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Properties of Large 2-Crossing-Critical Graphs

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Abstract. A c-crossing-critical graph is one that has crossing number at least c but each of its proper subgraphs has crossing number less than c. Recently, a set of explicit construction rules was identified by Bokal, Oporowski, Richter, and Salazar to generate all *large* 2-crossing-critical graphs (i.e., all apart from a finite set of small sporadic graphs). They share the property of containing a generalized Wagner graph V_{10} as a subdivision.

In this paper, we study these graphs and establish their order, simple crossing number, edge cover number, clique number, maximum degree, chromatic number, chromatic index, and treewidth. We also show that the graphs are linear-time recognizable and that all our proofs lead to efficient algorithms for the above measures.

Keywords. Crossing number, crossing-critical graph, chromatic number, chromatic index, treewidth.

1 Introduction

The first characterization of *planar* graphs is due to Kuratowski in 1930: A graph[1] is planar if and only if it neither contains a subgraph isomorphic to a subdivision of the $K_{3,3}$ nor the K_5 [31]. This result inspired several characterizations of graphs by forbidden subgraphs, which paved paths into

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[1]Multiple edges and loops arise naturally in the context of graph embeddings and graph drawings. Hence, in such context, a graph can have multiple edges and loops, and the term simple graph is employed whereever we emphasize that these features are not present. We follow this convention throughout this paper.

significantly different areas of graph theory. Extremal graph theory is concerned with forbidding any subgraph isomorphic to a given graph [11] and maximizing the number of edges under this constraint. Significant structural theory was developed when forbidden *induced* subgraphs were considered instead, for instance several characterizations of Trotter and Moore [49] and the remarkable weak and strong perfect graph theorems [33,38]. Wagner coined graph minor theory as another means of characterizing planar graphs [51]. It was later used to extend Kuratowski's theorem to higher surfaces: A seminal result by Robertson and Seymour states that all graphs embeddable into any prescribed surface are characterized by a finite set of forbidden minors [41]. While these minors are known for the projective plane [2], already on the torus, the number of forbidden minors reaches into tens of thousands and is as of now unknown [20]. Still, Mohar devised an algorithm to embed graphs on surfaces in linear time [35], that was later improved by Kawarabayashi, Mohar, and Reed [28]. Characterizations of graph classes by subdivisions received somewhat less renowned attention. Early on the above path, Chartrand, Geller, and Hedetniemi pointed at some common generalizations of forbidding a small complete graph and a corresponding complete bipartite subgraph as a subdivision, resulting in trees, outerplanar, and planar graphs [15]. More recently, Dvořák achieved a characterization of several other graph classes using forbidden subdivisions [19].

Another direction to generalize Kuratowski's theorem is the notion of *c-crossing critical* graphs, i.e., graphs that require at least $c \in \mathbb{N}$ crossings when drawn in the plane, but each of their subgraphs requires strictly less than c crossings. Allowing crossings in order to increase the degree of freedom rather than adding handles to the surface exhibits a richer structure compared to forbidden minors for embeddability on surfaces. Unlike the latter, it allows infinite families of topologically-minimal obstruction graphs, as first demonstrated by Širáň [48], who constructed an infinite family of 3-connected c-crossing-critical graphs for each $c > 2$. Kochol extended this result to simple, 3-connected graphs [30], for each $c > 1$, thus producing the first family of (simple) large 3-connected 2-crossing-critical graphs. Most importantly for our research is Bokal, Oporowski, Salazar, and Richter's [10] characterization of the *complete* list of minimal forbidden subdivisions for a graph to be realizable in the plane with only one crossing; that is, precisely the *2-crossing-critical* graphs. Bokal, Bračič, Dernar, and Hliněný characterized average degrees for infinite families of 2-crossing-critical graphs w.r.t. constraining the vertex-degrees that appear arbitrarily often [7]. For each restriction, the resulting average degrees form an interval. Hliněný and Korbela showed that if *all* degrees are prescribed, instead of just the frequent ones, the attainable average degrees are no longer intervals, but dense subsets of intervals [26]. Based upon [10], Bokal, Vegi-Kalamar, and Žerak defined a simple regular grammar describing *large* 2-crossing-critical graphs—i.e., all 3-connected 2-crossing-critical graphs except for a finite set of (small) sporadic graphs—and used it for counting Hamiltonian cycles in these graphs [9]. We build upon this grammar to study the graph theoretic properties of large 2-crossing-critical graphs.

We also briefly discuss recognizing c-crossing-critical graphs. 1-crossing critical graphs are precisely subdivisions of a K_5 or a $K_{3,3}$; they are thus trivial to recognize. For general $c \geq 2$, the problem is fixed-parameter tractable (FPT) w.r.t. c: Grohe [22] first showed that there is an algorithm to recognize graphs with $\mathrm{cr}(G) \leq c$ in time $\mathcal{O}(poly(|V(G)|) \cdot f(c))$ for some (at least doubly exponential but computable) function f. Kawarabayashi and Reed [29] improved this FPT-algorithm to an only linear dependency on $|V(G)|$. Despite the fact that these algorithms are infeasible in practice, they can theoretically be used as a building block to verify $\mathrm{cr}(G) \geq c$ and $\mathrm{cr}(G - e) < c$, for each $e \in E(G)$. Thus c-crossing-critical graphs can be recognized in FPT-time $\mathcal{O}(|V(G)| \cdot |E(G)| \cdot f(c))$, for some computable function f. We do not know of any further algorithmic results regarding the recognition problem.

Table 1: Overview on the properties of large 2-crossing-critical graphs studied in this paper.

property	characterization	values	see
graph size	complete	–	Observation 3.1
maximum degree	complete	$4, 5, 6$	Observation 3.2
clique number	complete	$2, 3$	Corollary 3.5
edge cover number	complete	$\lceil \lvert V(G)\rvert/2\rceil$	Observation 3.7
simple crossing number	complete	2	Theorem 4.2
chromatic number	partial	$2, 3, 4$	Theorems 5.5 & 5.6
chromatic index	complete*	$\Delta(G)$	Theorem 6.5
treewidth	complete*	4, 5	Corollary 7.9

* few (finitely many) graphs on only 3 elementary tiles can attain smaller (resp. larger) values for treewidth (resp. chromatic index). See corresponding sections.

Our Contribution. In Section 2, we recall the formal definition of large 2-crossing-critical graphs, their construction, and the recently proposed grammar to chiefly describe them. In Section 3, we proceed to determine some of their elementary properties, such as order, maximum degree, clique, and matching number. We also show that large 2-crossing-critical graphs are linear-time recognizable.

In Section 4, we establish that their *simple* crossing number is indeed also 2. We propose sufficient sets of *color propagations* (defined later) to find their chromatic number and index in Sections 5 and 6, respectively. Finally, in Section 7, we characterize the graphs' treewidth via the appearance of a single minor. Further, in each section, we propose natural linear time algorithms to compute the respective measures on any given large 2-crossing-critical graph.

Although the graphs under consideration form a structurally rich, yet countable infinite family, our results underline their structural cohesiveness: all investigated measures reside in a small range, some are even constant over all such graphs. Table 1 summarizes all considered properties and our results.

2 Large 2-Crossing-Critical Graphs

For standard graph theory terminology, such as (induced) subgraphs and graph minors, we refer to [17, 46]. A *drawing* of a graph G *in the plane* consists of two injective maps: One assigning each vertex $v \in V(G)$ to a point in $\mathbb{R}^2$, the other each edge $uv \in E(G)$ to a Jordan curve from u to v in $\mathbb{R}^2$ such that no curve has a vertex in its interior. In the context of crossing numbers, we typically restrict ourselves to *good* drawings: Each pair of curves has at most one interior point in common (if it exists, it is the *crossing* of this pair), adjacent curves have no common crossing, and the intersection of any three non-adjacent curves is empty.

Definition 2.1 *The* crossing number $\mathrm{cr}(G)$ *of a graph G is the smallest number of crossings over all of its drawings in the plane. Further, G is* c-crossing-critical *for some $c \in \mathbb{N}$, if* $\mathrm{cr}(G) \geq c$*, but every proper subgraph $H \subset G$ has $cr(H) < c$.*

Following this definition, we feel that some intuitive explanation of the context is in place before we formalize the details in the rest of this section. Note that the above definition defines

2-crossing-critical graphs, but does not say anything about how they actually look like. For 1-crossing-critical graphs, this was resolved by Kuratowski's theorem, which exposed K_5 and $K_{3,3}$ as the only two 3-connected 1-crossing-critical graphs, and all other 1-crossing-critical graphs as their subdivisions. As mentioned in the introduction, already 2-crossing-critical graphs—the next step beyond Kuratowski's Theorem—exhibit a significantly richer structure, and allow for an infinite family of 3-connected 2-crossing-critical graphs [30,48]. However, despite being infinite, this family has a tightly defined structure. The purpose of this section is to describe this structure, i.e., to use the characterization results of [10] to explain how (almost all) 3-connected 2-crossing-critical graphs actually look like. All 3-connected 2-crossing-critical graphs with sufficiently many vertices exhibit this structure; only finitely many do not (the Petersen graph being the most prominent example). Thus we call the graphs having this structure *large* 2*-crossing-critical graphs*.

Let us formally define the construction rules that generate the set of 2-crossing-critical graphs and give a brief overview of their history. The concept of *tiles* (to be defined in this section) was introduced by Pinontoan and Richter to answer a question of Salazar about average degrees of large families of c-crossing-critical graphs [37, 43]. Over a series of papers, it turned out to be a tool that gives surprisingly precise lower bounds on crossing numbers of several "tiled" graphs, see [8]. Dvořák, Hliněný, and Mohar showed that tiles form an essential ingredient of large c-crossing-critical graphs for every $c \geq 2$ [18]. In general, further structures (so-called *belts* and *wedges*) may also appear arbitrarily often, together with a bounded small graph that connects them [8]. For $c = 2$, however, Bokal, Oporowski, Richter, and Salazar proved that tiles are sufficient to describe almost all (i.e., all but finitely many) 2-crossing-critical graphs [10]. In fact, belts appear if and only if $c \geq 3$ and wedges if and only if $c \geq 13$ [8,24].

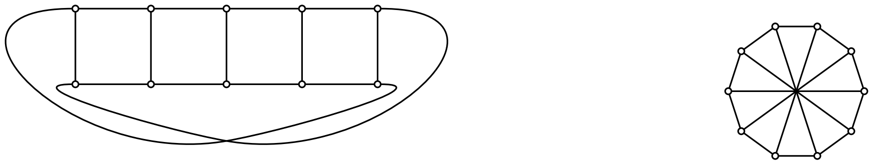
Intuitively, *tiles* are prespecified small graphs with vertex subsets at which we can glue (pairs of) tiles together. A *tiled graph* is a graph arising from cyclically glueing tiles together. Formally, we adopt the following notation from [43], which is illustrated in Figure 3:

Definition 2.2 *A* tile *is a triple* $T = (G, x, y)$*, consisting of a graph* G *and two non-empty sequences* $x = \langle x_1, x_2, \ldots, x_k\rangle$ *and* $y = \langle y_1, y_2, \ldots, y_l\rangle$ *of distinct vertices of* G*, with no vertex appearing in both* x *and* y*. The sequence* x *(sequence* y*) is* T*'s* left wall *(*right wall*, resp.). If* $|x| = |y| = k$*,* T *is a* k-tile*.*

Definition 2.3 Tiled graphs *are joins of cyclic sequences of tiles. We formalize this as follows:*

1. *A tile* $T = (G, x, y)$ *is* compatible *with a tile* $T' = (G', x', y')$ *if* $|y| = |x'|$*. Their* join $T \otimes T' := (G^*, x, y')$ *is a new tile, where* G^* *is obtained from the disjoint union of* G *and* G' *by identifying* y_i *with* x'_i *for each* $i = 1, \ldots, |y|$*.*
2. *A sequence* $\mathcal{T} = \langle T_0, T_1, \ldots, T_m\rangle$ *of tiles is* compatible *if* T_{i-1} *is compatible with* T_i *for each* $i = 1, 2, \ldots, m$*. The* join $\otimes\mathcal{T}$ *of a compatible sequence* $\mathcal{T}$ *is* $T_0 \otimes T_1 \otimes \cdots \otimes T_m$*.*
3. *For a* k*-tile* $T = (G, x, y)$*, the* cyclization *of* T *is the graph* $\bigcirc T$ *obtained from* G *by identifying* x_i *with* y_i *for each* $i = 1, \ldots, k$*. (Observe that in general,* T *may itself have arisen from a join of a compatible sequence.)*

With these tools, we are now ready to recall the constructive characterization of 2-crossing-critical graphs by tiles [10]. Thereby, we focus only on the graphs that belong to the theoretically relevant infinite family of these graphs. We disregard some finite set of special cases as well as graphs that are not 3-connected, as they add no relevant structural information. The latter ones can be trivially obtained from the 3-connected ones, and 3-connectivity is a typical restriction when



(a) V_{10} drawn as Möbius ladder. (b) V_{10} drawn according to definition.

Figure 1: The generalized Wagner graph V_{10}.

studying graphs from a topological perspective, such as crossing numbers. Put chiefly, we provide the characterization of all (except for finitely many) 3-connected 2-crossing-critical graphs, called large 2-crossing-critical graphs. In the course of this, we will also describe these graphs' alphabetic description [9], which associates unique and coherent names to each such graph.

Again, before we formally define the set $\mathcal{C}$ of large 2-crossing-critical graphs, we may give an intuitive definition. There are 42 planar 2-tiles $\mathcal{S}$ to choose from (to be described later). Each graph in $\mathcal{C}$ is a cyclization of a sequence of an odd number of these tiles. But thereby, every second tile will be used flipped top-to-bottom (which is not so important right now), and we reverse the order of the final right wall vertices prior to the cyclization. Without this final twist, the resulting graph would resemble a cyclic planar strip of tiles; due to the twist, the resulting graph becomes non-planar but can be embedded on the Möbius strip.

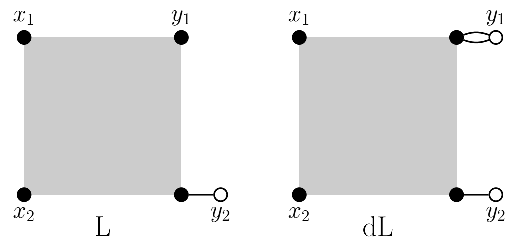
Figure 1 shows the general graph structure exhibited by this process: assume each tile is drawn within a square region, then Figure 1a represents the resulting Möbius strip, where one of the squares is twisted. The graph that is depicted is in fact a generalized Möbius ladder, also known as generalized Wagner graph, and it is instrumental in understanding 2-crossing-critical graphs. Formally, it is defined as the graph V_{2n}, $3 \leq n \in \mathbb{N}$, obtained from the cycle C_{2n} in which each pair of antipodal vertices is connected via an additional edge (a *spoke* of V_{2n}), see Figure 1b. The smallest Wagner graph V_6 is isomorphic to $K_{3,3}$.

Based on this structure, assuming each tile T_i in the sequence has some unique string s_i as its name, it is straight-forward to use the concatenation $s_1 s_2 \dots$ to describe the resulting graph. We call these strings *signatures*. The join of our tiles can also be understood such that we cyclically join tiles (without vertical flipping) by *always* reversing the order of the right wall vertices. While this understanding is not very helpful in terms of drawings with low crossing number, it shows that the graph-defining tile sequence is intrinsically cylic; consequently each graph's signature can be cyclically rotated as well, and for a graph with k tiles we obtain k potentially different signatures.

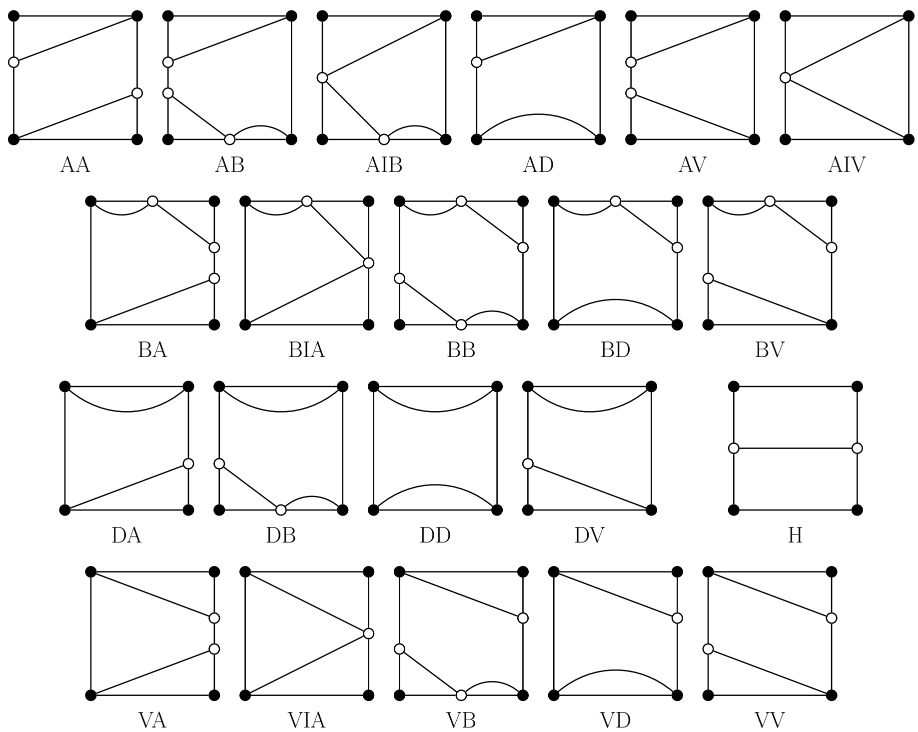
It remains to discuss the fundamental 42 planar 2-tiles themselves, as they are highly structured. Each tile can be understood to be composed of a *frame* and a *picture* within that frame. Formally, these are graphs, enriched with vertex markings. There are two different frames (Figure 2a), and 21 different pictures (Figure 2b). We will hence compose the signature of a tile as the concatenation of signatures of its picture and its frame. The names of the pictures arise from the graph structures along the top and bottom border of the tile (*top path* and *bottom path*, respectively) and their rough similarity to letters; see Figure 2c.

The example graph on five tiles in Figure 3 completes the informal definition of the construction of large 2-crossing-critical graphs. We will revisit this example graph in later sections to showcase the investigated properties. We are now ready to formally define our graph class.

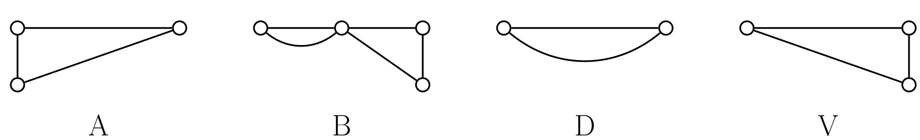
Definition 2.4 (based on [10]) *Large 2-crossing-critical graphs are defined as the set $\mathcal{C}$ in the following way:*



(a) The two frames and their names. Left wall vertices of the tiles obtained from these frames are x_1, x_2 and right wall vertices are y_1, y_2.

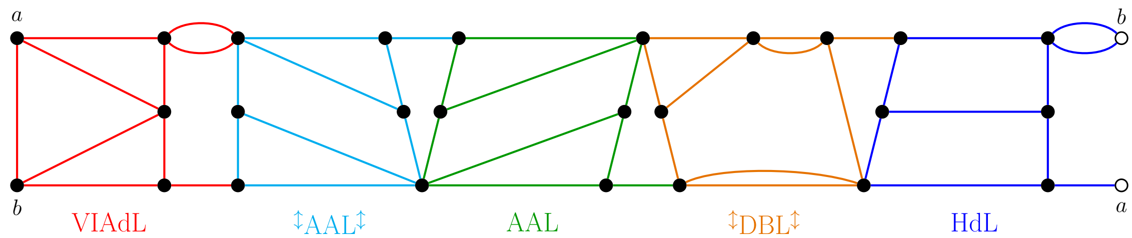


(b) The 21 pictures and their names. The black vertices of the pictures are identified with black vertices of the frames (without additional rotation) to yield tiles in the set $\mathcal{S}$.



(c) Explanation for the names of the pictures (except for *H* where the basis for the name is evident): We show the *top path* of the pictures, together with their names. *Bottom paths* are referred to equivalently but rotated by 180°. The amalgamation of these names yield the signatures of the pictures, using the additional letter *I*, if a vertex of the top path becomes identified with one of the bottom part.

Figure 2: Composition of tiles by pasting pictures into frames. The black vertices are identified when inserting a picture into a frame at the gray square. The tile's wall vertices are labeled.



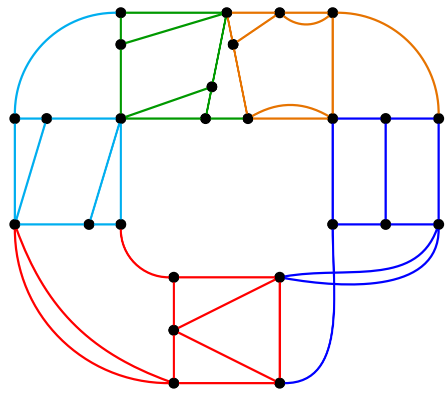


Figure 3: An example of a large 2-crossing-critical graph; its signature is *VIAdL AAL AAL DBL HdL*; colors represent the individual elementary tiles. On the top, the graph is drawn on the projective plane, where the labeled vertices are identified according to their names. On the bottom, the same graph is drawn equivalently but in the plane, resulting in two explicit crossings when twisting the dark blue HdL-tile.

1. *For a sequence x, let $\overline{x}$ denote the reversed sequence. The* right-inverted *(*left-inverted*) tile of a tile $T = (G, x, y)$ is the tile $T^{\updownarrow} := (G, x, \overline{y})$ (and ${}^{\updownarrow}T := (G, \overline{x}, y)$, respectively).*
2. *Let $\mathcal{S}$ be the set of tiles obtained as combinations of one of the two frames and one of the 21 pictures, shown in Figure 2, in such a way that a picture is inserted into a frame by identifying the gray area with it; the picture may not be rotated. While disregarding whether any wall order is reversed, we may call the tiles of $\mathcal{S}$* elementary tiles*.*
3. *Let $\mathcal{C}$ denote the set of all graphs of the form $\bigcirc(T_0^{\updownarrow} \otimes T_1^{\updownarrow} \otimes \ldots \otimes T_{2m}^{\updownarrow})$ with $m \geq 1$ and each $T_i \in \mathcal{S}$.*
4. *The* signature *$sig(T)$* of a tile *$T \in \mathcal{S}$ is the concatenation of the names of its picture and its frame. A* signature of a graph *G is based on its tile construction: $sig(G) := sig(T_0)sig(T_1)\ldots sig(T_{2m-1})sig(T_{2m})$.*

Observe that, by cyclic symmetry, the signature of a graph in $\mathcal{C}$ is not unique. Given two tiles T_a, T_b, also observe that $T_a^{\updownarrow} \otimes T_b$ is isomorphic to $T_a \otimes {}^{\updownarrow}T_b$. Thus we can rewrite $\bigcirc(T_0^{\updownarrow} \otimes T_1^{\updownarrow} \otimes \ldots \otimes T_{2m}^{\updownarrow}) = \bigcirc((T_0 \otimes {}^{\updownarrow}T_1^{\updownarrow} \otimes T_2 \otimes \ldots \otimes {}^{\updownarrow}T_{2m-1}^{\updownarrow} \otimes T_{2m})^{\updownarrow})$. While the former is formally more appealing and highlights the intrinsic symmetry, the latter implicitly tells us how to draw the graph with only 2 crossings: vertically flip every second tile to avoid all crossings until the last tile, where we require a simple twist.

Note that Definition 2.4 does not imply that these graphs are actually 2-crossing-critical, but the following theorem does:

Theorem 2.5 (Characterization by tiles [10, Theorems 2.18 & 2.19]) *Each element of $\mathcal{C}$ is 3-connected and 2-crossing-critical. Furthermore, all but finitely many 3-connected 2-crossing-critical graphs are contained in $\mathcal{C}$, and the set $\mathcal{C}$ contains all the 2-crossing-critical graphs that contain a V_{10} subdivision.*

Note that there may be small graphs of $\mathcal{C}$ that are 3-connected 2-crossing-critical, but do not have a V_{10} subdivision. Although this is a rather technical challenge, understanding it may simplify some approaches and the definition of $\mathcal{C}$, hence we pose it as an open problem:

Question 2.6 *List graphs of $\mathcal{C}$ with smallest number of vertices and edges. List graphs in $\mathcal{C}$ that do not contain a V_{10} subdivision or show that there are none.*

We denote the number of occurrences of a given symbol $X \in \{A, V, D, B, H, I\}$ in the signature of a 2-crossing-critical graph G by $\#X(G)$. We may omit the parameter G if it is clear from the context. It is trivial to test in linear time whether a supposed signature indeed describes a large 2-crossing-critical graph.

3 Elementary Properties

Given the characterization of large 2-crossing-critical graphs, we start our study by analyzing their elementary properties. We will later use these results to facilitate the study of more involved measures.

Observation 3.1 *The number of vertices and edges of a large 2-crossing-critical graph G is obtained using the following matrix-vector multiplication:*

$$\begin{bmatrix} |V(G)| \\ |E(G)| \end{bmatrix} := \begin{bmatrix} 3 & 5 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 0 & 1 \\ 2 & 3 \\ 2 & 4 \\ -1 & -1 \end{bmatrix}^{\intercal} \cdot \begin{bmatrix} \#L \\ \#d \\ \#A \\ \#V \\ \#D \\ \#H \\ \#B \\ \#I \end{bmatrix}$$

Proof: Considering each elementary tile, we count its number of vertices and edges. Since joining two tiles reduces the number of vertices by 2, we reduce the number of vertices for each tile by 2 (recall that this join is cyclic). It is straightforward to verify that each tile's signature generates the correct number of vertices and edges. □

Our example graph in Figure 3 with $sig(G) = VIAdLAALAALDBLHdL$ yields the graph-dependent vector $[5, 2, 5, 1, 1, 1, 1, 1]^{\intercal}$. Thus $[|V(G)|, |E(G)|]^{\intercal} = [3 \cdot 5 + 1 \cdot 2 + 1 \cdot 5 + 1 \cdot 1 + 0 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 - 1 \cdot 1, 5 \cdot 5 + 2 \cdot 2 + 2 \cdot 5 + 2 \cdot 1 + 1 \cdot 1 + 3 \cdot 1 + 4 \cdot 1 - 1 \cdot 1]^{\intercal} = [26, 48]^{\intercal}$.

Observation 3.2 *The maximum degree* Δ *of a large 2-crossing-critical graph* G *satisfies* $4 \leq \Delta \leq 6$*. In particular:*

- $\Delta(G) = 6$ *if and only if there are two consecutive elementary tiles* T_1, T_2*, such that* T_1*'s frame is* L*, its top path is* A *or* D*, as is the bottom path of* T_2 *(these paths are not necessarily equal).*
- $\Delta(G) = 5$ *if and only if* $\Delta(G) \neq 6$ *and* $\#\mathrm{A} + \#\mathrm{D} > 0$*.*
- $\Delta(G) = 4$ *if and only if* $\#\mathrm{A} + \#\mathrm{D} = 0$*.*

Proof: All elementary tiles with a *dL*-frame have a vertex of degree at least 4 where the frame's double edge connects. All elementary tiles with an *L*-frame have a vertex of at least degree 2 in the "top right", which gets identified with a vertex of the next tile with degree at least 2. Therefore, our graphs always contain vertices of degree at least 4.

Any path of $\{A, B, D, V, H\}$ increases the degree of vertices it connects to by at most 1. If a tile has top path A or D, its top right vertex has degree at least 5. The same applies to a tile's bottom left vertex, if it has bottom path A or D. Only by having a tile with an L frame and an upper path A or D followed by a tile with bottom path A or D, the identified vertex's degree becomes 6. □

A *clique* in a graph G is a subgraph of G that is complete. The *clique number* of G is the order of the maximum clique.

Observation 3.3 *An elementary tile contains a triangle if and only if its signature contains* A*,* V*, or* B *(cf. Figure 2). Moreover, each triangle in a large 2-crossing-critical graph* G *whose signature does not contain* A*,* V*, or* B*corresponds to a* DDLDD*-subsequence of sig*(G)*.*

In fact, this observation is sufficient to fully determine the clique number of a large 2-crossing-critical graph.

Observation 3.4 *A large 2-crossing-critical graph contains no* K_4*.*

Corollary 3.5 *A large 2-crossing-critical graph has clique number* 2 *if and only if all elementary tiles of* G *are one of* DDL*,* DDdL*,* HL *and* HdL*, and no subsequence* DDLDD *exists in sig*(G)*. Otherwise, its clique number is* 3*.*

Corollary 3.6 *Given the signature of a large 2-crossing-critical graph, its clique number can be determined in linear time.*

A *matching* in a graph G is a subset of pairwise non-adjacent edges. It is *perfect (near-perfect)* if its cardinality is $|V(G)|/2$ $((|V(G)|-1)/2$, resp.). From the fact that each large 2-crossing-critical graph contains a Hamiltonian cycle which can be computed in linear time [9], we obtain:

Observation 3.7 *Any large 2-crossing-critical graph* G *has a perfect matching if* $|V(G)|$ *is even, and a near-perfect matching otherwise. In both cases, the matching can be computed in linear time by choosing every second edge of a Hamiltonian cycle.*

Definition 3.8 *The* edge covering number *of a graph* G *is the minimal number of edges* F *in* G *such that each vertex* $v \in V(G)$ *is incident to an edge in* F*.*

Since a perfect matching yields a minimum edge cover, and a near-perfect matching requires only a single additional edge to become an edge cover, we have:

Algorithm 1: Large 2-crossing-critical graph recognition algorithm, deducing the signature in the positive case.

Input: graph G
Output: $sig(G)$ if G is a large 2-crossing-critical graph; $\varnothing$ otherwise
Define: $G{\downarrow}_8^v$ is the subgraph of G marked by a breadth-first search of bounded depth 8 starting at vertex v.

1 **if** $\Delta(G) > 6$ **then**
2   **return** $\varnothing$ *// G is not a large 2-crossing-critical graph*
3 choose any vertex $v \in V(G)$
4 let $\mathcal{X} := \{X \subseteq G{\downarrow}_8^v : X$ *is tile-isomorphic to a tile in* $\mathcal{S}\}$
5 **foreach** $X \in \mathcal{X}$ **do**
6     $sig(G) := sig(X)$
7     $G^* := G$ without the edges and non-wall vertices of X
8     $w_T, w_B :=$ the top and bottom right wall vertices of X
9   **loop**
10   Search for a subgraph $Y \subseteq G^*{\downarrow}_8^{x_1}$ that is tile-isomorphic to a tile in $\mathcal{S}$ using w_B as top left and w_T as bottom left wall vertex (note the reversed order) and such that all edges incident to w_B and w_T in G^* are in Y; prefer one with a dL-frame over one with an L-frame
11   **if** $\nexists\, Y$ **then break** *// continue with next X*
12   append $sig(Y)$ to $sig(G)$
13         $w_T, w_B :=$ the top and bottom right wall vertices of Y
14   remove from G^* all edges of Y and vertices of $V(Y) \setminus \{w_T, w_B\}$
15   **if** $V(G^*) \setminus \{w_T, w_B\} = \varnothing$ **then**
16   **if** w_B, w_T *are the top and bottom left wall vertices of* X *(note the reversed order) and* $\#L(sig(G))$ *is odd* **then**
17   **return** $sig(G)$ *// G is a large 2-crossing-critical graph*
18   **else break** *// continue with next X*
19 **return** $\varnothing$ *// G is not a large 2-crossing-critical graph*

Observation 3.9 *The edge covering number of any large 2-crossing-critical graph is* $\lceil |V(G)|/2 \rceil$.

Most importantly, large 2-crossing-critical graph are linear time recognizable. The general idea of Algorithm 1 is to restrict ourselves to a linear number of constantly sized graphs; in each of them, finding elementary tiles only requires constant time. In particular, this algorithms allows us to, in linear time, deduce the signature of a given graph if it belongs to the class; as such it will be the starting point for all subsequent algorithms to compute graph properties, as they can thus assume to be given the signature as input.

Theorem 3.10 *Algorithm 1 tests in linear-time whether a given graph G is a large 2-crossing-critical graph and, in the positive case, deduces a signature of G.*

Proof: We can reject graphs with maximum degree $\Delta(G) > 6$ in linear time (line 1). We say a subgraph H of G is *tile-isomorphic* to a tile T, if H is isomorphic to T, the non-wall vertices of H have no neighbors other than those described by T, and wall vertices from opposing walls of H are only adjacent if they are adjacent in T. We compute a subgraph $G{\downarrow}_8^v$ via a breadth-first search of bounded depth 8 starting at some arbitrary vertex v. This subgraph has constant size and can be found in constant time since $\Delta(G) \leq 6$. Furthermore, as the number $|\mathcal{S}|$ of possible tiles is constant, we can find the (constantly sized) set $\mathcal{X}$ of all subgraphs of $G{\downarrow}_8^v$ that are tile-isomorphic to a tile of $\mathcal{S}$ in constant time as well (line 4). The depth 8 is chosen so that, if G is a large 2-crossing-critical graph, it is guaranteed that $G{\downarrow}_8^v$ contains some subgraph H is tile-isomorphic to a tile of $\mathcal{S}$; thus $H \in \mathcal{X}$. The set $\mathcal{X}$ thus serves as a candidate list for T. We run the subsequent test (lines 6–18) for each $X \in \mathcal{X}$ (for-loop starting at line 5):

We remove X, retaining its wall vertices, and look for the neighboring tile Y to the right. Thereby, all edges incident to the wall vertices shared between X and Y have to lie in either of these tiles (in order to not overlook any unwanted incidencies between far-apart wall vertices). Again, this search only requires constant effort (line 10). In the positive case, after removing all of Y except its right wall vertices, we can iterate this process to identify all subsequent neighboring tiles, until we reattach – after an overall odd number of tiles – to the left wall of the initial tile X. By definition, we have to assure that the subsequent tiles use the common wall vertices in reverse order. If this process fails at any point, we reject the starting tile X and proceed with the next iteration of the for-loop, i.e., the next candidate from $\mathcal{X}$. If no iteration of the for-loop succeeds, we reject G.

In each iteration of the inner loop (lines 9–18) we either terminate the current for-loop iteration or remove a constant number of edges. Thus, the inner loop runs at most a linear number of times, each of its iterations requiring only constant time. This establishes the overall linear running time.

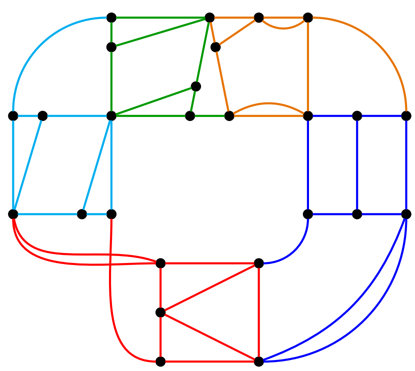
It is easy to see that if the algorithm returns a non-empty signature s, the large 2-crossing-critical graph constructed from s as per Definition 2.4 is isomorphic to G. On the other hand, suppose G is a large 2-crossing-critical graph and let s be a signature of G, such that v is in the first elementary tile $T \in \mathcal{S}$ of s. From the definition of $\mathcal{X}$, it follows that $T \in \mathcal{X}$. We only need to focus on the for-loop iteration in which $X = T$. If a graph has an elementary tile with a dL-frame as a subgraph, it also has an elementary tile with the same picture but an L-frame as a subgraph, but the converse is not true. Also given two elementary tiles with distinct pictures, at most one of them can be a subgraph that can be a right neighbor of the previously identified tile. Thus, in line 10, we obtain a unique potential candidate by prefering the new neighboring tile Y to have a dL-frame if possible. Based on the structure that tiles are cleanly separated by wall vertices (see Def. 2.3 and 2.4), we consequently have that our algorithm will indeed find signature s. □

4 Simple Crossing Number

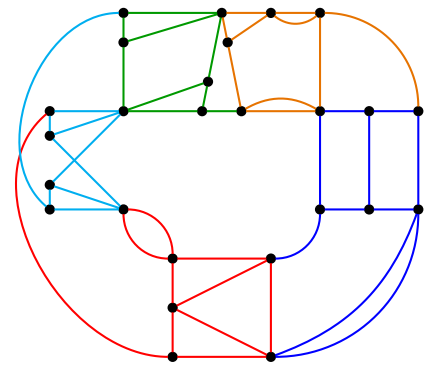
In this section, we prove that the simple crossing number of each large 2-crossing-critical graph equals its crossing number. To this end, we provide some definitions and briefly discuss their history. The study of 1-*planar* graphs was initiated more than half a century ago by Ringel in the context of graph coloring [39]. Buchheim et al. introduced the *simple crossing number*, while engineering the first general exact algorithms for computing crossing numbers [12].

Definition 4.1 *A* 1-planar drawing *of a graph G is a drawing of G in the plane such that each of its edges crosses at most one other edge. A graph that admits a* 1*-planar drawing is called* 1-planar*. The* simple crossing number $\mathrm{cr}^{\times}(G)$ *of G is the minimal number of crossings over all* 1*-planar drawings of G; we define* $\mathrm{cr}^{\times}(G) = \infty$ *if no such drawing exists.*

Albeit this crossing number variant is also known as *1-planar* crossing number, we prefer the term simple crossing number. This avoids confusion with the *k-planar* crossing number, $k \in \mathbb{N}$, as



(a) G still drawn as a Möbius strip but with the double crossing in the *VIAdL*-tile (red) instead of the *HdL*-tile (dark blue), to make it more similar to the figure to the right.



(b) A drawing showing $\mathrm{cr}^{\times}(G) = 2$ by twisting the left *AAL*-tile (light blue), as schematized in Figure 5a. Figure 5b shows how redrawing e.g. the *VIAdL*-tile would look like.

Figure 4: The example graph G from Figure 3 in the context of the simple crossing number.

defined by Owens (there, the graph is partitioned into k edge-sets and only the crossings in each set are counted) [36, 45].

By definition, $\mathrm{cr}(G) \leq \mathrm{cr}^{\times}(G)$. We remark that in general, $\mathrm{cr}(G) \neq \mathrm{cr}^{\times}(G)$ and there are graphs G with $\mathrm{cr}(G) = 2$ but $\mathrm{cr}^{\times}(G) > 2$ even on as few as 16 vertices [12]. By definition $\mathrm{cr}^{\times}(G) \in O(n^2)$, but for example $\mathrm{cr}(K_n) \in \Theta(n^4)$; in fact, already K_7 is not 1-planar (see, e.g., [45, 46]).

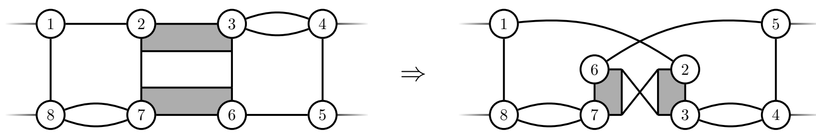
Theorem 4.2 *Any large 2-crossing-critical graph G has $\mathrm{cr}^{\times}(G) = 2$.*

Proof: Since $2 = \mathrm{cr}(G) \leq \mathrm{cr}^{\times}(G)$, the claim follows if each G admits a 1-planar drawing with 2 crossings.

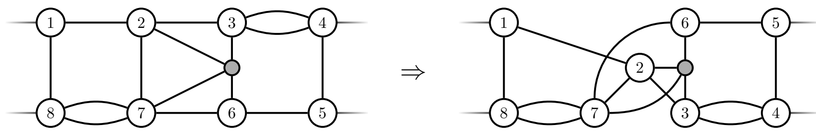
We achieve this by performing a *twist operation* at a single elementary tile X: In the natural drawing on the Möbius strip (cf. Figure 3) each tile is drawn planarly but we cannot identify the left-most with the right-most wall vertices in a planar fashion. Twisting a tile X within this drawing means to invert the vertical order of its left or right wall vertices, thereby incurring some crossings within X. Given this twisted tile, all subsequent tiles can be planarly drawn and we can now identify the left-most and right-most wall vertices planarly (cf. Figure 4). Thus we do not need any crossings except for those within X; we will discuss them below.

First, we consider a twisted tile consisting of a dL-frame and a picture without I. Figure 5a gives an abstract sketch (as well as its twist) of such tiles where the gray area hides crossing-free picture details. Hence, the twisting of these tiles can be drawn 1-planarly with 2-crossings.

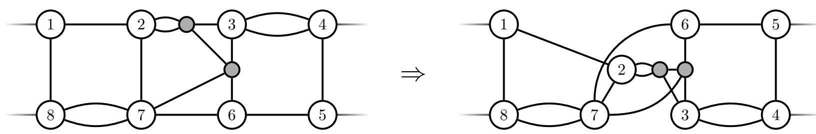
Next, we prove the claim for any twisted tile consisting of a dL-frame and a picture with identification. To this end, recall that there are only four pictures with identification: *VIA*, *BIA*, *AIV*, and *AIB*. We give 1-planar drawings of a twisted *VIAdL*-tile and a twisted *BIAdL*-tile in Figures 5b and 5c. The solutions for *AIVdL*- and *AIBdL*-tiles are identical up to mirroring. Contracting the double edges $(3, 4)$ and/or $(7, 8)$ in the given drawings maintains 1-planarity and the simple crossing number. Hence, the given drawings can be transformed to tiles with an L-frame. □



(a) Twisting a tile without I.



(b) Twisting a $VIAdL$-tile.



(c) Twisting a $BIAdL$-tile.

Figure 5: Twisting a tile.

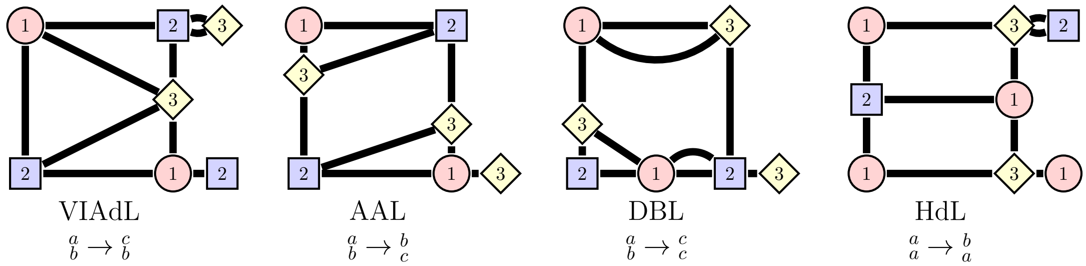
5 Chromatic Number

The question whether four colors are sufficient to color a map (in the sense of a separation of the plane into contiguous regions) such that no two adjacent regions (e.g., countries in a visual representation of national territories) are colored the same, plagued mathematicians and many other researchers since the late 19th century. It was finally, but not uncontroversially, positively answered in 1976 facilitating a computer-assisted proof [1]. Consequently, any planar graph has a vertex coloring that uses at most four colors. For general graphs however, it is NP-hard to decide whether a given number $k \geq 3$ of colors suffices (the smallest such k is the graph's *chromatic number*) [21] and even constant-factor approximations in polynomial time are impossible (unless P=NP) [52]. The chromatic number of graphs is of interest in applications like scheduling, register allocation, and pattern matching [14, 32, 34]. Ringel proved that 1-planar graphs can be colored using at most seven colors [39].

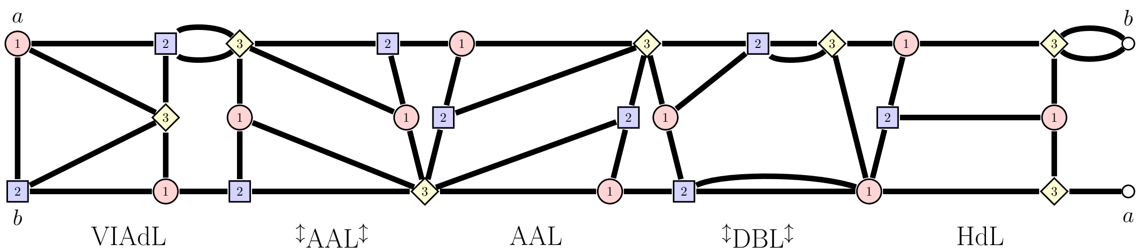
In this section, we study the chromatic number of large 2-crossing-critical graphs. We start by a characterization of bipartite, i.e., 2-colorable, such graphs in Theorem 5.5 and proceed to improve on Ringel's result by proving that each large 2-crossing-critical graph is 4-colorable, cf. Theorem 5.6. To show that this bound is tight (at least in some cases), we present an infinite family of large 2-crossing-critical graphs that are not 3-colorable. Finally, this is complemented by an infinite family of large 2-crossing-critical graphs with chromatic number 3.

Definition 5.1 *A* (vertex) coloring *of a graph G is a function $c\colon V(G) \to \mathbb{N}^+$, such that $c(v) \neq c(w)$ for every edge $vw \in E(G)$. Graph G is* k-colorable *if it admits a coloring using at most k colors and we call such a coloring a* k-coloring. *The* chromatic number *of G is the smallest k such that G is k-colorable.*

In Figure 6a we can see that each elementary tile of the example graph from Figure 3 can be colored with at most 3 colors on its own. But if we were to use these exact colorings in the full



(a) Individual 3-colorings of each elementary tile in the graph. While they are not compatible to each other as is, they can be thought of as propagations, see text.



(b) A 3-coloring of the example graph from Figure 3, based on the above propagations.

Figure 6: Using propagations to color a graph.

graph, the colorings of the wall nodes would clash. In the following, while we formally construct the graphs by joining tiles whose right wall is inverted (see Definition 2.4), it will be helpful to take the viewpoint already discussed following that definition, that we may vertically invert every second tile completely. This allows us to (mentally and in the figures) visualize each tile planarly.

We view every coloring of Figure 6a as a 2-*propagation* (to be formally defined below), and substitute explicit colors as needed. Intuitively (cf. Figure 6b), start with coloring the *VIAdL*-tile as proposed by its individual coloring. As its right wall vertices (with now fixed colors) are the left wall vertices of the neighboring *AAL*-tile, we cannot directly color the latter tile as desired by its individual coloring. Let x_1, x_2 and y_1, y_2 denote the two left and right wall vertices of this *AAL*-tile (observe that it is drawn as $^{\updownarrow}AAL^{\updownarrow}$ in G), respectively. We are only interested in the following properties of the *AAL*-coloring in Figure 6a: $c(y_1)$ is distinct from $c(x_1)$ and $c(x_2)$, and $c(y_2) = c(x_2)$. This allows us to substitute the color classes within this tile accordingly and proceed with the next tile. Put chiefly, a 2-propagation is the notion that, given a coloring of its left wall vertices, we know about the existence of a tile-coloring yielding certain coloring-properties on its right wall vertices. This concept can be formalized as follows:

Definition 5.2 *Let $T = (G, x, y)$ be a 2-tile. Consider a vertex coloring c of G. The colors of x_1, x_2 (y_1, y_2) are the* input colors *(*output colors*, respectively; each in that order) of T. Two k-colorings c, c' of T are* equivalent*, if $(c(v) = c(w)) \Leftrightarrow (c'(v) = c'(w))$ for each pair of wall vertices $v, w \in \{x_1, x_2, y_1, y_2\}$. We call the induced equivalence classes* (vertex-coloring-)propagations *and denote them by*

$$\begin{smallmatrix} c(x_1) \\ c(x_2) \end{smallmatrix} \rightsquigarrow \begin{smallmatrix} c(y_1) \\ c(y_2) \end{smallmatrix},$$

*using some representative coloring c. We may use the term k-*propagation *to specify that c is a k-coloring.*

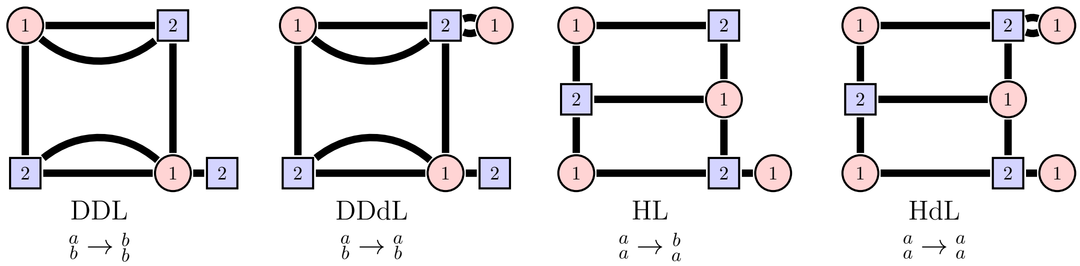


Figure 7: 2-vertex-colorings P_L^{DD} for DDL, P_{dL}^{DD} for $DDdL$, P_L^H for HL, and P_{dL}^H for HdL.

To aid comprehensibility, whenever we state propagations, we will denote each color by a unique letter from $\{a, b, c, d\}$ instead of a number. Observe that, while elementary tiles will yield the base cases of our propagations, the joins of tiles yield again tiles; therefore we may naturally concatenate propagations when joining two tiles, requiring only simple color substitutions in the second tile. Consider the first two tiles in the example graph Figure 6b: their propagations $\genfrac{}{}{0pt}{}{a}{b} \rightsquigarrow \genfrac{}{}{0pt}{}{c}{d}$ and $\genfrac{}{}{0pt}{}{a}{b} \rightsquigarrow \genfrac{}{}{0pt}{}{c}{d}$ lead to $\genfrac{}{}{0pt}{}{a}{b} \rightsquigarrow \genfrac{}{}{0pt}{}{c}{d} \rightsquigarrow \genfrac{}{}{0pt}{}{a}{b}$ and thus the propagation $\genfrac{}{}{0pt}{}{a}{b} \rightsquigarrow \genfrac{}{}{0pt}{}{a}{b}$ over the first two tiles.

We first consider small odd cycles that arise already when joining two elementary tiles.

Lemma 5.3 *Let $sig(G) = sig(T_0) \dots sig(T_{2m})$ be a signature of a large 2-crossing-critical graph G. Then G is not bipartite if $sig(T_{2m})sig(G)$ contains an element of* $\{\mathrm{DDdL\,H}, \mathrm{DDL\,DD}, \mathrm{HdL\,DD}, \mathrm{HL\,H}\}$ *as a substring.*

Proof: The subgraph corresponding to $DDL\,DD$ contains a triangle and the subgraphs corresponding to $HL\,H$, $DDdL\,H$, and $HdL\,DD$ each contain a 5-cycle. □

Recall that by Observation 3.3 every tile whose signature contains A, V, or B has a triangle and is therefore not bipartite.

Let us now consider the last "global" tile T whose cyclization yields a large 2-crossing-critical graph. We proceed to show that it is bipartite if none of the above local obstructions are present. Note that in the following lemma, we do *not* consider the final cyclization just yet.

Lemma 5.4 *Let $T = T_0^{\updownarrow} \otimes T_1^{\updownarrow} \otimes \dots \otimes T_{2m}^{\updownarrow}$ where $T_i \in \mathcal{S}$. The tile T is bipartite if $sig(T_i)sig(T_{(i+1) \bmod (2m+1)})$ starts with* DDdL DD, DDL H, HdL H, *or* HL DD *for each $i \in \mathbb{N}$ with $0 \le i \le 2m$.*

Proof: Figure 7 shows the existence of the following 2-propagations:

$$\begin{aligned} P_L^{DD} &:= \genfrac{}{}{0pt}{}{a}{b} \rightsquigarrow \genfrac{}{}{0pt}{}{b}{b}, \text{ for } DDL\text{-tiles},\\ P_{dL}^{DD} &:= \genfrac{}{}{0pt}{}{a}{b} \rightsquigarrow \genfrac{}{}{0pt}{}{a}{b}, \text{ for } DDdL\text{-tiles},\\ P_L^{H} &:= \genfrac{}{}{0pt}{}{a}{a} \rightsquigarrow \genfrac{}{}{0pt}{}{b}{a}, \text{ for } HL\text{-tiles},\\ P_{dL}^{H} &:= \genfrac{}{}{0pt}{}{a}{a} \rightsquigarrow \genfrac{}{}{0pt}{}{a}{a}, \text{ for } HdL\text{-tiles}. \end{aligned}$$

By assumption each elementary tile is from $\{DDL, DDdL, HL, HdL\}$ and $sig(T)$ is a subsequence of $((DDdL)^* DDL (HdL)^* HL)^+$ (using standard notation of regular expressions).

First we will look only at the case where each elementary tile of T is either DDL or HL. Then $sig(T)$ is a subsequence of $(DDL\,HL)^+$. Every subsequence $DDL\,HL$ of the latter admits the propagation ${a \atop b} \rightsquigarrow {b \atop b} \rightsquigarrow {b \atop a}$ by using P_L^H and P_L^{DD} (observe that the vertical order reverses in the second propagation as this tile is vertically inverted w.r.t. the first). Iterating these propagations yields a 2-coloring.

If we now also allow *HdL*-tiles, we see that each maximal $(HdL)^+$-subsequence admits the propagation ${a \atop a} \rightsquigarrow {a \atop a}$ by repeatedly using P_{dL}^H. Therefore each subsequence $(HdL)^+HL$ admits the same overall propagation as an individual *HL*-tile. Similarly, any $(DDdL)^+$-subsequence admits the propagation ${a \atop b} \rightsquigarrow {a \atop b}$ by repeatedly using P_{dL}^H, and thus $(DDdL)^+DDL$ admits the same overall propagation as an individual *DDL*-tile. Thus T is 2-colorable.

□

Finally, we can fully characterize bipartite large 2-crossing-critical graphs.

Theorem 5.5 *A large 2-crossing-critical graph G is 2-colorable if and only if its signature can be written as $sig(G) = sig(T_0) \dots sig(T_{2m})$, where T_i is an elementary tile for $0 \le i \le 2m$, such that:*

- *(i) tile T_0 contains an* H*-picture (defined in Figure 2b), and*
- *(ii) each $sig(T_i)sig(T_{(i+1) \bmod (2m+1)})$ starts with* DDdL DD, DDL H, HdL H, *or* HL DD *for $0 \le i \le 2m+1$; and*
- *(iii) the number of* L *frames in $\{T_{2i}\}_{0 \le i \le m}$ is odd.*

Proof: By Observation 3.3 and Lemma 5.3 each bipartite large 2-crossing-critical graph satisfies (ii). Moreover, since $(DDdL)^k$ for odd k is not bipartite, each bipartite large 2-crossing-critical graph contains an H-picture, implying (i).

Hence, it only remains to prove that a large 2-crossing-critical graph satisfying (i) and (ii) is bipartite if and only if it also satisfies (iii). To this end, assume G satisfies (i) and (ii).

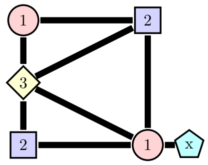
By Lemma 5.4, $(G', \langle x_1, x_2 \rangle, \langle y_2, y_1 \rangle) \coloneqq T_0^{\updownarrow} \otimes T_1^{\updownarrow} \otimes \dots \otimes T_{2m}^{\updownarrow}$ admits a 2-coloring c. Since such a coloring is unique (up to isomorphism and relabeling of colors), G is bipartite if and only if c induces a 2-coloring on G, i.e. $c(x_1) = c(y_1)$ and $c(x_2) = c(y_2)$.

It follows from (i) that $c(x_1) = c(x_2)$; by (i) and (ii) we have $c(y_1) = c(y_2)$. Thus, c induces a 2-coloring on G if and only if $c(x_1) = c(y_1)$. To this end we look at the parity of a path between x_1 and y_1. As G' is bipartite every such path has the same parity. Our path consists of the direct path between the (non-inverted) top wall nodes for tiles T_{2i} for $0 \le i \le m$ and the direct path between the (non-inverted) bottom wall nodes of tiles T_{2i+1} for $0 \le i < m$. We notice that the (edgewise) distance between the bottom wall nodes of each of our four elementary tiles is always 2. The same is the case for the distance between the top wall nodes for dL-framed elementary tiles but for L-framed elementary tiles the distance between the top wall nodes is 1. Therefore if and only if property (iii) holds, the distance between x_1 and y_1 is even and therefore c induces a 2-coloring on G. □

In Figure 6, we have already seen a 3-coloring of the example graph. Let us now generalize this way of coloring to show that, like any planar graph, indeed every large 2-crossing-critical graph requires at most 4 colors.

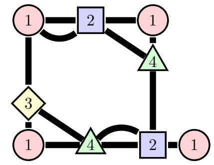
Theorem 5.6 *Every large 2-crossing-critical graph is* 4*-colorable.*

Proof: It is easy to verify that every elementary tile admits a 4-propagation ${a \atop b} \rightsquigarrow {c \atop d}$; we list all these propagations explicitly in Figures 13 and 14 in the appendix. Thus, any tile that consists of two joined elementary tiles, admits the 4-propagation ${a \atop b} \rightsquigarrow {a \atop b}$. We use ${a \atop b} \rightsquigarrow {a \atop b}$ on all but 3 consecutive



AIVL

Figure 8: The unique 3-coloring of an *AIVL*-tile (up to color-substitution and the free choice for 'x'$\neq 1$).



BBL

Figure 9: A 3-propagation $\genfrac{}{}{0pt}{}{a}{a} \rightsquigarrow \genfrac{}{}{0pt}{}{a}{a}$ of a *BBL*-tile.

elementary tiles and show that the join of these 3 tiles has an $\genfrac{}{}{0pt}{}{a}{b} \rightsquigarrow \genfrac{}{}{0pt}{}{b}{a}$ propagation: As also shown in Figures 13 and 14, each elementary tile also admits the 4-propagation $\genfrac{}{}{0pt}{}{a}{b} \rightsquigarrow \genfrac{}{}{0pt}{}{c}{b}$. Thus, three such tiles admit the required 4-propagation (recall that the tile in the middle is drawn vertically inverted w.r.t. the other two):

$$\genfrac{}{}{0pt}{}{a}{b} \rightsquigarrow \genfrac{}{}{0pt}{}{c}{b} \rightsquigarrow \genfrac{}{}{0pt}{}{c}{a} \rightsquigarrow \genfrac{}{}{0pt}{}{b}{a}.$$

□

Next, we present a class of large 2-crossing-critical graphs that are not 3-colorable. This shows that the bound presented above is tight for an infinite number of cases.

Observation 5.7 *Every large 2-crossing-critical graph where every elementary tile is an* AIVL*-tile is not* 3*-colorable.*

Proof: With Figure 8 it is straightforward to verify that each 3-propagation of an *AIVL*-tile is either $P_1 = \genfrac{}{}{0pt}{}{a}{b} \rightsquigarrow \genfrac{}{}{0pt}{}{b}{c}$ or $P_2 = \genfrac{}{}{0pt}{}{a}{b} \rightsquigarrow \genfrac{}{}{0pt}{}{b}{b}$. Since the two vertices on the left wall of an *AIVL*-tile have to be colored differently, each elementary tile uses propagation P_1. Thus, any join of an even number of elementary tiles propagates $\genfrac{}{}{0pt}{}{a}{b} \rightsquigarrow \genfrac{}{}{0pt}{}{a}{b}$. But then the last tile would have to propagate $\genfrac{}{}{0pt}{}{a}{b} \rightsquigarrow \genfrac{}{}{0pt}{}{b}{a} \neq P_1$. □

Complementing this, there are also infinitely many large 2-crossing-critical graphs with chromatic number 3.

Observation 5.8 *Every large 2-crossing-critical graph G where every elementary tile is a* BBL*-tile has chromatic number* 3*.*

Proof: By Theorem 5.5, G is not bipartite. On the other hand, Figure 9 shows that a *BBL*-tile admits a 3-propagation $\genfrac{}{}{0pt}{}{a}{a} \rightsquigarrow \genfrac{}{}{0pt}{}{a}{a}$ that we may use on all tiles. □

The previous observations point to the following open problem:

Question 5.9 *What is the full characterization of* 3*-colorable large 2-crossing-critical graphs?*

Although a graph-theoretic characterization of 3-colorable large 2-crossing-critical graphs is an open question, we can efficiently decide 3-colorability algorithmically:

Observation 5.10 *Using the fact that large 2-crossing-critical graphs have bounded treewidth (see Corollary 7.9 below), Courcelle's theorem [16] yields a linear-time algorithm to decide whether a given large 2-crossing-critical graph is 3-colorable.*

6 Chromatic Index

In this section, we investigate the chromatic index of large 2-crossing-critical graphs. The chromatic index is the minimum number of colors necessary to color edges of a graph, such that no two edges incident to the same vertex share a color. A trivial lower bound for the chromatic index is the maximum degree of the graph. Determining the chromatic index of a general graph is NP-hard [27]. However, there are classes of graphs for which the chromatic index can be shown to be close to the trivial lower bound. Simple graphs are said to be *class 1* if their chromatic index equals the maximum degree, and *class 2* otherwise, see e.g. [13]. However, the situation is more complicated for graphs that are not simple, as the graph's *density* (i.e., maximum ratio between the number of edges and vertices, over all induced subgraphs) is also a natural lower bound for the graph's chromatic index. This motivates the following slightly different definition [13]: a graph is *first class* when its chromatic index matches the lower bound given by the maximum degree or the density, and *second class* otherwise.

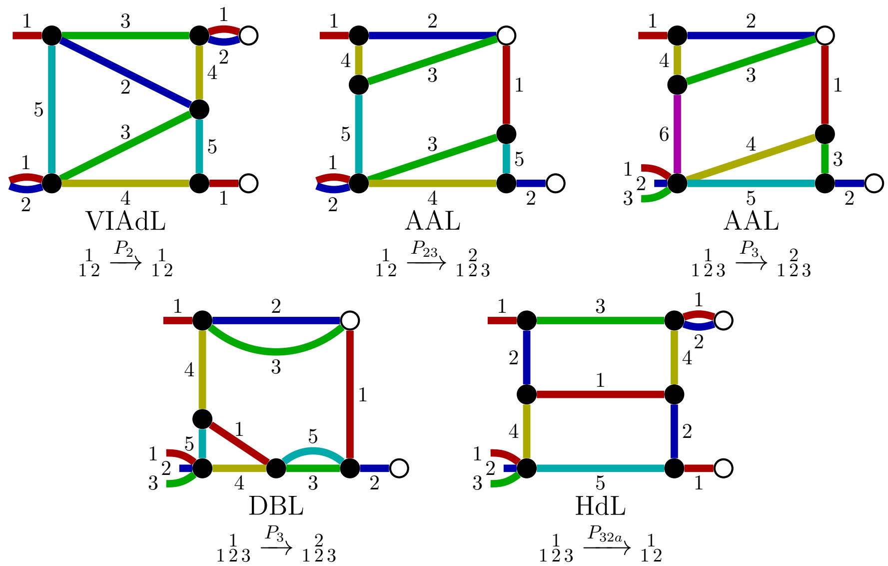
By construction, the density of large 2-crossing-critical graphs is low. In fact, we show that all large 2-crossing-critical graphs are first class by showing that they require only as many edge colors as their maximum degree. To this end, we exhibit such edge colorings for elementary tiles and combine them to a coloring of the full graph.

Definition 6.1 *An* edge coloring *of a graph G is a function $c\colon E(G) \to \mathbb{N}^+$ such that $c(e) \neq c(f)$ for each pair $e, f \in E(G)$ of adjacent edges. A* k-edge-coloring *is an edge coloring that uses at most k colors. The* chromatic index *of G is the smallest k such that a k-edge-coloring of G exists. In particular, if G admits a $\Delta(G)$-edge-coloring, G is said to be* first class.

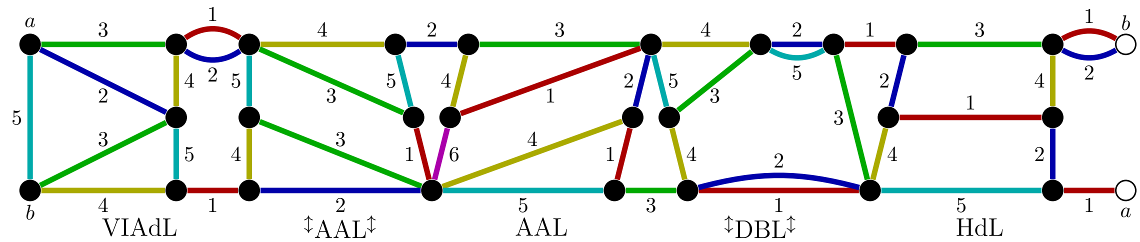
Similarly to our findings on chromatic numbers, we will use color propagations to investigate edge colorings. An example can be seen in Figure 10. Figure 10a shows five edge color propagations, one for each tile. Consider two neighboring tiles T_1, T_2 (T_1 left of T_2). For edge color propagations, the edges in T_1 incident to T_1's right wall are of interest, as they form restrictions for the edges in T_2 that are incident to T_2's left wall vertices. Thus, when showing a propagation for tile T_2, we also need to show these incident T_1-edges (the *input edges* of T_2), to the left of the wall vertices. The T_2-edges incident to T_2's right wall form the *output edges* of T_2. By substituting colors, we can now again assign these propagations to a list of joined tiles such that their colors match. Note that in our example graph we have two distinct color propagations for the two AAL-tiles, since in the full graph (Figure 10b) their left wall vertex y_2 becomes a vertex of degree 6 in the first, and degree 5 in the second case.

Let $T = (G, x, y)$ be a tile. The edges of T that are incident to vertices on T's right wall are the *output edges* of T. We observe that each tile T of a large 2-crossing-critical graph has either three or four output edges, where all but one edge e are pairwise adjacent. We call this edge e, which is the unique edge incident to the degree-1 vertex of the frame, the *single edge* of T. Consider an edge coloring of G, the colors of the output edges of T are its *output colors*. We denote output colors by ${}_{b}{}^{a}_{c}{}_{d}$, where a refers to the color of the single edge and b, c, d are the colors of the three adjacent edges (in no particular order). For those tiles that have only two instead of three adjacent such edges, we instead write ${}_{b}{}^{a}_{c}$. For a cyclic sequence of tiles that contains T, the output edges (output colors) of T's predecessor are the *input edges* (*input colors*, respectively) of T. We employ the same notation for input and output colors.

Definition 6.2 *Two k-edge-colorings c, c' of a tile T and its input edges are* equivalent *if $c(e) = c(f) \iff c'(e) = c'(f)$ for each pair $e, f \in X$, where X is the set of input and output edges*



(a) The raw propagations.



(b) A full edge coloring of maximum vertex degree of the example graph.

Figure 10: Edge color propagations in use.

of T. We call the induced equivalence classes (edge-coloring-k-)propagations *and denote them by $\mathcal{A} \longrightarrow \mathcal{B}$, where $\mathcal{A}$ ($\mathcal{B}$) are the input colors (output colors, respectively) of T.*

A tile (in a sequence) that has $k+1$ input and $\ell+1$ output colors, is a *k-ℓ-tile*. We observe that any large 2-crossing-critical graph is a cyclic sequence of elementary 2-2-, 2-3-, 3-3- and 3-2-tiles. Given a tile T in a large 2-crossing-critical graph G, we denote its *maximum degree* in G by $\Delta_G(T) = \max_{v \in V(T)} |\{e \in E(G) \mid v \in e\}|$.

We know from Observation 3.2 that the maximum degree of a large 2-crossing-critical graph is $4 \leq \Delta(G) \leq 6$. In order to show that all large 2-crossing-critical graphs are first class, we will first restrict ourselves to those with $\Delta(G) \geq 5$; thereafter, we will also consider the case $\Delta(G) = 4$.

Lemma 6.3 *Consider a (cyclic) sequence Q of elementary tiles corresponding to a large 2-crossing-critical graph G with maximum degree $\Delta(G) \geq 5$. Each 2-2-tile of Q admits the following propagation that uses 5 colors:*

$$P_2 \coloneqq \substack{1\\1\,2} \longrightarrow \substack{1\\1\,2}.$$

Each other tile T of Q has the below propagations, using at most $\Delta_G(T)$ colors:

$$P_{23} \coloneqq \genfrac{}{}{0pt}{}{1}{1\,2} \longrightarrow \genfrac{}{}{0pt}{}{2}{1\,2\,3} \text{ for 2-3-tiles,} \quad P_3 \coloneqq \genfrac{}{}{0pt}{}{1}{1\,2\,3} \longrightarrow \genfrac{}{}{0pt}{}{2}{1\,2\,3} \text{ for 3-3-tiles,}$$

$$P_{32b} \coloneqq \genfrac{}{}{0pt}{}{1}{1\,2\,3} \longrightarrow \genfrac{}{}{0pt}{}{2}{1\,2} \text{ for 3-2-tiles,} \quad P_{32a} \coloneqq \genfrac{}{}{0pt}{}{1}{1\,2\,3} \longrightarrow \genfrac{}{}{0pt}{}{1}{1\,2} \text{ for 3-2-tiles.}$$

Proof: This can (easily but tediously) be shown by demonstrating corresponding colorings for each elementary tile. Figures 15 to 19 list all cases in the appendix. □

Note that each elementary tile admits several propagations. In the example graph of Figure 10, there are two occurrences of AAL. They differ in that their left wall vertex y_2 has a degree of 5 or 6. We differentiate them by referring to the first one as a 2-3-tile, and the second as a 3-3-tile.by referring to the first one as a 2-3-tile, and the second as a 3-3-tile. The full sequence of propagations used is $\genfrac{}{}{0pt}{}{1}{1\,2} \xrightarrow{P_2} \genfrac{}{}{0pt}{}{1}{1\,2} \xrightarrow{P_{23}} \genfrac{}{}{0pt}{}{2}{1\,2\,3} \xrightarrow{P_3} \genfrac{}{}{0pt}{}{3}{1\,2\,3} \xrightarrow{P_3} \genfrac{}{}{0pt}{}{1}{1\,2\,3} \xrightarrow{P_{32a}} \genfrac{}{}{0pt}{}{1}{1\,2}$. This coloring uses $\Delta(G) = 6$ colors.

We will now use these propagations to obtain an edge coloring of arbitrary large 2-crossing-critical graphs with $\Delta(G) \geq 5$ and show that they are indeed first class.

Lemma 6.4 *Large 2-crossing-critical graphs G with $\Delta(G) \geq 5$ are first class.*

Proof: Throughout this proof, we only consider propagations using 5 colors for elementary 2-2 tiles and propagations using at most $\Delta(T)$ colors for each other elementary tile T.

First, assume G does not decompose into elementary 3-3-tiles only. Then, we prove the claim by decomposing G into (not necessarily elementary) tiles admitting a $\genfrac{}{}{0pt}{}{1}{1\,2} \longrightarrow \genfrac{}{}{0pt}{}{1}{1\,2}$ propagation. We decompose G into elementary 2-2-tiles (which allow these via P_2) and tiles of the form $T = \otimes(T_0, \dots, T_k)$ where T_0 is a 2-3-tile, T_k is a 3-2-tile, $T_1, \dots, T_{k-1}$ are 3-3-tiles, and each T_i is elementary. We only have to show that such a tile T admits a $\genfrac{}{}{0pt}{}{1}{1\,2} \longrightarrow \genfrac{}{}{0pt}{}{1}{1\,2}$-propagation.

Iteratively applying P_3 to $T_1 \dots T_{k-1}$ yields a $P_o \coloneqq \genfrac{}{}{0pt}{}{1}{1\,2\,3} \longrightarrow \genfrac{}{}{0pt}{}{2}{1\,2\,3}$-propagation if k is even and a $P_e \coloneqq \genfrac{}{}{0pt}{}{1}{1\,2\,3} \longrightarrow \genfrac{}{}{0pt}{}{1}{1\,2\,3}$-propagation otherwise. We obtain the following propagations for T:

$$\genfrac{}{}{0pt}{}{1}{1\,2} \xrightarrow{P_{23}} \genfrac{}{}{0pt}{}{2}{1\,2\,3} \xrightarrow{P_o} \genfrac{}{}{0pt}{}{1}{1\,2\,3} \xrightarrow{P_{32a}} \genfrac{}{}{0pt}{}{1}{1\,2} \quad \text{and} \quad \genfrac{}{}{0pt}{}{1}{1\,2} \xrightarrow{P_{23}} \genfrac{}{}{0pt}{}{2}{1\,2\,3} \xrightarrow{P_e} \genfrac{}{}{0pt}{}{2}{1\,2\,3} \xrightarrow{P_{32b}} \genfrac{}{}{0pt}{}{1}{1\,2}, \text{ respectively.}$$

Next, assume that G consists of elementary 3-3-tiles only. Note that using P_3, two subsequent such tiles admit the propagation $P_3^2 \coloneqq \genfrac{}{}{0pt}{}{1}{1\,2\,3} \longrightarrow \genfrac{}{}{0pt}{}{2}{1\,2\,3} \longrightarrow \genfrac{}{}{0pt}{}{1}{1\,2\,3}$; and three subsequent such tiles admit the propagation $P_3^3 \coloneqq \genfrac{}{}{0pt}{}{1}{1\,2\,3} \longrightarrow \genfrac{}{}{0pt}{}{2}{1\,2\,3} \longrightarrow \genfrac{}{}{0pt}{}{3}{1\,2\,3} \longrightarrow \genfrac{}{}{0pt}{}{1}{1\,2\,3}$. Since there is an odd number of elementary tiles, we can use P_3^3 for three subsequent elementary tiles and P_3^2 for the remaining pairs, obtaining a $\Delta(G)$-edge-coloring of G. □

Now that we have shown that large 2-crossing-critical graphs with $\Delta(G) \geq 5$ are first class, it remains to prove that those with $\Delta(G) = 4$ are also first class. There is only a constant number of 2-crossing-critical graphs with 3 elementary tiles with potentially sporadic behavior; we are interested in the remaining infinite class.

Theorem 6.5 *Large 2-crossing-critical graphs G with at least 5 elementary tiles are first class.*

Proof: By Lemma 6.4, it remains to consider $\Delta(G) = 4$. Let G consist of the elementary tile $T_0, \dots, T_{2m}$ in this order. Throughout this proof, we only consider propagations using 4 colors and assume $m \geq 2$.

We can find 4-propagations for all tiles with maximum degree 4 (all corresponding colorings are depicted in the appendix), in particular they can be categorized as follows:

$$\begin{aligned} P_2 &\coloneqq {}_{1}{}^{1}{}_{2} \longrightarrow {}_{1}{}^{1}{}_{2}, && \text{for } BVL \text{ and } VVdL \text{ (see Figure 19), and} \\ P_w &\coloneqq {}_{1}{}^{1}{}_{2} \longrightarrow {}_{1}{}^{3}{}_{3}, && \text{for } BVL \text{ and } VVdL \text{ (see Figure 20), and} \\ P_s &\coloneqq {}_{1}{}^{1}{}_{2} \longrightarrow {}_{1}{}^{1}{}_{3}, && \text{for all other tiles of maximum degree 4 (see Figure 20).} \end{aligned}$$

We call the tiles BVL and $VVdL$ *whimsical*, while the others are *sincere*. Let us prove the theorem by constructing a ${}_{1}{}^{1}{}_{2} \longrightarrow {}_{1}{}^{1}{}_{2}$-propagation for G. To this end, we consider the following three cases:

Case 1: Assume there is an odd number of whimsical tiles. Then the number of sincere tiles is even. We obtain the claim for G by using ${}_{1}{}^{1}{}_{2} \xrightarrow{P_2} {}_{1}{}^{1}{}_{2}$ for whimsical tiles, and alternate between two colorings using P_s.

Case 2: Now, assume there is an even number of whimsical tiles and only a single sincere tile. We use propagation P_2 on all but 3 consecutive whimsical tiles. These remaining whimsical tiles together propagate ${}_{1}{}^{1}{}_{2} \xrightarrow{P_w} {}_{1}{}^{4}{}_{4} \xrightarrow{P_w} {}_{3}{}^{3}{}_{4} \xrightarrow{P_w} {}_{1}{}^{1}{}_{3}$. Together with P_s for the sincere tile, we obtain the claimed propagation.

Case 3: Finally, assume we have an even number of whimsical tiles and at least 3 sincere tiles. Using P_2 for each whimsical tile, we only have to prove that we can construct a ${}_{1}{}^{1}{}_{2} \longrightarrow {}_{1}{}^{1}{}_{2}$-propagation for the sincere tiles. This is obtained by applying P_s to all but 3 of these tiles and using the following propagation on the remaining ones: ${}_{1}{}^{1}{}_{2} \xrightarrow{P_s} {}_{1}{}^{1}{}_{3} \xrightarrow{P_s} {}_{1}{}^{1}{}_{4} \xrightarrow{P_s} {}_{1}{}^{1}{}_{2}$.

Thus, any sufficiently large 2-crossing-critical graph G can be colored with $\Delta(G)$ colors and is first class. □

7 Treewidth

Treewidth is a central measure in graph theory and parameterized complexity [6]. It was first introduced by Bertelé and Brioschi under the term *dimension* but rediscovered twice in following years [4, 23, 40]. Robertson and Seymour coined the term *treewidth* and discovered a profound theory based on it that spawned a plethora of results. While it is known that the treewidth of 2-crossing-critical graphs is bounded from above by $2^{15\,361} - 2$ [25], the known bound is far from optimal. Lower bounds are known for $k \geq 3$ only [25].

Definition 7.1 *A* tree decomposition *of a connected graph G is a tree T and a function $f\colon V(T) \to 2^{V(G)}$ such that*

(1) for each edge $uv \in E(G)$, there exists a vertex $\alpha \in V(T)$ with $\{u, v\} \subseteq f(\alpha)$, and

(2) for each vertex $v \in V(G)$, the subgraph of T induced by $\{\alpha \in V(T) : v \in f(\alpha)\}$ is connected.

Each set $f(\alpha)$ is typically called a bag*. The* treewidth $\operatorname{tw}(G)$ *is the smallest $\gamma \in \mathbb{N}$, such that there exists a tree decomposition of G with $\max_{\alpha \in V(T)} |f(\alpha)| \leq \gamma + 1$.*

While the above definition is the classical one by Robertson and Seymour, there are several equivalent characterizations of treewidth. For our proofs, we use one by Seymour and Thomas that employs a game of cops and robber [47]: The cops and the robber stand on vertices of the graph G. The robber may move—at infinite speed—to any other vertex w unless every path from v to w contains a vertex with a cop located on it. Cops move by "helicopter", i.e., they are removed from their vertex and—at a later point in time—are placed on some other vertex. All participants know all positions and the graph at all times. The cops win if there exists a strategy such that after a finite number of cop movements, one of them is placed on the same vertex as the robber, independent of the robber's strategy. Otherwise, the robber has a strategy to avoid being caught indefinitely and wins. The treewidth of G is equal to the maximum number of cops such that the robber still wins. The intuitive connection between the original treewidth characterization and this game-theoretic approach is that cops would block all vertices of a bag in the decomposition tree T, locking the robber in some subtree of T; then, the cops can iteratively move over to the adjacent bag that is closer to the robber, essentially pushing the robber towards a leaf-bag, where he will eventually be catched. If there are too few cops, they will be unable to always lock the robber within a subtree, and the robber can flee ad infinitum. See [47] for details.

Similarly, since treewidth is minor-monotone, one may characterize graphs of treewidth at most k, also called *partial k-trees*, by a set of forbidden minors [3,42]. Since all 2-crossing-critical graphs are non-planar, it follows from Kuratowski's theorem that they contain the K_4 as a minor and their treewidth is at least 3 [5].

Definition 7.2 *A tile $T = (G, x, y)$ is* blocked*, if there are cops on G such that the robber—independent of his position— cannot move from a vertex on T's left wall x to a vertex on T's right wall y while using only edges of T, i.e., the graph $G[W]$ induced by the vertices W of G that are not occupied by a cop, contains no path from a vertex in x to a vertex in y.*

Lemma 7.3 *Any large 2-crossing-critical graph G with at least* 5 *elementary tiles has* $4 \leq \mathrm{tw}(G) \leq 5$.

Proof: Recall that the generalized Wagner graph V_8 is a cubic graph that is constructed from the cycle on 8 vertices $v_1, \ldots, v_8$ (in this order) by adding the edges $v_i v_{i+4}$, $1 \leq i \leq 4$ (cf. Figure 1 for the analogously defined V_{10}). The V_8 constitutes one of four obstructions in the characterization of graphs with treewidth ≤ 3 [3,44]. We obtain $\mathrm{tw}(G) \geq 4$ since any G with at least 5 elementary tiles contains the V_8 as a .

Let us now describe a strategy for catching a robber on any G with 6 cops: Using 2 cops, we may block any tile T by placing them on its left wall. Applying this operation iteratively, using 3 sets of 2 such cops each, we can force the robber into a single elementary tile T' (essentially using binary search), such that there is a cop on each wall vertex of T'. Checking each possible elementary tile individually, one can see that catching the robber within T' is then always possible with 6 cops. □

In fact, also the graphs on 3 elementary tiles contain V_8 as a minor unless each tile has the signature $\sigma\tau L$ with $\sigma, \tau \in \{A, D\}$. A treewidth-3 decomposition for the latter cases is easily obtained.[2] It remains to distinguish the large graphs with treewidth 4 from those with treewidth 5. Surprisingly, for this we only need to recognize one specific minor ⧖^{3}:

Definition 7.4 *The* hourglass graph ⧖ *is obtained from two disjoint triangles by identifying one vertex from the first with one vertex from the second triangle. The graph* ⧖^{3} *is obtained by cyclically joining three hourglass graphs, as given in Figure 11.*

We now consider a general refined cop strategy that will allow us to use less than 6 cops in some cases.

[2]The reader may check the central case $sig(G) = AAL\,AAL\,AAL$ either by hand or, e.g., using ToTo [50]. The other cases follow since they are minors of this G.

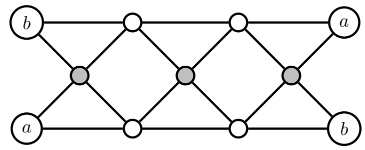


Figure 11: Drawing of the minimal forbidden minor ⧖^{3} for treewidth 4 in the projective plane as indicated by the identification of the same-labeled vertices. Vertices with the same labels are identified. Internal vertices are gray.

Definition 7.5 *Consider a tile $T = (G, x, y)$ in a large 2-crossing-critical graph with cops placed on some of its vertices. Let $W \subseteq V(G)$ denote the set of vertices occupied by cops. A vertex $v \in V(G)$ is* left-blocked *(*right-blocked*) if $G[V(G) \setminus W]$ contains no path from v to a vertex of its left wall (right wall, respectively). A vertex of G is* tracked *if itself or all its neighbors are occupied by cops. A* sweep *of T is a sequence of cop movements such that (1) the cops initially occupy the left wall, (2) after the sequence, the cops occupy the right wall; and (3) during the cop movements, each vertex in $V(G)$ enters the three states "left-blocked", "tracked", and "right-blocked" in that order such that each state is entered exactly once and at each point in time, at least one state applies.*

Observe that during a sweep, the respective tile always remains blocked since there is no vertex that is connected to both its walls. Further, a sweep is in fact applicable in both directions, i.e., the reverse sequence allows cops to move from the right to the left wall in the same manner.

Definition 7.6 *An elementary tile is* messy*, if it contains ⧖ as a minor. An elementary tile T is* neat *if there is a sweep of T that uses at most* 3 *cops.*

Lemma 7.7 *Each elementary tile is neat or messy.*

Proof: We show that elementary tiles with pictures $\mathcal{M} := \{VA, VIA, BA, BIA, H\}$ are messy and the remaining ones are neat. For this proof, we denote by $X \dot{\prec} Y$ that picture X is a minor of picture Y such that X and Y have the same frame vertices.

For the first part, we contract all but the center 4-cycle of each tile's frame. Picture H becomes ⧖ by contraction of the central edge. Clearly, $VIA \dot{\prec} VA$ and similarly $BIA \dot{\prec} BA$—each by contraction of a single edge. Also, $VIA \dot{\prec} BIA$ by contracting the double edge in B. Hence, all elements of $\mathcal{M}$ that are not H contain VIA as a minor that by removal of a single edge becomes ⧖.

It remains to show that tiles with pictures not in $\mathcal{M}$ are neat. These pictures are $\overline{\mathcal{M}} := \{DD, DV, DB, DA, VV, VB, BB, AA\}$. For the sweep, we may assume to start on the picture's vertices as it is trivial to move from any wall-vertex that is not part of the picture to its adjacent vertex in the picture. Note that we may also omit the mirrored pictures DA and AA since it suffices to show a sweep of their mirrored counterparts (DV and VV, respectively) that are also in $\overline{\mathcal{M}}$. The remaining pictures in $\overline{\mathcal{M}}$ are all minors of BB: $DD \dot{\prec} DV \dot{\prec} DB$ and $VV \dot{\prec} DB \dot{\prec} BB$ where each minor-relation, except for the last, is witnessed by contraction of a single edge. Hence, it suffices to provide a sweep on BB: we label this picture's vertices as v_i starting at the top left with v_0 in counter-clockwise order, see Figure 12. Assuming the cops arrive from the left side, they occupy v_0 and v_2. First, we move the 3rd cop to v_3. Observe that all neighbors of v_1 are now occupied by cops, i.e., v_1 is tracked. The remaining sweep goes as follows: $v_2 \to v_7, v_0 \to v_4, v_3 \to v_6$. Once again, v_5 is tracked by occupying its neighbors. □

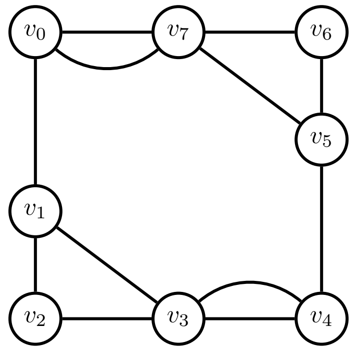


Figure 12: Picture BB with nodes labeled in counter-clockwise order.

Theorem 7.8 *A large* 2*-crossing-critical graph has treewidth* 5 *if and only if it contains at least* 3 *messy tiles, i.e, if and only if it contains* ⧖^{3} *as a minor.*

Proof: If there are no messy tiles, every elementary tile is neat and can be sweeped by 3 cops. Hence, we may block an arbitrary tile with 2 cops and sweep around the remaining graph with 3 cops. If during the sweep, the robber should remain on a vertex that is tracked by cops occupying its neighbors, we catch him using one of the first 2 cops.

Similarly, if there are up to two messy tiles, say X and Y, 4 cops are initially placed on X's walls. Then, the cops on the left wall of X start to sweep using a 5th cop c until they reach the wall of Y. Finally, the cops on the right wall of X sweep, again using c, until they reach the other wall of Y.

If, on the other hand, there are 3 messy tiles in G, then G contains the forbidden minor ⧖^{3}, as witnessed by contracting all edges that do not belong to the set of 3 messy tiles and contracting each messy tile to ⧖. On ⧖^{3}, however, there exists a simple strategy for the robber to win against 5 cops: There are two types of vertices in ⧖^{3}: 6 *rim* vertices and 3 *internal* ones, seen in Figure 11 as the (white) top/bottom and (gray) middle ones, respectively. The robber stays on an arbitrary rim vertex u until the last of its neighbors, say v, is about to be occupied by a cop. It then moves over v to a new rim vertex that is not adjacent to u.

If v is an internal vertex, it is adjacent to 4 rim vertices: u, a neigbor of u and two other vertices w_1, w_2 that are not adjacent to u. Since there are only 5 cops, w_1 or w_2 is not occupied and the robber may move to it. Conversely, if v is a rim vertex, the robber will, depending on the position of the remaining fifth cop, either move another edge along the rim or over the non-occupied internal vertex to a further rim vertex non-adjacent to u. Since any pair of non-adjacent rim vertices has exactly two common neighbors, not all neighbors of the new rim vertex are occupied even after the cop lands on v. □

Corollary 7.9 *Any large* 2*-crossing-critical graph on at least* 5 *elementary tiles has treewidth* 5 *if and only if it contains at least three elementary tiles with pictures from the set* {VA, VIA, BA, BIA, H}*. Otherwise, it has treewidth* 4*.*

8 Conclusions

For several graph classes, we have conjectures on their crossing numbers. But there are only very few classes for which we know their crossing numbers. Then, their structure is mostly rather simplistic. The class of 2-crossing-critical graphs seems to be the first graph class with known crossing numbers that still offers rich and non-trivial structure in terms of other graph measures as well.

In this paper, after some straight-forward graph properties as building blocks, we successfully discussed both their chromatic number and index, as well as their treewidth. We propose further investigation of general graph-theoretic properties of crossing-number related infinite graph families, to further the idea of interlinking the concepts of topological graph theory with other aspects of the field and further discovery of new applications. — On a more specific note, we recall Question 5.9 from above, which asks whether we can fully characterize 3-colorable large 2-crossing-critical graphs.

In all our proofs, knowing the structure of large 2-crossing-critical graphs was instrumental to proving the values of the above invariants. For further research, it would be of interest to obtain these values without referring to the structure of the graphs, possibly by just assuming the 3-connectivity, 2-crossing-criticality and (should it be needed), presence of a V_{10} subdivision. Such approaches to graph invariants on 2-crossing-critical graphs may then be generalizable to c-crossing-critical graphs for $c > 2$. Furthermore, there are other graph invariants and problems one could consider on these graphs, `www.graphclasses.org` sharing an extensive list. By investigating these invariants and specifically by obtaining proofs that require no knowledge about the structure of the underlying 2-crossing-critical graphs, one may find ways to simplify the characterization theorem of [10], or to identify an approach that would allow to list the finitely many 2-crossing-critical graphs that contain a V_8, but not a V_{10} subdivision, which is the final open step that would render their characterization completely constructive.

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Appendix

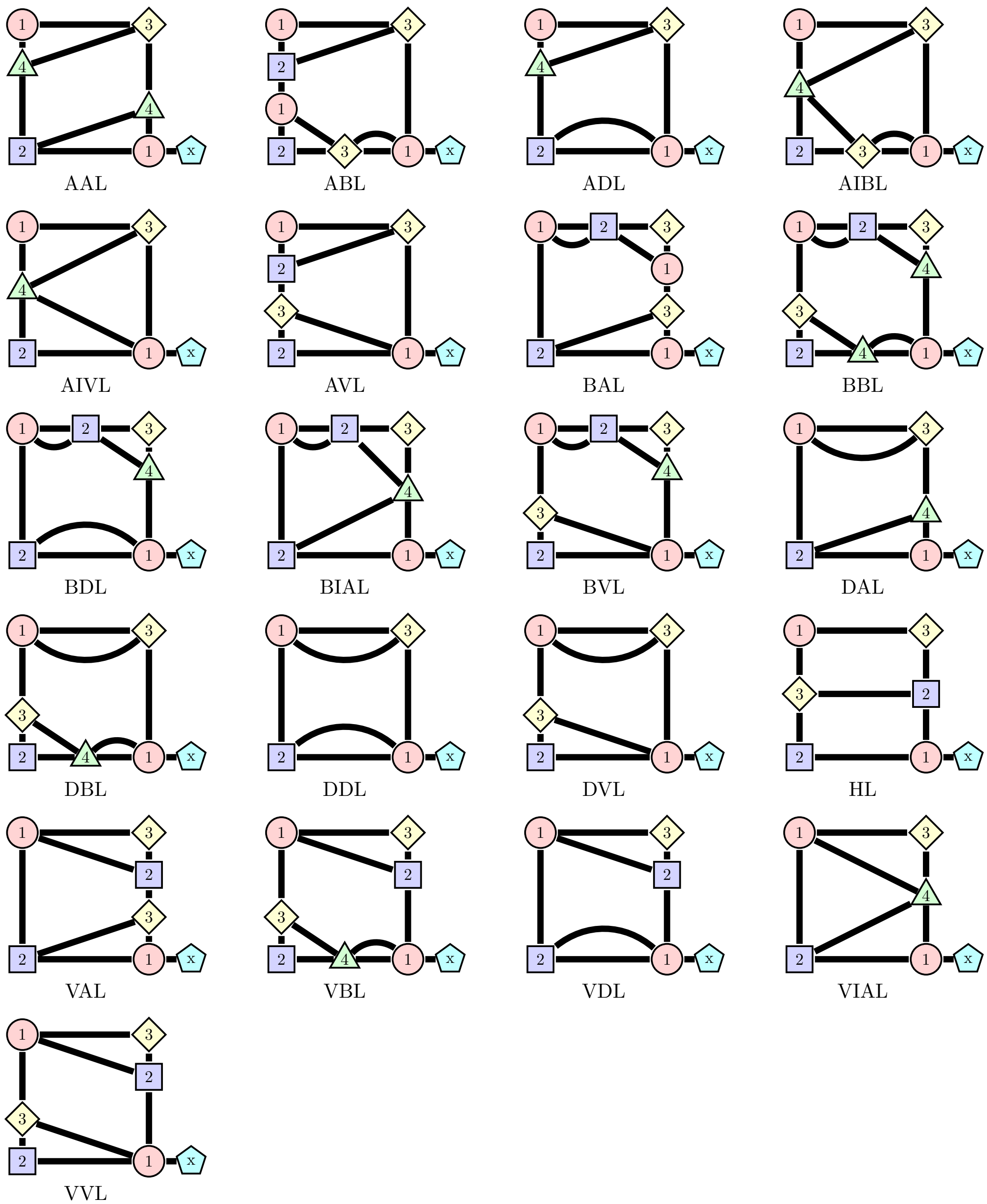

Figure 13: 4-vertex-coloring of every tile of $\mathcal{S}$ with L-frame with ${}^{a}_{b} \rightsquigarrow {}^{c}_{!a}$-propagation. The vertex marked as x can be colored in any color other than 1.

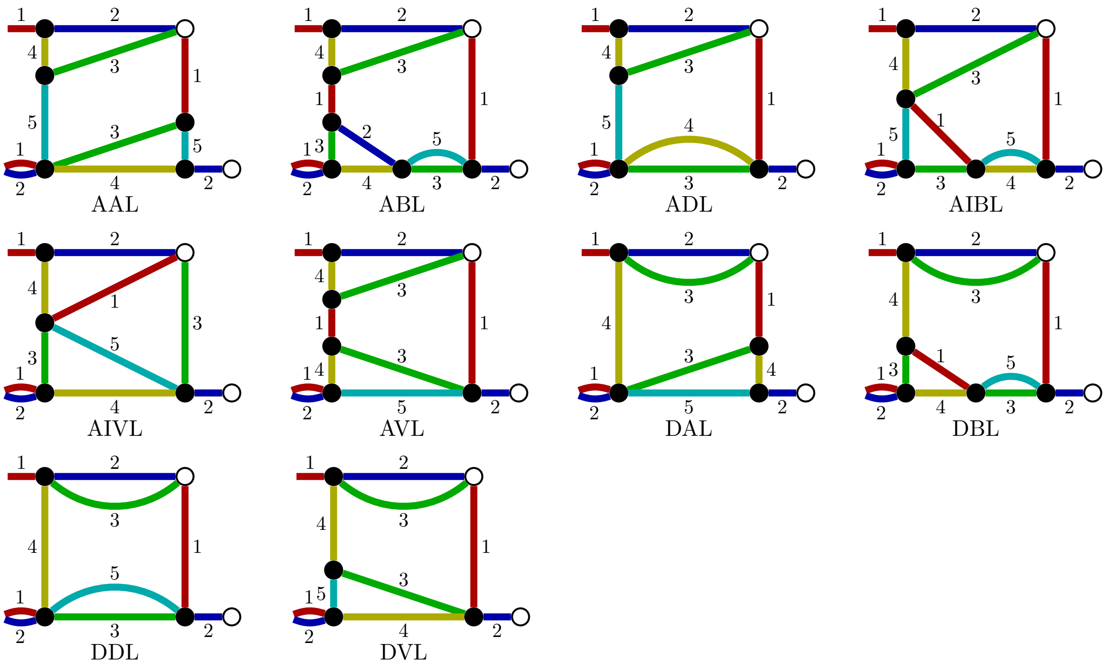

Figure 15: Edge color propagation P_{23} for all 2-3-tiles.

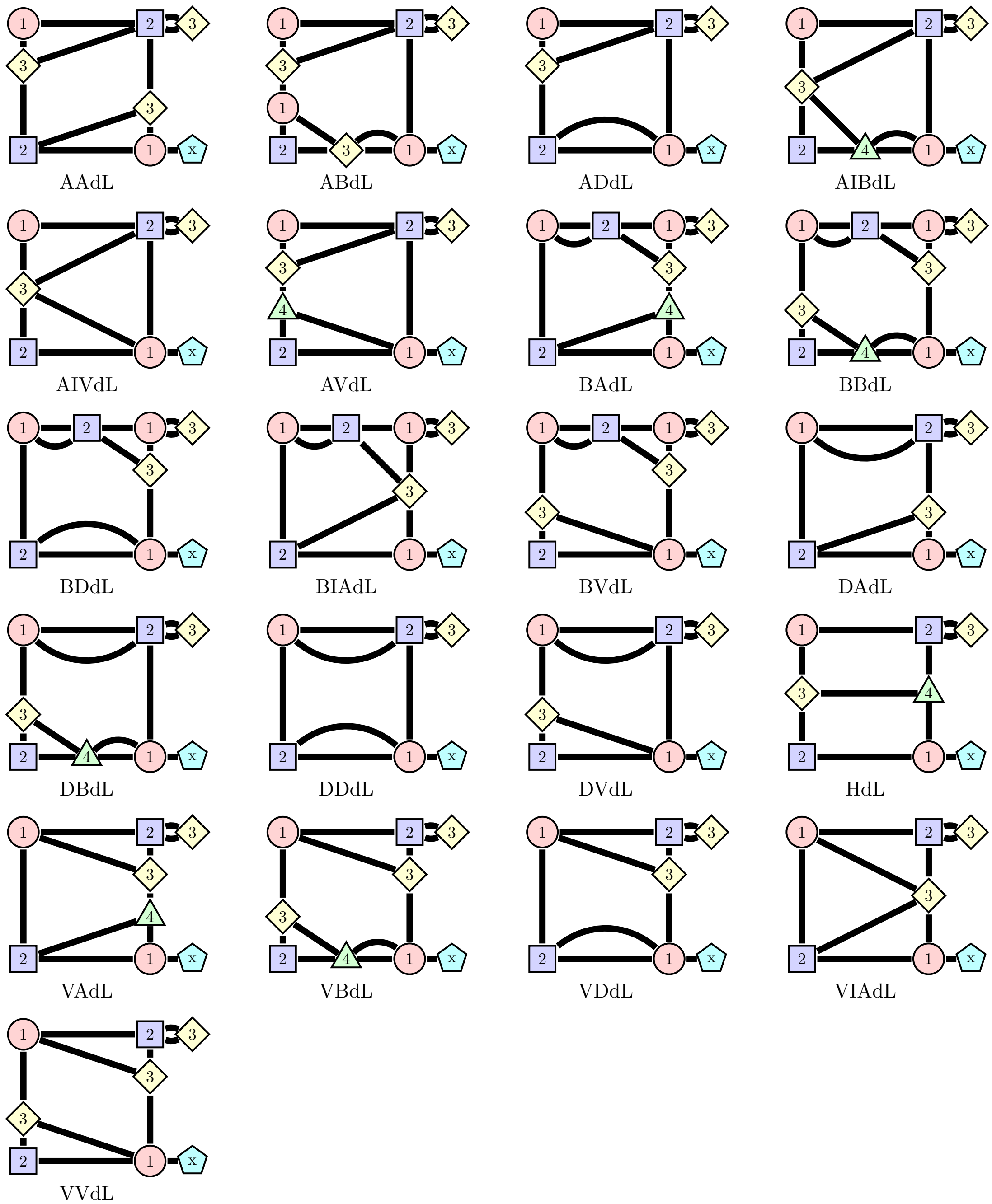

Figure 14: 4-vertex-coloring of every tile of $\mathcal{S}$ with dL-frame with ${}^{a}_{b} \rightsquigarrow {}^{c}_{!a}$-propagation. The vertex marked as x can be colored in any color other than 1.

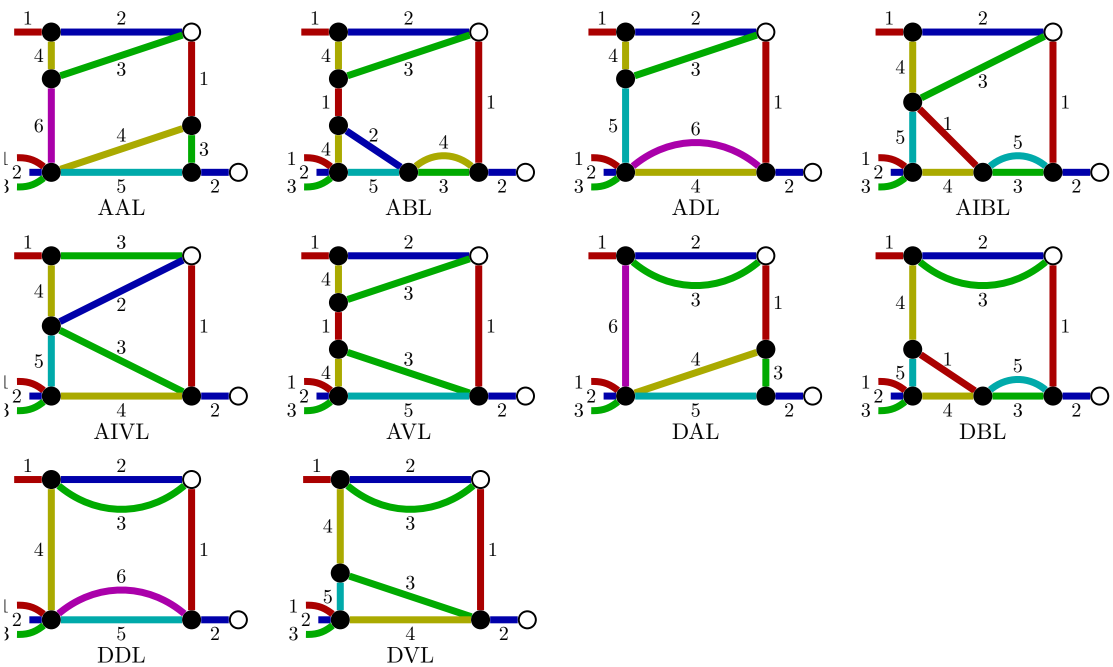

Figure 16: Edge color propagation P_3 for all 3-edge-tiles.

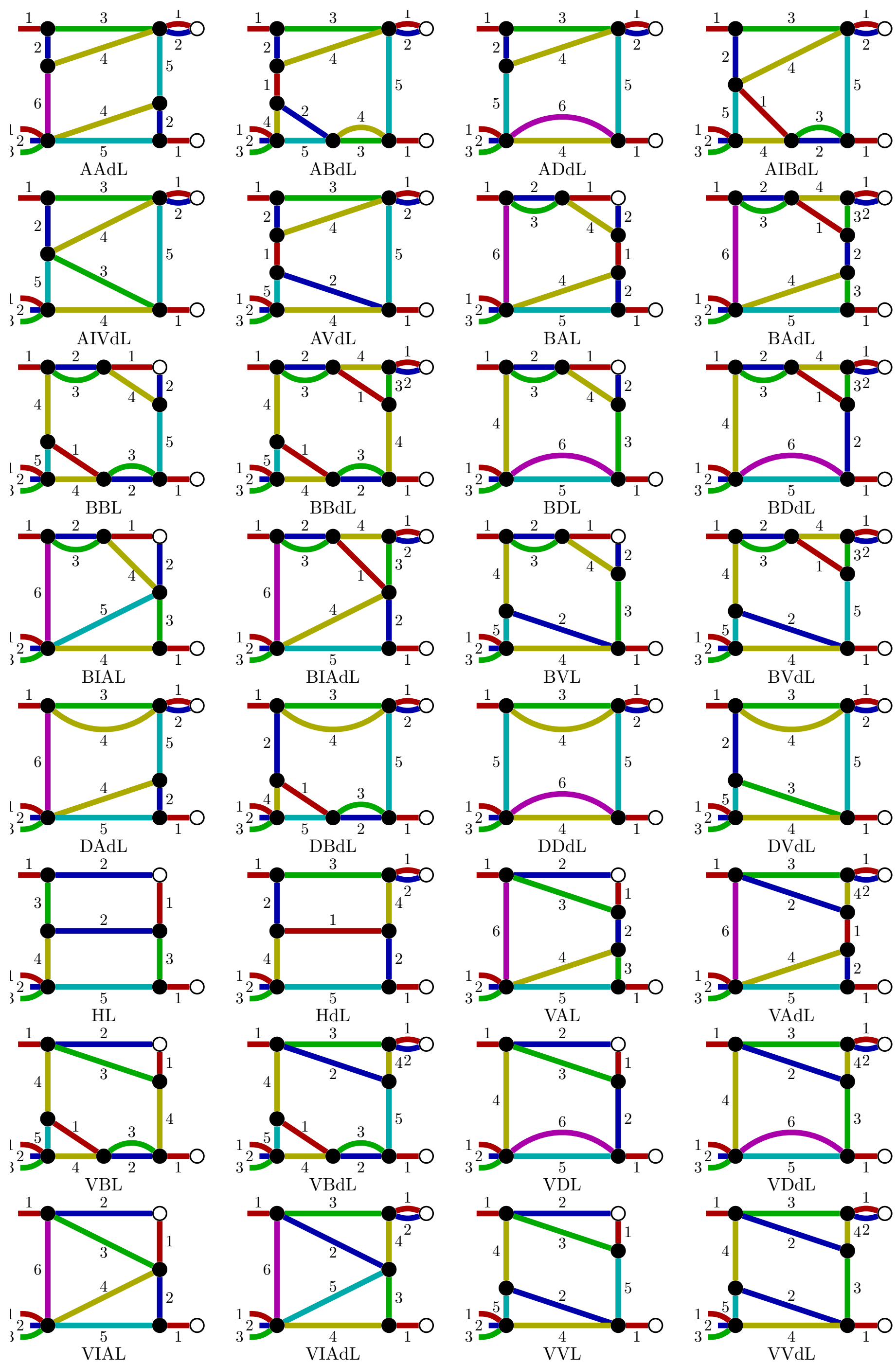

Figure 17: Edge color propagation P_{32a} for all 3-2-tiles.

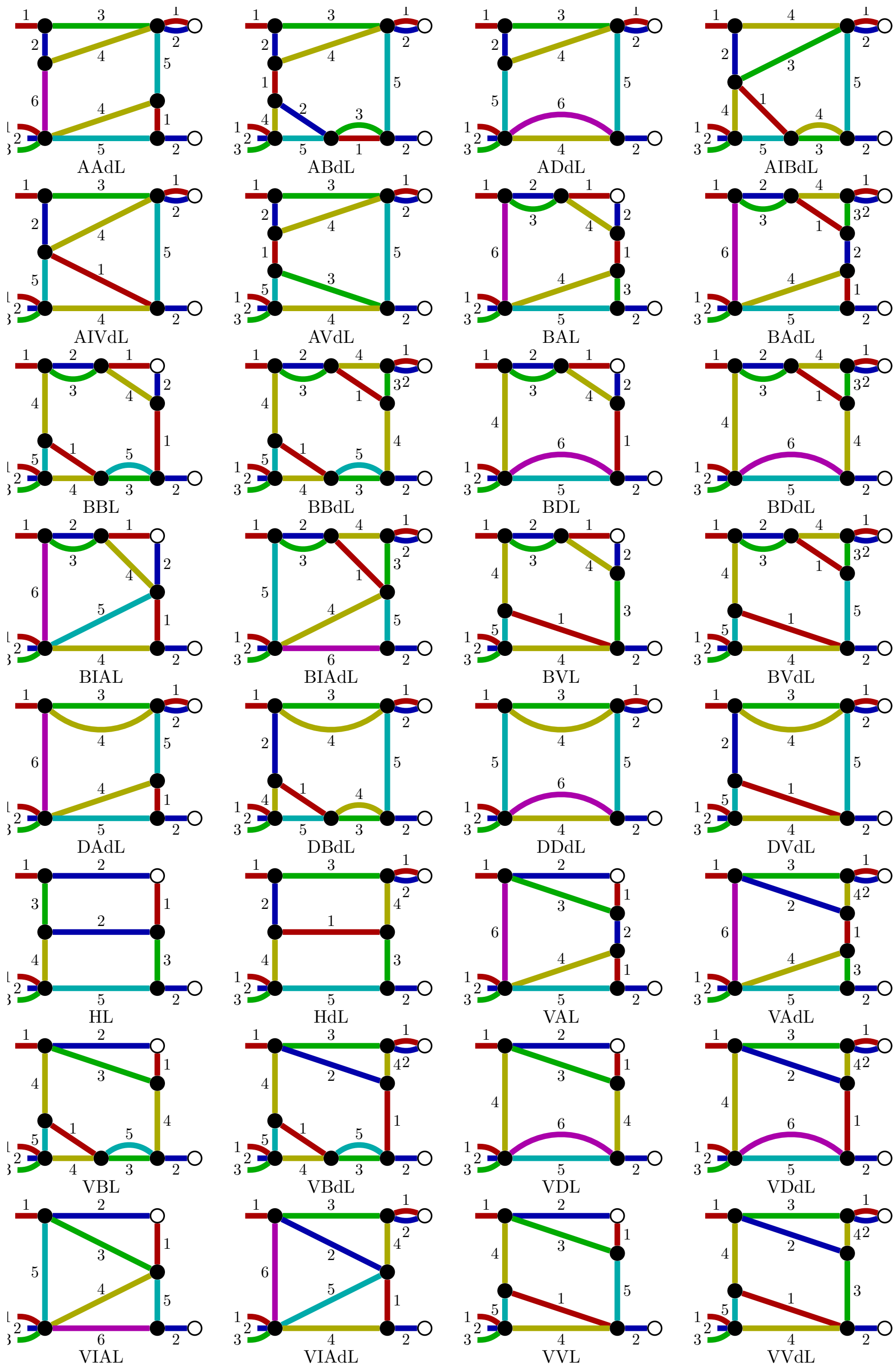

Figure 18: Edge color propagation P_{32b} for all 3-2-tiles.

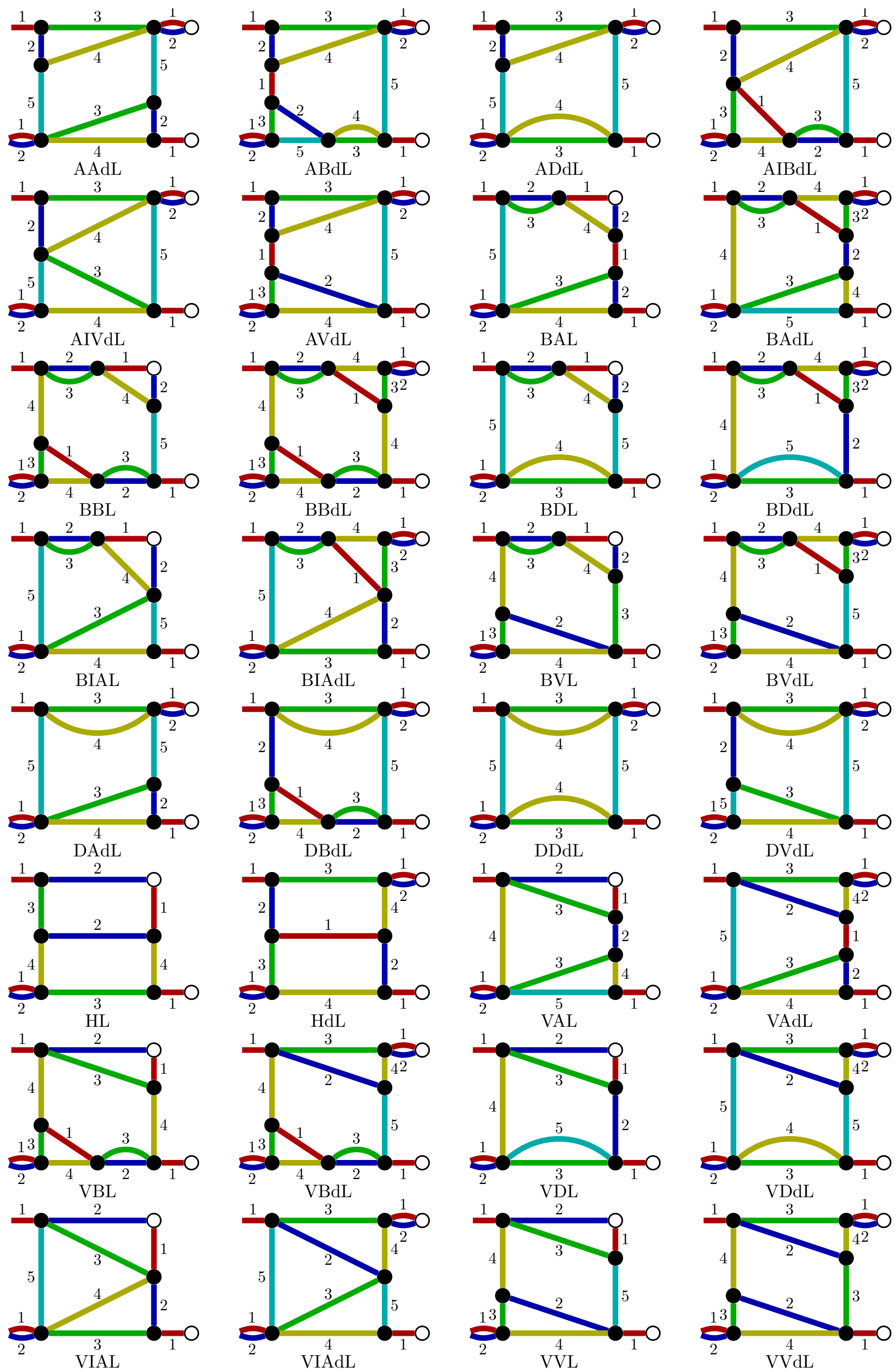

Figure 19: Edge color propagation P_2 for all 2-edge-tiles. Note that *BBL*, *VVL*, *BVdL*, *VBdL* require 5 colors for P_2 despite having maximum degree 4.

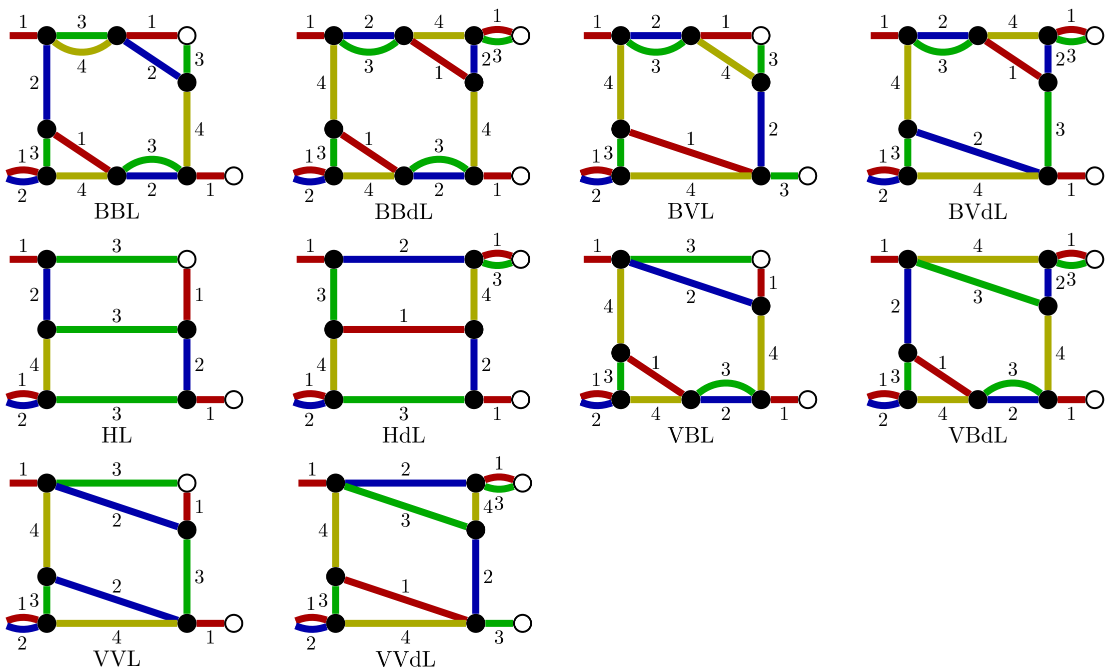

Figure 20: Edge color propagations P_w for BVL and $VVdL$ and P_s for all others of maximum degree 4.