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On Turn-Regular Orthogonal Representations

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Abstract. An interesting class of orthogonal representations consists of the so-called *turn-regular* ones, i.e., those that do not contain any pair of reflex corners that "point to each other" inside a face. For such a representation H it is possible to compute in linear time a minimum-area drawing, i.e., a drawing of minimum area over all possible assignments of vertex and bend coordinates of H. In contrast, finding a minimum-area drawing of H is NP-hard if H is non-turn-regular. This scenario naturally motivates the study of which graphs admit turn-regular orthogonal representations. In this paper we identify notable classes of biconnected planar graphs that always admit such representations, which can be computed in linear time. We also describe a linear-time testing algorithm for trees and provide a polynomial-time algorithm that tests whether a biconnected plane graph with "small" faces has a turn-regular orthogonal representation without bends.

1 Introduction

Computing *orthogonal drawings* of graphs is among the most studied problems in graph drawing (see, e.g., [9, 13, 14, 21, 25]), because of its direct application to several domains, such as software engineering, information systems, and circuit design (e.g., [2, 12, 15, 20, 23]). In an orthogonal

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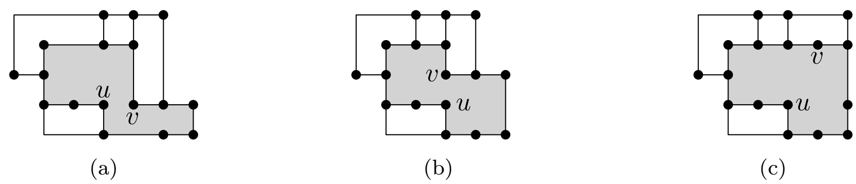

Figure 1: (a) Drawing of a non-turn-regular orthogonal representation H; vertices u and v point to each other in the gray face. (b) Another drawing of H with smaller area. (c) Drawing of a turn-regular orthogonal representation of the same graph.

drawing, the vertices of the graph are mapped to distinct points of the plane and each edge is represented as an alternating sequence of horizontal and vertical segments between its end-vertices. A point in which two segments of an edge meet is called a *bend*. An orthogonal drawing is a *grid* drawing if its vertices and bends have integer coordinates.

One of the most popular and effective strategies to compute a readable orthogonal grid drawing of a graph G is the so-called *topology-shape-metrics* approach [27], which consists of three consecutive steps: (i) compute a planar embedding of G by possibly adding dummy vertices to replace edge crossings if G is not planar; (ii) obtain an *orthogonal representation* H of G from the previously determined planar embedding; H describes the "shape" of the final drawing in terms of angles around the vertices and sequences of left/right bends along the edges; (iii) assign integer coordinates to vertices and bends of H to obtain the final non-crossing orthogonal grid drawing Γ of G.

If G is planar, the topology-shape-metrics approach computes a planar orthogonal grid drawing Γ of G. Such a planar drawing exists if and only if G is a *4-graph*, i.e., of maximum vertex-degree at most four. To increase the readability of Γ, a typical optimization goal of Step (ii) is the minimization of the number of bends. In Step (iii) the goal is to minimize the area or the total edge length of Γ; a problem referred to as *orthogonal compaction*. Unfortunately, while the computation of an embedding-preserving bend-minimum orthogonal representation H of a plane 4-graph is polynomial-time solvable [8, 27], the orthogonal compaction problem for a planar orthogonal representation H is NP-complete in the general case [26]. Nevertheless, Bridgeman et al. [6] showed that the compaction problem for the area requirement can be solved optimally in linear time for a subclass of orthogonal representations called *turn-regular*. A similar polynomial-time result for the minimization of the total edge length in this case is proved by Klau and Mutzel [22]. Esser showed that these two approaches are equivalent [16].

Informally speaking, a face of a planar orthogonal representation H is turn-regular if it does not contain a pair of reflex corners (i.e., turns of 270°) that point to each other (see Section 2 for the formal definition); H is turn-regular if all its faces are turn-regular. For a turn-regular representation H, every pair of vertices or bends has a unique orthogonal relation (left/right or above/below) in any planar drawing of H. Conversely, different orthogonal relations are possible for a pair of opposing reflex corners, which makes it computationally hard to optimally compact non-turn-regular representations. For example, Figs. 1(a) and 1(b) show two different drawings of a non-turn-regular orthogonal representation; the drawing in Fig. 1(b) has minimum area. Fig. 1(c) depicts a minimum-area drawing of a turn-regular orthogonal representation of the same graph.

The aforementioned scenario naturally motivates the problem of computing orthogonal representations that are turn-regular, so to support their subsequent compaction. To the best of our knowledge, this problem has not been studied so far (a related problem is studied for upward planar drawings only [5, 10, 11]). Heuristics have been described to make any given orthogonal representation H turn-regular, by adding a minimal set of dummy edges [6, 19]; however, such edges impose constraints that may yield a drawing of sub-optimal area for H.

Our contribution is as follows:

- We identify notable classes of biconnected planar graphs that always admit turn-regular orthogonal representations. We prove that biconnected planar 3-graphs and planar Hamiltonian 4-graphs (which include planar 4-connected 4-graphs [24]) admit turn-regular representations with at most two bends per edge and at most three bends per edge, respectively. For these graphs, a turn-regular representation can be constructed in linear time. We recall that a graph is Hamiltonian if it contains a simple path passing through all its vertices. We also prove that every biconnected planar graph admits an orthogonal representation that is *internally* turn-regular, i.e., its internal faces are turn-regular (Section 3). We leave open the question whether every biconnected planar 4-graph admits a turn-regular representation.

- For 1-connected planar graphs, including trees, there exist infinitely many instances for which a turn-regular representation does not exist. Motivated by this scenario, and since the orthogonal compaction problem remains NP-hard even for orthogonal representations of paths [17], we study and characterize the class of trees that admit turn-regular representations. We then describe a corresponding linear-time testing algorithm, which in the positive case computes a turn-regular drawing without bends (Section 4). Finally, we prove that such drawings are "convex" (i.e., all edges incident to leaves can be extended to infinite crossing-free rays). We remark that a linear-time algorithm to compute planar straight-line convex drawings of trees is described by Carlson and Eppstein [7]. However, in general, the drawings they compute are not orthogonal.

- We address the problem of testing whether a given biconnected plane graph admits a turn-regular *rectilinear* representation, i.e., a representation without bends. For this problem we give a polynomial-time testing algorithm that works for plane graphs with "small" faces, namely faces of degree at most eight. If the test is positive, a turn-regular rectilinear representation is computed (Section 5).

2 Preliminary Definitions and Basic Results

In this paper we only consider connected graphs. In the following we recall notation and definitions used in the remainder of the paper. We assume familiarity with basic concepts on graph connectivity and graph planarity [9].

A *k-graph* is a graph with vertex-degree at most k. We denote by $\deg(v)$ the degree of a vertex v. A *plane graph* is a planar graph with a given planar embedding. Let G be a plane graph and let f be a face of G. We always assume that the boundary of f is traversed counterclockwise, if f is an internal face, and clockwise, if f is the external face. Note that if G is not biconnected, an edge may occur twice and a vertex may occur multiple times on the boundary of f. The total number of vertices (or edges), counted with their multiplicity, is called the *degree of f* and is denoted as $\deg(f)$.

Figure 2: Illustration of (a) convex, flat and reflex corners, and (b) kitty corners.

Orthogonal Drawings and Representations. Let G be a planar 4-graph. A planar *orthogonal drawing* Γ of G is a planar drawing of G that represents each vertex as a point and each edge as an alternating sequence of horizontal and vertical segments between its end-vertices. A *bend* in Γ is a point of an edge in which a horizontal and a vertical segment meet. Informally speaking, an orthogonal representation of G is an equivalence class of orthogonal drawings of G having the same planar embedding and the same "shape", i.e., the same sequences of angles around the vertices and of bends along the edges.

More formally, let G be a plane graph, and let e_1 and e_2 be two (possibly coincident) edges incident to a vertex v of G that are consecutive in the clockwise order around v. We say that $a = \langle e_1, v, e_2 \rangle$ is an *angle at* v of G or simply an *angle* of G. Let Γ and Γ' be two embedding-preserving orthogonal drawings of G. We say that Γ and Γ' are *equivalent* if: (i) for every angle a of G, the geometric angle corresponding to a is the same in Γ and Γ', and (ii) for every edge $e = (u, v)$ of G, the sequence of left and right bends along e moving from u to v is the same in Γ and in Γ'. An *orthogonal representation* H of G is a class of equivalent orthogonal drawings of G. Representation H is completely described by the embedding of G, by the value $\alpha \in \{90°, 180°, 270°, 360°\}$ for each angle a of G (α defines the geometric angle associated with a), and by the ordered sequence of left and right bends along each edge (u, v), moving from u to v; if we move from v to u this sequence and the direction (left/right) of each bend are reversed. An orthogonal representation without bends is also called a *rectilinear representation.*

W.l.o.g., from now on we assume that an orthogonal representation H comes with a given orientation, i.e., we shall assume that for each edge segment $\overline{pq}$ of H (where p and q correspond to vertices or bends), it is fixed whether p is to the left, to the right, above, or below q in every orthogonal drawing that preserves H.

Let f be a face of H. We assume that the boundary of f is traversed counterclockwise (clockwise) if f is internal (external). The *rectilinear image* of H is the orthogonal representation $\overline{H}$ obtained from H by replacing each bend with a degree-2 vertex. For any face f of H, let $\overline{f}$ denote the corresponding face of $\overline{H}$. For each occurrence of a vertex v of $\overline{H}$ on the boundary of $\overline{f}$, let prec(v) and succ(v) be the edges preceding and following v, respectively, on the boundary of $\overline{f}$ (prec(v) = succ(v) if deg(v) = 1). Let α be the value of the angle internal to $\overline{f}$ between prec(v) and succ(v). We associate with v one or two *corners* based on the following cases: If $\alpha = 90°$, associate with v one *convex* corner; if $\alpha = 180°$, associate with v one *flat* corner; if $\alpha = 270°$, associate with v one *reflex* corner; if $\alpha = 360°$, associate with v an ordered pair of *reflex* corners. For example, in the (internal) face of Fig. 2(a), a convex corner is associated with v_1, a flat corner with v_2, a reflex corner with v_3, and an ordered pair of reflex corners with v_4.

Based on the definition above, there is a circular sequence of corners associated with (the

Figure 3: Directions of a reflex corner associated with a degree-2 vertex or with a bend: (a) up-left; (b) up-right; (c) down-left; (d) down-right. Directions of a degree-1 vertex: (e) upward; (f) downward; (g) leftward; (h) rightward.

boundary) of $\overline{f}$. For a corner c of $\overline{f}$, we define:

$$\text{turn}(c) = \begin{cases} 1 & \text{if } c \text{ is convex} \\ 0 & \text{if } c \text{ is flat} \\ -1 & \text{if } c \text{ is reflex} \end{cases}$$

For any ordered pair (c_i, c_j) of corners of $\overline{f}$, we define the following function: $\text{rot}(c_i, c_j) = \sum_c \text{turn}(c)$ for all corners c along the boundary of $\overline{f}$ from c_i (included) to c_j (excluded). For example, in the internal face of Fig. 2(a), let c_1, c_2, and c_3 be the corners associated with v_1, v_2, and v_3, respectively, and let (c_4, c'_4) be the ordered pair of reflex corners associated with v_4. For this internal face we have $\text{rot}(c_1, c_2) = 3$, $\text{rot}(c_3, c_4) = 1$, $\text{rot}(c_3, c'_4) = 0$, and $\text{rot}(c_3, c_1) = -3$. The properties below are consequences of results in [27, 28].

Property 1 *For each face $\overline{f}$ of $\overline{H}$ and for each corner c_i of $\overline{f}$, we have rot$(c_i, c_i) = 4$ if $\overline{f}$ is internal and rot$(c_i, c_i) = -4$ if $\overline{f}$ is external.*

Property 2 *For each ordered triplet of corners (c_i, c_j, c_k) of a face of $\overline{H}$, we have rot$(c_i, c_k) =$ rot$(c_i, c_j) +$ rot(c_j, c_k).*

Property 3 *Let c_i and c_j be two corners of $\overline{f}$. If $\overline{f}$ is internal then rot$(c_i, c_j) = 2 \iff$ rot$(c_j, c_i) = 2$. If $\overline{f}$ is external then rot$(c_i, c_j) = 2 \iff$ rot$(c_j, c_i) = -6$.*

Let c be a reflex corner of $\overline{H}$ associated with either a degree-2 vertex or a bend of H. Let s_h and s_v be the horizontal and vertical segments incident to c and let ℓ_h and ℓ_v be the lines containing s_h and s_v, respectively. We say that c (or equivalently its associated vertex/bend of H) points *up-left*, if s_h is to the right of ℓ_v and s_v is below ℓ_h. The definitions of c that points *up-right*, *down-left*, and *down-right* are symmetric (see Figs. 3(a)-3(d)). If v is a degree-1 vertex in H, then it has two associated reflex corners in $\overline{H}$. In this case, v points *upward* (*downward*) if its incident segment is vertical and below (above) the horizontal line passing through v. The definitions of a degree-1 vertex that points *leftward* or *rightward* are symmetric (see Figs. 3(e)-3(h)).

Two reflex corners c_i and c_j of a face of $\overline{H}$ are called *kitty corners* if $\text{rot}(c_i, c_j) = 2$ or $\text{rot}(c_j, c_i) = 2$. A face f of an orthogonal representation H is *turn-regular*, if the corresponding face $\overline{f}$ of $\overline{H}$ has no kitty corners. If every face of H is turn-regular, then H is *turn-regular*. For example, the orthogonal representation in Fig. 2(b) is not turn-regular as the faces f_1 and f_3 are turn-regular, while the internal face f_2 and the external face f_4 are not turn-regular (the pairs of kitty corners in each face are highlighted with dotted arrows). A graph G is *turn-regular*, if it admits a turn-regular orthogonal representation. If G admits a turn-regular rectilinear representation, then G is *rectilinear turn-regular*. The next lemma gives a sufficient condition for the existence of a kitty corner pair in the external face.

Lemma 1 *Let $\overline{H}$ be the rectilinear image of an orthogonal representation H of a plane graph G. Let (c_1, c_2) be two corners of the external face such that rot$(c_1, c_2) \geq 3$ or c_1 is a reflex corner and rot$(c_1, c_2) \geq 2$. Then, the external face contains a pair of kitty corners.*

Proof: Let π denote the path from c_1 to c_2 in a clockwise traversal of the external face. First assume that c_1 is not a reflex corner and $\text{rot}(c_1, c_2) \geq 3$. Let c be the corner that precedes c_1. If c is not a reflex corner, then $\text{rot}(c, c_2) = \text{rot}(c_1, c_2) + \text{turn}(c) \geq 3$, and if c is a reflex corner, then $\text{rot}(c, c_2) = \text{rot}(c_1, c_2) - 1 \geq 2$. We can hence iteratively expand the path π by adding preceding corners until we find a reflex corner c_1 such that $\text{rot}(c_1, c_2) \geq 2$. If c_2 is a reflex corner, and $\text{rot}(c_1, c_2) = 2$, we have kitty corners by definition. If $\text{rot}(c_1, c_2) > 2$, then, as $\text{rot}(c_1, c_1) = -4$ (by Property 1) and we only reduce the rotation value at reflex corners, we can keep extending π until we find a pair of corners (c_1, c_2') with $\text{rot}(c_1, c_2') = 2$ such that c_2' is a reflex corner. Similarly, if $\text{rot}(c_1, c_2) = 2$ but c_2 is not a reflex corner, we can extend π to the next reflex corner c_2' on the external face with $\text{rot}(c_1, c_2') = 2$. It follows that (c_1, c_2') is the claimed pair of kitty corners. □

The following corollary is an immediate consequence of Lemma 1.

Corollary 1 *Let H be an orthogonal representation of a plane graph G. If, after ignoring flat corners, the external face of $\overline{H}$ has three consecutive convex corners, H is not turn-regular.*

3 Turn-Regular Biconnected Graphs

The theorems in this section can be proven by modifying a well-known linear-time algorithm by Biedl and Kant [4] that produces an orthogonal drawing Γ with at most two bends per edge of a biconnected planar 4-graph G with a fixed embedding $\mathcal{E}$. Such an algorithm exploits an *st*-ordering $s = v_1, v_2, \ldots, v_n = t$ of the vertices of G, where s and t are two distinct vertices on the external face of $\mathcal{E}$. We recall that an *st*-ordering $s = v_1, v_2, \ldots, v_n = t$ is a linear ordering of the vertices of G such that any vertex v_i distinct from s and t has at least two neighbors v_j and v_k in G with $j < i < k$ [18]. The orthogonal drawing Γ is constructed incrementally by adding vertex v_k, for $k = 1, \ldots, n$, into the drawing Γ_{k-1} of $\{v_1, \ldots, v_{k-1}\}$, while preserving the embedding $\mathcal{E}$. Some invariants are maintained when vertex v_k is placed above Γ_{k-1}: (i) vertex v_k is attached to Γ_{k-1} with at least one edge incident to v_k from the bottom; (ii) after v_k is added to Γ_{k-1}, some extra columns are introduced into Γ_k to ensure that each edge (v_i, v_j), such that $i \leq k < j$ has a dedicated column in Γ_k that is reachable from v_i with at most one bend and without introducing crossings.

Theorem 1 *Every biconnected planar 3-graph admits a turn-regular representation with at most two bends per edge, which can be computed in linear time.*

Proof: Let G be a biconnected planar 3-graph and let $\mathcal{E}$ be any planar embedding of G. Let s and t be two distinct vertices on the external face of $\mathcal{E}$. As in [4], based on an *st*-ordering of G, we incrementally construct an orthogonal drawing Γ of G by adding v_k into the drawing Γ_{k-1} of graph G_{k-1}, for $k = 1, \ldots, n$. Besides the invariants (i) and (ii) described above, we additionally maintain the following invariant (iii): each reflex corner introduced in the drawing points either down-right or up-right with the possible exception of two reflex corners on the external face: one pointing down-left along the leftmost edge leaving v_1 and one pointing up-left along the leftmost edge entering v_n. Drawing Γ_1 consists of the single vertex v_1. Since $\deg(v_1) \leq 3$, the columns assigned to its three incident edges are the column where v_1 lies and the two columns immediately

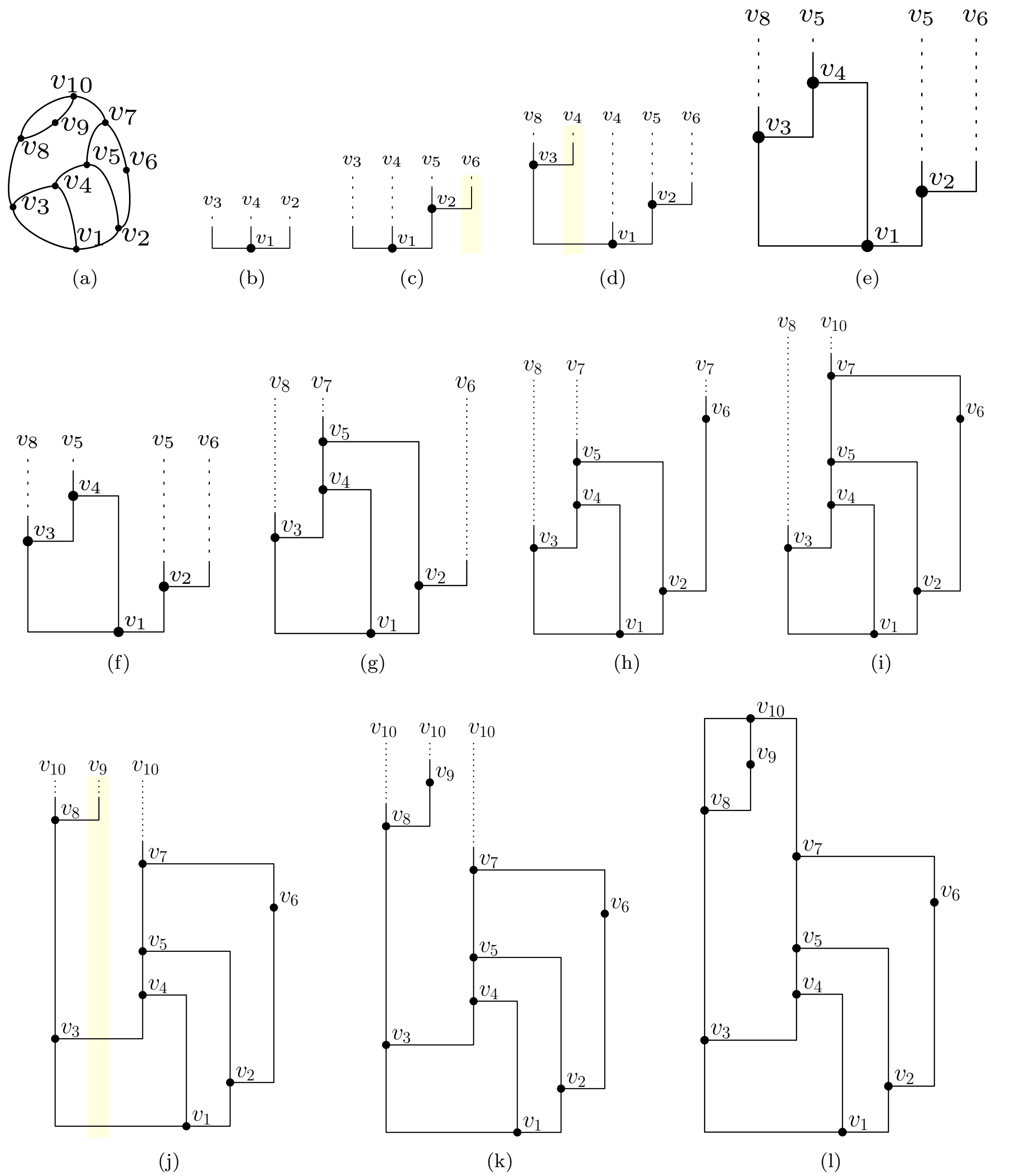

Figure 4: The steps of the algorithm in the proof of Theorem 1 for the construction of a turn-regular orthogonal drawing of a biconnected planar 3-graph shown in (a).

on its left and on its right (see Fig. 4(b)). These columns are assigned to the edges incident to v_1 in the order they appear in $\mathcal{E}$. Now, suppose you have to add vertex v_k to Γ_{k-1}. Observe that, since G has maximum degree three, v_k has a maximum of three edges (v_k, v_h), (v_k, v_i), and (v_k, v_j), where we may assume, without loss of generality, that $h < i < j$. To complete the proof, we consider three cases:

Case 1 $(h < k < i)$: We place v_k on the first empty row above Γ_{k-1} and on the column assigned to (v_k, v_h). Also, to preserve the invariant (ii), we introduce an extra column immediately to the right of v_k and we assign the column of v_k and the newly added extra column to (v_k, v_i) and (v_k, v_j) in the order that is given by $\mathcal{E}$. For example, Fig. 4(c) shows the placement of v_2 in the column reserved for the connection to v_2 in Fig. 4(b); one extra column is inserted to the right of v_2 and assigned to the edge (v_2, v_6).

Case 2 $(i < k < j)$: We place v_k on the first empty row above Γ_{k-1} and on the leftmost column between the columns assigned to (v_k, v_h) and (v_k, v_i). Also, we assign the column of v_k to (v_k, v_j), e.g., Fig. 4(e) shows the placement of v_4 on the leftmost column assigned to its incoming edges (v_3, v_4) and (v_1, v_4).

Case 3 $(j < k)$: Here, v_k is t. We place v_k on the first empty row above Γ_{k-1} and on the middle column among those assigned to edges (v_k, v_h), (v_k, v_i), and (v_k, v_j).

The discussion in [4] suffices to prove that Γ is a planar orthogonal drawing of G with at most two bends per edge. We claim that Γ is also turn-regular. In fact, the invariant (iii) guarantees that all internal faces have reflex corners pointing either down-right or up-right and, hence, are turn-regular. Regarding the external face, any pair of kitty corners necessarily involves either the reflex corner pointing down-left that possibly lies along the leftmost edge of v_1 or the reflex corner pointing up-left that possibly lies along the leftmost edge of v_n. In fact, the other reflex corners of the external face cannot form kitty corners among them since they point up-right or down-right. We prove that no kitty corner can be formed with the reflex corner pointing down-left (the proof for the reflex corner pointing up-left is analogous). Suppose by contradiction that the leftmost edge of v_1 has a corner c pointing down-left that forms a kitty corner on the external face with some other corner c' pointing up-right. By the definition of kitty corners either $\text{rot}(c, c') = 2$ or $\text{rot}(c', c) = 2$. In the first case by Property 3 $\text{rot}(c', c) = -6$. However, starting from c', which is a reflex corner pointing down-left, and traversing clockwise the external face before reaching c you necessarily traverse a reflex corner c'' with $\text{rot} c', c'' = -2$ and such that c'' points down-left as c, contradicting the fact that c is the only vertex on the external face pointing down-left. In the case in which $\text{rot}(c', c) = 2$ by Property 3 we have $\text{rot}(c, c') = -6$. Similarly as above, clockwise traversing the external face from c, which points down-left, towards c' you necessarily traverse a corner c''' such that $\text{rot}(c, c''') = -4$ and such that c''' points down-left as c, contradicting again the fact that c is the only corner pointing down-left.

About the time complexity, the fact that the algorithm can be executed in linear time follows arguments analogous to those given in [4]. □

The proofs of Theorems 2 and 3 exploit techniques similar to that used in the proof of Theorem 1.

Theorem 2 *Every planar Hamiltonian 4-graph G admits a turn-regular representation H with at most 3 bends per edge, and such that only one edge of H gets 3 bends and only if G is 4-regular. Given the Hamiltonian cycle, H can be computed in linear time.*

Proof: We use an iterative construction inspired by the algorithm by Biedl and Kant [4] where we replace the *st*-ordering of the input graph with the ordering given by the Hamiltonian cycle. Let G be a Hamiltonian planar 4-graph and let $\mathcal{E}$ be a planar embedding of G. If G has some vertices

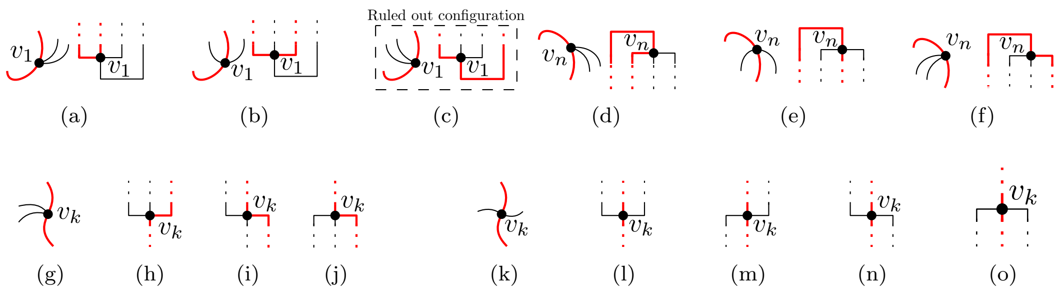

Figure 5: Drawing rules for the algorithm in the proof of Theorem 2. The Hamiltonian path is drawn red and thick. Figs. (g)–(j) are to be intended up to a horizontal flip.

of degree less than four, choose one of these to be v_1. Otherwise, if G is 4-regular, denote by v_1 a vertex of G such that the two edges of the Hamiltonian path incident to v_1 are not both on the external face. Such a vertex always exists since if G is 4-regular it cannot be also outerplanar, as an outerplanar graph has at least a vertex of degree two. We consider the vertices in the order $v_1, v_2, \dots, v_n$ given by the Hamiltonian cycle of G. We also assume that edge (v_n, v_1) is incident to the external face of $\mathcal{E}$ (otherwise we could change the external face preserving the rotation scheme of $\mathcal{E}$ and avoiding that (v_1, v_2) is also on the external face). We incrementally construct an orthogonal drawing Γ of G by adding v_k, for $k = 1, \dots, n$, to the drawing Γ_{k-1} of $\{v_1, \dots, v_{k-1}\}$, preserving the embedding of $\mathcal{E}$. In particular, vertex v_k is always placed above Γ_{k-1}.

After v_k is added to Γ_{k-1}, we introduce some extra columns into Γ_k so to maintain some invariants. Analogously to [4] and to the proof of Theorem 1, we maintain the invariant that each edge (v_i, v_j) such that $i \leq k < j$ has a dedicated column in Γ_k that is reachable from v_i with at most one bend without introducing crossings. Observe that, since the vertices are inserted in the order given by the Hamiltonian cycle, in the drawing Γ_{k-1} a special path can be identified, that we call the *spine*, connecting, for $i = 1, \dots, k-2$, vertex v_i to v_{i+1}. We maintain the invariant that all the reflex corners introduced in the drawing point down-left or up-left if they are contained into a face that is on the left side of the spine and point down-right or up-right if they are contained into a face that is contained on the right side of the spine, with the possible exception of the reflex corners on the external face occurring on edges incident to s or to t.

Suppose v_1 has degree 4. Then, its additional two edges may be: (a) both on the right side of the spine (Fig. 5(a)) or (b) one on the left side and the other on the right side of the spinfe (Fig. 5(b)). The case where the additional edges are both on the left side of the spine, depicted in Fig. 5(c), is ruled out by the choice of v_1. In the cases (a) and (b) the drawing of v_1 and the columns reserved for its outgoing edges are depicted in Figs. 5(a) and 5(b), respectively. If, instead, v_1 has degree three or two, the drawing of v_1 is obtained from Fig. 5(b) by omitting the missing edges.

For $k = 1, 2, \dots, n$, each vertex v_k added to the drawing has one incoming and one outgoing edge of the spine. The drawing of v_k follows simple rules that depend on the number of additional edges of v_k on the left side and on the right side of the spine in $\mathcal{E}$.

Now consider a vertex v_k with $1 < k < n$. Suppose v_k has degree four and that its additional edges are both on the left side of the spine (Fig. 5(g)). Denote by v_i and v_j the other endpoints of the additional edges of v_k and assume, without loss of generality, that $v_i < v_j$. There are three cases: $v_k < v_i < v_j$ (Fig. 5(h)), or $v_i < v_k < v_j$ (Fig. 5(i)), or $v_i < v_j < v_k$ (Fig. 5(j)). In all cases

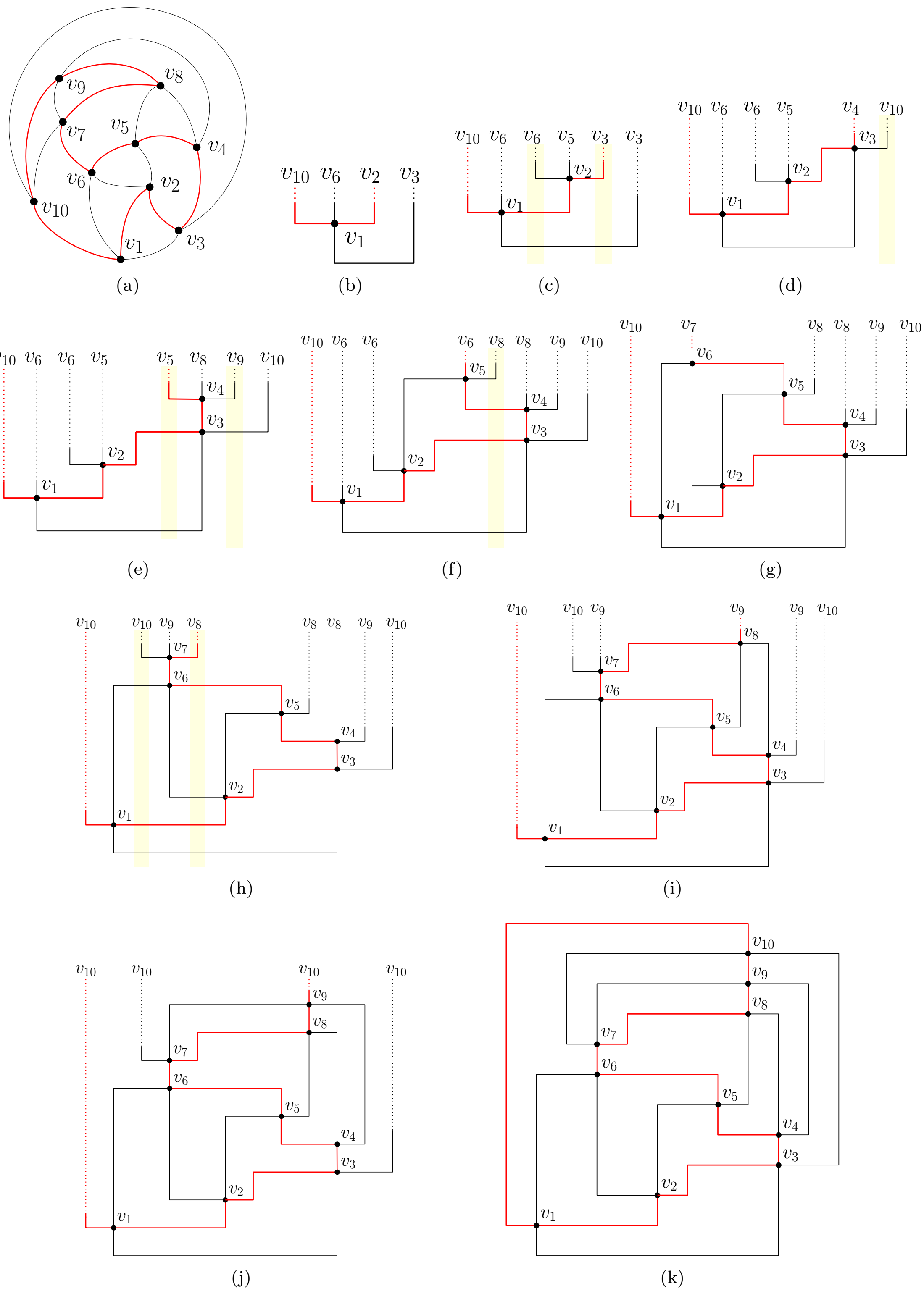

Figure 6: The drawing produced by the algorithm of the proof of Theorem 2 for the construction of a turn-regular orthogonal drawing of a Hamiltonian planar 4-graph. The Hamiltonian cycle is drawn red and thick.

the drawings of v_k depicted in Figs. 5(h)-5(j)) guarantee the invariants. If v_k has degree two or three, its drawing is that of Fig. 5(i) where the missing edges are omitted.

The case when v_k has degree four and its additional edges are both on the right of the spine can be handled analogously, horizontally mirroring the configurations of Figs. 5(h)-5(j).

Suppose now that v_k has degree four and that its additional edges are one on the left side and one on the right side of the spine (Fig. 5(k)). Again, depending on whether the vertices adjacent to v_k precede or follow v_k in the ordering given by the Hamiltonian cycle we can adopt for v_k one of the drawings depicted in Figs. 5(l)-5(n).

Finally, vertex v_n is added to the drawing by using one of the configurations shown in Figs. 5(d), 5(e), and 5(f); see Fig. 6 for a complete example.

Since the configurations in Figs 5(a)-5(n) all respect the invariant that in the faces on the left side (right side, resp.) of the spine only top-left and bottom-left (top-right and bottom-right, resp.) reflex corners are inserted, the drawing cannot have kitty corners and is turn regular. Also observe that each edge has a maximum of three bends per edge.

As for Theorem 1, the algorithm can be executed in linear time with an analysis similar to that given in [4], assuming that the Hamiltonian cycle of G is given as part of the input. □

Theorem 3 *Every biconnected planar 4-graph has a representation with $O(n)$ bends per edge and $O(n^4)$ area that is internally turn-regular and that is computed in $O(n)$ time.*

Proof: We modify the algorithm of Biedl and Kant [4] again, where instead of the standard bottom-up construction, we adopt a spiraling one. The vertices are inserted in the drawing according to an *st*-ordering, based on the rules depicted in Fig. 7 (a full example of the algorithm is in Fig. 8).

For an internal face f let $s(f)$ ($d(f)$, resp.) be the index of the first (last, resp.) inserted vertex incident to f. By construction, f is bounded by two paths P_ℓ and P_r that go from $v_{s(f)}$ to $v_{d(f)}$, where P_ℓ precedes P_r in the left-to-right list of outgoing edges of $v_{s(f)}$. The construction rules imply that P_r has only convex corners. On the other hand, each convex corner of P_ℓ is always immediately preceded or immediately followed by a reflex corner. This rules out kitty corners in f. Indeed, consider any two reflex corners c_i and c_j along P_ℓ and the path, contained into P_ℓ, that is traversed when turning clockwise around f from c_i to c_j. When computing $\text{rot}(c_i, c_j)$ a positive amount $+1$ is always followed by a negative amount -1, and the sum is never equal to 2. Since f is an internal face, by Property 1 $\text{rot}(c_i, c_j) + \text{rot}(c_j, c_i) = 4$, and $\text{rot}(c_i, c_j) \neq 2$ implies

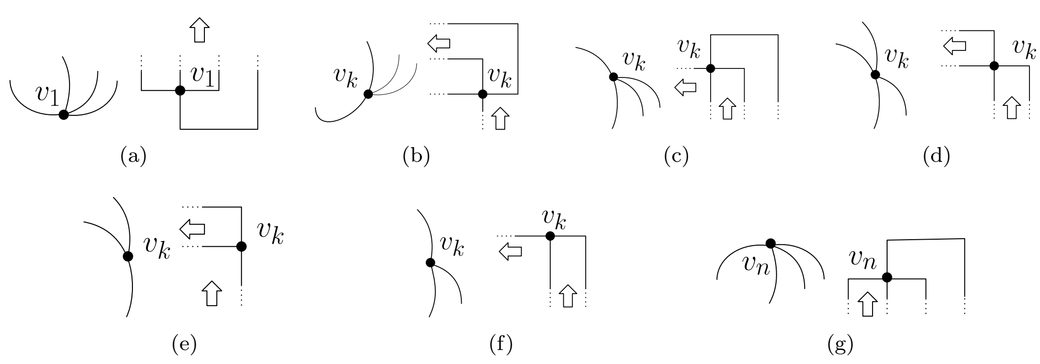

Figure 7: The construction rules for the algorithm in the proof of Theorem 3.

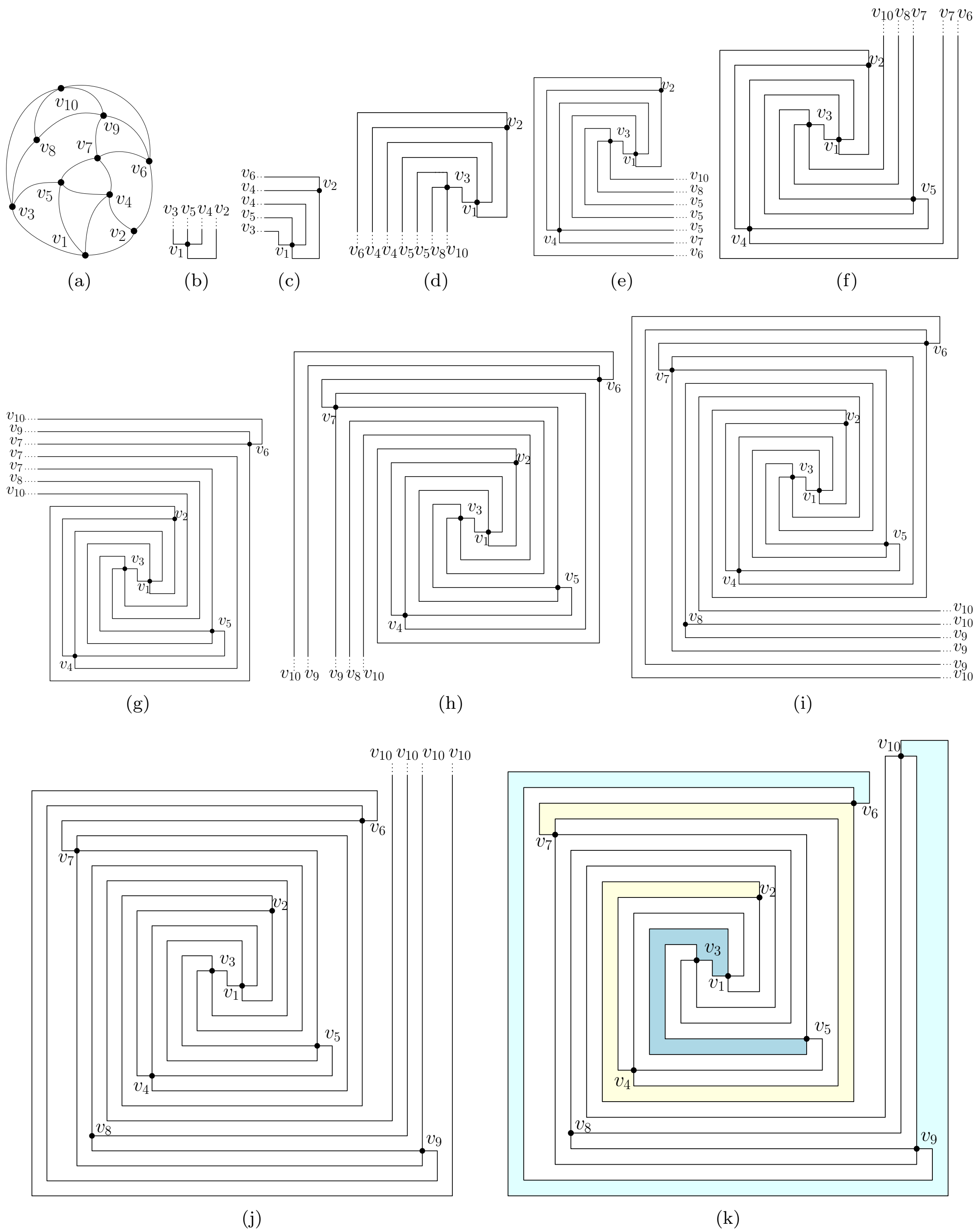

Figure 8: Example illustrating the algorithm in the proof of Theorem 3 for the construction of an orthogonal drawing with turn-regular internal faces.

$\text{rot}(c_j, c_i) \neq 2$. Note that an edge (v_i, v_j) contains $O(j-i)$ bends, which yields $O(n)$ bends per edge in the worst case.

Regarding the area of the obtained representation consider a worst case where an n-vertex 4-regular graph, where n can be assumed even, is such that the vertices $v_2, v_3, \dots, v_{n/2}$ have one incoming and three outgoing edges, while the vertices $v_{n/2+1}, v_{n/2+2}, \dots, v_{n-1}$ have three incoming and one outgoing edge. The number of edges that spiral around the drawing grows linearly while adding vertices $v_2, \dots, v_{n/2}$ and then decreases linearly afterwards. Hence, the height and the width of the drawing are both $O(n^2)$ and their product is $O(n^4)$.

The time complexity follows from the same analysis in [4]. □

4 Turn-regular Trees

We give a characterization of the trees that admit turn-regular representations, which we use to derive a corresponding linear-time testing and drawing algorithm. Let T be a tree that contains a degree-2 vertex v. Let (u, v) and (v, w) be the two edges incident to v in T. *Smoothing* v is the operation of removing v from T and replacing the two edges (u, v) and (v, w) with a single edge (u, w). We denote by smooth(T) the tree obtained from T by smoothing all degree-2 vertices of T (see, e.g., Fig. 9).

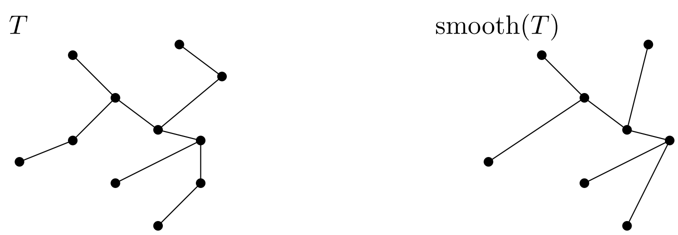

Figure 9: A tree T (left) and the tree smooth(T) (right).

We start with an auxiliary lemma which is central in our approach. It shows that in order to test whether a tree T is turn-regular, we can assume w.l.o.g. that T has no degree-2 vertices and then test for rectilinear turn-regularity.

Lemma 2 *T is turn-regular if and only if* smooth(T) *is rectilinear turn-regular.*

Proof: Suppose that T is turn-regular, and let H be a turn-regular representation of T (the other direction is obvious). Consider the rectilinear image $\overline{H}$ of H. We show that $\overline{H}$ can be transformed into a turn-regular representation $\overline{H'}$ with only flat corners at degree-2 vertices. Let u and v be two vertices of $\overline{H}$ such that $\deg(u) \neq 2$, $\deg(v) \neq 2$, and the path π_{uv} connecting u to v in $\overline{H}$ has only (possibly none) degree-2 vertices. W.l.o.g., we assume that π_{uv} does not contain two consecutive corners c_1 and c_2 such that $\text{turn}(c_1) = 1$ and $\text{turn}(c_2) = -1$ (i.e., corners that form a zig-zag pattern), because in this case we can replace both c_1 and c_2 with two flat corners. Hence, we can assume that all non-flat corners encountered along π_{uv}, while moving from u to v, always turn in the same direction, say to the right; i.e., all of them are convex corners. Let k be the number of convex corners along π_{uv}. Since the only face of H is the external face, by Corollary 1 we may assume that $1 \leq k \leq 2$, which yields two cases.

Assume first $k = 1$. Let c be the convex corner along π_{uv}. W.l.o.g., suppose that c points up-left. Since $\overline{H}$ is turn-regular, by Corollary 1, either u has no edge segment incident from the right (see

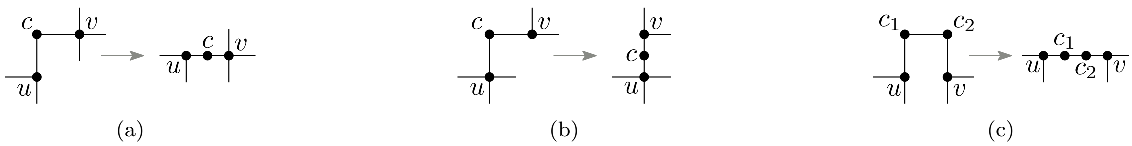

Figure 10: Illustrations for the proof of Lemma 2.

Fig. 10(a)) or v has no edge segment incident from below (see Fig. 10(b)). Hence, we can transform c into a flat corner by using one of these two directions to reroute π_{uv} around u or around v.

Let now $k = 2$. Let c_1 and c_2 be the two convex corners along π_{uv}. W.l.o.g., suppose that c_1 points up-left and c_2 points up-right (see Fig. 10(c)). Since $\overline{H}$ is turn-regular, by Corollary 1, u has no edge segment incident from the right and v has no edge segment incident from the left. Hence, again, we can transform c into a flat corner by using these two directions to reroute π_{uv} around u and v.

$\overline{H'}$ is obtained by applying the above transformation to each pair of vertices u and v that have the properties above. □

Unless otherwise specified, from now on we will assume by Lemma 2 that T is a tree without degree-2 vertices. We will further refer to a tree as turn-regular if and only if it is rectilinear turn-regular. The next property directly follows from Lemma 2 and the absence of degree-2 vertices.

Property 4 *Let H be a turn-regular rectilinear representation of a tree T. Then, the reflex corners of H are formed by the leaves of T.*

While turn-regularity is not a hereditary property in general graphs, the next lemma, shows that it is in fact hereditary for trees.

Lemma 3 *If a tree T is turn-regular then any subtree of T is turn-regular.*

Proof: Let H be a turn-regular rectilinear representation of tree T, and let T' be a subtree of T, i.e., T' is a connected subgraph of T. If T and T' consist of n and n' vertices, respectively, then T' can be obtained from T by repeatedly removing exactly one leaf of it, $n - n'$ times. For $i = 0, 1 \ldots, n - n'$, denote by T_i the subtree of T that is derived after the removal of the first i leaves, and by H_i the restriction of H to tree T_i; clearly, $T = T_0$, $H = H_0$ and $T' = T_{n-n'}$ hold. For $i = 1 \ldots, n - n'$, we will prove that H_i is turn-regular, under the assumption that H_{i-1} is turn-regular (note that here we cannot assume that the degree-2 vertices of T_i can be neglected in view of Lemma 2, as an application of the lemma to T_i may result in changing its representation, while H_i must be a restriction of H to T_i in our claim).This will imply that T' is turn-regular, as desired. Let u be the leaf that is removed from H_{i-1} to obtain H_i. We emphasize that H_{i-1} consists of a single face, i.e., the external. Hence, the removal of u from H_{i-1} implies the removal of a pair of reflex corners from its external face. We next focus on the case, in which the removal of u from H_{i-1} introduces a reflex corner in H_i that is not present in H_{i-1}, as otherwise H_i is clearly turn-regular.

Let v be the (unique) neighbor of u in T_{i-1}. If v has no neighbor in T_{i-1} other than u, then $i = n - 1$ holds, i.e., T_i consists of single vertex v and thus is turn-regular. Hence, we may assume that v has a neighbor, say w, in T_{i-1} that is different from v. W.l.o.g., we assume that (v, w) is vertical in H_{i-1} (and thus in H_i) with v being its top end-vertex. If $\deg(v) = 4$ in T_{i-1}, then the removal of u from T_{i-1} yields a flat corner in H_i, which implies H_i does not contain a reflex corner that does not exist in H_{i-1}; a contradiction. Hence, $\deg(v) \in \{2,3\}$ in T_{i-1}. We will focus on the case, in which $\deg(v) = 2$ in T_{i-1}; the case, in which $\deg(v) = 3$ in T_{i-1}, is analogous. Note that if (u,v) is horizontal in H_{i-1}, then (u,v) and (v,w) inevitably form a reflex corner ζ_v in H_{i-1}.

Assume for a contradiction that H_i is not turn-regular. If we denote by $\langle c_v, c'_v \rangle$ the ordered pair of reflex corners at v in H_i, then at least one of c_v and c'_v, say the former, forms a pair of kitty corners with another corner c of H_i (and thus of H_{i-1}). Denote by $\langle c_u, c'_u \rangle$ the ordered pair of reflex corners at u in H_{i-1}. If (u,v) is horizontal and u is to the left (right) of v in H_{i-1}, then we observe that c and c'_u (c and ζ_v, respectively) form a pair of kitty corners in H_{i-1}. On the other hand, if (u,v) is vertical, then c and c_u form a pair of kitty corners in H_{i-1}. In both cases, we obtain a contradiction to the fact that H_{i-1} is turn-regular. □

A *trivial tree* is a single edge; otherwise, it is *non-trivial*. For $k \in \{2,3\}$, a *k-fork* in a tree T consists of a vertex v whose degree is $k+1$ and at least k leaves adjacent to it in T. For $k \in \{2,3\}$, a *k-fork* at a vertex v in a tree T consists of vertex v whose degree is $k+1$ and at least k leaves adjacent to it in T. Due to the degree restriction, a 2-fork is not a 3-fork, and vice versa. Also, notice that by definition $K_{1,4}$ is a 3-fork. The next lemma follows from [7, Lem. 7]; for completeness, we give here a simplified proof.

Lemma 4 *A turn-regular tree has (i) at most four* 2*-forks and no* 3*-fork, or (ii) two* 3*-forks and no* 2*-fork, or (iii) one* 3*-fork and at most two* 2*-forks.*

Proof: To prove the lemma, we first state and formally prove two claims.

Claim 1 *Let H be a turn-regular rectilinear representation of a tree T. Let u and v be two leaves associated with two ordered reflex corner-pairs $\langle c_u, c'_u \rangle$ and $\langle c_v, c'_v \rangle$. If in a traversal of the external face of T from u to v there is no other leaf of T, then rot$(c'_u, c_v) \in \{-1,0,1\}$ (or equivalently rot$(c_u, c_v) \in \{0,-1,-2\}$).*

Proof of the claim: By Property 4, it follows that there exist no reflex corners between c'_u and c_v. By Lemma 1, it follows that $\text{rot}(c'_u, c_v) < 2$, which implies there exist at most two convex corners between c'_u and c_v. Hence, $\text{rot}(c'_u, c_v) \in \{-1,0,1\}$. This completes the proof of the claim.

Consider a leaf u of T and assume w.l.o.g. that u is pointing upward in H. Let v be the leaf that follows u in the traversal of the external face of T in H. Let also $\langle c_u, c'_u \rangle$ and $\langle c_v, c'_v \rangle$ be the two pairs of ordered reflex corners associated with u and v, respectively. By Claim 1, v does not point leftward, as otherwise $\text{rot}(c_u, c_v) = -3$ or $\text{rot}(c_u, c_v) = 1$. If v is pointing upward (i.e., $\text{rot}(c_u, c_v) = 0$), then we say that there exists no *change in direction* between u and v. Otherwise, we say that there exists a change in direction between u and v, which implies $\text{rot}(c_u, c_v) < 0$.

Claim 2 *Let H be a turn-regular rectilinear representation of a tree T. Then, the total number of changes in direction of the leaves of T in H is at most four.*

Proof of the claim: Assume to the contrary that there exist $s \geq 5$ pairs of consecutive leaves $\langle v_1, v_2 \rangle, \ldots, \langle v_{2s-1}, v_{2s} \rangle$ in a traversal of the external face of T, where a change in direction occurs. Note that $v_1, v_2, \ldots, v_{2s-1}, v_{2s}$ are not necessarily distinct. For $i = 1, \ldots, 2s$, let $\langle c_i, c'_i \rangle$ be the ordered pair of corners associated with v_i. Since $\langle v_{2i-1}, v_{2i} \rangle$ introduces a change in direction, $\text{rot}(c_{2i-1}, c_{2i}) \leq -1$. Summing up over i, we obtain $\sum_{i=1}^{s} \text{rot}(c_{2i-1}, c_{2i}) \leq -5$. Since the remaining leaves of T do not introduce a change in direction, $\text{rot}(u,u) = \sum_{i=1}^{s} \text{rot}(c_{2i-1}, c_{2i})$ holds for any leaf u in H, which is a contradiction to Property 1. This completes the proof of the claim.

Since 2- and 3-forks require one and two changes in direction, the proof of lemma follows directly from Claim 2. □

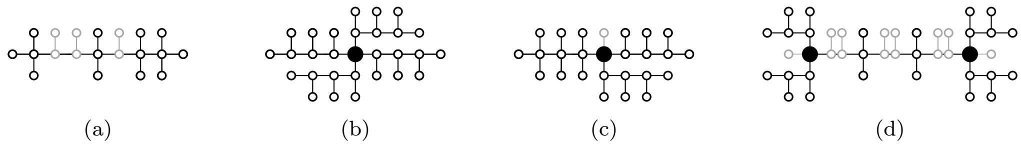

Figure 11: Illustration of (a) a 4-caterpillar, (b) a 4-windmill, (c) a 3-windmill, and (d) a double-windmill. Possible extensions are highlighted in gray.

Lemma 5 *A non-trivial tree T contains at least one 2- or 3-fork.*

Proof: Since T is non-trivial and contains no vertices of degree two, there exists a non-leaf vertex v with degree either three or four, such that v is adjacent to exactly two or three leaves, respectively. Thus, the claim follows. □

Corollary 2 *A turn-regular tree has at most four non-trivial disjoint subtrees.*

A vertex v of a tree T is a *splitter* if v is adjacent to at least three non-leaf vertices. The following lemma shows that a turn-regular tree cannot contain more than two splitters.

Lemma 6 *A turn-regular tree T contains at most two splitters.*

Proof: Assume to the contrary that T contains at least three splitters v_1, v_2 and v_3. We first claim that it is not a loss of generality to assume that v_1, v_2 and v_3 appear on a path in T. If this is not the case, then there is a vertex, say u, such that v_1, v_2 and v_3 lie in three distinct subtrees rooted at u. Hence, u is a splitter that lies on the path from v_1 to v_3. If we choose v_2 to be u, the claim follows.

Let P be the path containing v_1, v_2 and v_3 in T, and assume w.l.o.g. that v_1 and v_3 are the two end-vertices of P. Since v_1 is a splitter, it is adjacent to at least three vertices that are not leaves and two of them do not belong to P. Let T_1 and T_2 be the subtrees of T rooted at these two vertices, which by definition are non-trivial and do not contain v_2 and v_3. By a symmetric argument on v_3, we obtain two non-trivial subtrees T_3 and T_4 of T that do not contain v_1 and v_2. The third splitter v_2 may have only one neighbor that is not a leaf and does not belong to P. The (non-trivial) subtree T_5 rooted at this vertex contains neither v_1 nor v_3. Hence, $T_1, \ldots, T_5$ contradict Corollary 2. □

By Lemma 6, a turn-regular tree contains either zero or one or two splitters; we study each of these cases in Lemmas 7, 8 and 9, respectively. A *caterpillar* is a tree, whose leaves are within unit distance from a path, called *spine*. For $k \geq 3$, a *k-caterpillar* is a non-trivial caterpillar (i.e., not a single edge) whose spine vertices have degree at least 3 and at most k.

Lemma 7 *A tree T without splitters is a 4-caterpillar and turn-regular.*

Proof: In the absence of splitters in T, all inner vertices of T form a path. Hence, T is a 4-caterpillar and thus turn-regular; see Fig. 11(a). □

A tree with one splitter v is (i) a 4-*windmill*, if v is the root of four 3-caterpillars (Fig. 11(b)), (ii) a 3-*windmill*, if v is the root of two 3-caterpillars and one 4-caterpillar (Fig. 11(c)). Note that in the latter case, v can be adjacent to a leaf if it has degree four. The operation of *pruning* a rooted tree T *at* a degree-k vertex v with $k \in \{3, 4\}$ that is not the root of T, removes the $k-1$ subtrees of T rooted at the children of v without removing these children, and yields a new subtree T' of T, in which v and its children form a $(k-1)$-fork in T'.

Lemma 8 *A tree T with one splitter is turn-regular if and only if it is a* 3- *or* 4-*windmill.*

Proof: Clearly, every 3- and 4-windmill is turn-regular; this can be easily seen looking at the illustrations in Figs. 11(c) and 11(b), where no pairs of kitty corners are present. For the other direction, consider a turn-regular tree T and let u be the unique splitter of T. By definition, u has either four or three neighbors that are not leaves. In the former case, we will prove that T is a 4-windmill, while in the latter case that T is a 3-windmill. Note that u may be adjacent to a leaf, only if $\deg(u) = 4$.

We start with the case in which u has four neighbors that are not leaves, which implies that u is the root of exactly four non-trivial subtrees $T_1, \ldots, T_4$. Since u is the only splitter in T, by Lemma 7, it follows that each of $T_1, \ldots, T_4$ is a 4-caterpillar. To show that T is a 4-windmill, it remains to show that each of $T_1, \ldots, T_4$ cannot contain a degree-4 vertex. Assume to the contrary that T_1 contains a degree-4 vertex v. Since the $\deg(u) = 1$ in T_1, $u \neq v$ holds. We root T at u and we proceed by pruning T at v, which will result in a subtree T' of T that contains a 3-fork at v. Since T is turn-regular, by Lemma 3, T' is also turn-regular. Since $T_2, \ldots, T_4$ are non-trivial, by Lemma 5 each of them contains a fork. By Lemma 4, these three forks together with the 3-fork formed at v contradict the fact that T' is turn-regular.

We now consider the case in which u has three neighbors that are not leaves, that is, u is the root of exactly three non-trivial subtrees T_1, T_2 and T_3. Since u is the only splitter in T, we have again that each of these subtrees is a 4-caterpillar. We now claim that two of them must be 3-caterpillars, which also implies that T is a 3-windmill, as desired. Assume to the contrary that T_1 and T_2 are not 3- but 4-caterpillars, that is, there exist vertices v_1 and v_2 in T_1 and T_2 that are of degree 4, respectively. Note that $u \neq v_1$ and $u \neq v_2$. We assume that T is rooted at u and, as in the previous case, we prune T first at v_1 and then at v_2. The resulting subtree T' contains two 3-forks at v_1 and v_2 and one additional fork that is contained in T_3 (by Lemma 5). Hence, by Lemma 4, T' is not turn-regular, which is a contradiction to Lemma 5 (since T' is a subtree of the turn-regular tree T). □

A tree T with exactly two splitters u and v is a *double-windmill* if (i) the path from u to v in T forms the spine of a 4-caterpillar in T, (ii) each of u and v is the root of exactly three non-trivial subtrees, and (iii) the two non-trivial subtrees rooted at u (v) that do not contain v (u) are 3-caterpillars; see Fig. 11(d). The proof of the next lemma is similar to the one of Lemma 8.

Lemma 9 *A tree T with two splitters is turn-regular if and only if it is a double-windmill.*

Proof: Clearly, every double-windmill is turn-regular, as it can be easily seen looking at the illustration in Fig. 11(d). Now consider a turn-regular tree T and let u and v be the two splitters of T.

To prove that T has Property (i) of a double-windmill, consider the path P from u to v in T, and let w be an internal vertex of P (if any). Since w is an internal vertex of P, it is adjacent to two non-leaves of T (i.e., its neighbors in P). Note that w cannot be a splitter, because T has exactly two splitters (that is, u and v). Hence, the neighbors of w that are not in P are leaves, which implies that Property (i) of a double-windmill holds for T.

We now prove that T has Property (ii) of a double-windmill. Assume to the contrary that u is the root of four non-trivial subtrees. We root T at u and we denote by $T_1, \ldots, T_4$ the subtrees of

T that are rooted at the four children of u. It follows that $T_1, \ldots, T_4$ are non-trivial and disjoint. W.l.o.g., assume that T_1 contains v. Since v is a splitter, there exist at least two non-trivial subtrees of T_1, say T_1^1 and T_1^2, that are rooted at two children of v. Hence, T_1^1, T_1^2, T_2, T_3 and T_4 are disjoint subtrees of T. Since each of these subtrees is non-trivial, by Corollary 2 we have a contradiction to the turn-regularity of T. Hence, Property (ii) of a double-windmill holds for T.

By Property (ii), u has three non-leaf neighbors $u_1, \ldots, u_3$ in T. We again root T at u and we denote by $T_1, \ldots, T_3$ the subtrees of T that are rooted at $u_1, \ldots, u_3$. It follows that $T_1, \ldots, T_3$ are non-trivial and disjoint. As above, we assume w.l.o.g. that T_1 contains v, and we define in the same way the two subtrees, T_1^1 and T_1^2, of T_1. We now claim that none of T_1^1, T_1^2, T_2 and T_3 contains a degree-4 vertex z. Assume to the contrary that one, say T_3, contains a degree-4 vertex z. Since we have assumed T to be rooted at u, we proceed by pruning T at z. The resulting tree T', which is turn-regular by Lemma 3, contains a 3-fork at z and three additional forks that lie in the non-trivial trees T_1^1, T_1^2 and T_2, which contradicts Corollary 2. Hence, our claim follows. In particular, since u and v are the only splitters in T, our claim implies that each of T_2 and T_3 together with u, as well as, each of T_1^1 and T_1^2 together with v is a 3-caterpillar, which implies that Property (iii) of a double-windmill holds for T. This conclude the proof of this lemma. □

We are now ready to prove the main theorem of this section.

Theorem 4 *A tree T is turn-regular if and only if* smooth(T) *is (i) a 4-caterpillar, or (ii) a 3- or a 4-windmill, or (iii) a double-windmill. Moreover, recognition and drawing can be done in linear time.*

Proof: Since by Lemma 6 a turn-regular tree has at most two splitters, the correctness of our characterization follows from Lemmas 7, 8 and 9 (recall that we have assumed w.l.o.g. that T does not have degree-2 vertices). For the recognition, we first count how many splitter tree T contains, which can be done in $O(n)$ time. If there are more than two splitters, we reject the instance. If there are no splitters, we accept the instance and we report the representation described in Lemma 7. For the remaining cases, we first observe that one can trivially test whether a (sub-)tree is a 3- or a 4-caterpillar in time linear to its number of vertices. This observation directly implies that in linear time one can test whether T is turn-regular, when T has exactly one splitter. It remains to consider the case in which T contains exactly two splitters u and v. It follows that each internal vertex of the path from u to v is adjacent only to leaves of T. We now argue for vertex u; symmetric arguments hold for v. Two of the subtrees of T rooted at u that do not contain v have to be 3-caterpillars, while the third (if any) has to be a leaf. Both can be checked in time linear in the size of T, which completes the description of the proof. □

5 Turn-Regular Rectilinear Representations

Here we focus on rectilinear planar representations of biconnected plane graphs, and prove the following result.

Theorem 5 *Let G be an n-vertex biconnected plane graph with faces of degree at most eight. There exists an $O(n^{1.5})$-time algorithm that decides whether G admits an embedding-preserving turn-regular rectilinear representation and computes one in the positive case.*

Proof: We describe a testing algorithm based on a constrained version of Tamassia's flow network $N(G)$, which models the space of orthogonal representations of G within its given planar embedding [27]. Let V, E, and F be the set of vertices, edges, and faces of G, respectively. Tamassia's

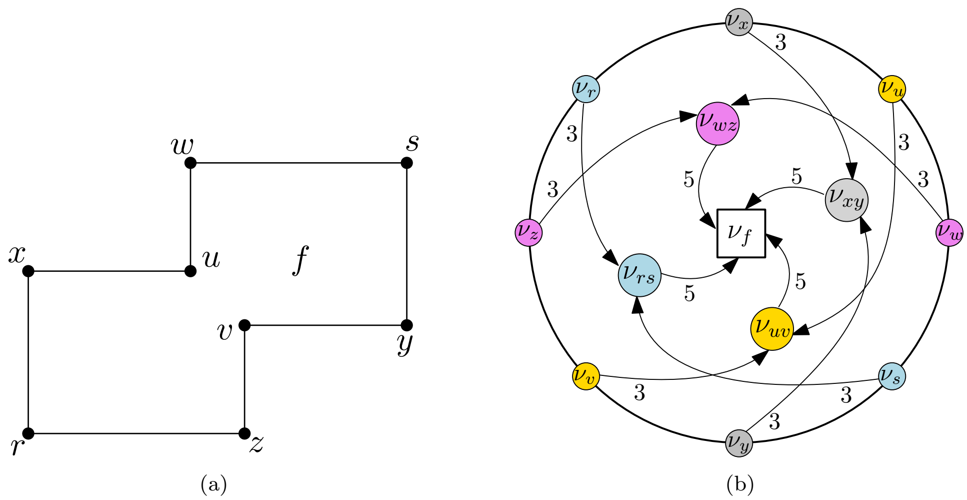

Figure 12: (a) A pair of kitty corners in a face of degree eight. (b) The modification of the flow network around a face-node corresponding to an internal face. The labels on the directed edge represent capacities.

flow network $N(G)$ is a directed multigraph having a *vertex-node* ν_v for each vertex $v \in V$ and a *face-node* ν_f for each face $f \in F$. $N(G)$ has two types of edges: (i) for each vertex v of a face f, there is a directed edge (ν_v, ν_f) with capacity 3; (ii) for each edge $e \in E$ incident to faces f and g, there is a directed edge (ν_f, ν_g) and a directed edge (ν_g, ν_f), both with infinite capacity.

A feasible flow on $N(G)$ corresponds to an orthogonal representation of G: a flow value $k \in \{1, 2, 3\}$ on an edge (ν_v, ν_f) represents an angle of $90 \cdot k$ degrees at v in f (since G is biconnected, there is no angle larger than $270°$ at a vertex); a flow value $k \geq 0$ on an edge (ν_f, ν_g) represents k bends on the edge of G associated with (ν_f, ν_g), and all these bends form an angle of $90°$ inside f. Hence, each vertex-node ν_v supplies 4 units of flow in $N(G)$, and each face-node ν_f in $N(G)$ demands an amount of flow equal to $c_f = (2\deg(f) - 4)$ if f is internal and to $c_f = (2\deg(f) + 4)$ if f is external. The value c_f represents the *capacity of* f. It is proved in [27] that the total flow supplied by the vertex-nodes equals the total flow demanded by the face-nodes; if a face-node ν_f cannot consume all the flow supplied by its adjacent vertex-nodes (because its capacity c_f is smaller), it can send the exceeding flow to an adjacent face-node ν_g, through an edge (ν_f, ν_g), thus originating bends.

Our algorithm has to test the existence of an orthogonal representation H such that: (a) H has no bend; (b) H is turn-regular. To this aim, we suitably modify $N(G)$ so that the possible feasible flows only model the set of orthogonal representations that verify Properties (a) and (b). To enforce Property (a), we just remove from $N(G)$ the edges between face-nodes. To enforce Property (b), we enhance $N(G)$ with additional nodes and edges. Consider first an internal face f of G. By hypothesis $\deg(f) \leq 8$. It is immediate to see that if $\deg(f) \leq 7$ then f cannot have a pair of kitty corners. If $\deg(f) = 8$, a pair $\{u, v\}$ of kitty corners necessarily requires three vertices along the boundary of f going from u to v (and hence also from v to u); see Fig. 12(a). Therefore, for such a face f, we locally modify $N(G)$ around ν_f as shown in Fig. 12(b). Namely, for each potential pair $\{u, v\}$ of kitty corners, we introduce an intermediate node ν_{uv}; the original edges (ν_u, ν_f) and (ν_v, ν_f) are replaced by the edges (ν_u, ν_{uv}) and (ν_v, ν_{uv}), respectively (each still having capacity 3); finally, an edge (ν_{uv}, ν_f) with capacity 5 is inserted, which avoids that u and v form a reflex corner inside f at the same time. For the external face f, it can be easily seen that a pair of kitty corners is possible only if the face has degree at least 10. Since we are assuming that $\deg(f) \leq 8$, we do not need to apply any local modification to $N(G)$ for the external face.

Hence, a rectilinear turn-regular representation of G corresponds to a feasible flow in the modified version of $N(G)$. Since $N(G)$ can be easily transformed into a sparse unit capacity network, this problem can be solved in $O(n^{1.5})$ time by applying a maximum flow algorithm (the value of the maximum flow must be equal to $4|V|$) [1]. □

6 Open Problems

This paper started the study of which planar graphs admit orthogonal planar drawings that are turn-regular. We showed notable classes of biconnected planar graphs that always admit such a drawing; also, we provided polynomial-time testing and drawing algorithms for trees and for biconnected plane graphs with faces of degree at most eight.

Our work raises several open research directions. We list three open problems that, in our opinion, are among the most interesting.

- **Problem 1.** One the main natural question not answered in this paper is whether all biconnected planar 4-graphs are turn-regular. We recall that Theorem 3 gives a positive answer only if we disregard the external face.
- **Problem 2.** The proof of Theorem 3 shows how to construct an internally turn-regular drawing of an n-vertex biconnected planar 4-graph in $O(n^4)$ area. It would be interesting to know whether biconnected planar 4-graphs admit internally turn-regular drawings in a more restricted area.
- **Problem 3.** As a special case of the previous problem, one can restrict to the class of triconnected planar 4-graphs. We suspect that graphs in this class are always turn-regular.
- **Problem 4.** For rectilinear representations, Theorem 5 gives a polynomial-time turn-regularity testing algorithm when the graph has faces of degree at most eight. It would be interesting to extend this result to more general classes of plane graphs.

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