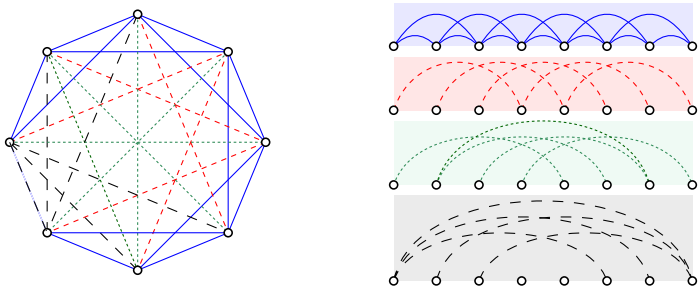


Figure 1: A 4-queue layout of K_8.

A rich body of literature is concerned with the study of upper bounds for the queue number of several planar and non-planar graph families (see, e.g., [1, 4, 5, 9, 14, 19, 21, 22, 31, 41] and also [23] for additional references). For instance, a graph of treewidth w has queue number at most $\mathcal{O}(2^w)$ [41], while every proper minor-closed class of graphs (including planar graphs) has constant queue number [21].

Of particular interest to us is the corresponding recognition problem, which we denote by Queue Number: Given a graph G and a positive integer h, decide whether G admits an h-queue layout. In 1992, in a seminal paper, Heath and Rosenberg proved that 1-Queue Number, i.e., the restriction of Queue Number to instances with $h = 1$, is NP-complete [32]. In particular, they characterized the graphs that admit queue layouts with only one queue as the arched leveled-planar graphs, and showed that the recognition of these graphs is NP-complete [32].

Since Queue Number is NP-complete even for a single queue, it is natural to ask under which conditions the problem can be solved efficiently. For instance, it is known that if the linear order of the vertices is given (and the aim is thus to simply partition the edges of the graph into h queues), then the problem becomes solvable in polynomial time [31]. We follow up on recent work made for the stack number [7] and initiate the study of the parameterized complexity of Queue Number by asking under which parameterizations the problem is fixed-parameter tractable. In other words, we are interested in whether (1-)Queue Number can be solved in time $f(k) \cdot n^{\mathcal{O}(1)}$ for some computable function f of the considered structural parameter k of the n-vertex input graph G. Parameterized complexity is a modern algorithmic paradigm that allows us to obtain a more fine-grained understanding of the complexity of difficult problems, and it has recently gained increasing attention in the graph drawing community; see the recent Dagstuhl seminar for more information about the paradigm's limitations and applicability [27]. Parameterized complexity has been successfully applied on graph drawing problems related to 1-planarity [3, 26], crossing minimization [33], layered graph drawing [20], linear layouts [7, 9], orthogonal planarity [16], upward planarity [10], and others.

As our main result, we show 1-Queue Number is fixed-parameter tractable parameterized by the *treedepth* of the input graph (Section 3). We remark that treedepth is a fundamental graph parameter with close ties to the theory of graph sparsity (see, e.g., [35]). The main technique used by the algorithm is iterative pruning, where we recursively identify irrelevant parts of the input and remove these until we obtain a bounded-size equivalent instance (a *kernel*) solvable by

brute force. While the iterative pruning technique has already been used in a few other algorithms that exploit treedepth [28–30], the unique challenge here lay in establishing that the removal of seemingly irrelevant parts of the graph cannot change NO-instances to YES-instances. The proof of this claim, formalized in Lemma 1, uses a new type of block decomposition of 1-queue layouts.

For our second result, we turn to the general Queue Number problem. Here, we establish fixed-parameter tractability when parameterized by a larger parameter, namely the *vertex cover number* (Section 4). This result is also achieved by kernelization and forms a natural counterpart to the recently established fixed-parameter tractability of computing the stack number under the same parameterization [7], see also recent work on upward book thickness [9], although the technical arguments and steps of the proof differ due to the specific properties of queue layouts.

2 Preliminaries

We adopt standard notation and terminology from graph theory [17]. We can assume that our input graphs are connected, as the queue number of a graph is the maximum queue number over all its connected components. Given a graph $G = (V, E)$ and a vertex $v \in V$, let $N(v)$ be the set of neighbors of v in G. Also, for $r \in \mathbb{N}$, we denote by $[r]$ the set $\{1, \ldots, r\}$. An *h-queue layout* of G is a pair $\langle \prec, \sigma \rangle$, where $\prec$ is a linear order of V, and $\sigma \colon E \to [h]$ is a function that maps each edge of E to one of h sets, called queues. In an h-queue layout $\langle \prec, \sigma \rangle$ of G, it is required that no two independent edges in the same queue *nest*, that is, for no pair of edges $uv, wx \in E$ with four distinct end-vertices and $\sigma(uv) = \sigma(wx)$, the vertices are ordered as $u \prec w \prec x \prec v$. Given two distinct vertices u and v of G, u is to the *left* of v if $u \prec v$, else u is to the *right* of v. Note that a 1-queue layout of G is simply defined by a linear order $\prec$ of V and $\sigma \equiv 1$.

We assume familiarity with basic notions in parameterized complexity [12, 18]. We consider two graph parameters for our algorithms: treedepth and vertex cover number.

2.1 Treedepth

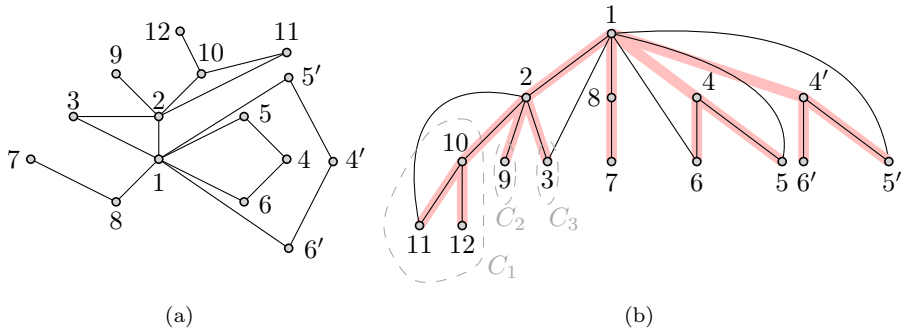
Treedepth is a parameter closely related to treewidth, and the structure of graphs of bounded treedepth is well understood [35]. A useful way of thinking about graphs of bounded treedepth is that they are (sparse) graphs with no long paths. We formalize a few notions needed to define treedepth, see also Fig. 2 for an illustration. A *rooted forest* $\mathcal{F}$ is a disjoint union of rooted trees. For a vertex x in a tree T of $\mathcal{F}$, the *height* (or *depth*) of x in $\mathcal{F}$ is the number of vertices in the path from the root of T to x. The *height of a rooted forest* is the maximum height of a vertex of the forest. Let $V(T)$ be the vertex set of any tree $T \in \mathcal{F}$.

Definition 1 (Treedepth) *Let the* closure *of a rooted forest $\mathcal{F}$ be the graph $clos(\mathcal{F}) = (V_c, E_c)$ with the vertex set $V_c = \bigcup_{T \in \mathcal{F}} V(T)$ and the edge set $E_c = \{xy \mid x$ is an ancestor of y in some $T \in \mathcal{F}\}$. A* treedepth decomposition *of a graph G is a rooted forest $\mathcal{F}$ such that $G \subseteq clos(\mathcal{F})$. The* treedepth *$td(G)$ of a graph G is the minimum height of any treedepth decomposition of G.*

An optimal treedepth decomposition can be computed by an FPT algorithm.

Proposition 1 ([39]) *Given an n-vertex graph G and an integer k, it is possible to decide whether G has treedepth at most k, and if so, to compute a treedepth decomposition of G of height at most k in time $2^{\mathcal{O}(k^2)} \cdot n$.*

Proposition 2 ([35]) *Let G be a graph and $td(G) \leq k$. Then G has no path of length 2^k.*



(a) (b)

Figure 2: (a) A graph G and (b) a treedepth decomposition T of G of height 4, in which T has light red edges. In particular, $P_2 = \{1, 2\}$, $A_2 = \{C_1, C_2, C_3\}$, and $m_2 = 3$. The tree is highlighted in red.

2.2 Vertex cover number

A *vertex cover* C of a graph $G = (V, E)$ is a subset $C \subseteq V$ such that each edge in E has at least one incident vertex in C. The *vertex cover number* of G, denoted by $\tau(G)$, is the size of a minimum vertex cover of G. Observe that $td(G) \leq \tau(G) + 1$: it suffices to build $\mathcal{F}$ as a single path with vertex set C and with the leaves $V \setminus C$ all placed below the last vertex of this path. Computing an optimal vertex cover of G is FPT.

Proposition 3 ([11]) *Given an n-vertex graph G and a constant τ, it is possible to decide whether G has vertex cover number at most τ, and if so, to compute a vertex cover C of size τ of G in time $\mathcal{O}(2^\tau + \tau \cdot n)$.*

3 Parameterization by Treedepth

In this section, we establish our main result: the fixed-parameter tractability of 1-Queue Number parameterized by treedepth. We formalize the statement below.

Theorem 1 *Let G be a graph with n vertices and constant treedepth k. We can decide in $\mathcal{O}(n)$ time whether G has queue number one, and, if this is the case, we can also output a 1-queue layout of G in the same time.*

3.1 Algorithm Description

Since we assume G to be connected, any treedepth decomposition of G consists of a single tree T. Now, suppose that a treedepth decomposition T of G of depth k is given. For a vertex t of T, let P_t be the set of ancestors of t including t, let A_t be the set of connected components of $G - P_t$ that contain a child of t, and m_t be the maximum number of vertices in a component in A_t; see also Fig. 2(b). Notice that $|A_t|$ is precisely the number of children of t in T.

Observation 1 *For every component $C \in A_t$ and for every vertex $v \in C$, it holds that $N(v) \subseteq C \cup P_t$. Thus, $|C \cup P_t| \leq m_t + k$.*

Now, we define the following equivalence over components in A_t. Components $C, D \in A_t$ satisfy $C \sim D$ if and only if there exists a bijective renaming function $\eta_{C,D} : C \to D$ over (the vertices of) C, D such that each vertex $c_i \in C$ has a *counterpart* $\eta_{C,D}(c_i) = d_i \in D$ that satisfies: (i) $N(c_i) \cap P_t = N(d_i) \cap P_t$ and (ii) c_i is adjacent to $c_j \in C$ if and only if d_i is adjacent to its counterpart d_j. When C, D are clear from the context, we may drop the subscript of η for brevity. For an example of two equivalent components, see the subtrees rooted at vertices 4 and $4'$ in Fig. 2(b).

By Observation 1, the number of equivalence classes of $\sim$ is upper-bounded by the number of possible graphs on $k + m_t$ vertices, which is at most $2^{(k+m_t)^2}$. The next observation allows us to propagate the bounds formalized by the notation above from children towards the root.

Observation 2 *If for a vertex t of T there exist integers a, b such that each child q of t satisfies $|A_q| \leq a$ and $m_q \leq b$, then $m_t \leq (a \cdot b) + 1$.*

The main component of our treedepth algorithm is Lemma 1, stated below. Intuitively, applying Lemma 1 bottom-up on T (together with Observation 2) allows us to iteratively remove subtrees from T while preserving the (non-)existence of a hypothetical solution—in particular, we will be able to prune subtrees of parents with a very large number of children until we reach an equivalent instance where each vertex has a bounded number of children. To formalize the meaning of "very large", we define the following function for $i \geq 2$ (recalling that k is the depth of T):

$$\#children(k, i) = \left(\left((2^{(k+1)} + 1)^{size(k,i)^2} + 1\right) \cdot (size(k, i) + k)!\right) \cdot 2^{(k+size(k,i))^2},$$

where $size(k, i)$ is a recursively defined function that captures the size bound given by Observation 2 as follows:

- $size(k, i) = (size(k, i-1) \cdot \#children(k, i-1)) + 1$ for $i \geq 2$, and
- $size(k, 1) = \#children(k, 1) = 0$.

As an example, we note that while $size(k, 2) = 1$, the value of $\#children(k, 2)$ is already in $k^{\Theta(k)}$ and both functions experience an exponential jump with each increase of i from there on. Intuitively, the precise values of the functions are set to guarantee that if one has successfully completed pruning for subtrees on lower levels of the treedepth decomposition, and if at the same time the number of children of a vertex at depth i is greater than $\#children(k, i)$, we will find an equivalence class at level i that is sufficiently large to guarantee the correctness of the pruning step. This intuition is formalized in the aforementioned Lemma 1 (and readers are invited to compare the definitions of these functions with the way they are used in its proof):

Lemma 1 *Assume G has a vertex t at depth i in T with the property that $|A_t| \geq \#\text{children}(k, i)$, but $m_t \leq \text{size}(k, i)$ and every descendant q of t in T satisfies that $|A_q| \leq \#\text{children}(k, i-1)$. Then there exists a component B of A_t such that $G - B$ has queue number one if and only if G has queue number one. Moreover, B can be computed in time $\mathcal{O}(\text{size}(k, i)! \cdot \#\text{children}(k, i)^2)$.*

The proof of the lemma is deferred to Section 3.2. Before proceeding, we show how Lemma 1 is used to obtain Theorem 1.

Proof: [of Theorem 1] We start by applying Proposition 1 to compute a treedepth decomposition T of G of depth at most k. Consider now vertices at depth $k - 1$ in T, i.e., vertices whose children are all leaves in T, and set $i = 2$. Observe that every vertex v at this depth satisfies

$m_v \leq \mathit{size}(k,2)$ since $\mathit{size}(k,2) = 1$ and $m_v = 1$. If $|A_v| \geq \#\mathit{children}(k,2)$, we apply Lemma 1 to obtain an equivalent graph with fewer vertices and restart on that graph. Otherwise, every vertex v at depth $k-1$ satisfies $|A_v| < \#\mathit{children}(k,2)$.

We now iterate the above argument for depths smaller than $k-1$ (i.e., for vertices closer to the root). In particular, assume that for some depth d where $1 \leq d \leq k-1$, every vertex v at depth d satisfies $|A_v| < \#\mathit{children}(k,i)$, where $i = k-d+1$. Then we can set $d' := d-1$, $i' := i+1$, and recall from Observation 2 that every vertex v' at depth d' satisfies $m_{v'} \leq \mathit{size}(k,i')$. Consider now a speciifc vertex v' with too many children—in particular, $|A_{v'}| \geq \#\mathit{children}(k,i')$. If such a vertex exists, we will once again apply Lemma 1 to obtain an equivalent smaller instance and then restart the algorithm. Repeating this procedure for d' will eventually stop, and at that point it will hold that $|A_{v'}| < \#\mathit{children}(k,i')$ for every v' at depth d', in turn allowing us to continue the procedure at the next level of the decomposition (i.e., at depth $d'' \leq d'$).

Once the above procedure terminates for the last time, the root r of T satisfies $|A_r| < \#\mathit{children}(k,k)$ and $m_r \leq \mathit{size}(k,k)$. At that point, we have a kernel G' [12, 18]—an equivalent graph that has size bounded by a function of k, notably by $f(k) = \#\mathit{children}(k,k) \cdot \mathit{size}(k,k) + 1$. To prove Theorem 1, it suffices to decide whether G' admits a 1-queue layout by a brute-force algorithm that runs in time $\mathcal{O}(f(k)! \cdot f(k)^2)$. Since Lemma 1 is applied $\mathcal{O}(n)$ times and the runtime of the associated algorithm is $\mathcal{O}(\mathit{size}(k,k) \cdot \#\mathit{children}(k,k)^2)$, the total runtime is upper-bounded by a function of k times n. □

Finally, we note that while it would be possible to provide a term upper-bounding the dependency on k of the running time of Lemma 1, it is clear that such a term must necessarily be non-elementary—indeed, the recursive definition of the two functions $\#\mathit{children}(k,k)$ and $\mathit{size}(k,k)$ results in a tower of exponents of height k.

3.2 Proof of Lemma 1

Since we have

$$|A_t| \geq \left(\left(2^{(k+1)}+1\right)^{\mathit{size}(k,i)^2}+1\right) \cdot (\mathit{size}(k,i)+k)! \cdot 2^{(k+\mathit{size}(k,i))^2} = \#\mathit{children}(k,i)$$

and the number of equivalence classes of $\sim$ is upper-bounded by $2^{(k+m_t)^2} \leq 2^{(k+\mathit{size}(k,i))^2}$, there must exist an equivalence class, denoted $A_t^{\sim} \subseteq A_t$, containing at least $\left(\left(2^{(k+1)}+1\right)^{\mathit{size}(k,i)^2}+1\right) \cdot (\mathit{size}(k,i)+k)!$ connected components in A_t which are pairwise equivalent w.r.t. $\sim$. The reason we need this many components in the equivalence class is that this will allow us to argue that every hypothetical solution must contain two components which behave "in the same way" (as will become clear later). Moreover, this equivalence class can be computed in time at most $\mathit{size}(k,i)! \cdot \#\mathit{children}(k,i)^2$ by simply brute-forcing over all potential renaming functions η between arbitrarily chosen $\#\mathit{children}(k,i)$-many components in A_t to construct the set of all equivalence classes of these components. Let B be an arbitrarily selected component in $A_t^{\sim}$. First, observe that if G is a YES-instance then so is $G-B$, as deleting vertices and edges cannot increase the queue number. On the other hand, assume there is a 1-queue layout of $G-B$ with linear order $\prec$. Our aim for the rest of the proof is to obtain a linear order $\prec'$ of G that extends $\prec$ and yields a valid 1-queue layout of G.

A Refined Equivalence. Recall that we are, at this stage, proceeding under the assumption that there exists a 1-queue layout of $G-B$ with linear order $\prec$. Let $\equiv_{\prec}$ be an equivalence over components in $A_t^{\sim}$ which takes this hypothetical order $\prec$ into account and is defined as follows.

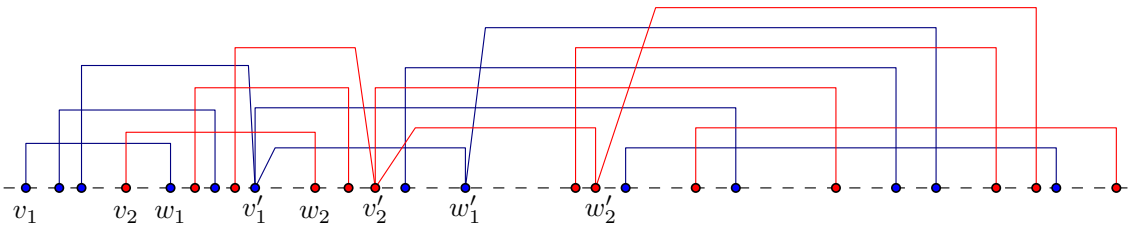


Figure 3: Two delimiting components C_1 and C_2 (blue and red), with two counterpart (and hence interleaving) edges labeled. Notice that no two counterpart edge pairs are separate.

For two components $C, D \in A_t^{\sim}$, $C \equiv_{\prec} D$ if and only if the following holds: the linear order $\prec$ restricted to $P_t \cup \eta_{C,D}(C)$ is the same as $\prec$ restricted to $P_t \cup C$. In other words, $\equiv_{\prec}$ is a refinement of $\sim$ restricted to $A_t^{\sim}$ which groups components based on the order in which their vertices appear (also taking into account which subinterval they appear in w.r.t. P_t). Note that $\equiv_{\prec}$ has at most $(m_t + k)! \leq (size(k,i) + k)!$ many equivalence classes, since $|P_t| \leq k$; hence, by the virtue of $A_t^{\sim}$ having size at least $\left((2^{(k+1)} + 1)^{size(k,i)^2} + 1\right) \cdot (size(k,i) + k)!$, there must exist an equivalence class U of $\equiv_{\prec}$ containing at least $(2^{(k+1)} + 1)^{size(k,i)^2} + 1$ components of $A_t^{\sim}$.

We adopt the following terminology for U: we will denote the components in U as $C_1, C_2, \ldots, C_u$, where $u = |U|$, we will identify the vertices in a component C_i by using the lower index i, and for each such vertex v, say $v = v_i \in C_i$, use v_j to denote its counterpart $\eta_{C_i,C_j}(v_i)$.

Identifying Delimiting Components. Consider two adjacent vertices v_i, w_i in C_i. We say that component C_j is *vw-separate* from C_i if edges v_iw_i and v_jw_j neither nest nor cross each other. On the other hand, C_j is *vw-interleaving* (respectively, *vw-nesting*) with C_i if v_iw_i and v_jw_j cross each other (respectively, if one of v_iw_i and v_jw_j nests the other). By the definition of $\equiv_{\prec}$ and U, these three cases are exhaustive. Moreover, if v_iw_i is an edge then so is v_jw_j and hence C_j cannot be vw-nesting with C_i.

Our next aim will be to find two components—we will call them *delimiting components*—that are not vw-separate for *any* edge vw, see, e.g., Fig. 3. To this end, for some two adjacent vertices v_i, w_i of C_i, denote by D_1 the component whose counterpart to v_i (say v_1) is placed leftmost in $\prec$ among all components in U. We now define a sequence of components as follows: D_ℓ is the unique component that is (i) vw-separate from $D_{\ell-1}$ and (ii) whose vertex v_ℓ is placed to the right of $v_{\ell-1}$, and (iii) v_ℓ is placed leftmost among all components satisfying properties (i) and (ii). Let d be the maximum integer such that D_d exists.

Lemma 2 $d \leq 2^{k+1} + 1$.

Proof: Consider, for a contradiction, that there exists a component D_ℓ such that $\ell > 2^k$ and $\ell < d - 2^k$, i.e., that there is a sequence of at least 2^k pairwise vw-separated components to the left as well as to the right of D_ℓ. By the connectivity of G, there must be a path from v to some vertex in P_t, say p. However, by the definition of $\equiv_{\prec}$ every vertex in P_t lies either to the left of v_1 or to the right of w_d, and hence a path from v to p would need to pass through a sequence of 2^k edges forming disjoint intervals in the linear order $\prec$. Since nestings are not allowed, such a path must have at least one vertex inside each of these intervals, and hence its length is at least 2^k, which contradicts Proposition 2. □

Moreover, each component C_q in U can be uniquely assigned to one component D_ℓ as defined above (w.r.t. the chosen edge vw) as follows: If $C_q = D_\ell$ for some ℓ, then C_q is assigned to itself; otherwise, D_ℓ is the component whose vertex v_ℓ is to the left of and simultaneously closest to the corresponding vertex v_q in C_q among all components $D_1, \ldots, D_d$.

Lemma 3 *Let C_q and C_p be two components assigned to the same component D_ℓ w.r.t. the edge vw. Then C_q and C_p are vw-interleaving.*

Proof: Assume without loss of generality that w_ℓ is placed to the right of v_ℓ; recall that w_ℓ and v_ℓ are the counterparts of w and v, respectively, in D_ℓ. Since both C_q and C_p are assigned to D_ℓ, the counterparts v_q and v_p to v_ℓ must be placed to the left of w_ℓ (by the definition of assignment). Because edges cannot nest on the same queue, this implies that the counterparts w_q and w_p to w_ℓ must be placed to the right of w_ℓ. Hence C_q and C_p cannot be vw-separate, and the observation follows by recalling that C_q and C_p cannot be vw-nesting either. □

We are now ready to construct our delimiting components. Recall that at this point, $|U| \geq (2^{(k+1)} + 1)^{size(k,i)^2} + 1$ while the maximum number of edges inside a component in U is upper-bounded by $m_t^2 \leq size(k,i)^2$. Hence by the pigeon-hole principle and by applying the bound provided in Lemma 2 for each edge inside the components of U, there must exist two components in U, say C_x and C_y, that are assigned to the same component D_ℓ^{vw} for each edge vw. By Lemma 3 it now follows that they are vw-interleaving for every edge vw.

Using Delimiting Components. Before we use C_x and C_y to insert the component B of A_t as required by Lemma 1, we can show that the way they interleave with each other is "consistent" in $\prec$.

Lemma 4 *Assume without loss of generality that some vertex v_x is to the left of v_y. Then for each vertex w_x it holds that w_x is to the left of w_y.*

Proof: Consider for a contradiction that there is a vertex w_x to the right of w_y. Consider a v_x-w_x path P_x in the subgraph of G induced on the vertices of C_x, and let P_y be the v_y-w_y path in the subgraph of G induced on the vertices of C_y consisting of the counterparts of P_x. Let $a_x b_x$ be the first edge on P_x such that a_x is placed to the left of a_y but b_x is placed to the right of b_y. Then the edges $a_x b_x$ and $a_y b_y$ would be nesting, contradicting the correctness of $\prec$. □

We remark that it is not the case that C_x must be vw-interleaving with C_y if vw is not an edge—this is, in fact, a major complication that we will need to overcome to complete the proof.

Without loss of generality and recalling Lemma 4, we will hereinafter assume that every vertex $v_x \in C_x$ is placed to the left of its counterpart $v_y \in C_y$. The following definition allows us to partition the vertices of C_x into subsequences that should not be interleaved with vertices of B.

Definition 2 (Block) *A* block $L = \{v_x^1, v_x^2, \ldots, v_x^h\}$ *of C_x is a maximal set of vertices of C_x such that: (1) there is no vertex v_y^i (the counterpart in C_y of v_x^i), with $1 \leq i \leq h$, between two vertices of L in $\prec$; and (2) there are no two vertices of L such that one has a neighbor to its left and one has a neighbor to its right.*

We observe that, as an immediate consequence of Definition 2, no two vertices of L are adjacent (an edge uv in L would imply that u has a neighbor to its right and v has a neighbor to its left, or vice versa).

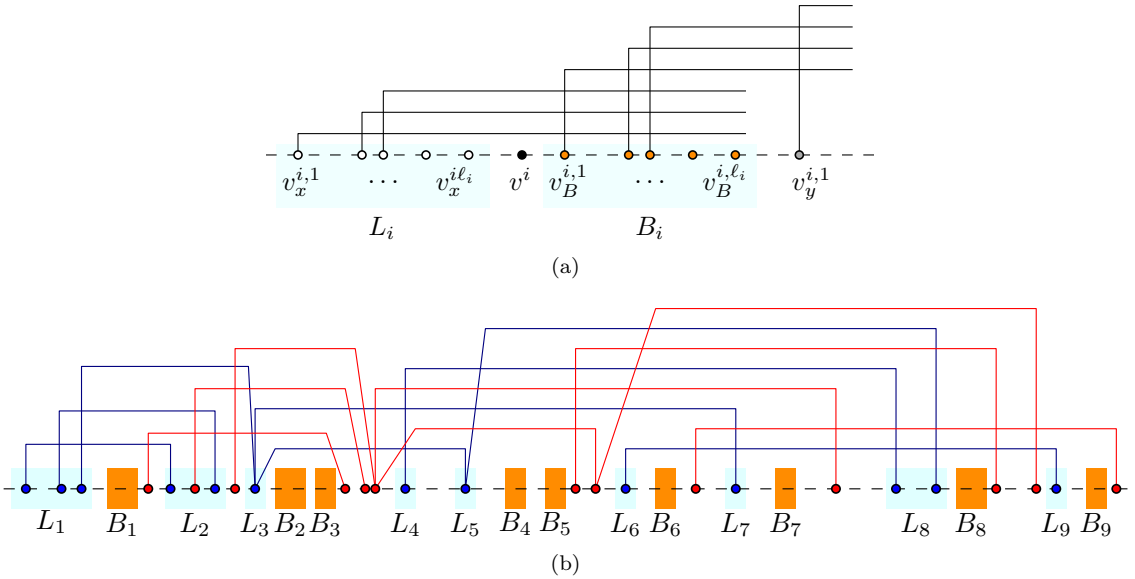


Figure 4: Reinsertion of B_i: (a) A schematic illustration, and (b) an example where blue and red vertices belong to C_x and C_y, respectively.

For each block $L = \{v_x^1, v_x^2, \ldots, v_x^h\}$ of C_x, there is a corresponding set of vertices $\{v_B^1, v_B^2, \ldots, v_B^h\}$ of B, i.e., the set containing the counterparts of L in B. We will obtain a linear order of G by processing the blocks of C_x one by one as encountered in a left-to-right sweep of $\prec$, and for each block L, we will extend $\prec$ by suitably inserting the corresponding vertices of B.

Consider the i-th encountered block $L_i = \{v_x^{i,1}, v_x^{i,2}, \ldots, v_x^{i,\ell_i}\}$ of C_x, refer to Fig. 4 for an illustration. Note that, because C_x and C_y are equivalent components, it holds $v_y^{i,1} \prec v_y^{i,2} \cdots \prec v_y^{i,\ell_i}$ (even though such vertices might not be consecutive). Also, let v^i be the first vertex to the left of $v_y^{i,1}$ in $\prec$ (possibly $v^i = v_x^{i,\ell_i}$). We insert all vertices in the corresponding block B_i of B such that: $v_i \prec v_B^{i,1} \prec v_B^{i,2} \prec \ldots v_B^{i,\ell_i} \prec v_y^{i,1}$. After processing the last block of C_x, we know that all vertices of C_x have been considered and hence all vertices of B have been reinserted, that is, we extended $\prec$ to a linear order $\prec'$ of the whole graph G. The next observation immediately follows by the procedure described above.

Observation 3 *For every vertex v_x, it holds that $v_x \prec' v_B \prec' v_y$.*

We now establish the correctness of $\prec'$, completing the proof of Lemma 1.

Lemma 5 *The linear order $\prec'$ yields a valid 1-queue layout of G.*

Proof: To prove the statement, we argue that no two edges of G nest in the 1-queue layout defined by $\prec'$. We recall that $\prec'$ extends $\prec$, hence we do not need to argue about pairs of edges in $G - B$. Moreover, by construction, $\prec'$ restricted to C_x is the same as $\prec'$ restricted to B (up to the renaming function η). Consequently, no two edges having both endpoints in B can nest. To complete the proof, it suffices to consider the two cases of an edge having only one endpoint or both endpoints in B (i.e., the "newly added" edges), and show that no such edge can be involved in any nesting.

- We first consider any edge v_Bw for $w \in P_t$ and $v_B \in B$, and assume $v_B \prec' w$ (else the argument is symmetric). Suppose, for a contradiction, that v_Bw nests another edge ab. Recall that since C_x and B are equivalent components, if v_B is to the left of w, the same holds for v_x. By Observation 3, we know $v_x \prec' v_B \prec' w$, which implies that ab is nested by v_xw as well, a contradiction with the correctness of $\prec$. Similarly, if v_Bw is nested by an edge ab, then we know $v_B \prec' v_y \prec' w$, which implies that ab nests v_yw as well, again a contradiction.

- We now consider any edge v_Bw_B, with $v_B \prec' w_B$. We further distinguish whether, for a contradiction, v_Bw_B nests an edge ab or is nested by an edge ab.

  – Assume v_Bw_B nests an edge ab. Since Definition 2 ensures that a block cannot contain a pair of adjacent vertices, we know that v_x and w_x belong to different blocks, say L_i and L_j (with $i < j$) respectively. Therefore, we can rename the vertices as $v_x = v_x^{i,i'}$ and $w_x = v_x^{j,j'}$, and similarly $v_B = v_B^{i,i'}$ and $w_B = v_B^{j,j'}$; refer to Fig. 5(a) for an illustration. By Observation 3, it holds $v_x^{i,i'} \prec' v_B^{i,i'} \prec' v_y^{i,i'}$ and $v_x^{j,j'} \prec' v_B^{j,j'} \prec' v_y^{j,j'}$. Moreover, the correctness of $\prec$ implies that $v_B^{i,i'} \prec' a \prec' v_y^{i,i'}$ (since $v_y^{i,i'}v_y^{j,j'}$ cannot nest ab) and $v_x^{j,j'} \prec' b \prec' v_B^{j,j'}$ (since $v_x^{i,i'}v_x^{j,j'}$ cannot nest ab). Because a is between $v_B^{i,i'}$ and $v_y^{i,i'}$, either there exists another vertex $v_y^{i,1}$ (the counterpart to the first vertex in block L_i, where possibly $v_y^{i,1} = a$) such that $v_B^{i,i'} \prec' v_y^{i,1} \preceq' a \prec' v_y^{i,i'}$, or $a = v_y^{i,i'}$.

  Suppose first $a \neq v_y^{i,1}$ and $a \neq v_y^{i,i'}$. Observe that $v_x^{i,1}$ has at least one neighbor in C_x (because C_x is connected), and that $v_x^{j,j'}$ is to the right of $v_x^{i,i'}$, hence, by Definition 2, $v_x^{i,1}$ also has a neighbor to its right, say v_x^{l,j^*}. Because no two edges nest in $\prec$, it must be: (i) $v_x^{i,1} \prec' v_x^{i,i'}$, (ii) $v_x^{l,j^*} \prec' b$, and (iii) $v_y^{l,j^*} \prec' b$ (possibly $v_y^{l,j^*} = b$). Altogether, this implies that $v_x^{j,j'}$ and v_x^{l,j^*} are in the same block (i.e., $l = j$) and hence $v_B^{j,j'} \prec' v_y^{j,j^*} \prec' b$, which contradicts $b \prec' v_B^{j,j'}$. If instead $a = v_y^{i,1}$ or $a = v_y^{i,i'}$, then b is either a vertex of C_y or a vertex of P_t. If $b \in C_y$, the argument is similar, as we can set $b = v_y^{j,j^*}$ and observe that $v_B^{j,j'}$ should be to the left of v_y^{j,j^*}, see Fig. 5(b). If $b \in P_t$, we would have $v_x^{j,j'} \prec' b \prec' v_y^{j,j'}$, which contradicts the fact that C_x and C_y are equivalent components, see Fig. 5(c).

  – Assume now that v_Bw_B is nested by an edge ab. Again we can rename the vertices as $v_x = v_x^{i,i'}$ and $w_x = v_x^{j,j'}$, and similarly $v_B = v_B^{i,i'}$ and $w_B = v_B^{j,j'}$. By the position of b we can deduce either that $b = v_y^{j,j'}$ (possibly $j' = 1$) or that edge $v_y^{i,i^*}v_y^{j,1}$ exists. In the latter case either $v_y^{i,i^*}v_y^{j,1}$ is also nested by ab or $v_B^{i,i'} \prec' a$, and in both cases we obtain a contradiction; refer to Fig. 6(a) for an illustration. In the former case, we should again distinguish whether $a \in C_y$ or $a \in P_t$. If $a \in C_y$, it should be $v_B^{i,i'} \prec a = v_y^{i,i^*}$, see Fig. 6(b). If $a \in P_t$, we would have $v_x^{i,i'} \prec' a \prec' v_y^{i,i'}$, which again contradicts the fact that C_x and C_y are equivalent components, see Fig. 6(c).

□

4 Parameterization by Vertex Cover Number

We now turn to the general Queue Number problem and show that it is fixed-parameter tractable when parameterized by the vertex cover number. We formalize our result as follows.

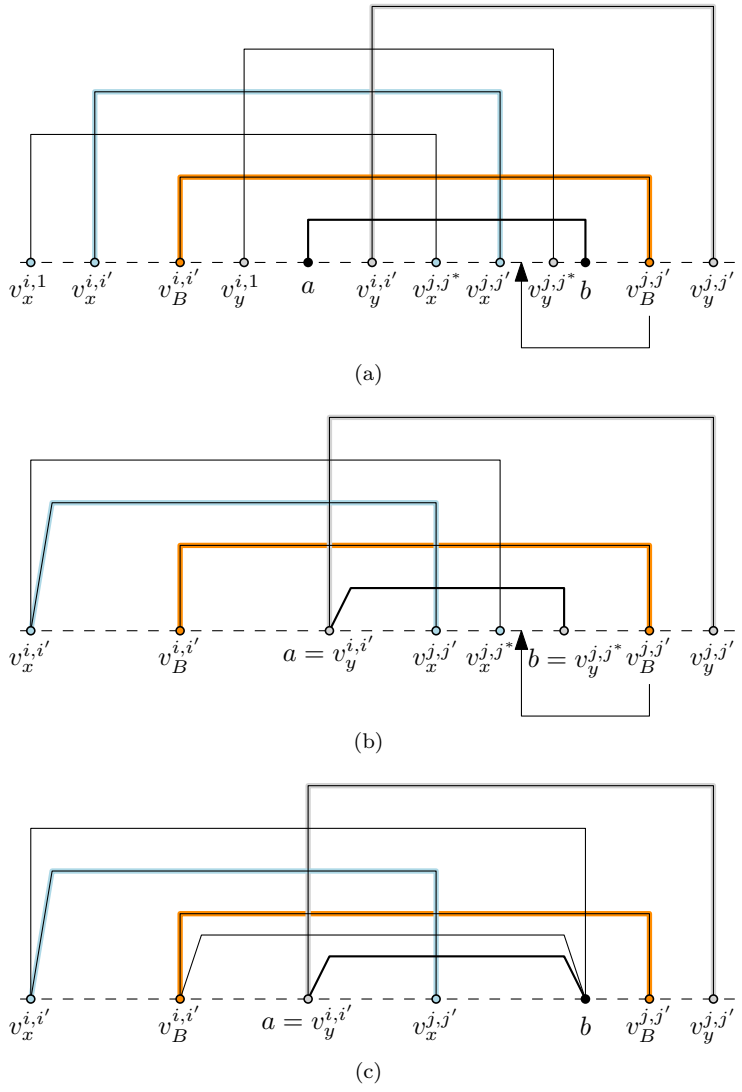


Figure 5: Illustration for the proof of Lemma 5: $v_B^{i,i'}v_B^{j,j'}$ nests ab.

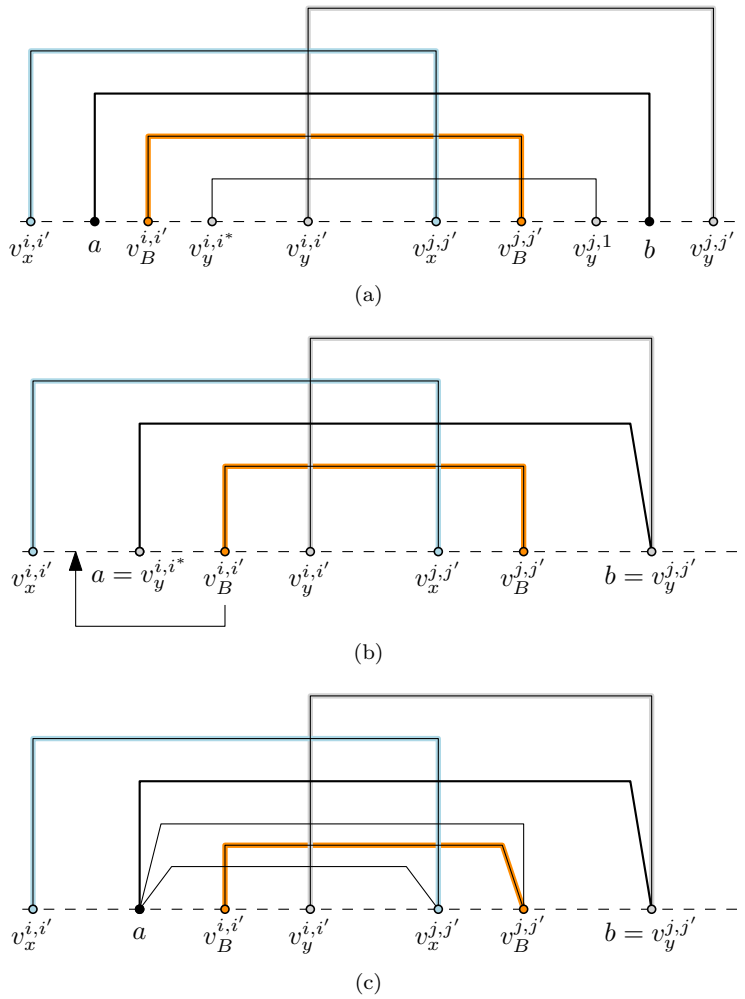


Figure 6: Illustration for the proof of Lemma 5: $v_B^{i,i'}\, v_B^{j,j'}$ is nested by ab.

Theorem 2 *Let G be a graph with n vertices and vertex cover number $\tau = \tau(G)$. A queue layout of G with the minimum number of queues can be computed in $\mathcal{O}(2^{\tau^{\mathcal{O}(\tau)}} + \tau \log \tau \cdot n)$ time.*

4.1 Algorithm Description

Before describing the algorithm behind Theorem 2, we make an easy observation (which matches an analogous observation in [7]).

Lemma 6 *Every n-vertex graph $G = (V, E)$ with a vertex cover C of size τ admits a τ-queue layout. Moreover, if G and C are given as input, such a τ-queue layout can be computed in $\mathcal{O}(n + \tau \cdot n)$ time.*

Proof: Denote by $c_1, \ldots, c_\tau$ the τ vertices of C and let $\prec$ be any linear order of G such that $c_i \prec c_{i+1}$, for $i = 1, 2, \ldots, \tau - 1$. A queue assignment σ of G on h queues can be obtained as follows. Let $U = V \setminus C$. For each $i \in [\tau]$ all edges uc_i with $u \in U \cup \{c_1, \ldots, c_{i-1}\}$ are assigned to queue i. Now, consider the edges assigned to any queue $i \in [\tau]$. By construction, they are all incident to vertex c_i, and thus no two of them nest each other. Therefore, the pair $\langle \prec, \sigma \rangle$ is a τ-queue layout of G and can be computed in $\mathcal{O}(n + \tau \cdot n)$ time. □

Let C be a vertex cover of size τ of graph G. For any subset U of C, a vertex $v \in V \setminus C$ is of *type* U if $N(v) = U$. This defines an equivalence relation on $V \setminus C$ and in particular partitions $V \setminus C$ into at most $\sum_{i=1}^{\tau} \binom{\tau}{i} = 2^\tau - 1 < 2^\tau$ distinct types. Denote by V_U the set of vertices of type U.

Lemma 7 *Let $h \in \mathbb{N}$ and $v \in V_U$ such that $|V_U| \geq 2 \cdot h^\tau + 2$. Then G admits an h-queue layout if and only if $G' = G - \{v\}$ does. Moreover, an h-queue layout of G' can be extended to an h-queue layout of G in linear time.*

The proof of Lemma 7 is deferred to Section 4.2.

Proof: [of Theorem 2] By Proposition 3, we can determine the vertex cover number τ of G and compute a vertex cover C of size τ in time $\mathcal{O}(2^\tau + \tau \cdot n)$. With Lemma 7 in hand, we can then apply a binary search on the number of queues $h \leq \tau$ as follows. If $h > \tau$, by Lemma 6 we can immediately conclude that G admits a τ-queue layout and compute one in $\mathcal{O}(n + \tau \cdot n)$ time. Hence we shall assume that $h \leq \tau$. We construct a kernel G^* from G of size $h^{\mathcal{O}(\tau)}$ as follows. We first classify each vertex of G based on its type. We then remove an arbitrary vertex from each set V_U with $|V_U| > 2 \cdot h^\tau + 1$ until $|V_U| \leq 2 \cdot h^\tau + 1$. Thus, constructing G^* can be done in $\mathcal{O}(2^\tau + \tau \cdot n)$ time, since 2^τ is the number of types and $\tau \cdot n$ is the maximum number of edges of G. From Lemma 7 we conclude that G admits an h-queue layout if and only if G^* does.

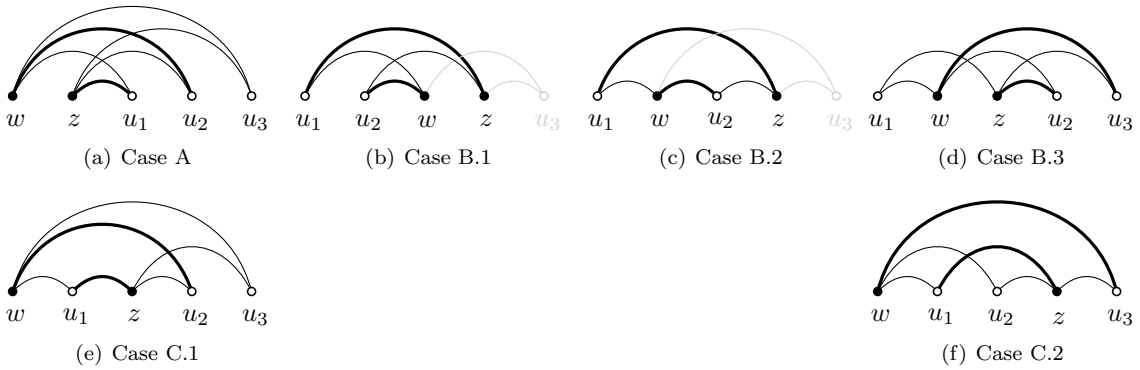
Given a linear order $\prec^*$ of G^*, a queue assignment σ^* such that $\langle \prec^*, \sigma^* \rangle$ is an h-queue layout of G^* exists if and only if σ^* contains no h-rainbow [31], i.e., h independent edges that pairwise nest, which can be easily checked (and computed if it exists) in $h^{\mathcal{O}(\tau)}$ time [31]. Consequently, determining whether G^* admits an h-queue layout can be done by first guessing all linear orders, and then for each of them by testing for the existence of an h-rainbow. Since we have 2^τ types, and each of the at most $2 \cdot h^\tau + 1$ elements of the same type are equivalent in the queue layout (that is, the position of two elements of the same type can be exchanged in $\prec^*$ without affecting σ^*), the number of linear orders can be upper bounded by $(2^\tau)^{\mathcal{O}(h^\tau)} = 2^{\tau^{\mathcal{O}(\tau)}}$. Thus, whether h queues suffice for G^* can be determined in $2^{\tau^{\mathcal{O}(\tau)}} \cdot h^{\mathcal{O}(\tau)} = 2^{\tau^{\mathcal{O}(\tau)}}$ time. An h-queue layout of G^* (if any) can be extended to one of G by iteratively applying the constructive procedure of Lemma 7,

in $\mathcal{O}(\tau \cdot n)$ time. Finally, by applying a binary search on h we obtain an overall time complexity of $\mathcal{O}(2^{\tau^{\mathcal{O}(\tau)}} + \tau \log \tau \cdot n)$, as desired. □

4.2 Proof of Lemma 7

One direction follows easily, since removing a vertex from an h-queue layout still gives an h-queue-layout of the resulting graph. So let $\langle \prec, \sigma \rangle$ be an h-queue layout of G'. We prove that an h-queue layout of G can be constructed by inserting v immediately to the right of a suitable vertex u in V_U and by assigning the edges of v to the same queues as the corresponding edges of u.

We say that two vertices $u_1, u_2 \in V_U$ are *queue equivalent*, if for each vertex $w \in U$, the edges u_1w and u_2w are both assigned to the same queue according to σ. Each vertex in V_U has degree exactly $|U|$, hence this relation partitions the vertices of V_U into at most $h^{|U|} \leq h^\tau$ sets. Let $V_U^* = V_U \setminus \{v\}$. Since $|V_U^*| \geq 2 \cdot h^\tau + 1$, at least three vertices of this set, which we denote by u_1, u_2, and u_3, are queue equivalent. Consider now the graph induced by the edges of these three vertices that are assigned to a particular queue. By the above argument, such a graph is a $K_{l,3}$, for some $l > 0$. However, $K_{3,3}$ does not admit a 1-page queue layout, because any graph with queue number 1 is planar [32]. As a consequence, $l \leq 2$, that is, each $u_i \in V_U^*$ has at most two edges on each queue. Denote such two edges by u_iw and u_iz and assume without loss of generality that $u_1 \prec u_2 \prec u_3$ and $w \prec z$. We now claim that $w \prec u_1 \prec u_2 \prec u_3 \prec z$, else two edges would nest. We can distinguish a few cases based on the position of u_1 (recall that $u_1 \prec u_2 \prec u_3$), refer to Fig. 7 for an illustration.



(a) Case A (b) Case B.1 (c) Case B.2 (d) Case B.3

(e) Case C.1 (f) Case C.2

Figure 7: Illustration for the proof of Lemma 7.

- **Case A:** $w \prec z \prec u_1$, then the nesting edges are zu_1 and wu_2.
- **Case B:** $u_1 \prec w \prec z$, then we distinguish three more subcases.
  - **Case B.1:** $u_2 \prec w$, then the nesting edges are u_1z and u_2w.
  - **Case B.2:** $w \prec u_2 \prec z$, then the nesting edges are u_1z and wu_2.
  - **Case B.3:** $z \prec u_2$, then the nesting edges are zu_2 and wu_3.
- **Case C:** $w \prec u_1 \prec z$, if $w \prec u_2 \prec u_3 \prec z$ the claim follows. Else, we have two more subcases based again on the position of u_2.

– **Case C.1:** $w \prec z \prec u_2$, then the nesting edges are wu_2 and u_1z.
– **Case C.2:** $w \prec u_2 \prec z \prec u_3$, then the nesting edges are wu_3 and u_1z.

It follows that we can extend $\prec$ by introducing v as the first vertex to the right of u_1 and, for each edge vw such that $w \in U$, we can assign vw to the same queue as u_1w. This operation does not introduce any nesting. Namely, if vw is assigned to a queue containing only one edge of u_1, the graph induced by the edges in this queue is a star with center w and no two edges can nest. If vw is assigned to a queue containing two edges of u_1, say u_1w and u_1z, then we know that all vertices of V_U are between w and z in $\prec$ and again no two edges nest.

5 Conclusions and Open Problems

We proved that h-Queue Number is fixed-parameter tractable parameterized by treedepth for $h = 1$, and by the vertex cover number for arbitrary $h \geq 1$. Several interesting questions arise from our research, among them:

1. A first natural question is to understand whether Theorem 1 can be extended to the general case ($h \geq 1$). In particular, our arguments establishing the existence of interleaving components already fail for $h = 2$.

2. Extending Theorem 1 to graphs of bounded treewidth is also an interesting problem; here the main issue is to be able to forget information about vertices in a partial order, thus an approach based on testing arched leveled-planarity might be more suitable.

3. Finally, we mention the possibility of studying the parameterized complexity of *mixed* linear layouts, using both queues and stacks, see [2, 13, 25, 32, 38].

It is worth noting that the preliminary version of this manuscript [8] has already led to interesting follow-up work [34] which uses analogous techniques to generalize Theorem 2.

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