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Angle Covers: Algorithms and Complexity

William Evans[1] *Ellen Gethner*[2] *Jack Spalding-Jamieson*[1] *Alexander Wolff*[3]

[1]Dept. of Computer Science, Univ. of British Columbia,
Vancouver, B.C., Canada.
[2]Dept. of Computer Science and Engineering
University of Colorado Denver, U.S.A.
[3]Universität Würzburg, Würzburg, Germany

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Abstract. Consider a graph with a rotation system, namely, for every vertex, a circular ordering of the incident edges. Given such a graph, an *angle cover* maps every vertex to a pair of consecutive edges in the ordering – an *angle* – such that each edge participates in at least one such pair. We show that any graph of maximum degree 4 admits an angle cover, give a poly-time algorithm for deciding if a graph without a degree-3 vertex has an angle cover, and prove that, given a graph of maximum degree 5, it is NP-hard to decide whether it admits an angle cover. We also consider extensions of the angle cover problem where every vertex selects a fixed number $a > 1$ of angles or where an angle consists of more than two consecutive edges. We show an application of angle covers to the problem of deciding if the 2-blowup of a planar graph has isomorphic thickness 2.

1 Introduction

A well-known problem in combinatorial optimization is *vertex cover*: given an undirected graph, select a subset of the vertices such that every edge is incident to at least one of the selected vertices.

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E-mail addresses: will@cs.ubc.ca (William Evans) ellen.gethner@ucdenver.edu (Ellen Gethner) jack-etsj@alumni.ubc.ca (Jack Spalding-Jamieson)

The aim is to select as few vertices as possible. The problem is one of Karp's 21 NP-complete problems [9] and remains NP-hard even for graphs of maximum degree-3 [1]. Moreover, vertex cover is APX-hard [2] and while it is straightforward to compute a 2-approximation (take all endpoints of a maximal matching), the existence of a $(2-\varepsilon)$-approximation for any $\varepsilon > 0$ would contradict the so-called *Unique Games Conjecture* [10]. Vertex cover is the "book example" of a fixed-parameter tractable problem.

Note that in vertex cover, a vertex covers *all* its incident edges. In this paper we alter the problem by restricting the covering abilities of the vertices. We assume that the input graph has a given *rotation system*, that is, for every vertex, a circular ordering of its incident edges. In the basic version of our problem, the *angle cover problem*, at each vertex we can cover one pair of its incident edges that are consecutive in the ordering (i.e., form an *angle* at the vertex), and every edge must be covered. An example of a planar graph with a vertex cover and an angle cover is shown in Fig. 1.

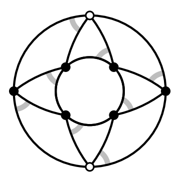


Figure 1: A graph with a minimum vertex cover (black vertices) and an angle cover (gray arcs).

In this paper, we mainly treat the decision version of angle cover, but various optimization versions are interesting as well; see Sections 7 and 8. Clearly, a graph that admits an angle cover cannot have too many edges, both locally and globally. We say that a graph G has *low edge density* if, for all k, every k-vertex subgraph has at most $2k$ edges. Observe that only graphs of low edge density can have an angle cover. Whether a graph G with vertex set $V(G)$ and edge set $E(G)$ has low edge density can be easily checked by testing whether the following bipartite auxiliary graph G_{mat} has a matching of size at least $|E(G)|$. The graph G_{mat} has one vertex for each edge of G, two vertices for each vertex of G, and an edge for every pair (v, e) where v is a vertex of G and e is an edge of G incident to v. A matching of size $|E(G)|$ in G_{mat} implies that every edge in G can be assigned to one of its endpoints without assigning more than two edges to an endpoint, implying low edge density. Further, by Hall's Theorem, if G has low edge density then such a matching exists. Note that this implies that any graph with low edge density has *some* rotation system that admits an angle cover. Indeed, for each vertex v, we simply have to make the at most two edges that are matched to v in G_{mat} consecutive in the ordering around v.

Example classes of graphs with low edge density are outerplanar graphs and maximum-degree-4 graphs (both of which always admit angle covers; see below), Laman graphs (graphs such that for all k, every k-vertex subgraph has at most $2k-3$ edges), and pointed pseudo-triangulations. Given a set P of points in the plane, a *pseudo-triangulation* is a plane graph with vertex set P and straight-line edges that partition the convex hull of P into pseudo-triangles, that is, simple polygons with exactly three convex angles. A pseudo-triangulation is *pointed* if at each vertex there is an angle greater than π. Note that these large angles do not necessarily form an angle cover. It is known that every pointed pseudo-triangulation is a planar Laman graph [14] and that every planar Laman graph can be realized as a pointed pseudo-triangulation [4]. We show that not all (plane) Laman graphs admit an angle cover.

Our interest in angle covers arose from the study of graphs that have isomorphic thickness 2.

The *thickness*, $\theta(G)$, of a graph G is the minimum number of planar graphs whose union is G. By allowing each edge to be a polygonal line with bends, we can draw G on $\theta(G)$ parallel planes where each vertex appears in the same position on each plane and within each plane the edges do not cross [7]. The *isomorphic thickness*, $\iota(G)$, of a graph G is the minimum number of *isomorphic* planar graphs whose union is G. Horák and Širáň [6] credit Wessel with the introduction of this variant of thickness. The *k-blowup* of a graph G is the graph $B^k(G)$ with $k|V|$ vertices $\bigcup_{a=1}^{k} V_a$ and edges $\bigcup_{1\le a,b\le k} E_{ab}$, where $V_a = \{v_a : v \in V(G)\}$, and $E_{ab} = \{(u_a, v_b) : (u,v) \in E(G)\}$. The k-blowup of G is also known as the lexicographic product of G and a graph containing k independent vertices. As we will show, it is NP-hard to determine whether a graph has isomorphic thickness 2, but all graphs that are the 2-blowup of a plane graph with an angle cover have isomorphic thickness 2.

Given a graph G with a rotation system, the existence of an angle cover can be expressed by the following integer linear program (ILP) without objective function. We have two 0–1 variables $x_{u,v}$ and $x_{v,u}$ for every edge uv. The intended meaning of $x_{u,v} = 1$ is that vertex u selects edge uv as one of the two edges of its angle. We denote the set of vertices adjacent to a vertex u by $N(u)$.

$$\sum_{v\in N(u)} x_{u,v} \le 2 \text{ for } u \in V(G) \tag{1}$$

$$x_{u,v} + x_{u,w} \le 1 \text{ for } u \in V(G) \text{ and } v, w \in N(u) \text{ not consecutive around } u \tag{2}$$

$$x_{u,v} + x_{v,u} \ge 1 \text{ for } uv \in E(G) \tag{3}$$

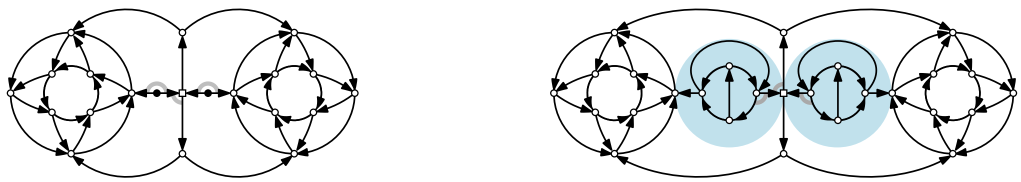
$$x_{u,v} \in \{0,1\} \text{ for } u \in V(G) \text{ and } v \in N(u) \tag{4}$$

If G_{mat} has a perfect matching, we can require equality in the edge constraints (3). The ILP formulation has $2 \cdot |E|$ variables and $O\big(\sum_{v\in V} \deg^2(v)\big)$ constraints.

Our contribution. We first consider a few concrete examples that show that not every planar graph of low edge density admits an angle cover for a fixed (planar) embedding. This is even true if we restrict the maximum degree to 5 or if we consider planar Laman graphs (see Section 2). Next we turn to our original motivation for studying angle covers. We show that the 2-blowup of any plane graph with an angle cover has isomorphic thickness 2 (Section 3). We then give efficient algorithms for special cases (Section 4). In particular, we show that all outerplane graphs and all graphs of maximum degree 4 admit angle covers. (Recall that an *outerplane* graph is an outerplanar graph with a given outerplanar embedding and, hence, fixed rotation system.) For graphs without vertices of degree 3, the above ILP somewhat surprisingly leads to an efficient algorithm for testing the existence of an angle cover (see Theorem 12). Then we prove that, given a graph of maximum degree 5 with a fixed rotation system, it is NP-hard to decide whether it admits an angle cover (Section 5). We also consider three extensions of the angle cover problem where (i) each vertex can select a different number of angles and the aim is to use the minimum number of angles to cover all edges (Section 6), and (ii) every vertex is associated with a fixed number $a > 1$ of angles (Section 7), and (iii) an angle consists of more than two consecutive edges (Section 8). We conclude with some open problems (Section 9).

2 Observations and Examples

In this section we show examples of graphs from very restricted families of low density graphs that do not admit an angle cover. All our examples are *plane* graphs, that is, they are planar and their rotation system corresponds to a planar drawing.



(a) example with vertex degrees 2–5 (b) example with vertex degrees 3–5

Figure 2: Two plane graphs that do not admit angle covers. Edges are oriented such that each vertex has outdegree 2 (hence both graphs have low edge density).

Observation 1 *There is a plane graph (Fig. 2a) with low edge density and maximum degree 5 that does not admit an angle cover.*

Proof: Consider the graph in Fig. 2a. We have oriented its edges so that each vertex has outdegree 2. Hence, the graph has low edge density. The graph has $n = 21$ vertices and $2n$ edges. Due to the way the two degree-2 vertices (filled black) are arranged around the central vertex (square) of degree 4, one of the horizontal edges incident to the central vertex is covered twice. This implies that the n vertices cover at most $2n - 1$ edges. Thus, there is no angle cover. □

Note that the counterexample critically exploits the use of degree-2 vertices. Next we show that there are also counterexamples without such vertices.

Observation 2 *There is a biconnected plane graph (Fig. 2b) with low edge density and vertex degrees in* $\{3, 4, 5\}$ *that does not admit an angle cover.*

Proof: Consider the graph in Fig. 2b. Again, we oriented the edges such that every vertex has outdegree 2, which shows that the graph has low edge density. The graph is the same as the one in Fig. 2a except that we replaced the two degree-2 vertices by copies of K_4 (light blue). Since the number of edges is $2n$ and K_4 has four vertices and six edges, the two edges that connect each copy of K_4 to the rest of the graph, must be covered in any angle cover by a vertex in the copy of K_4. As a result, the two copies of K_4 behave like the degree-2 vertices in Fig. 2a: in any angle cover, they, along with the central vertex, must cover one of the horizontal edges incident to the central vertex twice. Thus, there is no angle cover. □

Observation 3 *There is a planar maximum-degree-5 graph (Fig. 3) with two embeddings such that one admits an angle cover, but the other does not.*

Proof: Note that the embedded graph of Fig. 3 contains four subgraphs isomorphic to the graph in Fig. 1, each with 8 vertices and 16 edges. Hence all the angles of the vertices of these subgraphs must be used on for the edges within these subgraphs. This forces the angles drawn in the remainder of the graph. As a result, only the right embedded graph admits an angle cover. □

Thus, when determining whether a graph (of maximum degree greater than 4) has an angle cover, we must consider a particular embedding, which determines a rotation system. This applies to non-planar graphs as well. However, if we have a topological embedding of a non-planar graph, we can decide whether it has an angle cover by considering its planarization. By a *topological graph* we mean a graph together with a drawing of that graph where any pair of edges (including their

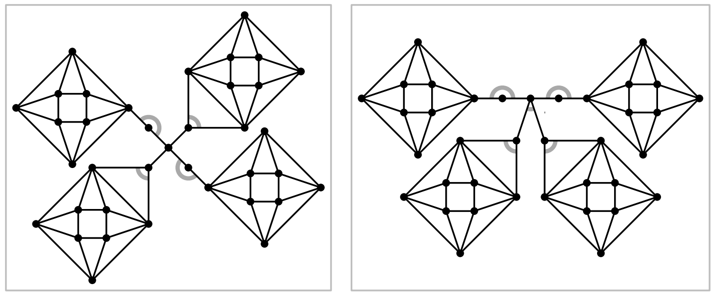


Figure 3: A graph with two planar embeddings; one without and one with an angle cover.

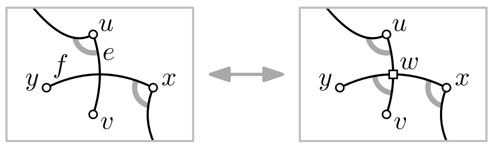


Figure 4: Any topological graph (left) admits an angle cover if and only if its planarization (right) admits an angle cover.

endpoints) has at most one (crossing not touching) point in common and any point of the plane is contained in at most two edges. By the *planarization* of a topological graph we mean the plane graph that we get if we replace, one by one, in arbitrary order, each crossing by a new vertex that is incident exactly to the four pieces of the two edges that defined the crossing. We define the order of the four new edges around the new vertex to be the same as the order of the four pieces of the old edges around the crossing.

Observation 4 *Any topological graph admits an angle cover if and only if its planarization admits an angle cover.*

Proof: We show the equivalence for the first step of the planarization procedure defined above. Then, induction proves our claim.

Let G be a topological graph, and let G' be the graph that we obtain from G by replacing an arbitrary crossing of two edges $e = uv$ and $f = xy$ by a new vertex w that is incident to u, v, x, and y; see Fig. 4. (Let the order of the endpoints around the crossing in G and around the new vertex in G be $\langle u, x, v, y\rangle$.)

Suppose that G has an angle cover α. Edges e and f must be covered by angles incident to, say, vertices u and x. Then it is simple to extend α to G' by mapping w to the angle $\{wv, wy\}$ incident to w.

Now suppose that G' has an angle cover α' with, say, $\alpha'(w) = \{vw, yw\}$. Since w does not cover uw and xw, u must cover uw and x must cover xw. Now we restrict α' to G: we replace uw by uv and xw by xy. Hence, both uv and xy are covered. Finally, we remove w (with vw and yw). Clearly, the resulting map is an angle cover for G. □

Laman graphs are a natural class of graphs to consider for the angle cover problem because their *size characterization* ensures low edge density: an n-vertex graph G is Laman if G has exactly $2n - 3$ edges and, for every $k > 1$, every k-vertex subgraph of G has at most $2k - 3$ edges. Note that by this characterization all outerplanar graphs are subgraphs of Laman graphs. Another

Figure 5: A planar Laman graph with a planar embedding that does not (left) and a planar embedding that does (right) admit an angle cover.

characterization due to Henneberg [5] is that Laman graphs (with at least two vertices) are those graphs that can be constructed by starting with an edge and repeatedly either

(S1) adding a new vertex to the graph and connecting it to two existing vertices, or

(S2) subdividing an edge of the graph and adding an edge connecting the newly created vertex to a third vertex

Their low edge density suggests the possibility that Laman graphs admit an angle cover. While this is true for outerplanar graphs (with an outerplanar rotation system; see Theorem 10), it is not the case for all Laman graphs and rotation systems.

Observation 5 *There is a plane Laman graph not admitting an angle cover.*

Proof: Consider the graph with embedding in Fig. 5 (left). The graph is Laman since it admits a Henneberg construction using step (S1) to first create all the square black vertices, and then all the circular red vertices.

At each square black vertex, every pair of thick black edges is separated by at least one thin red edge in the planar embedding. Therefore, each square black vertex can cover at most one thick black edge. However, there are five thick black edges and only four square black vertices, which are the only vertices adjacent to thick black edges. Hence no angle cover exists. □

3 Isomorphic Thickness

Our original motivation for considering angle covers of a graph is that they provide a method for showing that the 2-blowup of the graph has isomorphic thickness at most 2. This connection is described in the proof of the following theorem.

Theorem 6 *If a plane graph G has an angle cover, then the 2-blowup of G has isomorphic thickness at most 2.*

Proof: For $a \in \{1,2\}$, let $V_a = \{v_a : v \in V(G)\}$. For $a,b \in \{1,2\}$, let $E_{ab} = \{(u_a, v_b) : (u,v) \in E(G)\}$. Let λ be an angle cover of G, that is, $\lambda(v) = \{(v,x),(v,y)\}$ for some consecutive pair of edges (v,x) and (v,y) around v. Some edges in G may be "double-covered" by λ. Later in this proof, we will need to avoid an edge that is covered by both its endpoints. Thus, we define $\lambda'(v) \subseteq \lambda(v)$ such that for any $e \in E(G)$, there is a unique $v \in V$ where $e \in \lambda'(v)$. Let H be the graph with vertex set $V_1 \cup V_2$ and edge set $E_{11} \cup \bigcup_{v \in V(G), u \in V(G)} \{(v_2, u_1) : (v,u) \in \lambda'(v)\}$. Note that since λ is an angle cover for G, for every edge $(u,v) \in E(G)$, either (u_2, v_1) or (u_1, v_2) is an edge in H. Let $\widetilde{H}$ be the graph isomorphic to H where vertex v_a maps to v_b with $b = 3 - a$. We claim that H and $\widetilde{H}$ are planar graphs whose union is the 2-blowup of G.

To see that H (and hence $\widetilde{H}$) is planar, fix a planar straight-line drawing of G realizing the embedding for which λ is the angle cover for G. Let v also denote the point representing vertex v in this embedding. Place v_2 close enough to v in the angle formed by the edges $\lambda(v) = \{(v, x), (v, y)\}$, so that the segments (v_2, x) and (v_2, y) do not cross any edge segments of G. Such a placement exists since the segments (v, x) and (v, y) do not cross any edge segment of G. Relabel each vertex u in G as u_1. Thus the edges (u_1, v_1) do not cross and the remaining edges in H (of the form (v_2, u_1)) do not cross these segments from the embedding of G and do not cross each other since only one of the two crossing segments (v_2, u_1) and (v_1, u_2) is in H (as no edge is covered by both endpoints in λ'). This creates a planar drawing of H; see Fig. 6.

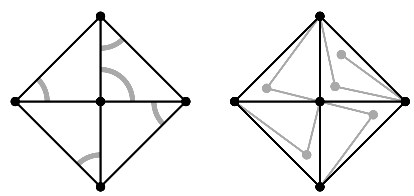


Figure 6: An example of how the existence of an angle cover in a graph G implies that the 2-blowup of G has isomorphic thickness 2.

The graph H contains the edge (u_1, v_1) for every $(u, v) \in E(G)$. It also contains the edge (u_1, v_2) or (u_2, v_1) for every $(u, v) \in E(G)$ since λ is an angle cover. Hence, $\widetilde{H}$ contains, for every $(u, v) \in E(G)$, the edge (u_2, v_2) and the edge (u_1, v_2) or (u_2, v_1) that is not in H. Thus the union of H and $\widetilde{H}$ equals the 2-blowup of G. □

Note that this proof easily generalizes: For any $k \geq 2$, if a plane graph G has an angle cover, then the k-blowup of G has isomorphic thickness at most k. This is accomplished simply by using $k - 2$ additional copies of V_2 and, for each copy, the edges incident to V_2. This implies that the k-blowup of a plane graph with an angle cover has isomorphic thickness at most k.

As one would expect, not every planar graph whose 2-blowup has isomorphic thickness 2 has a plane embedding that admits an angle cover.

Observation 7 *There is a graph G whose 2-blowup has isomorphic thickness 2 but none of the planar embeddings of G admits an angle cover.*

Proof: We show that the graph G presented in Fig. 7 has all the desired properties. Note that G

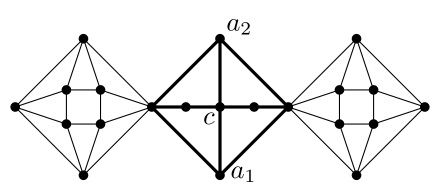


Figure 7: A graph G whose 2-blowup has isomorphic thickness 2. The graph has no planar embedding admitting an angle cover. The graph G', described in the proof of Observation 7, is bold.

contains two disjoint induced copies H_1 and H_2 of the graph H presented in Fig. 1. The graph H has an angle cover and, since it has 8 vertices and 16 edges, any angle cover will cover exactly the edges in H. Indeed, a planar embedding of G has an angle cover if and only if the embedded subgraphs H_1 and H_2 have angle covers and after contracting both H_1 and H_2 to single vertices the resulting graph G' (see Fig. 7) has a partial angle cover where the contracted vertices do not cover an angle. This is equivalent to finding a planar embedding for G' in which the edges (c, a_1) and (c, a_2) are adjacent around the central vertex c (see Fig. 7) since the covered edges of all other vertices are forced (and include all edges except (c, a_1) and (c, a_2)). Since G' is the subdivision of a 3-connected planar graph, (c, a_1) and (c, a_2) are not adjacent in any planar embedding of G' [15].

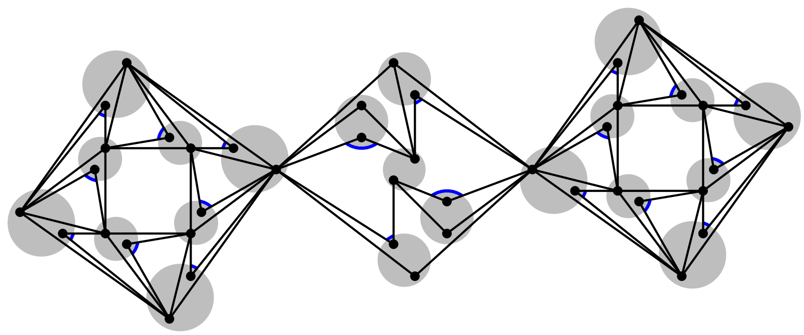


Figure 8: A planar embedding of a spanning subgraph of the 2-blowup of the graph G in Fig. 7. The gray circles denote pairs of vertices corresponding to single vertices in G. The graph was generated using the process outlined in the proof of Theorem 6 for the angle cover shown as thick dark blue arcs *except* at the vertex labeled c in Fig. 7.

Finally, we refer to the graph in Fig. 8 to show that the isomorphic thickness of the 2-blowup of G is at most 2. Since the graph in the figure is a spanning subgraph of the 2-blowup, we can label the vertices of the graph such that they match some labelling of the 2-blowup. Call the resulting labelled graph B_1. Create a second labelled graph, where the labels of pairs of vertices inside the same gray circle in the figure are swapped, and call this B_2. Since B_1 and B_2 differ only in their labels, the two graphs are isomorphic. One can visually observe that the union of the two labelled graphs B_1 and B_2 is exactly the 2-blowup of G. □

Theorem 8 *It is NP-complete to decide whether a graph has isomorphic thickness* 2*.*

Proof: The problem of deciding if a graph (V, E) has isomorphic thickness 2 is in NP since, given two subsets E_1 and E_2 of the edges E and a permutation π of the vertices V, we can check in polynomial time that $E_1 \cup E_2 = E$, the graphs (V, E_1) and (V, E_2) are isomorphic, using π as the isomorphism, and that (V, E_1) is planar.

To show that it is NP-hard, we describe a reduction from the problem of determining whether a graph has thickness 2, which is known to be NP-hard [12]. Given a graph $G = (V, E)$, we construct a copy $G' = (V', E')$ of G on a set of vertices V' disjoint from V. We claim that the union $G + G'$ has isomorphic thickness 2 if and only if G has thickness 2.

If $G + G'$ has isomorphic thickness 2 then, since G is a subgraph of $G + G'$, G has thickness 2. Indeed, if H and $\widetilde{H}$ are two isomorphic planar graphs with vertex sets $V \cup V'$ whose union is $G + G'$, then $H[V]$ and $\widetilde{H}[V]$ (the subgraphs of H and $\widetilde{H}$ induced by V) are two planar graphs whose union is G.

If G has thickness 2, then there exist two planar graphs G_1 and G_2 whose union is G. Since G' is a copy of G (on a disjoint set of vertices), G' is the union of G'_1 and G'_2, with G'_1 isomorphic to G_1 and G'_2 isomorphic to G_2. Thus, $H = G_1 + G'_2$ and $\widetilde{H} = G'_1 + G_2$ are isomorphic and their union is $G + G'$. □

4 Algorithms for Outerplane Graphs and Graphs with Degree Restrictions

In this section we show that outerplane and maximum-degree-4 graphs always admit angle covers by describing efficient algorithms that compute the covers. For graphs without degree-3 vertices, we show how to decide (in quadratic time) whether or not they admit an angle cover for a given rotation system.

Lemma 9 *Let G be a graph with a rotation system that admits an angle cover. Then any subgraph G' of G has an angle cover (for the rotation system that G' inherits from G).*

Proof: An angle cover in G implies an angle cover in G' where the assigned angle at a vertex in G' is the angle whose span contains the span of the assigned angle at the same vertex in G. □

Any n-vertex outerplanar graph has at most $2n - 3$ edges (if $n > 1$), and outerplanarity is hereditary, so outerplanar graphs have low edge density. Therefore, outerplanar graphs are natural candidates for admitting angle covers.

Theorem 10 *Every outerplane graph has an angle cover, and such a cover can be found in linear time.*

Proof: Assume G is a maximal outerplane graph, that is, no edge can be added to G without violating outerplanarity. Consider an outerplane drawing of G, and let n be the number of vertices of G. It is well known that any outerplane graph has an *ear*, that is, a vertex of degree at most 2 [11]. For our purposes we can assume that our graph (as long as it has at least three vertices) contains an ear of degree exactly 2 (which is true for maximal outerplane graphs). If we repeatedly remove ears $v_1, v_2, \ldots, v_{n-2}$ from G, note that the two edges incident to v_i (for $1 \le i \le n-2$) in $G_i = G[\{v_i, \ldots, v_n\}]$ are consecutive in the rotation system of G. This is due to the fact that these two edges are part of the boundary of the outer face of G_i, which is a closed curve. Hence, if α_i is the angle formed by the two edges in the interior of the closed curve and v_i is adjacent to a vertex v_j with $j < i$, then the edge $v_i v_j$ cannot enter v_i through α_i because v_j lies outside of the closed curve. Let α_{n-1} be the degenerate angle that consists of the single edge $v_{n-1}v_n$. Then $\alpha_1, \ldots, \alpha_{n-1}$ is an angle cover of G.

We can turn the above existence proof into an algorithm for any outerplane graph G. First, we augment G to a maximal outerplane graph by introducing dummy edges where necessary. This can be done in linear time [8]. Second, if we bucket the vertices by degree then, after linear-time preprocessing, we can find in constant time, for $i \in \{1, \ldots, n-2\}$, the next ear v_i with its two incident edges forming the angle α_i. Third, we remove the dummy edges and use Lemma 9 to adapt the angle cover to the original graph G. In total, this takes $O(n)$ time. □

Theorem 11 *Any graph of maximum degree 4 with any rotation system admits an angle cover, and such a cover can be found in linear time.*

Proof: We can assume that the given graph is connected since we can treat each connected component independently. If the given graph is not 4-regular, we arbitrarily add dummy edges between vertices of degree less than 4 until the resulting (multi)graph is 4-regular or there is a single vertex, say v, of degree less than 4. If one vertex remains with degree less than 4, we add self-loops to that vertex until it has degree 4. This is always possible since all other vertices have degree 4, which is even, hence the last vertex must also have even degree.

We find a collection of directed cycles in the now 4-regular graph, in a manner similar to the algorithm for finding an Eulerian cycle. We follow the rule to exit a degree-4 vertex using the edge that is not consecutive to the edge by which we entered. Whenever we close a cycle and there are still edges that we have not traversed yet, we start a new cycle from one of these edges. In this way we never visit an edge of the input graph twice, which establishes the linear running time.

The algorithm yields a partition of the edge set into (directed) cycles with the additional property that pairs of cycles may cross each other (or themselves), but they never touch without crossing. Hence, in every vertex the two outgoing edges are always consecutive in the circular ordering around the vertex. We assign to each vertex the angle formed by this pair of edges. Due to Lemma 9, this angle cover is also an angle cover for the original graph, which is a subgraph of the 4-regular graph that we have constructed. □

Theorem 12 *Given an n-vertex graph without vertices of degree 3, the angle cover problem can be solved in $O(n^2)$ time.*

Proof: In the ILP formulation in Section 1, condition (1) for vertex u is implied by the satisfaction of condition (2) for vertex u as long as u is not of degree 3. Without condition (1), the ILP becomes an instance of 2-SAT.

The number of clauses is bounded from above by $|E| + |E|^2$, where $|E| \in O(n)$. Since 2-SAT can be solved in time linear in the number of clauses [3], the algorithm takes $O(n^2)$ time. □

5 NP-Hardness for Maximum-Degree-5 Graphs

Theorem 13 *The angle cover problem is NP-hard even for graphs of maximum degree 5.*

Proof: We reduce from 3-colouring. Given a graph $G = (V, E)$, we construct a graph $H = (U, F)$ with a rotation system f where $f(u)$ is the sequence of edges in circular order around vertex $u \in U$, such that (H, f) admits an angle cover if and only if G has a 3-colouring.

For each vertex $v \in V$, let $E_1(v), \ldots, E_{\deg(v)}(v)$ be its adjacent edges in some arbitrary order. We create a graph, called a *gadget*, for vertex v that contains $1+9\deg(v)$ vertices. The centre of the gadget is a vertex $c(v)$ that is adjacent to the first vertex in three paths, $e_1^k(v), e_2^k(v), \ldots, e_{\deg(v)}^k(v)$, one for each of the three colours $k \in \{0, 1, 2\}$. Each vertex $e_j^k(v)$, for $j = 1, \ldots, \deg(v)$, is adjacent to two degree-1 vertices $a_j^k(v)$ and $b_j^k(v)$ (as well as its neighbours in the path) that are part of the gadget (see Fig. 9). In addition, if $E_i(u) = E_j(v)$, that is, (u, v) is an edge in G and is the ith edge adjacent to u and the jth edge adjacent to v, then the vertex $e_j^k(v)$ is adjacent to $e_i^k(u)$ (see Fig. 10). The circular order of edges around $e_j^k(v)$ is $[e_{j-1}^k(v), a_j^k(v), e_{j+1}^k(v), e_i^k(u), b_j^k(v)]$ where $e_{j-1}^k(v)$ is $c(v)$ if $j = 1$ and $e_{j+1}^k(v)$ does not exist if $j = \deg(v)$. The *separator* edges $(e_j^k(v), a_j^k(v))$ and $(e_j^k(v), b_j^k(v))$ prevent an angle cover at $e_j^k(v)$ from (i) covering both $(e_j^k(v), e_{j-1}^k(v))$ and $(e_j^k(v), e_{j+1}^k(v))$ or (ii) covering both $(e_j^k(v), e_{j-1}^k(v))$ and $(e_j^k(v), e_i^k(u))$. The graph containing all of the gadgets and the edges between them along with the specified rotation system is then

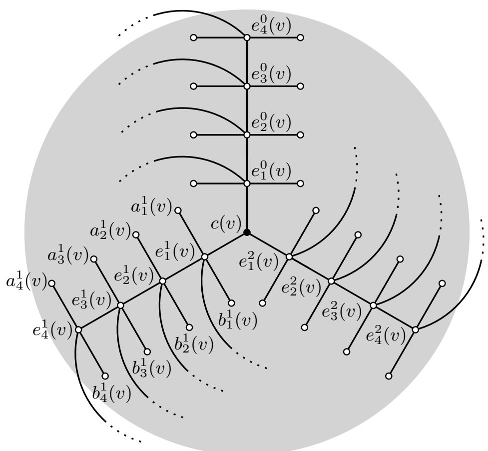


Figure 9: Gadget for a degree-4 vertex v; the edge incident to $c(v)$ that is not covered by the angle cover corresponds to the colour of v.

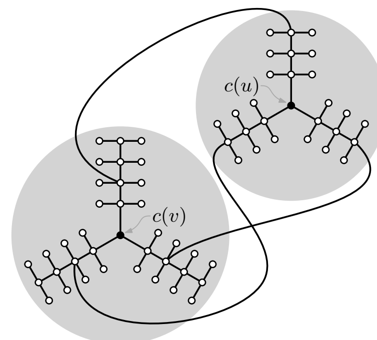


Figure 10: The edge (u, v) of G is represented by the three curved edges in H. Here, (u, v) is the third edge of u and the second edge of v; $\deg(u) = 3$ and $\deg(v) = 4$.

$(H = (U, F), f)$. Observe that the maximum degree of H is 5 and that the construction takes polynomial time.

It remains to show that G is 3-colourable if and only if (H, f) has an angle cover. We start with the "only if" direction.

G is 3-colourable implies that (H, f) has an angle cover:
Let $t\colon V \to \{0, 1, 2\}$ be a 3-colouring of G. We construct an angle cover $\alpha\colon U \to F \times F$. For each vertex $v \in V$, if $t(v) = k$, then set $\alpha(c(v)) = ((c(v), e_1^{k+1}(v)), (c(v), e_1^{k+2}(v)))$, where all superscripts are taken modulo 3. Also, for $j = 1, \ldots, \deg(v)$, set $\alpha(e_j^k(v)) = \left((e_j^k(v), e_{j-1}^k(v)), (e_j^k(v), a_j^k(v))\right)$, where $e_{j-1}^k(v)$ is $c(v)$ for $j = 1$. Furthermore, for $\ell \neq k$ (i.e., $\ell \in \{k+1, k+2\}$), and $(u, v) \in E$, set $\alpha(e_j^\ell(v)) = \left((e_j^\ell(v), e_{j+1}^\ell(v)), (e_j^\ell(v), e_i^\ell(u))\right)$, where $E_i(u) = E_j(v)$. Since any vertex of degree at most 2 covers all its adjacent edges, all edges in the construction, except for possibly $(e_j^k(v), e_i^k(u))$ where $E_i(u) = E_j(v)$, are covered. Since t is a 3-colouring, we know that $t(u) \neq t(v) = k$. Therefore, the edge $(e_j^k(v), e_i^k(u))$ is covered by $e_i^k(u)$ by the construction above, so all edges are covered by the constructed angle cover.

(H, f) has an angle cover implies that G is 3-colourable:
For every vertex $v \in V$, if $c(v)$ covers edges $(c(v), e_1^{k+1}(v))$ and $(c(v), e_1^{k-1}(v))$ in the angle cover then set $t(v) = k$ (i.e., the colour given by the edge not covered by $c(v)$). Suppose for the sake of contradiction that an edge (u, v) in G is not properly coloured and $t(u) = t(v) = k$. The edge $(c(u), e_1^k(u))$ is not covered by $c(u)$ and the edge $(c(v), e_1^k(v))$ is not covered by $c(v)$. Then, those edges must be covered by $e_1^k(u)$ and $e_1^k(v)$, respectively, so $e_1^k(u)$ and $e_1^k(v)$ cannot cover the edges $(e_1^k(u), e_2^k(u))$ and $(e_1^k(v), e_2^k(v))$, respectively, nor the edges $(e_1^k(u), e_i^k(w))$ where $E_1(u) = E_i(w) = (u, w)$ and $(e_1^k(v), e_j^k(x))$ where $E_1(v) = E_j(x) = (v, x)$, respectively. Let $E_{i^*}(u) = E_{j^*}(v) = (u, v)$. Repeating this argument, we see that the edge $(e_{i^*}^k(u), e_{j^*}^k(v))$ is neither

covered by $e^k_{i^*}(u)$ nor by $e^k_{j^*}(v)$. This is a contradiction since we assumed that (H, f) has an angle cover. □

We now apply Observation 4, which reduces general angle cover to angle cover for planar graphs, to a drawing of the graph in the above reduction.

Corollary 14 *The angle cover problem is NP-hard even for* planar *graphs of maximum degree 5.*

6 Angle Allocation

It is perhaps helpful in understanding what makes the angle cover problem hard to consider a relaxation of the problem where each vertex can select a different number of angles, and still, all edges must be covered. We call this an *angle allocation* and it is *optimal* if it uses the minimum number of angles among all allocations. Unlike the angle cover problem, we know an efficient solution to the optimal angle allocation problem. In fact, it is closely related to the problem of finding optimal edge covers.

Theorem 15 *Given a graph G with a rotation system, an optimal angle allocation can be computed in $O(|E(G)|^{3/2})$ time.*

Proof: Consider the *medial graph* G_{med} associated with the given graph G and its rotation system. The vertices of G_{med} are the edges of G, and two vertices of G_{med} are adjacent if the corresponding edges of G are incident to the same vertex of G and consecutive in the circular ordering around that vertex. The medial graph is always 4-regular. If G has no degree-1 vertices, G_{med} has no loops. If G has minimum degree 3, G_{med} is simple.

An *edge cover* in G_{med}, i.e., a set of edges such that every vertex is incident to at least one edge in the set, corresponds to an angle allocation in G since every edge of G_{med} corresponds to an angle in G and every vertex of G_{med} corresponds to an edge of G. Thus a minimum edge cover in G_{med} is an optimal angle allocation in G. Since a minimum edge cover can be found by greedily augmenting a maximum matching in G_{med}, this can be done in $O(\sqrt{|V(G_{\mathrm{med}})|} \cdot |E(G_{\mathrm{med}})|)$ time [13]. To complete the proof, it remains to note that $V(G_{\mathrm{med}}) = E(G)$ and $|E(G_{\mathrm{med}})| = \sum_{v \in V(G)} \deg(v) = 2 \cdot |E(G)|$. □

7 Multi-Angle Cover

In this and the following section, we consider two natural generalizations of the basic angle cover problem. First we consider the *a-angle cover problem* where every vertex v covers a angles, where an angle is (as before) a pair of edges incident to v that are consecutive in the circular ordering around v. We start with a positive result.

Theorem 16 *For even $\Delta > 0$, any graph of maximum degree at most Δ with any rotation system admits an a-angle cover for $a \geq \Delta/2 - \lfloor \Delta/6 \rfloor$, and such an angle cover can be found in linear time.*

Proof: The proof is similar to that of Theorem 11. As in that proof, we can assume that the given graph is connected and Δ-regular.

We again direct the edges of the graph to form a directed Eulerian cycle. To this end, we number the edges incident to each vertex from 0 to $\Delta - 1$ in circular order. For $i = 0, \ldots, \lfloor \Delta/6 \rfloor - 1$, we

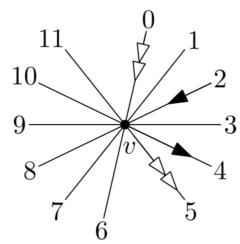


Figure 11: In the proof of Theorem 16, assuming the first edge into v is edge 2, the Euler cycle would continue out on edge 4. When it re-enters v in the $i = 0$ sextet, say on edge 0, it continues out on edge 5 (or 3). Note that no matter on what edge it first re-enters v in the $i = 0$ sextet (directing that edge toward v), there still exists an undirected edge consecutive to edge $e = 4$.

call the group of edges $6i, \ldots, 6i+5$ a *sextet*. In creating the directed cycle, when entering a vertex v, our goal is to exit (directing an outgoing edge) in such a way that we obtain two consecutive outgoing edges in each sextet at v. This proves the theorem since every vertex is then able to cover all of its $\Delta/2$ outgoing edges (and thus all edges in the graph are covered) using at most $\Delta/2 - \lfloor \Delta/6 \rfloor$ angles since in each of the $\lfloor \Delta/6 \rfloor$ sextets, two outgoing edges are covered by a single angle.

The rule we follow to ensure that every sextet contains two consecutive outgoing edges is when an incoming edge to v enters a sextet for the first time, we exit v on an edge e, in the same sextet, that lies between two undirected edges (edges that are not part of the cycle yet). Since the sextet contains six edges, either three consecutive edges of the sextet precede or follow the first incoming edge to the sextet and such an edge e exists. When the cycle next enters v on an edge in this sextet, we exit on one of the remaining undirected edges immediately before or after e. See Fig. 11. □

Theorem 16 shows that as long as the number of angles each vertex can cover is large enough, we can find an angle cover efficiently for any graph with maximum degree Δ, but the bound is not always tight. Theorem 16 only implies that all graphs with maximum degree $\Delta = 4, 6, 8$ have a-angle covers with $a = 2, 2, 3$, respectively. However, for $\Delta = 4$, Theorem 11 shows that $a = 1$ suffices. For $\Delta = 6$, $a = 1$ certainly does not suffice (see Fig. 2a for an example with $\Delta < 6$) so $a = 2$ is optimal. In general, Theorem 17 shows that if $\Delta \geq 4a + 1$ (i.e., $a \leq (\Delta - 1)/4$), the a-angle cover problem is NP-hard. This value of $4a + 1$ comes from the fact that K_{4a+1} is the smallest maximum degree complete graph with n vertices and $2an$ edges, which is useful for the proof. In Theorem 18, we obtain a slight improvement to this for $\Delta = 8$, which implies that the bound $a = 3$ (from Theorem 16) is optimal for $\Delta = 8$.

Theorem 17 *For any $a \geq 1$ the a-angle cover problem is NP-hard even for graphs of maximum degree $4a + 1$.*

Proof: The case of $a = 1$ is proven in Theorem 13. If every vertex can select $a > 1$ angles, we can use the same NP-hardness reduction described in the proof of Theorem 13 but attach $2(a-1)$ adjacent edges, each connected to its own copy of K_{4a+1}, to every vertex $e_j^k(v)$ to force the vertex to "waste" $a - 1$ of its angles on these edges. These edges precede the edge $(e_j^k(v), a_j^k(v))$ in the circular order at vertex $e_j^k(v)$. Now, $\deg(e_j^k(v)) = 2a + 3 < 4a + 1$. For the centre vertex $c(v)$, we similarly attach $2(a - 1)$ edges, each connected to its own copy of K_{4a+1}, to $c(v)$ so that these edges lie between $(c(v), e_1^0(v))$ and $(c(v), e_1^1(v))$. Now, $\deg(c(v)) = 2a + 1 < 4a + 1$.

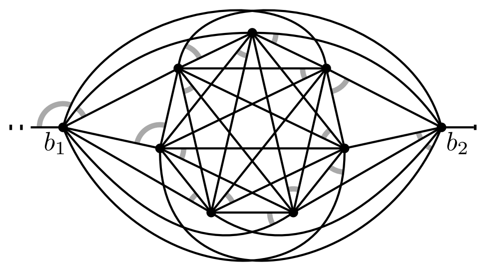


Figure 12: The graph T with a 2-angle cover that covers all edges except the outgoing edge from b_2. A symmetric cover leaves only the outgoing edge from b_1 uncovered.

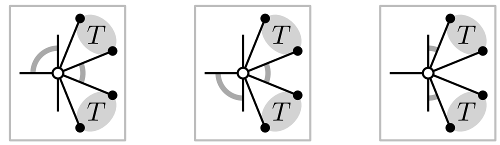


Figure 13: The three cases of $c(v)$ (the white vertex) with the newly added edges to copies of T.

Finally, since the maximum degree in each of these attached K_{4a+1} copies is $4a+1$, the maximum degree of the graph is $4a+1$. □

Theorem 18 *The 2-angle cover problem is NP-hard even for graphs of maximum degree* 8*.*

Proof: The maximum degree of 9 given by the construction described in the proof of Theorem 17 comes from the use of K_9. To decrease the maximum degree to 8, instead of attaching two copies of K_9 to a vertex, we attach one copy of the following graph, T, using two edges. The graph T contains a copy of K_7 and two vertices b_1 and b_2 that are connected to all seven vertices of the copy of K_7. The graph T is attached to an external vertex v (not in T) by edges (b_1, v) and (b_2, v). Thus every vertex in T has degree 8. Including these edges, T contains 37 edges and nine vertices. Since each vertex can cover at most four edges, at least one edge must be covered by v in any valid angle cover. Furthermore, all edges except for one of the external outgoing edges can be covered; see Fig. 12.

For each vertex $e_j^k(v)$ in the original NP-hardness reduction, we attach three new edges, using a copy of T, and another isolated vertex x. As in the high-degree construction, these edges, in the order $(e_j^k(v), x)$, $(e_j^k(v), b_1)$, $(e_j^k(v), b_2)$, directly precede the edge to the degree-1 vertex $(e_j^k(v), a_j^k(v))$ in the circular order at vertex $e_j^k(v)$. The circular order of just these four edges is then $(e_j^k(v), x)$, $(e_j^k(v), b_1)$, $(e_j^k(v), b_2)$, $(e_j^k(v), a_j^k(v))$, so we observe that the two edges connecting to b_1 and b_2 are between edges connected to degree-1 vertices. As a result, $e_j^k(v)$ must "waste" one of its angles on either $(e_j^k(v), b_1)$ or $(e_j^k(v), b_2)$, and hence, it cannot use this angle on any other edge connected to a non-isolated vertex. Now $\deg(e_j^k(v)) = 8$. To the centre vertex $c(v)$, we similarly attach four edges using two copies of T so that their edges lie between $(c(v), e_1^0(v))$ and $(c(v), e_1^1(v))$. Now $\deg(c(v)) = 7$. If $c(v)$ covers two of these new edges, then $c(v)$ cannot cover all three of its original edges, since it can only cover a total of four. Furthermore, assuming that the edges to each copy of T are consecutive in the edge-ordering of $c(v)$, we can cover any two of the original edges alongside one edge from each copy of T. All three cases are depicted in Fig. 13. The full vertex gadget is depicted in Fig. 14. □

A key ingredient in proving the hardness results in Theorems 17 and 18 was the use of a subgraph (such as K_9 or the graph T in Fig. 12) that requires one of its adjacent edges to be covered by a vertex v outside the subgraph. In fact, we can use any graph without an a-angle cover for some rotation system to serve this purpose.

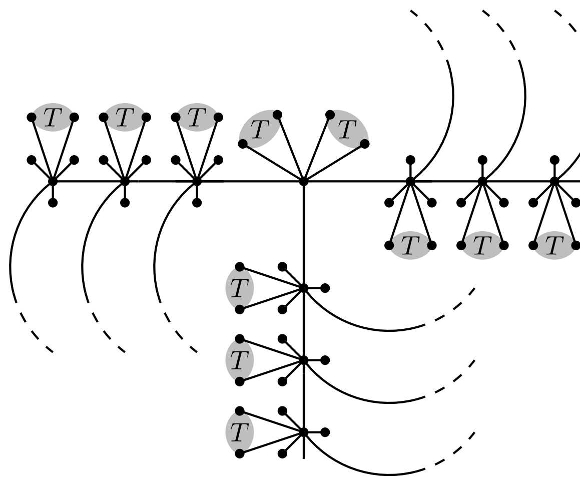


Figure 14: How the graph T is used in the vertex gadget construction.

Theorem 19 *For all $a \geq 1$, if there exists a graph with maximum degree at most $2a+3$ and a rotation system that does not admit an a-angle cover, then the a-angle cover problem for graphs with maximum degree at most $2a+3$ is NP-hard.*

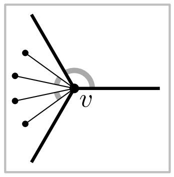
Proof: Fix $a \in \mathbb{N}$. Suppose a graph H has a rotation system without an a-angle cover and has maximum degree at most $2a+3$. We will use a 2-step reduction.

First, we will show that the following variant of the a-angle cover problem is NP-hard: Restricted a-angle Cover Problem: Given a graph G of maximum degree at most $2a+3$, with a rotation system, and a subset S of degree-1 vertices in G, does the graph admit an a-angle cover where none of the vertices in S provide angles?

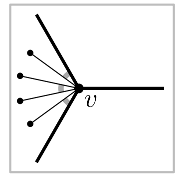
To show NP-hardness for this variant of the problem, we reduce from the 1-angle cover problem for graphs of maximum degree 5, which was shown to be NP-hard in Theorem 13. In the second step, we reduce the restricted variant to the standard a-angle cover problem, using H to create a gadget.

Start with an input to the 1-angle cover problem, i.e., a graph G of maximum degree at most 5. Create a new graph G' containing G as an induced subgraph with a consistent rotation system. For each vertex $v \in V(G)$, there are $2(a-1)$ extra adjacent degree-1 vertices in G', all consecutive in the circular order around v. Let $S = V(G') \setminus V(G)$. Note that in this construction, the maximum degree of G' is at most $5 + 2(a-1) = 2a+3$ An example of this construction for a degree-3 vertex can be seen in Fig. 15.

Assume that G admits an angle cover. Each vertex $v \in V(G') \setminus S = V(G)$ is forced to "waste" $a-1$ of its angles on the edges connecting to vertices in S, in a very similar manner to the proof of Theorem 17. Let the sequence $u_1, s_1, \ldots, s_{2(a-1)}, u_2$ be a consecutive subsequence of vertices in circular order adjacent to v, where $s_1, \ldots, s_{2(a-1)}$ are the vertices in S. We can now construct a restricted a-angle cover of G'. If the angle used for v in the angle cover of G covered edges to u_1 and u_2, then the a angles covered should be $(u_1, s_1), (s_2, s_3), \ldots, (s_{2(a-1)}, u_2)$ as in Fig. 15b for the particular angle $\{x, y\}$ shown. Otherwise, if the angle for v in the angle cover of G covered edges to x and y, such that $\{x, y\} \neq \{u_1, u_2\}$, then in the a-angle cover of G', v should cover the angles $(x, y), (s_1, s_2), (s_3, s_4), \ldots, (s_{2(a-1)-1}, s_{2(a-1)})$ as in Fig. 15a. In such a covering, all the



(a)



(b)

Figure 15: Given a degree-3 vertex v, the reduction to the RESTRICTED a-ANGLE COVER PROBLEM for $a = 3$ adds the small vertices and thin edges to the graph, and the small vertices to S. Subfigures (a) and (b) show the two possible ways that angles of v can cover the thin edges. Note that this does not show all the possible ways that thick edges may be covered.

edges present in G are covered, as are all the edges in $E(G') \setminus E(G)$, and so this is a restricted a-angle cover.

Now, instead assume that G' admits a restricted a-angle cover, and we will construct an angle cover of G. Again let $u_1, s_1, \ldots, s_{2(a-1)}, u_2$ be the consecutive subsequence of vertices in circular order adjacent to v, where $s_1, \ldots, s_{2(a-1)} \in S$. Since each vertex $v \in V(G') \setminus S$ has $2(a-1)$ adjacent vertices in S, at least $a-1$ angles of v in the a-angle cover must be used in order to cover these edges. If $a-1$ angles each cover two edges to vertices in S, then the remaining angle must also be a valid angle in G, and that is the angle we use at v in the angle cover of G. Otherwise, we use the angle at v to cover the edges to u_1 and u_2, which are consecutive around v in G. This suffices to cover all edges in G.

We now use the restricted a-angle cover problem with maximum degree $2a+3$ to show that the standard a-angle cover problem with the same maximum degree is NP-hard. Let D be the set of edges left uncovered by an a-angle assignment covering the maximum number of edges in H. Let G be an input to the restricted a-angle cover problem with a rotation system and a subset S of its degree-1 vertices. We construct a graph G' containing $2|D|$ copies of G as induced subgraphs, as well as $|S|$ copies of $H-D$ as induced subgraphs. Each induced subgraph also has a consistent rotation system. In each of the subgraphs of $H-D$, we mark the *gaps* in the circular order around a vertex where an edge in D was removed. In total, there are $2|D| \cdot |S|$ such gaps. For each vertex u of the $2|D| \cdot |S|$ degree-1 vertices in the copies of G, we choose a corresponding gap, and identify u with the vertex v that has the gap so that the single edge adjacent to u fills the position of the gap in the circular order around v. It is easy to check that G' has maximum degree at most $2a+3$.

Since D was generated by the edges not covered in some a-angle assignment covering a maximum number of edges in H, it must be the case that any a-angle cover of G' induces an a-angle cover in $H-D$ covering the exact same set of edges. In particular, in any a-angle cover, the vertices in the $H-D$ subgraphs cannot also cover edges outside of the $H-D$ subgraph, since this would induce an a-angle assignment covering a larger number of edges. Thus, all the vertices outside the induced subgraphs of $H-D$, which correspond exactly to the vertices in induced subgraphs of $G \setminus S$, must cover all the edges not in an induced subgraph of $H-D$. This is equivalent to asking if G contains a restricted a-angle cover for S. $\square$

8 Wide-Angle Cover

Another obvious generalization of angle covers is to consider "wider" angles. In the *m-wide angle cover problem* every vertex v can cover m consecutive edges in their circular order around v.

Theorem 20 *For $m \geq 3$, the m-wide angle cover problem is NP-hard even for graphs of maximum degree $3m-3$.*

Proof: We can again modify the NP-hardness reduction described in the proof of Theorem 13. We now attach $m-1$ separator edges to $e_j^k(v)$ in place of $(e_j^k(v), a_j^k(v))$ and $m-1$ separator edges to $e_j^k(v)$ in place of $(e_j^k(v), b_j^k(v))$ for all $v \in V$, $k \in \{0,1,2\}$, and $j = 1, \ldots, \deg(v)$. In addition, we attach $m-2$ separator edges to $c(v)$ between $(c(v), e_1^k(v))$ and $(c(v), e_1^{k+1}(v))$ for all $v \in V$ and $k \in \{0,1,2\}$.

As in the construction of Theorem 13, a consecutive set of $m-1$ separator edges prevents $e_i^k(u)$ from covering both $(e_i^k(u), e_{i-1}^k(u))$ and either $(e_i^k(u), e_{i+1}^k(u))$ or $(e_i^k(u), e_j^k(w))$ (where $E_i(u) = E_j(w) = (u,w)$). In addition, at $c(u)$ the three sets of $m-2$ separator edges prevent $c(u)$ from covering more than two of the edges $(c(u), e_1^k(u))$ with $k \in \{0,1,2\}$. □

9 Open Problems

The angle cover problems that we introduced in this paper all assume that we are given a graph together with a rotation system of this graph. Figures 3 and 5 show planar graphs where the existence of an angle cover depends on the chosen planar embedding. This leads us to pose the following question: Can we test efficiently whether a given planar graph has *some* planar embedding that admits an angle cover? Note that when this is true is exactly when we can apply Theorem 6.

This question is even interesting for the special case of planar Laman graphs. We have seen that there is a plane Laman graph (actually a 2-tree) that does not admit an angle cover; see Fig. 5 (left). But does every planar Laman graph have *some* planar embedding that admits an angle cover (which would allow us to apply Theorem 6)? At least for the planar Laman graph in Fig. 5 this is true; see Fig. 5 (right). Generally, every 2-tree has such an embedding: choose a stacking order such that the new vertex can always be placed in the outer face.

For even Δ, we have shown that every maximum-degree-Δ graph with a rotation system admits an a-angle cover for $a \approx \lceil \Delta/3 \rceil$; see Theorem 16. This is optimal for $\Delta = 6$ and $\Delta = 8$ (see Theorem 18). For $\Delta = 4$, however, we need only *one* angle per vertex, so we pose the following questions:

For even Δ, does every graph of maximum degree Δ with a rotation system admit an a-angle cover for $a \approx \lceil c\Delta \rceil$, where $c < 1/3$? What about graphs of maximum degree Δ if Δ is odd? Figure 16 gives a more detailed picture of our knowledge with respect to the complexity of the a-angle cover problem.

Finally, is there a kind of relaxation of the angle cover problem such that the 2-blowup of a graph G has isomorphic thickness 2 if and only if G has a relaxed angle cover?

Acknowledgements We thank the anonymous reviewers for their assistance in clarifying our presentation and for their observation that our result for 2-blowups (Theorem 6) extends to the isomorphic thickness of the k-blowup of plane graphs for values of k greater than 2.

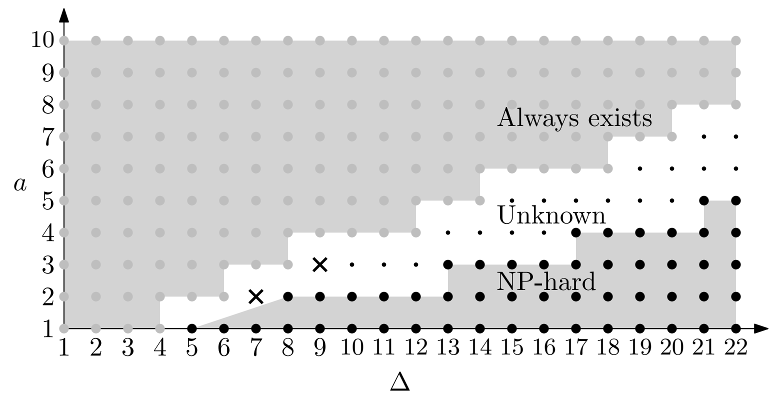


Figure 16: A diagram showing the variants of the a-angle problem for given maximum degree Δ that have known complexity (gray areas). Any integral point (Δ, a) in the upper gray area or on its boundary corresponds to a case for which an a-angle cover always exists by Theorems 11 and 16. Any integral point in the lower gray area or on its boundary corresponds to a case for which the a-angle cover problem is NP-hard by Theorems 17 and 18. The two crosses ($\times$) indicate the cases for which only Theorem 19 applies: in these cases either the problem is NP-hard, or an angle cover always exists. The complexity of all remaining integral points is unknown.

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