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The Stub Resolution of 1-planar Graphs

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Abstract. The resolution of a drawing plays a crucial role when defining criteria for its quality. In the past, grid resolution, edge-length resolution, angular resolution and crossing resolution have been investigated. In this paper, we investigate the *stub resolution*, a recently introduced criterion for nonplanar drawings. Intersection points divide edges into parts, called stubs, which should not be too short for the sake of readability. Thus, the stub resolution of a drawing is defined as the minimum ratio between the length of a stub and the length of the entire edge, over all the edges of the drawing. We consider 1-planar graphs and we explore scenarios in which near optimal stub resolution, i.e., arbitrarily close to $\frac{1}{2}$, can be obtained in drawings with zero, one or two bends per edge, as well as further resolution criteria, such as angular and crossing resolution. In particular, our main contributions are as follows: (i) Every IC-planar graph, i.e., every 1-planar graph with independent crossing edges, has a straight-line drawing with near optimal stub resolution; (ii) Every 1-planar graph has a 1-bend drawing with near optimal stub resolution.

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(a) (b)

Figure 1: Two RAC drawings of the same 1-planar graph. The drawing in (b) has better stub resolution (equal to $\frac{1}{2}$) than the one in (a).

1 Introduction

The question of drawing graphs with high resolution is one of the most prominent when it comes to better understanding of a diagram. We quote from an early graph drawing tutorial by Cruz and Tamassia (1994): "Display devices and the human eye have only finite resolution". This viewpoint inspired the convention to use an underlying integer grid for the drawings, which guarantees a certain minimum distance between any two vertices, as well as criteria like the ratio between the shortest and the longest edge (known as *edge-length resolution*) [37].

The *angular resolution* of a drawing is the minimum angle that occurs at a vertex (often called *vertex angle*). This branch has been started by Formann et al. [27]. Important contributions on planar graphs have been made by Malitz and Papakostas [38], and by Duncan and Kobourov [25]. An early work by Di Battista and Vismara [19] characterized the realizability of planar straight-line drawings for a given set of vertex angles and lead the way for the minimization of the largest vertex angle. Special graph classes, e.g., trees, allow more direct approaches to get a good angular resolution, especially with respect to the used area (see, e.g., [24, 28]). From there, the research line on the planar slope number developed, where only a fixed set of slopes can be used to draw the edges of a graph. While this approach does not lead to good angular resolution for planar straight-line drawings [35], Angelini et al. [3] showed how to compute planar drawings with one bend per edge using a set of slopes that guarantees asymptotically optimal angular resolution.

Huang et al. [30] experimentally showed the detrimental effect on readability when crossing angles are "sharp". This, together with the seminal paper by Didimo et al. [21], started a line of research on nonplanar graph drawings where sharp angles are forbidden, i.e., with good *crossing resolution*. The ultimate goal are *right-angle crossing (RAC)* drawings, where crossing edge segments always form right angles. Angelini et al. [4] studied the effect of drawing planar graphs with large or right crossing angles. Di Giacomo et al. [20] considered RAC drawings on 2 parallel lines. Most notably on the RAC model are the results on maximum edge density when allowing zero, one or two bends per edge [7, 21], as well as the NP-hardness result by Argyriou et al. [5].

Vertex and crossing resolutions have been considered together only for very restricted types of graphs and drawings [6, 22], and more recently in terms of edge density and recognition [1].

In this paper we investigate a recent criterion for nonplanar drawings, called *stub resolution* [33]. Intersection points divide edges into parts, called *stubs*, which should not be too short to guarantee adequate readability. Hence, the stub resolution of a drawing is defined as the minimum ratio between the length of the shortest stub of an edge and the length of the entire edge. As we indicate in Figure 1, not only is the crossing resolution helpful for the sake of readability, but good stub resolution is essential as well. An earlier research direction in the same spirit is *partial edge* *drawing* (see, e.g., [9, 11, 12, 13, 14, 15, 16, 31]), which follows the idea that for the effective display of a crossed edge, only long enough end segments are important, while the crossings might lead to visual clutter and could be omitted. Similar optimization problems are also studied in [26].

Contribution and paper organization. After a formal introduction of the model and an overview of our approach (Section 2), we consider 1-planar graphs as a meaningful graph class where crossings are naturally involved. A graph is 1-*planar* if it can be drawn with at most one crossing per edge (refer to [36] for a survey). This family of graphs is among the most investigated ones in the rapidly growing literature about graph drawing beyond planarity [23]. A natural question is whether 1-planar graphs admit 1-planar drawings with bounded stub resolution. As a preliminary result, we proved in [33] that a class of maximal 1-planar graphs allows straight-line 1-planar drawings with stub resolution $\frac{1}{5}$.

Our contribution is as follows.

1. We first study 1-planar straight-line drawings (Section 3), and we show that stub resolution equal to $\frac{1}{2}$ (i.e., optimal) cannot be always achieved, while stub resolution arbitrarily close to $\frac{1}{2}$ is possible for *IC-planar graphs*, i.e., for 1-planar graphs with independent crossings.

2. We then study 1-planar drawings with at most one bend per edge (Section 4), and we show that stub resolution arbitrarily close to $\frac{1}{2}$, or angular resolution that is lower bounded by a function of the maximum vertex degree of the graph (similar as the one in [38]) is always possible. Note that the study of 1-bend drawings is also motivated by the fact that there exist 1-planar graphs that do not admit a 1-planar straight-line drawing [29, 40], while 1-planar 1-bend RAC drawings exist for all 1-planar graphs [8, 17].

3. Finally, we study 1-planar drawings with at most two bends per edge (Section 4), and we show that stub resolution arbitrarily close to $\frac{1}{2}$ and right-angle crossings can be achieved simultaneously.

2 Preliminaries and Proof Strategy

Drawings and embeddings. We consider simple undirected graphs. A *drawing* Γ of a graph G maps the vertices of G to distinct points in the plane and the edges of G to simple Jordan arcs connecting their endpoints. Γ is *planar* if no edges cross, and *1-planar* if each edge is crossed at most once. Γ is *IC-planar* if it is 1-planar and there are no two crossed edges that share a vertex (i.e., the set of crossing edges is a matching in G). A graph G is *planar* (1-*planar*, *IC-planar*) if it admits a planar (respectively, 1-planar, IC-planar) drawing. In the following, we shall not distinguish between a vertex (an edge) of G and its corresponding point (arc) in Γ.

A planar drawing Γ of a graph G induces an *embedding*, which is the class of topologically equivalent drawings. In particular, an embedding specifies the regions of the plane, called *faces*, whose boundary consists of a cyclic sequence of edges. The unbounded face is called the *outer face*. For a 1-planar drawing, we can still derive an embedding by allowing the boundary of a face to consist also of edge segments from a vertex to a crossing point. A graph with a given planar (1-planar, IC-planar) embedding is called a *plane* (*1-plane, IC-plane*) graph. A *kite* $K = \{a, b, c, d\}$ is a graph isomorphic to K_4 with an embedding such that there is a crossing-free 4-cycle $\langle a, b, c, d\rangle$, and the two edges (a, c) and (b, d) cross inside this cycle; see Figure 2(a). Let G be a 1-plane graph, and let $K = \{a, b, c, d\}$ be a kite such that $K \subseteq G$. K is an *empty kite*, if there is no vertex of G inside the 4-cycle $\langle a, b, c, d\rangle$. An *outer kite* $K = \{a, b, c, d\}$ is a graph isomorphic to K_4 with an

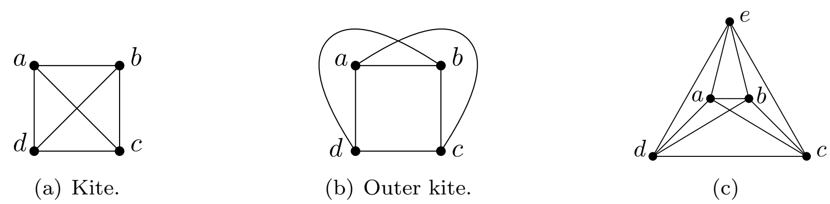


Figure 2: (a)-(b) Crossing configurations. (c) Unique 1-planar embedding of K_5.

embedding such that there is a crossing-free 4-cycle $\langle a, b, c, d\rangle$, and the two edges (a, c) and (b, d) cross outside this cycle; see Figure 2(b).

Drawing resolutions. A drawing Γ of a graph G is *straight-line* if all edges are segments, while it is *b-bend* ($b > 0$) if each edge is a polyline with at most $b + 1$ line segments. Drawing Γ is *right-angle crossing (RAC)* if the angles at any crossing point are right angles. The *angular resolution* of Γ is the minimum angle that any two incident edges form at a vertex. Note that for a graph with maximum vertex degree Δ, the angular resolution cannot be greater than $\frac{2\pi}{\Delta}$. We recall the following result concerning the angular resolution of planar drawings. A planar graph is *triangulated* if it is maximal, i.e., it has $3n - 6$ edges over n vertices.

Lemma 1 (Theorem 2.2 in [38]) *Every triangulated planar graph with maximum vertex degree Δ admits a planar straight-line drawing with angular resolution $\Omega(0.15^\Delta)$.*

We shall assume (and ensure) that no more than two edges cross at any point of a drawing Γ. An edge e of Γ that is crossed k times is divided into $k+1$ parts called *stubs*. Let l_e and s_e be the length of e and of its shortest stub, respectively. The *stub resolution* of e is $\mathrm{sr}_e = \frac{s_e}{l_e}$. The *stub resolution* of Γ is the minimum stub resolution over all edges of Γ, i.e., $\mathrm{sr}_\Gamma = \min_{e\in\Gamma} \mathrm{sr}_e$. The next observation follows immediately.

Observation 1 *A drawing in which the maximum number of crossings per edge is $k \geq 0$ has stub resolution at most $\frac{1}{k+1}$.*

3 Straight-line Drawings

We first show that K_5 has no 1-planar straight-line drawing with optimal stub resolution, and so this is the case for any 1-planar graph having K_5 as a subgraph.

Observation 2 *Let Γ be a 1-planar straight-line drawing of a graph G with $\mathrm{sr}_\Gamma = \frac{1}{2}$, and let (a, c) and (b, d) be a pair of edges crossing in Γ. Then vertices a, b, c, d form a parallelogram in Γ.*

Lemma 2 *Let Γ be a straight-line drawing of K_5. Then $\mathrm{sr}_\Gamma < \frac{1}{2}$.*

Proof: As K_5 is not planar, Γ contains at least one crossing. Let $k > 0$ be the maximum number of crossings per edge in Γ. If $k \geq 2$, the statement follows by Observation 1. Suppose that $k = 1$, and assume for a contradiction that $\mathrm{sr}_\Gamma = \frac{1}{2}$. As shown by Suzuki[39], there is a unique 1-planar embedding of K_5 (up to the choice of the outer face), which is depicted in Figure 2(c). Note that vertex e is drawn outside the quadrilateral Q representing the 4-cycle $\langle a, b, c, d\rangle$ in Γ.

Also, to realize the edges incident to e as straight-line segments, the quadrilateral Q cannot be a parallelogram, which contradicts Observation 2. □

The next theorem proves that IC-planar graphs can be realized with 1-planar straight-line drawings with worst-case optimal stub resolution. We remark that IC-planar graphs also admit straight-line RAC drawings [10].

Theorem 1 *Every IC-planar graph G has a 1-planar straight-line drawing Γ with stub resolution $\mathrm{sr}_\Gamma = \frac{1}{2} - \varepsilon$, for any fixed $\varepsilon > 0$.*

Proof: If G is a subgraph of K_5, the statement immediately follows as we can use the embedding of Figure 2(c) and draw $\langle a, b, c, d\rangle$ almost as a square (placing vertex e sufficiently far). Hence, we assume that G has at least six vertices. Start from an IC-planar embedding of $G = (V, E)$ and use the transformation by Brandenburg et al. [10, Lemma 1] to obtain a 3-connected planar-maximal IC-plane graph $G' = (V, E')$ with $E \subseteq E'$ such that each pair of crossing edges induces an empty kite (hence there is no outer kite) and all faces are triangles. (Such transformation is based on the rerouting of edges that form so-called B-configurations in the embedding, and on the triangulation of large faces.)

We prove that G' admits a 1-planar straight-line drawing Γ' with stub resolution $\frac{1}{2} - \varepsilon$, with the additional property that the outer face of Γ' is a prescribed triangle T. Removing the edges in $E' \setminus E$ from Γ' cannot decrease the stub resolution of the drawing, and hence the drawing obtained by removing these edges is the desired representation of G.

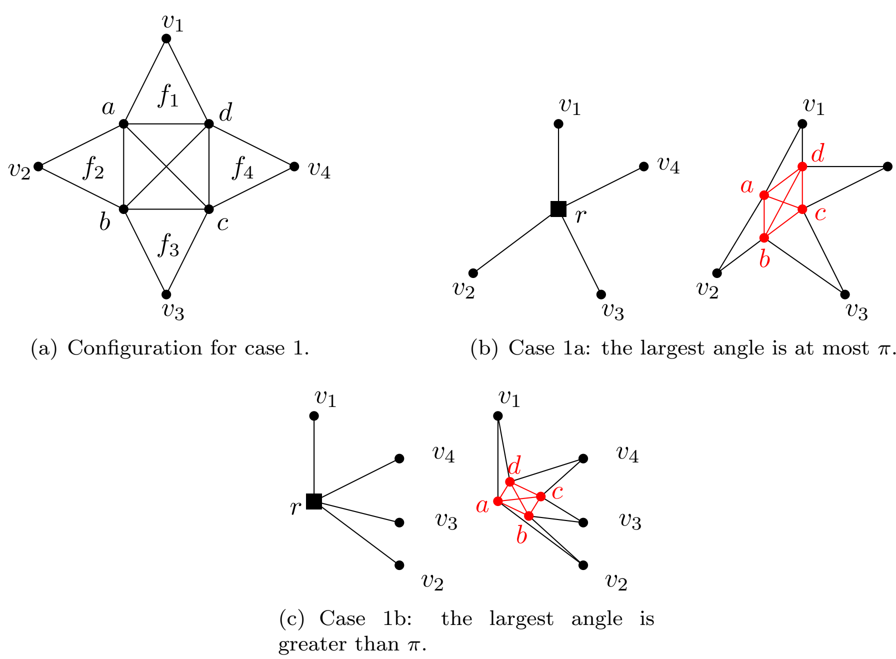
The proof is by induction on the number of empty kites (recall that each kite of G' is empty due to the applied transformation). In the base case G' has no empty kites, thus G' contains no crossings, i.e., it is a plane graph. Then we can apply the algorithm by Chiba et al. [18] to compute a planar straight-line drawing Γ' of G' such that the outer face of G' corresponds to the prescribed triangle T. By induction, if G' has $k \geq 0$ empty kites, then our claim holds. We shall prove that the claim still holds in the case where G' has $k + 1$ empty kites. We first distinguish two cases, based on whether G' contains a separating triangle or not.

CASE A: G' contains a separating triangle $C = \{a, b, c\}$. We claim that the three edges of C are all crossing-free or they can be redrawn (interpreting an embedding as a drawing) to be crossing-free. Suppose for a contradiction that one edge of C, say (b, c), is crossed and it cannot be redrawn without crossings. Observe that in this case no other edge of C is crossed, as otherwise G' would not be IC-plane. Let c_1 and c_2 be two components of $G' \setminus C$ such that c_1 contains the edge that crosses (b, c). Since the other two edges of C are not crossed, c_2 lies completely inside or outside the closed curve defined by C. Consider the face of $c_2 \cup C$ that contains (b, c) and an edge of c_2. Since we cannot reroute edge (b, c) inside this face without creating new crossings, it means that an edge of c_2 is crossed by an edge of another component. Hence there exists a kite merging the two components; contradicting the fact that the two components are distinct. Denote by C_{in} (C_{out}) the subgraph of G' that lies in the interior (exterior) of C. Note that C is the outer face (an empty face) of C_{in} (C_{out}). If C_{out} has no empty kites, then we draw it with the algorithm of Chiba et al. [18] inside the prescribed triangle T. Note that C_{in} contains $k + 1$ empty kites, and it must be drawn inside the triangle defined by C, that is, we can assume that G' corresponds to C_{in}, and that the prescribed triangle T is C. Similarly, if C_{in} has no empty kites, then we assume that G' corresponds to C_{out} and that it must be drawn inside the prescribed triangle T. Once we obtain a drawing of C_{out}, we can again use the algorithm of Chiba et al. [18] for C_{in} with the drawing of C as prescribed outer face. By the above discussion, we can assume that both C_{in} and C_{out} have at least one empty kite. Then by the induction hypothesis C_{out} and C_{in} can be drawn with stub resolution $\frac{1}{2} - \varepsilon$, as desired.

CASE B: G' has no separating triangles. We distinguish two further cases depending on whether there exists an empty kite K such that none of its edges is part of the outer face of G' or not. Note that if an edge of K belongs to the outer face of G', then this edge is not crossed.

CASE B.1: Suppose first that G' contains an empty kite $K = \{a, b, c, d\}$ such that none of its crossing-free edges belongs to the outer face of G'. Let f_1, f_2, f_3, and f_4 denote the faces incident to the crossing-free edges of K (refer to Figure 3(a)). Recall that these faces are triangles, and denote as v_i the vertex of f_i that does not belong to K, for $i = 1, 2, 3, 4$. Some of these vertices can coincide, but no three of them can be the same vertex, otherwise K would be a K_5 in G' and there would be a separating triangle in G' (recall that no edge of K belongs to the outer face of G'). For the same reason, and because G' is IC-plane, any crossing-free edge of K belongs to at most one triangle of G', and this triangle is a face of G' distinct from the outer face. The general idea is that we collapse K into a single vertex r. The derived graph G'' has fewer empty kites than G' and therefore we can obtain a drawing Γ'' of G'' with stub resolution $\frac{1}{2} - \varepsilon$ and straight-line edges, inside the prescribed triangle T. Then, we can reinsert the kite as a parallelogram and connect its vertices to their neighbours with crossing-free straight-line segments; to this aim, we distinguish three main cases, based on whether some of $v_1, \ldots, v_4$ coincide. Note that, in order to reinsert kite K as a parallelogram, it may need to be scaled down to an appropriate size.

Case 1: None of $v_1, \ldots, v_4$ coincide; refer to Figure 3. We distinguish two subcases depending on the largest angle between any two edges at r. Without loss of generality, the largest angle is between rv_1 and rv_2.



(a) Configuration for case 1.

(b) Case 1a: the largest angle is at most π.

(c) Case 1b: the largest angle is greater than π.

Figure 3: Case 1: None of the vertices coincide.

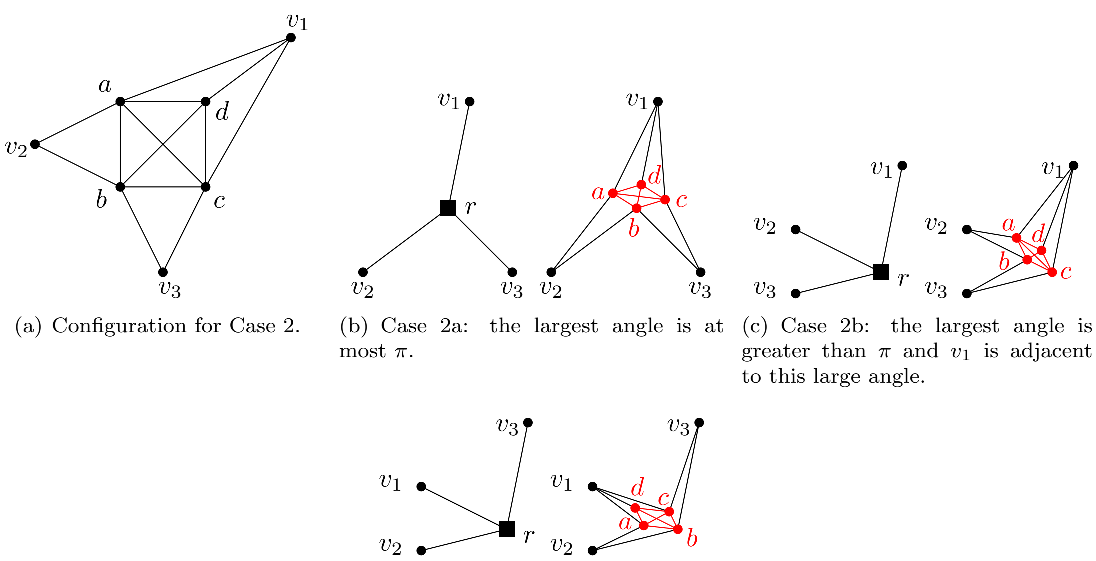
a) The largest angle at vertex r is at most π. Vertex c is placed at the position of r. The edge (c, d) of the parallelogram is placed on the line rv_1, the edge (c, b) on rv_2. The length of (c, d)

and (c, b) can be the same and sufficiently small so that the reinserted parallelogram does not cross any other edge.

b) The largest angle is greater than π. Vertex a is placed at the position of r. Vertex c is placed on the bisector of $\angle v_3 r v_4$. We create a parallelogram so that its edges are parallel to the bisector of $\angle v_1 r v_4$, and the bisector of $\angle v_2 r v_3$ (in particular vertex d is along the bisector of $\angle v_1 r v_4$, while vertex b is along the bisector of $\angle v_2 r v_3$); see Figure 3(c).

In both subcases, we scale the parallelogram, so that it is empty, and so that faces f_1, f_2, f_3 and f_4 are also empty.

Case 2: Two of $v_1, \ldots, v_4$ coincide, say v_1 and v_4; refer to Figure 4. We distinguish three subcases depending on the largest angle between any two edges at r.



(a) Configuration for Case 2.

(b) Case 2a: the largest angle is at most π.

(c) Case 2b: the largest angle is greater than π and v_1 is adjacent to this large angle.

(d) Case 2c: the largest angle is greater than π and v_1 is not adjacent to this large angle.

Figure 4: Case 2: Two of the vertices coincide.

a) The largest angle is at most π; refer to Figure 4(b). Vertex b is placed at the position of r and the diagonal bd of the parallelogram is placed on the line rv_1. We place vertex a along the bisector of $\angle v_1 r v_2$, and vertex c along the bisector of $\angle v_1 r v_3$.

b) The largest angle is greater than π and v_1 is adjacent to this large angle. Then we assume that the largest angle is between rv_1 and rv_3 (the case where the largest angle is between rv_1 and rv_2 is symmetric); refer to Figure 4(c). Then vertex c is placed at the position of r. The edge (b, c) of the parallelogram is placed on the line rv_2 and vertex d along the bisector of $\angle v_1 r v_2$.

c) The largest angle is greater than π and v_1 is not adjacent to this large angle; refer to Figure 4(d). Vertex b is placed at the position of r. The diagonal bd of the parallelogram is placed on the line rv_1. We place vertex a along the bisector of $\angle v_1 r v_2$, and vertex c along the bisector of $\angle v_1 r v_3$.

Similarly as in the first case, in all subcases we may need to sufficiently scale the parallelogram.

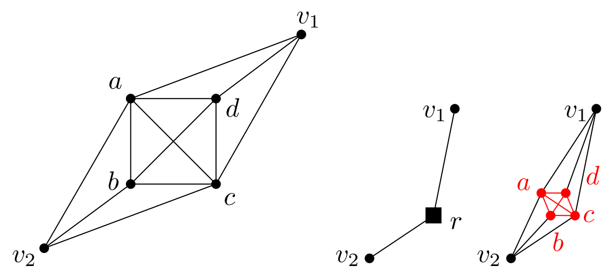


Figure 5: Case 3: Two pairs of vertices coincide.

Case 3: Two pairs of vertices coincide; refer to Figure 5. Vertex c is placed at the position of r. The diagonal ac of the parallelogram is placed on the bisector of $\angle v_1 r v_2$ (inside the angle that is less than π). We place vertex d along the bisector of $\angle v_1 r a$, and vertex b along the bisector of $\angle v_2 r a$. Again we may need to sufficiently scale the parallelogram.

CASE B.2: Suppose now that every empty kite of G' has a non-crossing edge on the outer face of G'. Since every face of G' is a triangle and G' is IC-planar, it follows that in this case G' has only one empty kite $K = \{a, b, c, d\}$ so that an edge of K, say (c, d) belongs to the outer face of G'; see Figure 6(a). Let e be the third vertex of the outer face of G'. Also, let u_1 be the common neighbor of vertices a and d, and u_2 be the common neighbor of vertices b and c. Note that u_1 and u_2 are distinct as otherwise there would be a separating triangle. Then, we contract edge (a, d) to a single vertex w_1 and edge (b, c) to vertex w_2. After removing parallel edges, the derived graph G'' has no kites and remains fully triangulated. Hence, by the base case of our induction, we can compute a planar drawing Γ'' of G'' with vertices w_1, w_2 and e on its outer face.

Suppose that edge (w_1, w_2) is drawn as a horizontal segment (up to a rotation of the drawing); refer to Figure 6(b). We want to draw kite $K = \{a, b, c, d\}$ as an isosceles trapezoid $\mathcal{P}$ with $|ab| < |cd|$ as its two parallel bases and so that a and b are drawn at the points of vertices w_1 and w_2 in Γ''. Consider vertex w_1. In a clockwise traversal of the edges incident at w_1 and starting from edge (w_1, e), first we encounter the neighbors of d in the same order as they appear in G', and once we encounter u_1 the neighbors of a follow in the same order as they appear in G'. We need to ensure that we can redraw the edges of d as straight-line segments and without introducing any crossing in the drawing. Let $C_1(w_1, r_1)$ be a circle with center w_1 and radius r_1, for a value of r_1 that is sufficiently small as explained below; refer to Figure 6(c). Consider the sector of C_1 that is bounded above by the line through w_1 and w_2 and the line through w_1 and u_1 (gray shaded in Figure 6(c)). We choose r_1 to be sufficiently small such that by drawing vertex d onto (the middle of) the arc of this sector, we have that d can be connected to all its neighbors in G' with straight-line segments without introducing any new crossings (blue thick edges in Figure 6(c)). Observe that such a choice is always feasible because as r_1 decreases, the difference between the slopes of the edges incident to d, and the slopes of the edges incident to w_1 also decreases. A similar argument holds for vertex w_2. We draw a circle $C_2(w_2, r_2)$, where r_2 is again sufficiently small such that there exists an arc of C_2 where we can draw vertex c and connect c to its neighbors in G' without introducing any crossings. Let $r < \min\{r_1, r_2\}$ and let ϕ be the smaller angle of the two arcs (on C_1 and C_2). We draw an isosceles trapezoid $\mathcal{P}$ so that $|ab| < |cd|$ are its two parallel bases, and its base angles are equal to $\frac{\phi}{2}$; refer to Figure 6(d). We draw edges (a, c) and (b, d) as straight-line segments and let m be their crossing point. Since $\mathcal{P}$ is isosceles, we have that

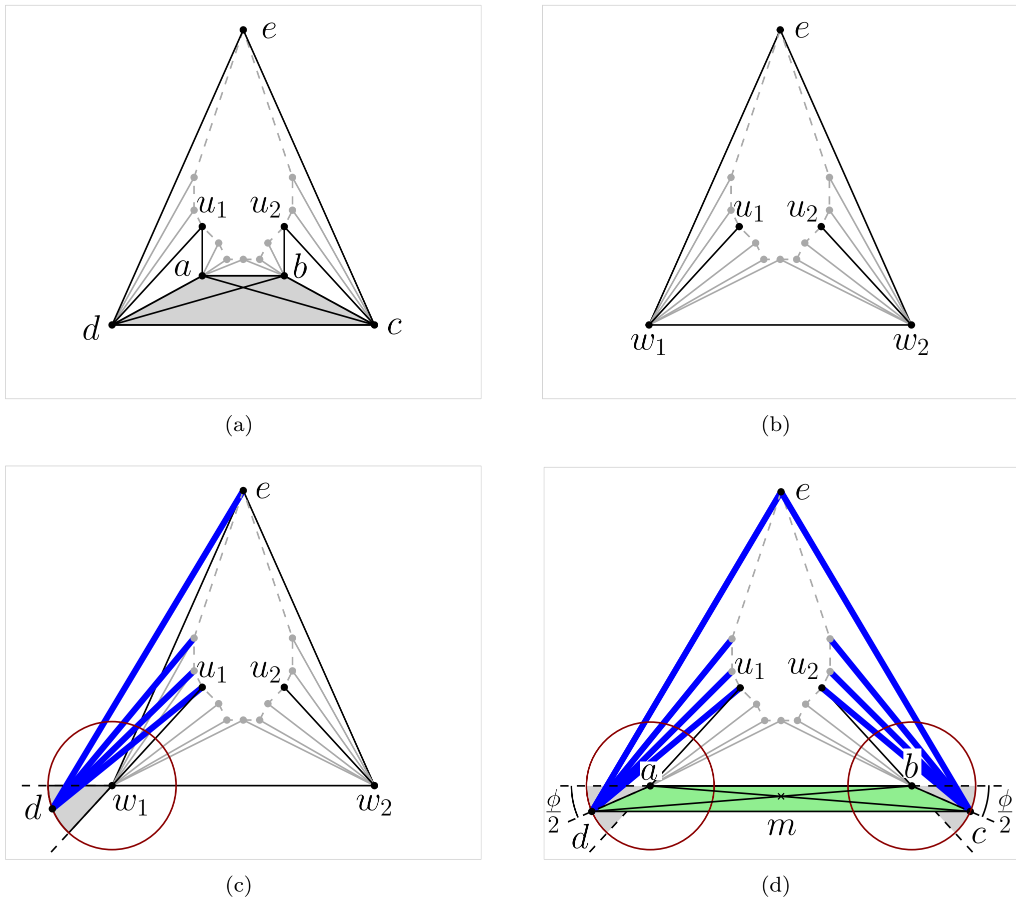


Figure 6: Illustration for **CASE B.2**.

$$\frac{|am|}{|mc|} = \frac{|bm|}{|md|} = \frac{|ab|}{|cd|}.$$

We want to prove that by appropriately choosing the radius r, the stub-resolution of edges (a, c) and (b, d) is at least $\frac{1}{2} - \varepsilon$. Note that $|ab|$ is fixed, while $|cd|$ depends on the choice of r, namely $|cd| = |ab| + 2r \cos\frac{\phi}{2}$. By setting $r = \frac{2\varepsilon'|ab|}{(1-2\varepsilon')\cos\frac{\phi}{2}}$ for some $\varepsilon' \leq \varepsilon$, with some manipulations we obtain $|ab| = |cd|\frac{1-2\varepsilon'}{1+2\varepsilon'}$, and hence the stub-resolution is equal to $\frac{1}{2} - \varepsilon' \geq \frac{1}{2} - \varepsilon$, as desired. On the other hand, by choosing ε' small enough we can ensure that $r < \min\{r_1, r_2\}$ (observe that r decreases as ε' decreases). This concludes the proof of Theorem 1. □

4 Polyline Drawings

While there exist 1-planar graphs that do not admit a 1-planar straight-line drawing [29, 40], every 1-planar graph has a 1-planar 1-bend RAC drawing [8, 17]. This section shows that for

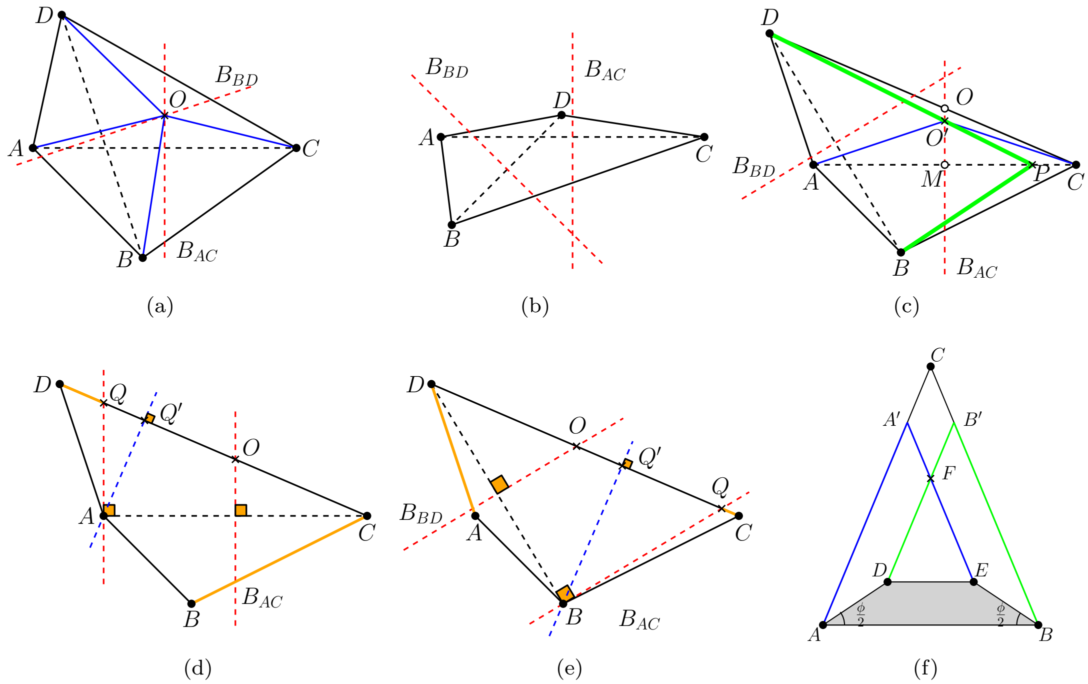


Figure 7: (a)–(e) Configurations of Lemma 3. (f) Configuration of Lemma 4.

1-planar 1-bend drawings, it is possible to optimize also the stub-resolution (Theorem 2), and the angular resolution (Theorem 3). In particular, the main contribution of this section is the following theorem.

Theorem 2 *Every 1-planar graph has a 1-planar 1-bend drawing Γ with stub resolution $\mathrm{sr}_\Gamma \geq \frac{1}{2} - \varepsilon$, for any fixed $0 < \varepsilon < \frac{1}{2}$. Furthermore all crossing-free edges are drawn straight-line.*

The proof is based on a constructive argument that uses the next two technical lemmas as building blocks.

Lemma 3 *Let $K = \{a, b, c, d\}$ be an empty kite and let $\mathcal{P}$ be a convex polygon with four corners. There exists an embedding-preserving drawing Γ of K such that: i) $\mathrm{sr}_\Gamma = \frac{1}{2}$; ii) Vertices $\{a, b, c, d\}$ are placed at the corners of $\mathcal{P}$; iii) The two crossing edges (a, c) and (b, d) are drawn with at most one bend each, while the crossing-free edges are straight-line; iv) The bend point of one of the two crossing edges is at the crossing point.*

Proof: Let $\{A, B, C, D\}$ be the corners of $\mathcal{P}$. We start by placing vertices a, b, c, d at corners A, B, C, D respectively, and draw the crossing-free edges of K on the boundary of $\mathcal{P}$ as straight-line segments. Let B_{AC} and B_{BD} be the perpendicular bisectors of the diagonals AC and BD, respectively. We consider two cases depending on whether B_{AC} and B_{BD} cross in the interior of $\mathcal{P}$ or not.

If B_{AC} and B_{BD} cross in the interior of $\mathcal{P}$ at point O (refer to Figure 7(a)), we draw edge (a, c) and edge (b, d) with a bend at point O. Since $O \in B_{AC}$ and $O \in B_{BD}$, we have that $|AO| = |OC|$

and $|BO| = |OD|$. Hence (a,c) and (b,d) cross at O with stub resolution equal to $\frac{1}{2}$. Furthermore, the bend of both crossing edges is at their crossing point as claimed.

Suppose now that B_{AC} and B_{BD} cross in the exterior of $\mathcal{P}$ (note that they cannot be parallel because P is convex); refer to Figure 7(c). In the following we prove that we can draw edges (a,c) and (b,d) with one bend each so that: (i) the bend point, say O, of one of the two edges, say (a,c), is along its bisector B_{AC}, and (ii) edge (b,d) crosses (a,c) at O with stub resolution $\frac{1}{2}$. For the sake of contradiction, suppose that this is not true. W.l.o.g. we can assume that bisector B_{AC} separates vertices A and D from vertex C (note that we do not make any assumption about the relative position of vertex B and bisector B_{AC}). We claim that we can also assume that B_{BD} separates B and C from D as in Figure 7(c). Suppose momentarily that this is not true, i.e., B_{BD} separates B from D and C; see Fig 7(b). This case is symmetric to the previous one, if we rename points $\{A,B,C,D\}$ as $\{B,C,D,A\}$.

Hence we have the configuration of Figure 7(c). This implies that for any point X of B_{AC} in the interior of $\mathcal{P}$, bisector B_{BD} separates X and B from D and $|XB| < |XD|$ holds. Similarly, for any point X' of B_{BD} in the interior of $\mathcal{P}$ it is $|X'A| < |X'C|$. Furthermore, B_{AC} and B_{BD} both cross the boundary edge CD of $\mathcal{P}$.

Consider first the edge (a,c). Let O be the crossing point of B_{AC} with CD and M its crossing point with AC. For any point $O' \in MO$ we draw edge (b,d) as follows: we start from point D with a straight line through point O' until we cross diagonal AC, we add a bend point, say P, on AC and continue with a straight line up to B (see the thick green edge in Figure 7(c)). If $|DO'| < |O'P| + |PB|$, we have that the midpoint of edge (b,d) is not on DO'. As we move the bend point of (b,d) from P towards O', the midpoint of (b,d) also changes, and when the bend point coincides with O' the midpoint of (b,d) is on DO' (since $|DO'| > |O'B|$). Hence, one can find a bend-point on $O'P$, so that the midpoint of (b,d) is O' and the lemma holds.

Hence, we can assume that $|DO'| > |O'P| + |PB|$ for any point $O' \in MO$. For $O' = O$ the bend point P coincides with C and we have $|DO| > |OC| + |CB|$. We draw the parallel of B_{AC} through point A that crosses the line defined by points C and D at point Q; refer to Figure 7(d). We claim that point Q is between points C and D. Since triangle $\{A,C,Q\}$ is similar to triangle $\{M,C,O\}$, we have $\frac{|QC|}{|OC|} = \frac{|AC|}{|MC|} = 2$. This implies that $|QO| = |OC|$, and from the previous inequality $|DO| > |QO|$, i.e., Q is between points C and D as claimed. In particular we have that $|DO| > |QO| + |CB| \Rightarrow |DQ| > |CB|$. We also draw the perpendicular line of CD through point A crossing CD at point Q'. From the orthogonal triangles $\{A,C,Q'\}$ and $\{A,C,Q\}$ we have $|CQ'| < |AC|$ and $|AC| < |CQ|$. Hence $|CQ'| < |CQ| \Rightarrow |DQ'| > |DQ|$. Considering the orthogonal triangle $\{A,D,Q'\}$ it is $|AD| > |DQ'|$. Combining the above:

$$|AD| > |DQ'| > |DQ| > |CB| \Rightarrow |AD| > |CB|. \tag{1}$$

Arguing similarly for edge (b,d), we can either conclude that the lemma holds or that $|CB| > |AD|$ as shown in Figure 7(e), contradicting Equation 1. □

Lemma 4 *Let $K = \{a,b,c,d\}$ be an outer kite, let $T = \{A,B,C\}$ be an isosceles triangle, and let $0 < \varepsilon < \frac{1}{2}$. There exists an embedding-preserving drawing Γ of K such that: i) $\mathrm{sr}_\Gamma = \frac{1}{2} - \varepsilon$; ii) Vertices $\{a,b,c,d\}$ are placed at the corners of a trapezoid $\mathcal{P}$ such that its larger base coincides with AB, and its smaller base is inside T; iii) The two crossing edges (a,c) and (b,d) are drawn with at most one bend each, while the crossing-free edges are straight-line.*

Proof: Suppose that T is drawn so that its base AB is horizontal, and let ϕ be the value of its base angles. We draw an isosceles trapezoid $\mathcal{P} = \{A,B,D,E\}$ so that $DE < AB$ are its two parallel bases, and $\angle B, A, D = \angle A, B, E = \frac{\phi}{2}$; refer to Figure 7(f). Vertices a, b, c, d of K are placed on the corners A, B, E, D of $\mathcal{P}$ respectively, so that uncrossed edges of K are drawn on the boundary of $\mathcal{P}$ as straight-line segments. In order to draw edge (a, c) we start from point E parallel to BC until we cross AC at point A', we bend at A' and follow $A'A$ up to point A. Edge (b, d) is drawn symmetrically, and let F be the crossing point of (a, c) and (b, d). Since triangle $T' = \{D, E, F\}$ is similar to triangle T, the stub resolution of (a, c) and (b, d) is the same. Now edge (a, c) has two stubs: the first one consists of segments AA' and $A'F$, and the second one only of segment FE, where $|AA'| + |A'F| > |FE|$. We want to prove that by appropriately choosing the height h of trapezoid $\mathcal{P}$, the stub-resolution of (a, c) is equal to $\frac{1}{2} - \varepsilon$. Since $|A'F| = |A'C|$ the stub-resolution equals $\frac{|FE|}{|FE|+|AC|}$. As $\cos\phi = \frac{|DE|}{2|FE|} = \frac{|AB|}{2|AC|}$ we have $\frac{|AC|}{|FE|} = \frac{|AB|}{|DE|}$. Then the stub-resolution equals $\frac{|FE|}{|FE|+|AC|} = \frac{1}{1+\frac{|AC|}{|FE|}} = \frac{1}{1+\frac{|AB|}{|DE|}} = \frac{|DE|}{|DE|+|AB|}$. For $h = \frac{2\varepsilon|AB|}{1+2\varepsilon}\tan\frac{\phi}{2}$, we have that $|DE| = |AB|\frac{1-2\varepsilon}{1+2\varepsilon}$, and stub-resolution is equal to $\frac{1}{2} - \varepsilon$, therefore completing the proof. □

We first describe how to construct drawings for 3-connected 1-planar graphs, and then extend our technique to all 1-planar graphs. We assume an embedding is given in input, although our technique may need to change it.

Lemma 5 *Every* 3*-connected 1-plane graph G has a* 1*-planar* 1*-bend drawing* Γ *with* $\mathrm{sr}_\Gamma = \frac{1}{2}$*, except for at most one pair of crossing edges whose stub resolution is* $\frac{1}{2} - \varepsilon$*, for any fixed* $0 < \varepsilon < \frac{1}{2}$*. All crossing-free edges are drawn straight-line.*

Proof: After possibly augmenting G with crossing-free edges and changing its embedding, we may assume that all pairs of crossing edges of G induce an empty kite, except for at most one pair of crossing edges that are part of the outer face and form an outer kite [2].

Let G' be the plane graph obtained from G by removing all pairs of crossing edges. We say that a quadrangular face $f = \{u, w, v, z\}$ of G' is *marked*, if (u, v) and (w, z) are two crossing edges of G. We first compute a straight-line drawing Γ' of G' by using the algorithm of Chiba et al. [18]. The algorithm in [18] has two main properties. First, it produces a drawing in which all faces are convex. Second, it allows to specify any convex polygon P to represent the outer face of the input graph. We now describe how to specify the outer polygon. If the outer face of G' is not marked, then we can use any convex polygon. Else, the outer face of G' is the 4-cycle of an outer kite K of G. In this case we let T be any isosceles triangle, and we apply Lemma 4 to obtain a drawing of K. This drawing fixes a trapezoid P for the four vertices of K, which we use as input polygon. It remains to show how to reinsert all edges in $G \setminus G'$ that belong to a marked inner face. For each such face $f = \{u, w, v, z\}$ of Γ', we reinsert the pair of crossing edges (u, v) and (w, z) by applying Lemma 3, where the convex polygon is the drawing of f in Γ'. This concludes the proof. □

Proof for Theorem 2: Bekos et al. [8] proved that a 1-plane graph G can be augmented by adding both vertices and edges, such that the resulting multigraph G^* has the following properties: (i) It is 1-plane; (ii) All faces have length three (and hence all pairs of crossing edges induce an empty kite); (iii) Possible parallel edges are crossing-free and pairwise non-homotopic; (iv) If there is a set of $k > 0$ parallel edges between two vertices u and v, then $\{u, v\}$ is a separation pair for G^* (and also for G). See Figure 8(a) for an example.

By the above definition, every maximal induced subgraph S of G^* that does not contain parallel edges (except possibly in its outer face) is a 3-connected 1-plane graph. Note that this graph either corresponds to G^*, or it is enclosed between a pair of parallel edges e_1 and e_2 with end-vertices u and v, such that $\{u, v\}$ is a separation pair for G^*. Removing S from G^*, except for u and v, and replacing e_1 and e_2 with a single edge, results in a new 1-plane multigraph with fewer parallel

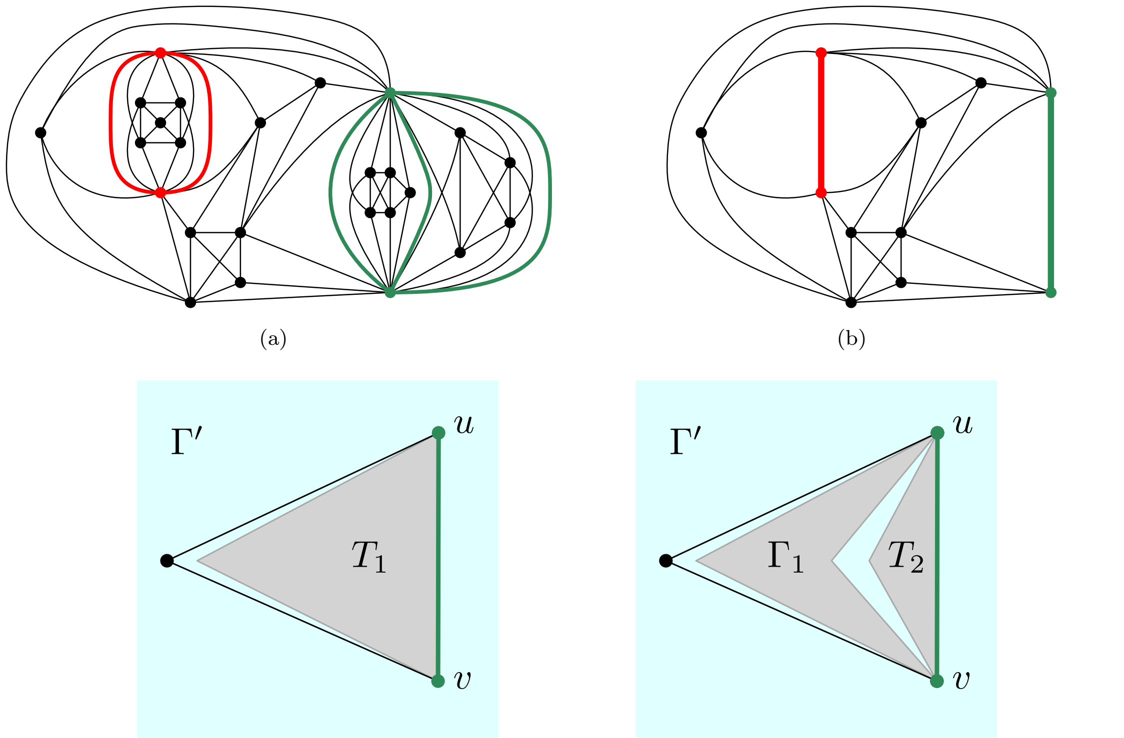


Figure 8: Illustration for Theorem 2.

edges. By iteratively applying this procedure, we obtain a hierarchical decomposition of the graph in 3-connected components, as proved in [8]. See Figure 8(b) for an example.

We adopt a similar approach as in [8]. Following a top-down traversal, for each 3-connected component S of G^*, we apply a modified version of the algorithm used to prove Lemma 5. In particular, we still use the algorithm by Chiba et al. [18] to draw the graph S_P obtained from S by removing all pairs of crossing edges. Also, we use Lemma 3 to reinsert those pairs of crossing edges that belong to marked inner faces. However, we use a different strategy to specify the outer polygon of S. In particular, we choose this polygon such that we can merge the drawing Γ of S with the drawing Γ' of its parent component S'. Components attached to distinct pairs of vertices of S' can be merged independently. Let $S_1, \dots, S_k$ be a set of $k \geq 1$ components attached to a same pair of vertices, denoted by u and v, of S'. Let T_1 be an isosceles triangle having the drawing of (u, v) in Γ' as base and whose height is such that we could replace (u, v) in Γ' with T_1 without introducing crossings; see, e.g., Figure 8(c). If the outer face of S_1 does not contain a crossing, then we use T_1 as outer polygon to draw S_1. Else, we apply Lemma 4 with T_1 as prescribed triangle (with $A = u$ and $B = v$), to draw the outer kite of S_1, and we use the polygon defined by this drawing as outer polygon to draw S_1. In this way we can merge the drawing Γ_1 of S_1 with Γ'. In the resulting drawing, consider the (interior) triangular face having on its boundary (u, v) and either a crossing or a vertex of S_1. This face defines a triangle T_2'. Let T_2 be an isosceles triangle

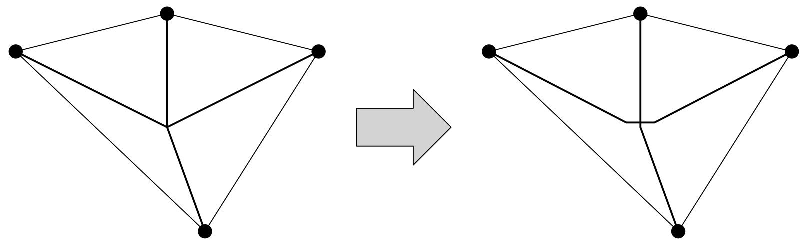


Figure 9: Illustration for the proof of Theorem 4

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having the drawing of (u, v) as base and whose height is smaller than the height of T_2'; see, e.g., Figure 8(d). Again, if the outer face of S_2 does not contain a crossing, then we use T_2 as outer polygon to draw S_2. Else, we apply Lemma 4 with T_2 as prescribed triangle (again with $A = u$ and $B = v$) to draw the outer kite of S_2, and we use the polygon defined by this drawing as outer polygon to draw S_2. We repeat this procedure for all S_i, $i = 3, \ldots, k$. □

The next theorem states that angular resolution bounded from below by a function of the maximum vertex degree of the graph and independent of its size can be obtained.

Theorem 3 *Every 1-plane graph G with maximum degree Δ has a 1-planar 1-bend drawing with angular resolution $\Omega(0.15^{6\Delta})$. Also, all crossing-free edges are drawn straight-line.*

Proof: Let G_p be the plane graph obtained from G by replacing each crossing with a dummy vertex. Let G_p^* be the triangulated plane graph obtained from G_p by applying the edge-augmentation procedure by Kant and Bodlaender[32], which produces a graph with maximum degree $\Delta^* \leq \lceil 3/2\Delta \rceil + 11 < 6\Delta$ (for $\Delta \geq 3$). We compute a planar straight-line drawing Γ^* of G_p^* by applying Lemma 1. We finally remove from Γ^* all the edges in $G_p^* \setminus G_p$, and we replace the dummy vertices of G_p with bend points. This results in a drawing of G with angular resolution $\Omega(0.15^{6\Delta})$, in which all crossing-free edges are drawn straight-line. Also, note that the 1-planar embedding of G is preserved by this drawing. □

We conclude by showing that if two bends per edge are allowed, right-angle crossings and stub resolution close to $\frac{1}{2}$ can be simultaneously achieved. This last result holds for 3-connected 1-plane graphs only and extending it to all 1-plane graphs remains an interesting open problem. In particular, we could not follow a similar approach as done for Theorem 2 because it is not clear how to merge different components attached at the same separation pair.

Theorem 4 *Every 3-connected 1-plane graph G has a 1-planar 2-bend RAC drawing Γ with $\mathrm{sr}_\Gamma = \frac{1}{2}$, except for at most one pair of edges whose stub resolution is $\frac{1}{2} - \varepsilon$, for any fixed $0 < \varepsilon < \frac{1}{2}$. All crossing-free edges are straight-line.*

Proof: Construct a 1-planar 1-bend drawing Γ' of G by using Lemma 5. Recall that all pairs of crossing edges form an empty kite except at most one that forms an outer kite. To draw the pairs of crossing edges forming an empty kite we used Lemma 3, and hence at least one edge for each pair has a bend at the corresponding crossing point. We replace this bend point with two bends sufficiently close and at opposite sides of the crossing point such that the two edges cross perpendicularly; see also Figure 9. Note that if both edges bend at the crossing point a slight perturbation is needed to achieve a right-angle crossing. This perturbation however can be sufficiently small to guarantee stub resolution $\frac{1}{2} - \varepsilon$. To draw the possible pair of crossing edges that form an outer kite, we used Lemma 4 with any isosceles triangle T. In this case we choose T such that its base angles are $\frac{\pi}{4}$. With this choice the technique of Lemma 4 draws one of the two crossing segments with slope $+1$, while the other with slope -1, and thus the two edges cross perpendicularly. This yields to the desired drawing. □

5 Conclusions and Open Problems

We investigated the stub resolution as an aesthetic for nonplanar drawings. We developed drawing techniques for 1-planar graphs with optimal or near-optimal stub resolution that achieve interesting trade-offs between the number of bends per edge and restrictions on the set of 1-planar graphs, as well as angular resolution and crossing resolution.

Interesting open problems arise from our research, among them:

- Is there a constant $\delta > 2$ such that every straight-line drawable 1-planar graph has a 1-planar straight-line drawing with stub resolution at least $\frac{1}{\delta}$?
- Does every 1-planar graph with maximum vertex degree Δ admit a 1-planar 1-bend drawing with $\Omega(\frac{1}{\Delta})$ angular resolution?
- Can we generalize our results to k-planar graphs? In this direction, we have preliminary results showing that 2-planar drawings with bounded stub resolution are possible for optimal 2-planar graphs (i.e., for 2-planar graphs that achieve the maximum density of $5n-10$ edges over n vertices) if we allow a constant number of bends for the crossing edges.
- Finally, it would be interesting to study stub resolution in combination with other aesthetics, such as compact area or few slopes for the edge segments.

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References

[1] O. Aichholzer, M. Korman, Y. Okamoto, I. Parada, D. Perz, A. van Renssen, and B. Vogtenhuber. Graphs with large total angular resolution. In D. Archambault and C. D. Tóth, editors, *GD 2019*, volume 11904 of *LNCS*, pages 193–199. Springer, 2019. doi:10.1007/978-3-030-35802-0_15.

[2] M. J. Alam, F. J. Brandenburg, and S. G. Kobourov. Straight-line grid drawings of 3-connected 1-planar graphs. In S. K. Wismath and A. Wolff, editors, *GD 2013*, volume 8242 of *LNCS*, pages 83–94. Springer, 2013. doi:10.1007/978-3-319-03841-4_8.

[3] P. Angelini, M. A. Bekos, G. Liotta, and F. Montecchiani. Universal slope sets for 1-bend planar drawings. *Algorithmica*, 81(6):2527–2556, 2019. doi:10.1007/s00453-018-00542-9.

[4] P. Angelini, G. Di Battista, W. Didimo, F. Frati, S. Hong, M. Kaufmann, G. Liotta, and A. Lubiw. Large angle crossing drawings of planar graphs in subquadratic area. In A. Márquez,

P. Ramos, and J. Urrutia, editors, *EGC 2011*, volume 7579 of *LNCS*, pages 200–209. Springer, 2011. doi:10.1007/978-3-642-34191-5_19.

[5] E. N. Argyriou, M. A. Bekos, and A. Symvonis. The straight-line RAC drawing problem is NP-hard. *J. Graph Algorithms Appl.*, 16(2):569–597, 2012. doi:10.7155/jgaa.00274.

[6] E. N. Argyriou, M. A. Bekos, and A. Symvonis. Maximizing the total resolution of graphs. *Comput. J.*, 56(7):887–900, 2013. doi:10.1093/comjnl/bxs088.

[7] K. Arikushi, R. Fulek, B. Keszegh, F. Moric, and C. D. Tóth. Graphs that admit right angle crossing drawings. *Comput. Geom.*, 45(4):169–177, 2012. doi:10.1016/j.comgeo.2011.11.008.

[8] M. A. Bekos, W. Didimo, G. Liotta, S. Mehrabi, and F. Montecchiani. On RAC drawings of 1-planar graphs. *Theor. Comput. Sci.*, 689:48–57, 2017. doi:10.1016/j.tcs.2017.05.039.

[9] C. Binucci, G. Liotta, F. Montecchiani, and A. Tappini. Partial edge drawing: Homogeneity is more important than crossings and ink. In N. G. Bourbakis, G. A. Tsihrintzis, M. Virvou, and D. Kavraki, editors, *IISA 2016*, pages 1–6. IEEE, 2016. doi:10.1109/IISA.2016.7785427.

[10] F. J. Brandenburg, W. Didimo, W. S. Evans, P. Kindermann, G. Liotta, and F. Montecchiani. Recognizing and drawing IC-planar graphs. *Theor. Comput. Sci.*, 636:1–16, 2016. doi:10.1016/j.tcs.2016.04.026.

[11] T. Bruckdorfer, S. Cornelsen, C. Gutwenger, M. Kaufmann, F. Montecchiani, M. Nöllenburg, and A. Wolff. Progress on partial edge drawings. *J. Graph Algorithms Appl.*, 21(4):757–786, 2017. doi:10.7155/jgaa.00438.

[12] T. Bruckdorfer and M. Kaufmann. Mad at edge crossings? Break the edges! In E. Kranakis, D. Krizanc, and F. L. Luccio, editors, *FUN 2012*, volume 7288 of *LNCS*, pages 40–50. Springer, 2012. doi:10.1007/978-3-642-30347-0_7.

[13] T. Bruckdorfer, M. Kaufmann, and S. Leibßle. PED user study. In E. Di Giacomo and A. Lubiw, editors, *GD 2015*, volume 9411 of *LNCS*, pages 551–553. Springer, 2015. doi:10.1007/978-3-319-27261-0_47.

[14] T. Bruckdorfer, M. Kaufmann, and F. Montecchiani. 1-bend orthogonal partial edge drawing. *J. Graph Algorithms Appl.*, 18(1):111–131, 2014. doi:10.7155/jgaa.00316.

[15] M. Burch, H. Schmauder, A. Panagiotidis, and D. Weiskopf. Partial link drawings for nodes, links, and regions of interest. In *IV 2014*, pages 53–58. IEEE Computer Society, 2014. doi:10.1109/IV.2014.45.

[16] M. Burch, C. Vehlow, N. Konevtsova, and D. Weiskopf. Evaluating partially drawn links for directed graph edges. In M. J. van Kreveld and B. Speckmann, editors, *GD 2011*, volume 7034 of *LNCS*, pages 226–237. Springer, 2011. doi:10.1007/978-3-642-25878-7_22.

[17] S. Chaplick, F. Lipp, A. Wolff, and J. Zink. Compact drawings of 1-planar graphs with right-angle crossings and few bends. *Comput. Geom.*, 84:50–68, 2019. doi:10.1016/j.comgeo.2019.07.006.

[18] N. Chiba, T. Yamanouchi, and T. Nishizeki. Linear algorithms for convex drawings of planar graphs. *Progress in graph theory*, pages 153–173, 1984.

[19] G. Di Battista and L. Vismara. Angles of planar triangular graphs. *SIAM J. Discrete Math.*, 9(3):349–359, 1996. doi:10.1137/S0895480194264010.

[20] E. Di Giacomo, W. Didimo, P. Eades, and G. Liotta. 2-layer right angle crossing drawings. *Algorithmica*, 68(4):954–997, 2014. doi:10.1007/s00453-012-9706-7.

[21] W. Didimo, P. Eades, and G. Liotta. Drawing graphs with right angle crossings. *Theor. Comput. Sci.*, 412(39):5156–5166, 2011. doi:10.1016/j.tcs.2011.05.025.

[22] W. Didimo, M. Kaufmann, G. Liotta, Y. Okamoto, and A. Spillner. Vertex angle and crossing angle resolution of leveled tree drawings. *Inf. Process. Lett.*, 112(16):630–635, 2012. doi:10.1016/j.ipl.2012.05.006.

[23] W. Didimo, G. Liotta, and F. Montecchiani. A survey on graph drawing beyond planarity. *ACM Comput. Surv.*, 52(1):4:1–4:37, 2019. doi:10.1145/3301281.

[24] C. A. Duncan, D. Eppstein, M. T. Goodrich, S. G. Kobourov, and M. Nöllenburg. Drawing trees with perfect angular resolution and polynomial area. *Discr. Comp. Geom.*, 49(2):157–182, 2013. doi:10.1007/s00454-012-9472-y.

[25] C. A. Duncan and S. G. Kobourov. Polar coordinate drawing of planar graphs with good angular resolution. *J. Graph Algorithms Appl.*, 7(4):311–333, 2003. doi:10.7155/jgaa.00073.

[26] D. Eppstein, M. J. van Kreveld, E. Mumford, and B. Speckmann. Edges and switches, tunnels and bridges. *Comput. Geom.*, 42(8):790–802, 2009. doi:10.1016/j.comgeo.2008.05.005.

[27] M. Formann, T. Hagerup, J. Haralambides, M. Kaufmann, F. T. Leighton, A. Symvonis, E. Welzl, and G. J. Woeginger. Drawing graphs in the plane with high resolution. *SIAM J. Comput.*, 22(5):1035–1052, 1993. doi:10.1137/0222063.

[28] A. Garg and R. Tamassia. Planar drawings and angular resolution: Algorithms and bounds (extended abstract). In J. van Leeuwen, editor, *ESA 1994*, volume 855 of *LNCS*, pages 12–23. Springer, 1994. doi:10.1007/BFb0049393.

[29] S. Hong, P. Eades, G. Liotta, and S. Poon. Fáry's theorem for 1-planar graphs. In J. Gudmundsson, J. Mestre, and T. Viglas, editors, *COCOON 2012*, volume 7434 of *LNCS*, pages 335–346. Springer, 2012. doi:10.1007/978-3-642-32241-9_29.

[30] W. Huang, P. Eades, and S. Hong. Larger crossing angles make graphs easier to read. *J. Vis. Lang. Comput.*, 25(4):452–465, 2014. doi:10.1016/j.jvlc.2014.03.001.

[31] M. Hummel, F. Klute, S. Nickel, and M. Nöllenburg. Maximizing ink in partial edge drawings of k-plane graphs. In *GD 2019*, volume 11904 of *LNCS*, pages 323–336. Springer, 2019. doi:10.1007/978-3-030-35802-0_25.

[32] G. Kant and H. L. Bodlaender. Triangulating planar graphs while minimizing the maximum degree. In O. Nurmi and E. Ukkonen, editors, *SWAT 1992*, volume 621 of *LNCS*, pages 258–271. Springer, 1992. doi:10.1007/3-540-55706-7_22.

[33] M. Kaufmann, J. Kratochvíl, F. Lipp, F. Montecchiani, C. N. Raftopoulou, and P. Valtr. Bounded stub resolution for some maximal 1-planar graphs. In B. Panda and P. Goswami, editors, *CALDAM 2018*, volume 10743 of *LNCS*, pages 214–220. Springer, 2018. doi:10.1007/978-3-319-74180-2_18.

[34] M. Kaufmann, J. Kratochvíl, F. Lipp, F. Montecchiani, C. N. Raftopoulou, and P. Valtr. The stub resolution of 1-planar graphs. In *WALCOM 2020*, volume 12049 of *LNCS*, pages 170–182. Springer, 2020. doi:10.1007/978-3-030-39881-1_15.

[35] B. Keszegh, J. Pach, and D. Pálvölgyi. Drawing planar graphs of bounded degree with few slopes. *SIAM J. Discrete Math.*, 27(2):1171–1183, 2013. doi:10.1137/100815001.

[36] S. G. Kobourov, G. Liotta, and F. Montecchiani. An annotated bibliography on 1-planarity. *Computer Science Review*, 25:49–67, 2017. doi:10.1016/j.cosrev.2017.06.002.

[37] M. Krug and D. Wagner. Minimizing the area for planar straight-line grid drawings. In S.-H. Hong, T. Nishizeki, and W. Quan, editors, *GD 2007*, volume 4875 of *LNCS*, pages 207–212. Springer, 2007. doi:10.1007/978-3-540-77537-9_21.

[38] S. M. Malitz and A. Papakostas. On the angular resolution of planar graphs. *SIAM J. Discrete Math.*, 7(2):172–183, 1994. doi:10.1137/S0895480193242931.

[39] Y. Suzuki. Re-embeddings of maximum 1-planar graphs. *SIAM J. Discrete Math.*, 24(4):1527–1540, 2010. doi:10.1137/090746835.

[40] C. Thomassen. Rectilinear drawings of graphs. *J. Graph Theory*, 12(3):335–341, 1988. doi:10.1002/jgt.3190120306.