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Packing Trees into 1-planar Graphs

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Abstract. We introduce and study the *1-planar packing problem*: Given k graphs with n vertices $G_1, \ldots, G_k$, find a 1-planar graph that contains the given graphs as edge-disjoint spanning subgraphs. We mainly focus on the case when each G_i is a tree and $k = 3$. We prove that a triple consisting of three caterpillars or of two caterpillars and a path may not admit a 1-planar packing, while two paths and a special type of caterpillar always have one. We then study 1-planar packings with few crossings and prove that three paths (resp. cycles) admit a 1-planar packing with at most seven (resp. fourteen) crossings. We finally show that a quadruple consisting of three paths and a perfect matching with $n \geq 12$ vertices admits a 1-planar packing, while such a packing does not exist if $n \leq 10$.

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1 Introduction

In the *graph packing problem* we are given a collection of n-vertex graphs $G_1, \ldots, G_k$ and we are requested to find a graph G that contains the given graphs as edge-disjoint spanning subgraphs. Various settings of the problem can be defined depending on the type of graphs that have to be packed and on the restrictions on the packing graph G. The most general case is when G is the complete graph on n vertices and there is no restriction on the input graphs. Sauer and Spencer [17] prove that any two graphs with at most $n-2$ edges can be packed into K_n; Woźniak and Wojda [19] give sufficient conditions for the existence of a packing of three graphs. The setting when G is K_n and each G_i is a tree ($i = 1, 2, \ldots, k$) has been intensively studied. Hedetniemi *et al.* [9] show that two non-star trees can always be packed into K_n. Notice that, the condition that the trees are not stars is necessary for the existence of the packing because each vertex must have degree at least one in each tree, which is not possible if a vertex is adjacent to all other vertices as it is the case for a star. Wang and Sauer [18] give sufficient conditions for the existence of a packing of three trees into K_n, while Mahéo *et al.* [13] characterize the triples of trees that admit such a packing.

García *et al.* [6] consider the *planar packing problem*, that is the case when the graph G is required to be planar. They conjecture that the result of Hedetniemi *et al.* extends to this setting, *i.e.*, that every pair of non-star trees can be packed into a planar graph. Notice that, when G is required to be planar, two is the maximum number of trees that can be packed (because three trees have more than $3n-6$ edges). García *et al.* prove their conjecture for some restricted cases, namely when one of the trees is a path and when the two trees are isomorphic. In a series of subsequent papers the conjecture has been proved true for other pairs of trees. Oda and Ota [14] prove it when one tree is a caterpillar or it is a spider of diameter four. Frati *et al.* [5] extend the last result to any spider, while Frati [4] considers the case when both trees have diameter four. Geyer *et al.* show that a planar packing always exists for a pair of binary trees [7] and for a pair of non-star trees [8], thus finally settling the conjecture.

In the present paper we initiate the study of the *1-planar packing problem*, *i.e.*, the problem of packing a set of graphs into a 1-planar graph. A 1-planar graph is a graph that can be drawn so that each edge has at most one crossing. 1-planar graphs have been introduced by Ringel [16] and have received increasing attention in the last years in the research area called *beyond planarity* (see, *e.g.*, [10, 3]). Since any two non-star trees admit a planar packing, a natural question is whether we can pack more than two trees into a 1-planar graph. On the other hand, since each 1-planar graph has at most $4n-8$ edges [15], it is not possible to pack more than three trees into a 1-planar graph. Thus, our main question is whether any three trees with maximum vertex degree $n-3$ admit a 1-planar packing. The restriction to trees of degree at most $n-3$ is necessary because a vertex of degree larger than $n-3$ in one tree cannot have degree at least one in the other two trees. Our results are as follows.

- We show that there exist triples of structurally simple trees that do not admit a 1-planar packing (Section 3). These triples consist of three caterpillars with at least 10 vertices and of two caterpillars and a path with 7 vertices.

- Motivated by the above results, we study triples consisting of two paths and a caterpillar (Section 4). We characterize the triples consisting of two paths and a 5-legged caterpillar (a caterpillar where each vertex of the spine has no leaves attached or it has at least five leaves) that admit such a packing. We also characterize the triples that admit a 1-planar packing and which consist of two paths and a caterpillar whose spine has exactly two vertices.

- The packing technique of the results above is constructive and it gives rise to 1-plane graphs (*i.e.*, 1-planar embedded graphs) with a linear number of crossings. This naturally raises the question about the number of edge crossings required by a 1-planar packing. We show that any three paths with at least six vertices can be packed into a 1-plane graph with seven edge crossings in total (Section 5). We also extend this technique to three cycles obtaining 1-plane graphs with fourteen crossings in total.
- We finally consider the 1-planar packing problem for quadruples of acyclic graphs (Section 6). Since, as already observed, four paths cannot be packed into a 1-planar graph, we consider three paths and a perfect matching. We show that when $n \geq 12$ such a quadruple admits a 1-planar packing and that when $n \leq 10$ a 1-planar packing does not exist.

The rest of the paper is organized as follows. Preliminary definitions are given in Section 2. Instances for which a 1-planar packing does not exist are described in Section 3. Section 4 contains results about packing two paths and a caterpillar, while Section 5 describes 1-planar packings with a constant number of crossings. Results about three paths and a perfect matching are presented in Section 6. Conclusions and open problems are reported in Section 7.

2 Preliminaries

Given a graph G and a vertex v of G, $\deg_G(v)$ denotes the vertex degree of v in G. Let $G_1, \ldots, G_k$ be k graphs with n vertices; a *packing* of $G_1, \ldots, G_k$ is an n-vertex graph G that has $G_1, \ldots, G_k$ as edge-disjoint spanning subgraphs. We consider the case when G is a 1-*planar graph*, that is a graph that admits a drawing in the plane such that each edge has at most one crossing. Such a drawing is called a 1-*planar drawing* of G. In this case we say that G is a *1-planar packing* of $G_1, \ldots, G_k$. If $G_1, \ldots, G_k$ admit a (1-planar) packing G, we also say that $G_1, \ldots, G_k$ *can be packed into* G. We mainly concentrate on the case when each G_i is a tree ($1 \leq i \leq k$). In this case (and generally when each G_i is connected), we have restrictions on the values of k and n for which a packing exists.

Property 1 *A 1-planar packing of k connected n-vertex graphs $G_1, \ldots, G_k$ exists only if $k \leq 3$ and $n \geq 2k$. Moreover, $\deg_{G_i}(v) \leq n - k$ for each vertex v.*

Proof: If each G_i is connected, then it has at least $n-1$ edges and therefore any packing of $G_1, \ldots, G_k$ has at least $k(n-1)$ edges; since the complete graph with n vertices has $\frac{n(n-1)}{2}$ edges it holds that $k(n-1) \leq \frac{n(n-1)}{2}$, that is $n \geq 2k$. On the other hand, a 1-planar graph has at most $4n-8$ edges, and therefore it holds that $k(n-1) \leq 4n-8$, which implies $k \leq 3$. Moreover, if each G_i is connected then $\deg_{G_i}(v) \geq 1$ for each v, and since $\sum_{i=1}^{k} \deg_{G_i}(v) \leq n-1$ it holds that $\deg_{G_i}(v) \leq n-k$. □

A *caterpillar* T is a tree such that removing all the leaves results in a path called the *spine*. A *backbone* of T is a path $v_0, v_1, v_2, \ldots, v_l, v_{l+1}$ of T where $v_1, v_2, \ldots, v_l$ is the spine of T and v_0 and v_{l+1} are two leaves in T adjacent to v_1 and v_l, respectively. T is *h-legged* if every vertex of its spine has degree either 2 or at least $h+2$ in T.

3 Trees That Do Not Admit 1-planar Packings

In this section we describe triples of trees that do not admit a 1-planar packing.

Theorem 1 *For every $n \geq 10$, there exists a triple of caterpillars that does not admit a 1-planar packing.*

Proof: The triple consists of three isomorphic caterpillars T_1, T_2, T_3 with $n \geq 10$ vertices. Each T_i has a backbone of length 5 and $n-5$ leaves all adjacent to the middle vertex of the spine, which we call the *center* of T_i. Notice that each T_i is such that $\deg_{T_i}(v) \leq n-3$ for each vertex v, since the vertex with largest degree in T_i is its center, which has degree $n-3$. Thus, the necessary condition on the degree stated in Property 1 is verified.

Let G be any packing of T_1, T_2, and T_3 and let v_1, v_2, and v_3 be the three vertices of G where the three centers of T_1, T_2, T_3, respectively, are mapped. The three vertices v_1, v_2, and v_3 must be distinct because otherwise they would have degree larger than $n-1$ in G, which is impossible. For each v_i we have $\deg_{T_i}(v_i) = n-3$ and $\deg_{T_j}(v_i) \geq 1$, for $j \neq i$. Hence, $\deg_G(v_i) = \deg_{T_1}(v_i) + \deg_{T_2}(v_i) + \deg_{T_3}(v_i) \geq n-1$. Since $\deg_G(v_i)$ cannot be larger than $n-1$, it must be $\deg_G(v_i) = n-1$ for each v_i. In other words, each v_i is adjacent to all the other vertices of G. Thus, G contains $K_{3,n-3}$ as a subgraph. Since $n \geq 10$ and $K_{3,7}$ is not 1-planar [2], G is not 1-planar. □

Motivated by Theorem 1, we consider triples where one of the caterpillars is a path. Also in this case there exist triples that do not have a 1-planar packing.

Theorem 2 *There exists a triple consisting of a path and two caterpillars with $n = 7$ vertices that does not admit a 1-planar packing.*

Proof: Let T_i $(i = 1, 2)$ be a caterpillar with a backbone of length four such that one of the two internal vertices has degree three and the other one has degree four. Let G be a packing of T_1, T_2 and a path P of 7 vertices. Let v_1, v_2, v_3, and v_4 be the four vertices of G where the internal vertices of the backbones of T_1 and T_2 are mapped to. We first observe that v_1, v_2, v_3, and v_4 must be distinct. Suppose, as a contradiction, that two of them coincide, say v_1 and v_2; then $\deg_{T_1}(v_1) + \deg_{T_2}(v_1) \geq 6$. On the other hand $\deg_P(v_1) \geq 1$, and therefore $\deg_G(v_1) \geq 7$, which is impossible (since G has only 7 vertices). Denote by $G_{1,2}$ the subgraph of G containing only the edges of T_1 and T_2. Two vertices among v_1, v_2, v_3, and v_4, say v_1 and v_2, have degree 5 in $G_{1,2}$, while the other two have degree 4 in $G_{1,2}$. Consider now the edges of P. Since the maximum vertex degree in a graph of seven vertices is six, v_1 and v_2 must be the end-vertices of P, while v_3 and v_4 are internal vertices. This means that they all have degree 6 in G. The vertices distinct from v_1, v_2, v_3, and v_4 have degree 2 in $G_{1,2}$ and degree 4 in G. Thus in G there are four vertices of degree 6 and three vertices of degree 4. The only graph of seven vertices with this degree distribution is the graph obtained from K_7 by deleting all the edges of a 3-cycle, which is known to be non-1-planar [11]. □

4 1-planar Packings of Two Paths and a Caterpillar

In this section we prove that a triple consisting of two paths P_1 and P_2 and a 5-legged caterpillar T with at least six vertices admits a 1-planar packing. In order to obtain this result, stated in Theorem 4, we need to prove intermediate lemmas. The high-level idea of our approach can be described as follows. Let P be the backbone of T and let P'_1 and P'_2 be two paths with the same length as P. We first show how to construct a 1-planar packing of P, P'_1 and P'_2. We then modify the computed packing to include the leaves of the caterpillar so to obtain a 1-planar packing of

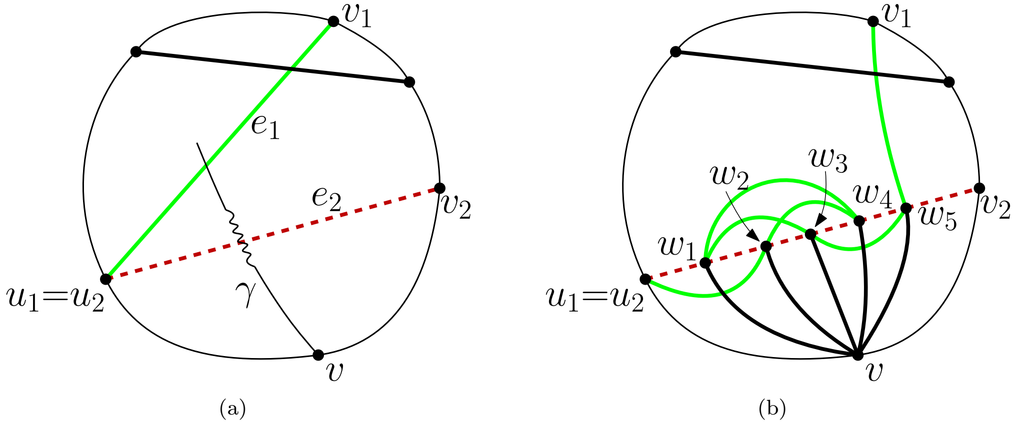


Figure 1: A 5-leaf addition operation. The cutting curve is shown with a zig-zag pattern on it.

P_1, P_2 and T; this requires transforming some edges of P'_1 and P'_2 to sub-paths that pass through the added leaves.

Let e be an edge of a given 1-planar drawing Γ, possibly with parallel edges. If e has one crossing c, then each of the two parts in which e is divided by c are called *sub-edges* of e; if e has no crossing, e itself is called a *sub-edge* of e. Let v be a vertex of Γ; a *cutting curve* of v is a simple open curve γ such that: (i) γ has v as an end-point; (ii) γ intersects two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ (possibly $u_1 = u_2$ and/or $v_1 = v_2$); (iii) γ does not intersect any other edge of Γ; (iv) e_1 and e_2 do not cross each other; (v) if e_1 and e_2 are parallel edges (*i.e.*, $u_1 = u_2$ and $v_1 = v_2$), they have no crossings. The *stub* of e_i with respect to γ is the sub-edge of e_i intersected by γ ($i = 1, 2$).

Given a cutting curve γ of a vertex v, and an integer $\ell \geq 5$, an *ℓ-leaf addition* operation adds ℓ vertices $w_1, w_2, \dots, w_\ell$ and the edges $(v, w_1), (v, w_2), \dots, (v, w_\ell)$ to Γ in such a way that: (i) the added vertices subdivide the stubs of both e_1 and e_2 with respect to γ; (ii) the subgraph induced by $u_1, u_2, v_1, v_2, w_1, w_2, \dots, w_\ell$ has no parallel edges (see Figure 1 for an example). In other words, a leaf addition adds a set of vertices adjacent to v and replaces the stubs of e_1 and e_2 with two edge-disjoint paths. This operation will be used to modify the 1-planar packing of P, P'_1 and P'_2 to include the leaves of the caterpillar. When the value of ℓ is not relevant, an ℓ-leaf addition will be simply called a *leaf addition*.

Lemma 1 *Let Γ be a 1-planar drawing possibly with parallel edges, let v be a vertex of Γ and let γ be a cutting curve of v. It is possible to execute an ℓ-leaf addition for every $\ell \geq 5$ in such a way that the resulting drawing is still 1-planar.*

Proof: Denote by e_1 and e_2 the two edges crossed by γ. If one of them or both are crossed in Γ replace their crossing points with dummy vertices. Let e'_i be the stub of e_i with respect to γ (if e_i is not crossed in Γ, e'_i coincides with e_i). After the replacement of the crossings with the dummy vertices the two stubs e'_1 and e'_2 have no crossing. Since γ does not cross any edge distinct from e_1 and e_2, the drawing Γ' obtained by removing e'_1 and e'_2 has a face f whose boundary contains the vertex v and all the end-vertices of e'_1 and of e'_2 (there are at least two and at most four such vertices).

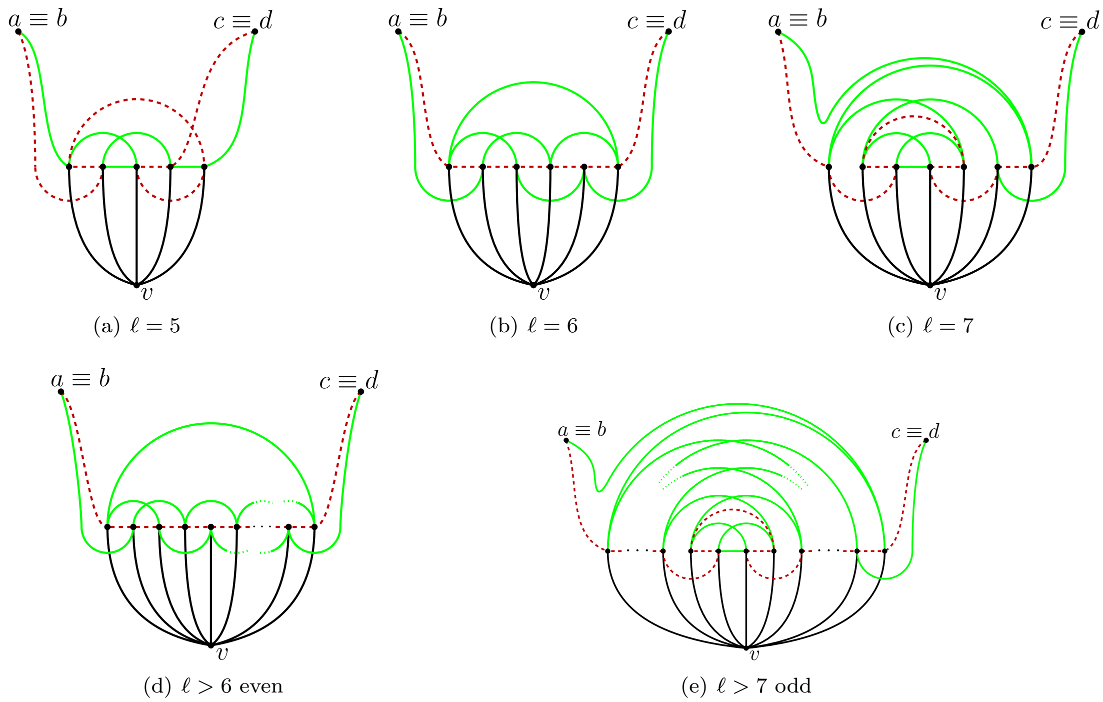
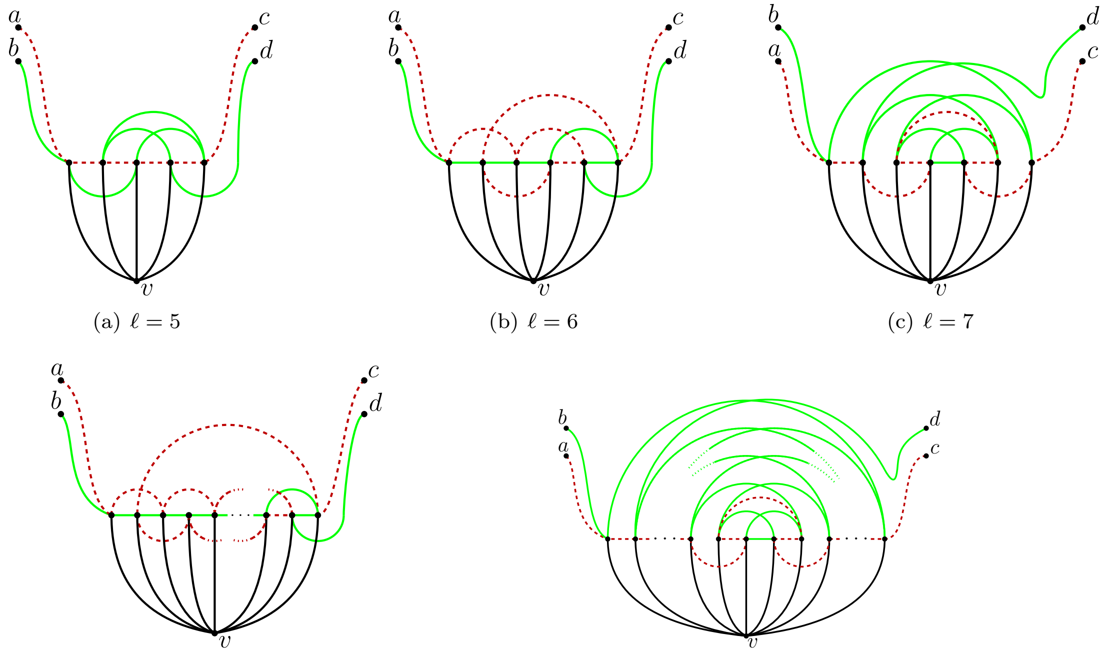


Figure 2: Gadgets used for parallel edges in the proof of Lemma 1.

The idea now is to insert into the face f, without creating any crossing, a gadget that realizes the ℓ-leaf addition for the desired value of $\ell \geq 5$. A gadget has ℓ vertices that will be added to Γ, a vertex that will be identified with v, and four vertices a, b, c, and d that will be identified with the end-vertices of e'_1 and e'_2. The four vertices a, b, c, and d will be called *attaching vertices* and the edges incident to them will be called *attaching edges*. In order to guarantee that the leaf addition is valid and that the drawing Γ'' obtained by the insertion of the gadget inside f is 1-planar, we have to pay attention to two aspects: (i) if an attaching edge is crossed in the gadget, then its attaching vertex cannot be identified with a dummy vertex (otherwise when we remove the dummy vertex we obtain an edge that is crossed twice); (ii) if two attaching vertices of the gadget coincide (because e'_1 and e'_2 have a vertex in common), then the corresponding attaching edges must not have the second end-vertex in common in the gadget (otherwise the leaf addition is not valid because it creates parallel edges).

We use different gadgets depending on whether e_1 and e_2 are parallel edges or not. If they are parallel edges, we use the gadgets of Figure 2. Notice that in this case, e_1 and e_2 are not crossed by definition of cutting curve. It follows that f has no dummy vertex and (i) is guaranteed. On the other hand, both end-vertices of e_1 and e_2 coincide and therefore the end-vertices of the attaching edges that are not attaching vertices must be distinct. This is true for the gadgets used in this case. If e_1 and e_2 are non-parallel, we use the gadgets of Figure 3. All these gadgets have only one attaching edge that is crossed (the one incident to vertex d in the figure); also, vertex d can be identified with vertex c without creating parallel edges. If e_1 and e_2 are non-parallel, at most two end-vertices of e'_1 and e'_2 are dummy; they cannot belong to the same stub, and they cannot



(a) $\ell = 5$ (b) $\ell = 6$ (c) $\ell = 7$

(d) $\ell > 6$ even (e) $\ell > 7$ odd

Figure 3: Gadgets used for non-parallel edges in the proof of Lemma 1.

coincide (because e_1 and e_2 do not cross each other). Thus we can identify d with a non-dummy vertex and we can identify c and d if needed. □

It is worth remarking that the leaf addition operation is not guaranteed to work with less than five leaves. Namely, in order to perform an ℓ-leaf addition operation, we have to add $m = 2(\ell - 1)$ edges, each connecting two of the leaves. Since the number of possible edges connecting pairs of leaves is $m' = \frac{\ell \cdot (\ell-1)}{2}$, the operation is possible only if $m \leq m'$.

For $\ell \leq 3$ we have $m' < m$ and the construction is not possible. If $\ell = 4$, $m' = m$ and the number of available edges would be enough. However, it can be seen that, no matter how we draw the edges, two of the four dangling edges have a crossing (see, for example, the red dashed edges in Figure 4). This makes the leaf addition operation not working for specific instances.

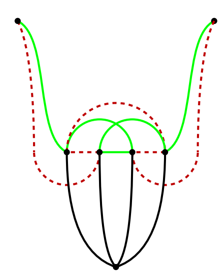


Figure 4

We are now ready to describe our construction of a 1-planar packing of P_1, P_2, and T. We use different techniques for different lengths of the backbone of T.

Lemma 2 *Two paths and a 5-legged caterpillar whose backbone contains $n' \geq 6$ vertices admit a 1-planar packing.*

Proof: We start with the construction of a 1-planar packing of the three paths P_1', P_2' and P. Let n' be the number of vertices of P_1', P_2' and P, assume first that $n' \geq 8$ and $n' \equiv 0 \pmod 4$. A

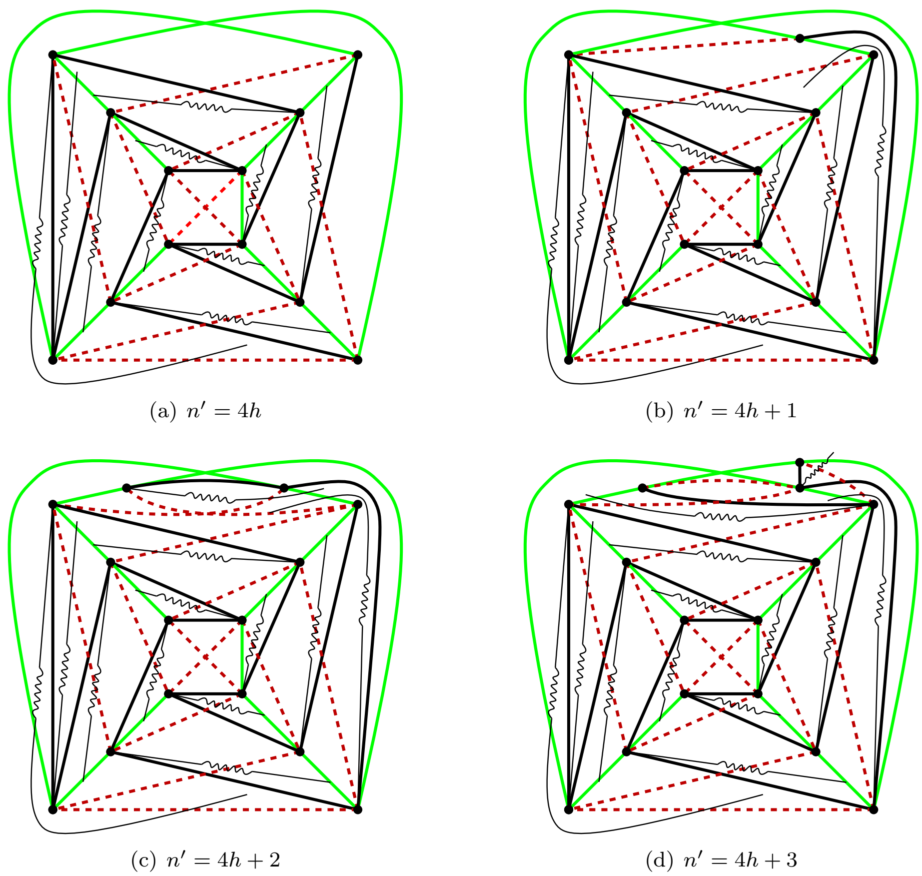
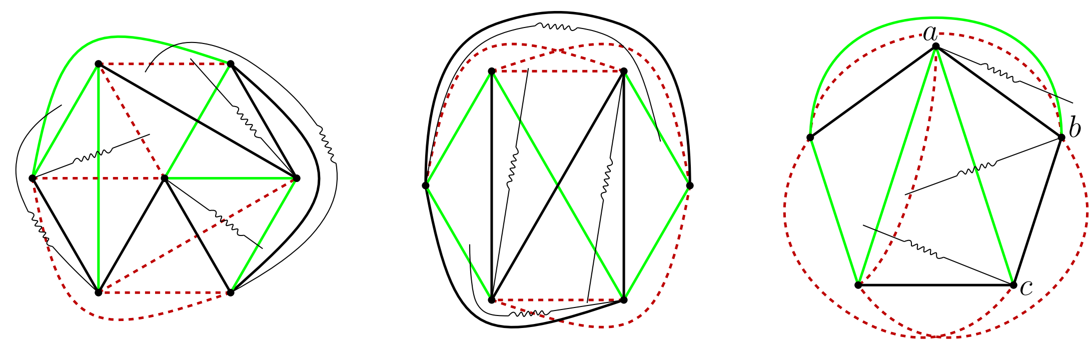


Figure 5: 1-planar packings of three paths with $n' \geq 8$ vertices (case h=3); A cutting curve is shown (zig-zag pattern) for each internal vertex of the black path.

1-planar packing of P_1', P_2' and P for this case is shown in Figure 5(a) for $n' = 12$ and it is easy to see that it can be extended to any n' multiple of 4. Assume that the backbone P of T is the path shown in black in Figure 5(a). To add the leaves of T to the construction we define a cutting curve for each vertex v that has some leaves attached; we then execute a leaf addition operation for each such vertex. By Lemma 1, it is possible to execute each leaf addition so to guarantee the 1-planarity of the resulting drawing. The cutting curve for each internal vertex of P is shown in Figure 5(a) with a zig-zag pattern. Note that, regardless of the order in which the leaf additions are executed, the cutting curves remain valid.

Suppose now that $n' \geq 8$ and $n' \not\equiv 0 \pmod 4$. In this case we first construct a 1-planar packing of three paths with $n'' = 4h$ vertices (with $h = \lfloor \frac{n'}{4} \rfloor$) using the same construction as in the previous case and then we add one, two or three vertices as shown in Figures 5(b)-5(d), where we also show the cutting curves for each internal vertex of P. If n' is equal to 6 or 7, we use the same approach; the only difference is in the construction of the 1-planar packing of P_1', P_2' and P. The construction for such a packing and the cutting curves for the internal vertices of P are shown in Figures 6(a) and 6(b). □



(a) $n' = 7$ (b) $n' = 6$ (c) $n' = 5$

Figure 6: 1-planar packings of three paths with $n' \in \{5, 6, 7\}$ vertices, with a cutting curve (zig-zag pattern) for each internal vertex of the black path.

Lemma 3 *Two paths and a 5-legged caterpillar T whose backbone contains $n' = 5$ vertices admit a 1-planar packing, unless T is a path.*

Proof: If T is a path, then P_1, P_2 and T are all paths of length five, and by Property 1, a 1-planar packing of P_1, P_2 and T does not exist. Suppose therefore that at least one internal vertex of the backbone P of T has some leaves attached. We use an approach similar to the one of Lemma 2. However, as just explained, a 1-planar packing of P'_1, P'_2 and P does not exist in this case. We start with a 1-planar packing with two pairs of parallel edges. For each pair, one edge belongs to P'_1 and the other one to P'_2. We will remove the parallel edges by performing the leaf addition operations. To this aim we must guarantee that there is a cutting curve for each pair of parallel edges. The 1-planar packing P'_1, P'_2 and P and the cutting curves for the internal vertices of P are shown in Figure 6(c), for the case when at least two vertices have leaves attached. Indeed, if only two vertices have leaves attached, they are either consecutive along the backbone or not. In the first case, these two vertices are mapped to the vertices labeled a and b in Figure 6(c) and the depicted cutting curves will remove the parallel edges; in the second case, the two vertices are mapped to the vertices labeled a and c and also in this case the depicted cutting curves will remove the parallel edges.

If only one vertex of P has leaves attached, we have only one cutting curve and thus it is not possible to intersect both pairs of parallel edges. To handle this case we distinguish two cases. If the only vertex with leaves attached is the middle vertex of the backbone, then we can adapt the technique used above as follows. Consider the 1-planar packing of P'_1, P'_2 and P shown in Figure 7(a), where we have two parallel edges (a, b) and two parallel edges (b, c). Consider now the cutting curve γ shown in Figure 7(a). This curve intersects the two parallel edges (a, b), thus, performing a leaf addition operation using that curve, we obtain a 1-planar packing of P_1, P_2 and T with the two parallel edges (b, c) (see Figure 7(b)). These two parallel edges can be removed by modifying the drawing as follows (see also Figure 7(c) for an illustration). Since the two edges crossed by γ are parallel edges, the leaf addition operation used must be one of those shown in Figures 2(a)–2(d) and 2(e). No matter which of the cases applies, one of the two edges incident to vertex a is non-crossed and can be disconnected from a and connected to c without introducing any crossing. Call this edge e. The parallel edge (c, b) with the same color as e can be disconnected

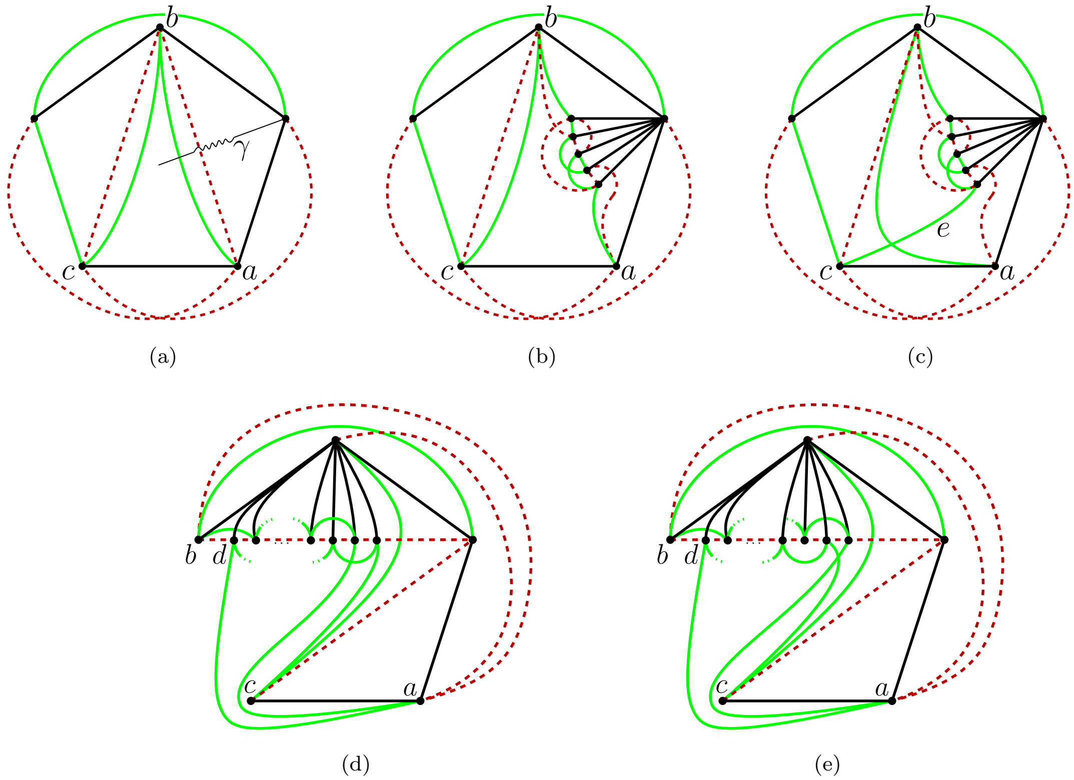


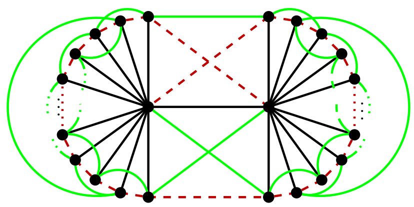
Figure 7: Illustration for the proof of Lemma 3.

from c and connected to a only crossing e. With this modification we obtain the desired 1-planar packing. If the only vertex with leaves attached is the second (or fourth) vertex of the backbone, we need an ad-hoc technique to compute a 1-planar packing of P_1, P_2 and T, which is shown in Figures 7(d) and 7(e) for an even or an odd number of leaves, respectively. The caterpillar T and the path P_1 (shown red dashed in the figures) are drawn without crossings. The path P_2 (shown green solid in the figures) is drawn such that the leaves of T alternately belong to the sub-path $\pi(b, a)$ from vertex b to vertex a and to the sub-path $\pi(d, c)$ from vertex d to vertex c (note that $\pi(b, a)$ is drawn so to cross some of the edges connecting the backbone of T to the leaves, while $\pi(d, c)$ is drawn without crossing the edges of T). Depending on whether the number of leaves is even or odd, the last leaf of T belongs to $\pi(b, a)$ or to $\pi(d, c)$, and the last but one leaf belongs to the other sub-path. This creates a crossing between two different edges of P_2. $\square$

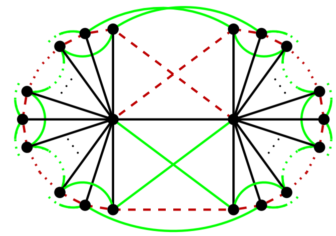
The next theorem gives a complete characterization for the case in which the backbone of T has length four.

Theorem 3 *Two paths and a caterpillar T whose backbone contains $n' = 4$ vertices admit a 1-planar packing if and only if $n \geq 6$ and $\deg_T(v) \leq n - 3$ for every vertex v.*

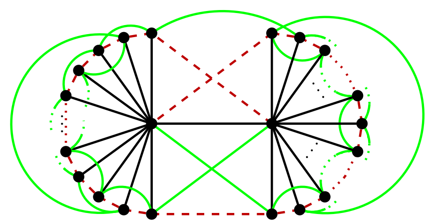
Proof: Since the length of the backbone is four, we have exactly two non-leaf vertices v_1 and v_2. Denote by n_i the number of leaves adjacent to v_i $(i = 1, 2)$ and assume $n_1 \leq n_2$. We distinguish



(a) even-even



(b) odd-odd



(c) even-odd

Figure 8: Illustration for the proof of Theorem 3.

different cases depending on the values of n_1 and n_2. If $n_1 = 1$, then we have $\deg_T(v_2) = n - 1$ and by Property 1 a 1-planar packing of P_1, P_2 and T does not exist. Assume now that $n_1 \geq 2$.

We start with the case when $n_1 \geq 5$. In this case we construct a 1-planar packing according to different techniques depending on the parity of n_1 and n_2. Figures 8(a), 8(b), and 8(c) show the construction for the cases when n_1 and n_2 are both even, when they are both odd, and when they have different parity, respectively. If $n_1 < 5$ we have different ad-hoc constructions that depend on the values of n_1 and n_2. All cases are shown in Figure 9. □

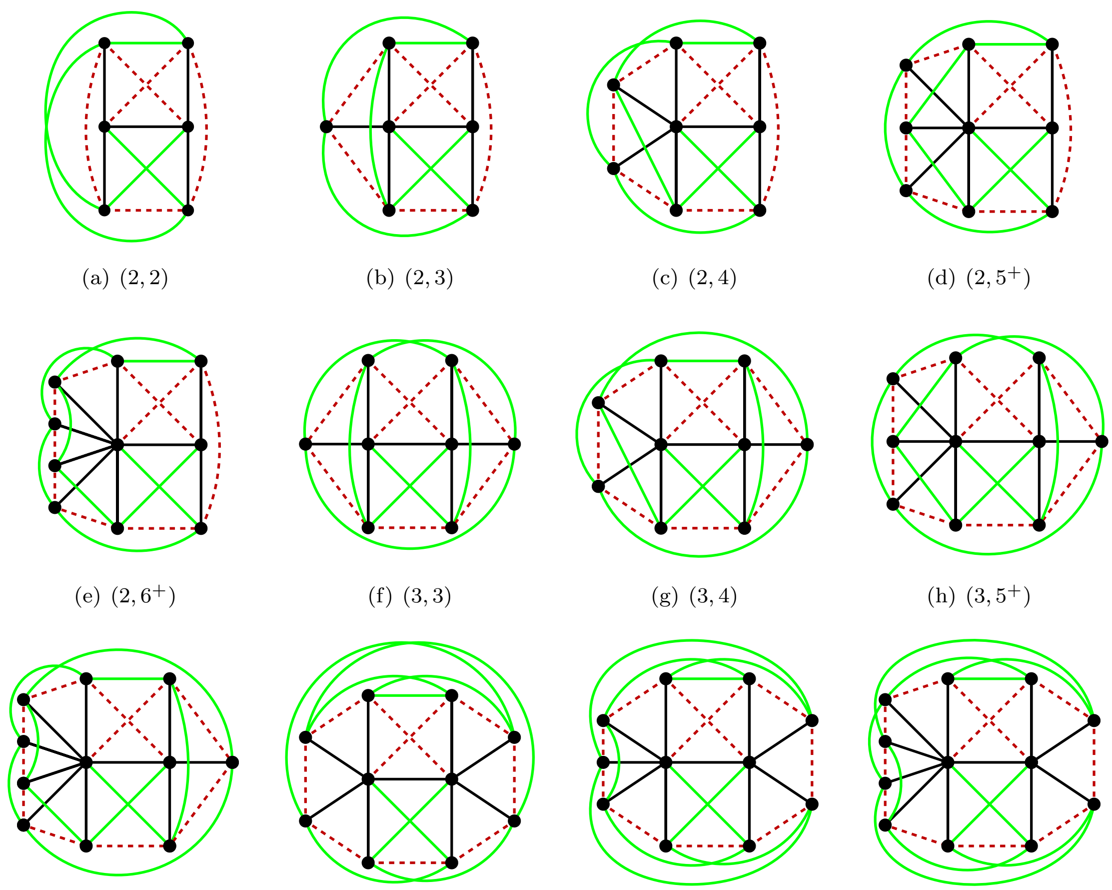
Lemmas 2 and 3, together with Theorem 3 imply the next theorem.

Theorem 4 *Two paths and a 5-legged caterpillar T with n vertices admit a 1-planar packing if and only if $n \geq 6$ and $\deg_T(v) \leq n - 3$ for every vertex v.*

5 1-planar Packings with Constant Edge Crossings

The technique described in the previous section constructs 1-planar drawings that have a linear number of crossings. A natural question is whether it is possible to compute a 1-planar packing with a constant number of crossings. In this section we prove that seven (resp. fourteen) crossings suffice for packing three paths (resp. cycles). It is worth remarking that a 1-planar packing of three paths has at least three crossings (because it has $3n - 3$ edges), while a 1-planar packing of three cycles has at least six crossings (because it has $3n$ edges).

Theorem 5 *Three paths with $n \geq 6$ vertices can be packed into a 1-plane graph with at most 7 edge crossings.*



(a) $(2,2)$ (b) $(2,3)$ (c) $(2,4)$ (d) $(2,5^+)$

(e) $(2,6^+)$ (f) $(3,3)$ (g) $(3,4)$ (h) $(3,5^+)$

(i) $(3,6^+)$ (j) $(4,4)$ (k) $(4,5^+)$ (l) $(4,6^+)$

Figure 9: Illustration for the proof of Theorem 3. Constructions for the cases when $n_1 < 5$. For each case the values (n_1, n_2) are indicated; 5^+ means $n_2 \geq 5$ with n_2 odd, while 6^+ means $n_2 \geq 6$ with n_2 even.

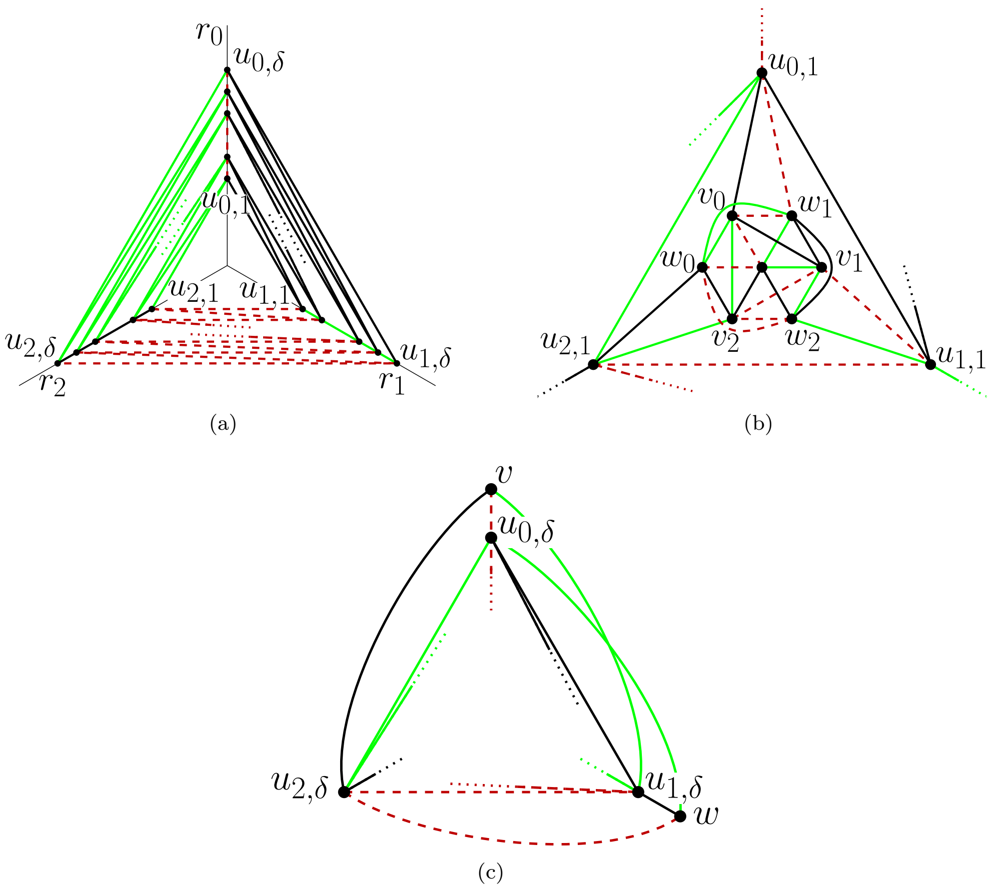


Figure 10: Illustration for the proof of Theorem 5.

Proof: We prove the statement by showing how to construct a 1-planar drawing with at most 7 crossings of a graph that is the union of three paths. Suppose first that $n = 7 + 3\delta$ for $\delta \in \mathbb{N}$. If $\delta = 0$, we draw the union of the three paths with 7 vertices as shown in Figure 6(a). The drawing is 1-planar and has three crossings in total. Suppose now that $\delta > 0$. We consider three rays r_0, r_1, r_2 with a common origin pairwise forming a 120° angle and we place δ vertices on each line. We denote by $u_{i,1}, u_{i,2}, \ldots, u_{i,\delta}$ the vertices of line r_i $(i = 0, 1, 2)$ in the order they appear along r_i starting from the origin (see Figure 10(a)). In the following, indices will be taken modulo 3 when working with the indices of the rays r_i. To draw path P_i $(i = 0, 1, 2)$ we draw the edges $(u_{i,1}, u_{i+1,1})$, $(u_{i,j}, u_{i+1,j-1})$, and $(u_{i,j}, u_{i+1,j})$ (for $j = 2, \ldots, \delta$) as straight-line segments. Notice that, these edges form a zig-zagging path between the vertices of rays r_i and r_{i+1}, so P_i passes through all vertices of r_i and r_{i+1} but not through the vertices of r_{i+2}. To include these missing vertices in P_i, we add to P_i edges $(u_{i+2,j}, u_{i+2,j+1})$ (for $j = 1, 2, \ldots, \delta - 1$).

In this way we draw two disjoint sub-paths for each path P_i, namely a zig-zagging path between r_i and r_{i+1} and a straight-line path along r_{i+2}. Moreover, we only draw 3δ edges and therefore there are still 7 missing vertices (and 8 missing edges) in each path. To add the missing vertices

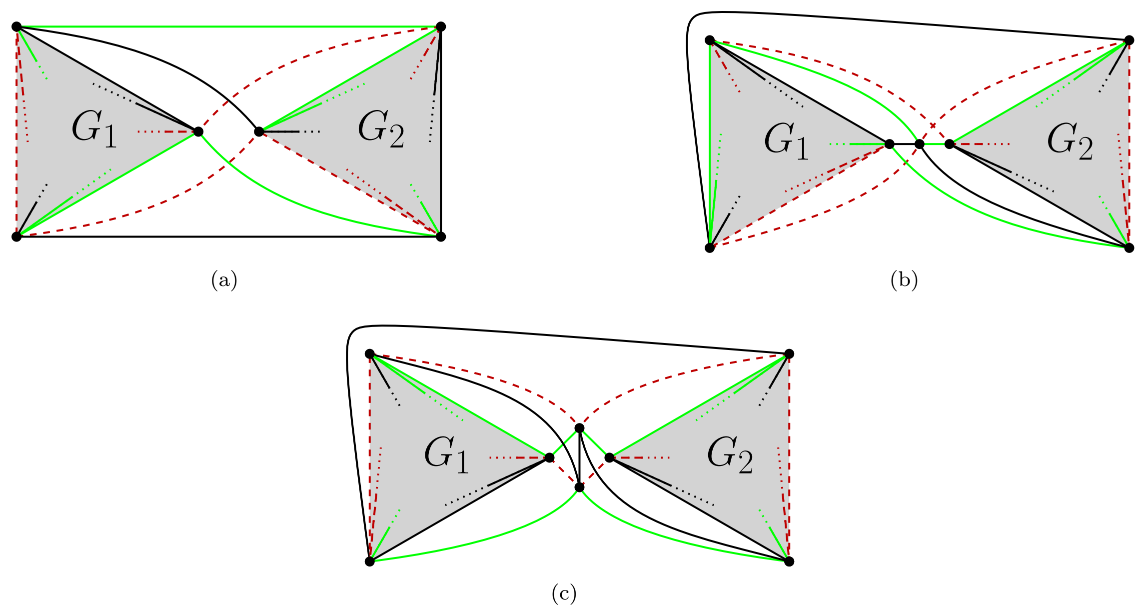


Figure 11: Illustration for the proof of Theorem 6.

and edges and to connect the two sub-paths of each path, we construct a drawing Γ_0 of three paths P'_0, P'_1, P'_2 with seven vertices as the one shown in Figure 6(a) (the same used when $\delta = 0$). Denote with v_i and w_i the end-vertices of P'_i in Γ_0. We place Γ_0 inside the triangle $u_{0,1}, u_{1,1}, u_{2,1}$ and add the edges $(v_i, u_{i,1})$ and $(w_i, u_{i+2,1})$. It is easy to see (see also Figure 10(b)) that these six edges can be added so that the drawing is still 1-planar and the total number of crossings is 6. This concludes the proof for $n = 7 + 3\delta$.

If $n = 7 + 3\delta + 1$ we start with the same construction as in the previous case and then add an extra vertex v outside the triangle $u_{1,\delta}, u_{2,\delta}, u_{3,\delta}$. Notice that each of these three vertices is the end-vertex of two of the three paths with $7 + 3\delta$ vertices. Thus we can extend each path to include v by connecting it to each of the three vertices $u_{1,\delta}, u_{2,\delta}, u_{3,\delta}$ without creating any crossing (see Figure 10(c) ignoring vertex w). Hence, the resulting drawing has still six crossings.

If $n = 7 + 3\delta + 2$, then we add two extra vertices outside the triangle $u_{0,\delta}, u_{1,\delta}, u_{2,\delta}$ and connect both of them to the three vertices $u_{0,\delta}, u_{1,\delta}, u_{2,\delta}$ (recall that each of these three vertices is the end-vertex of two distinct paths with $7 + 3\delta$ vertices). In this case however the addition of the two extra vertices causes the creation of one crossing. Thus the final drawing is 1-planar and the total number of crossings is at most 7 (see Figure 10(c)). This concludes the proof for $n \geq 7$.

If $n = 6$ we construct a 1-planar packing of three paths with three crossings in total as shown in Figure 6(b). □

The construction of Theorem 5 can be extended to three cycles.

Theorem 6 *Three cycles with $n \geq 20$ vertices can be packed into a 1-plane graph with at most 14 edge crossings.*

Proof: Suppose first that $n \equiv 2 \pmod 3$. In this case, we partition the set V of the n vertices in two groups V_1 and V_2 of size $7+3\delta$ and $7+3\lambda$, with $\delta, \lambda \geq 1$. We compute a 1-planar packing of G_1 and a 1-planar packing of G_2 as described in the proof of Theorem 5, so to obtain three edge-disjoint paths for each G_i ($i = 1, 2$) with $7+3\delta$ and $7+3\lambda$ vertices respectively. Each G_i has 6 crossings and is embedded so that each path has both end-vertices on the external face (each of the three vertices of the external face is the end-vertex of two distinct paths). We create a 1-planar packing of three cycles with n vertices by connecting the two end-vertices of each path in G_1 with the two end-vertices of a path in G_2. This requires the addition of six edges that can be embedded so to form two crossings (see Figure 11(a)). Thus, the total number of crossings in the final 1-planar packing is 14.

Note that, if δ or λ are equal to 0, the technique of Theorem 5 produces a 1-planar packing of G_1 or G_2 like the one shown in Figure 6(a). In this case, the property that each path has both end-vertices on the external face does not hold. Thus, our technique does not work for δ or λ equal to 0.

In the cases when $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$, we proceed in a way similar to the previous case. If $n \equiv 0 \pmod 3$, we create two 1-planar packings G_1 and G_2 with $7+3\delta$ and $7+3\lambda$ vertices ($\delta, \lambda \geq 1$) leaving out one vertex. When G_1 and G_2 are connected to create the 1-planar packing of three cycles we also add the missing vertex as shown in Figure 11(b). Similarly, if $n \equiv 1 \pmod 3$ we create two 1-planar packings G_1 and G_2 leaving out two vertices, and then we connect G_1 and G_2 to create the 1-planar packing of three cycles by adding the two missing vertices as shown in Figure 11(c). Also in these cases, when connecting G_1 and G_2 we have two additional crossings and a total of 14 crossings in the final 1-planar packing. □

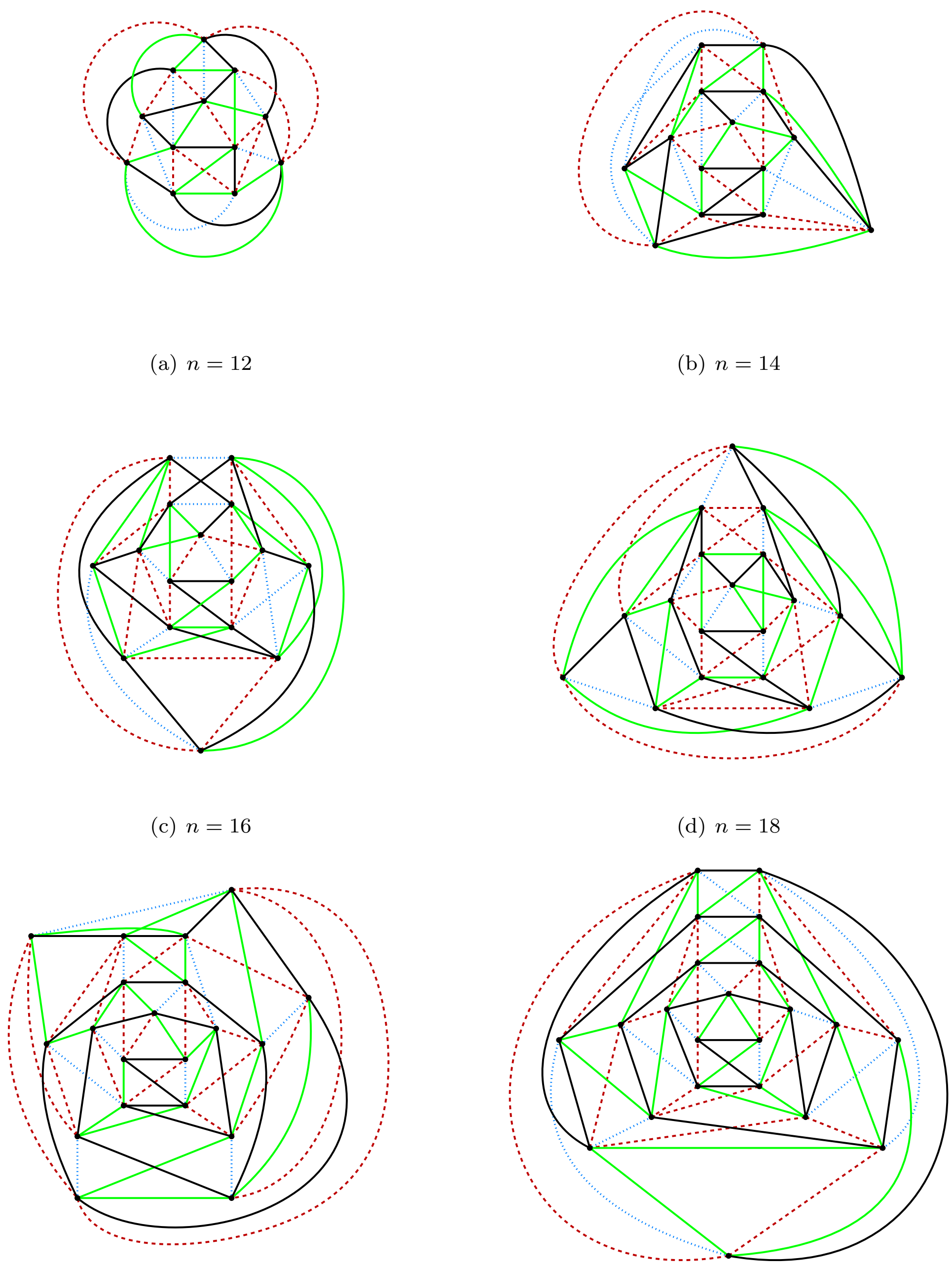
6 From Triples to Quadruples

In this section we extend the study of 1-planar packings from triples of graphs to quadruples of graphs. By Property 1, a 1-planar packing of four graphs does not exist if all graphs are connected, because the number of edges of the four graphs is higher than the number of edges allowed in a 1-planar graph. We consider therefore a quadruple consisting of three paths and a perfect matching. Notice that, in this case the number of vertices n has to be even.

Theorem 7 *Three paths and a perfect matching with $n \geq 12$ vertices admit a 1-planar packing. If $n \leq 10$, the quadruple does not admit a 1-planar packing.*

Proof: Three paths and a perfect matching have a total of $3(n-1) + \frac{n}{2} = \frac{7n}{2} - 3$ edges. Since a 1-planar graph has at most $4n-8$ edges, a 1-planar packing of three paths and a perfect matching exists only if $\frac{7n}{2} - 3 \leq 4n-8$, *i.e.*, if $n \geq 10$. If $n = 10$, we have $\frac{7n}{2} - 3 = 32$ and $4n-8 = 32$, which means that any 1-planar packing of three paths and a perfect matching with $n = 10$ vertices is an optimal 1-planar graph. It is known that every optimal 1-planar graph has at least eight vertices of degree exactly six [1]. On the other hand, in any 1-planar packing of three paths and a perfect matching all vertices, except the at most six end-vertices of the three paths, have degree seven, which implies that a 1-planar packing of three paths and a perfect matching does not exist.

We now prove that a 1-planar packing exists if $n \geq 12$. All instances having $12 \leq n \leq 22$ are pictorially proved by Figure 12. Concerning the remaining cases (*i.e.*, $n \geq 24$) we proceed as follows. Based on the fact that in any 1-planar packing of three paths and a perfect matching at least $n-6$ vertices have degree seven, we construct the desired 1-planar packing starting from a 1-planar graph G such that at least $n-6$ vertices have degree at least seven; we then partition the edges of G into five sets; three of these sets form a spanning path each, the fourth one forms

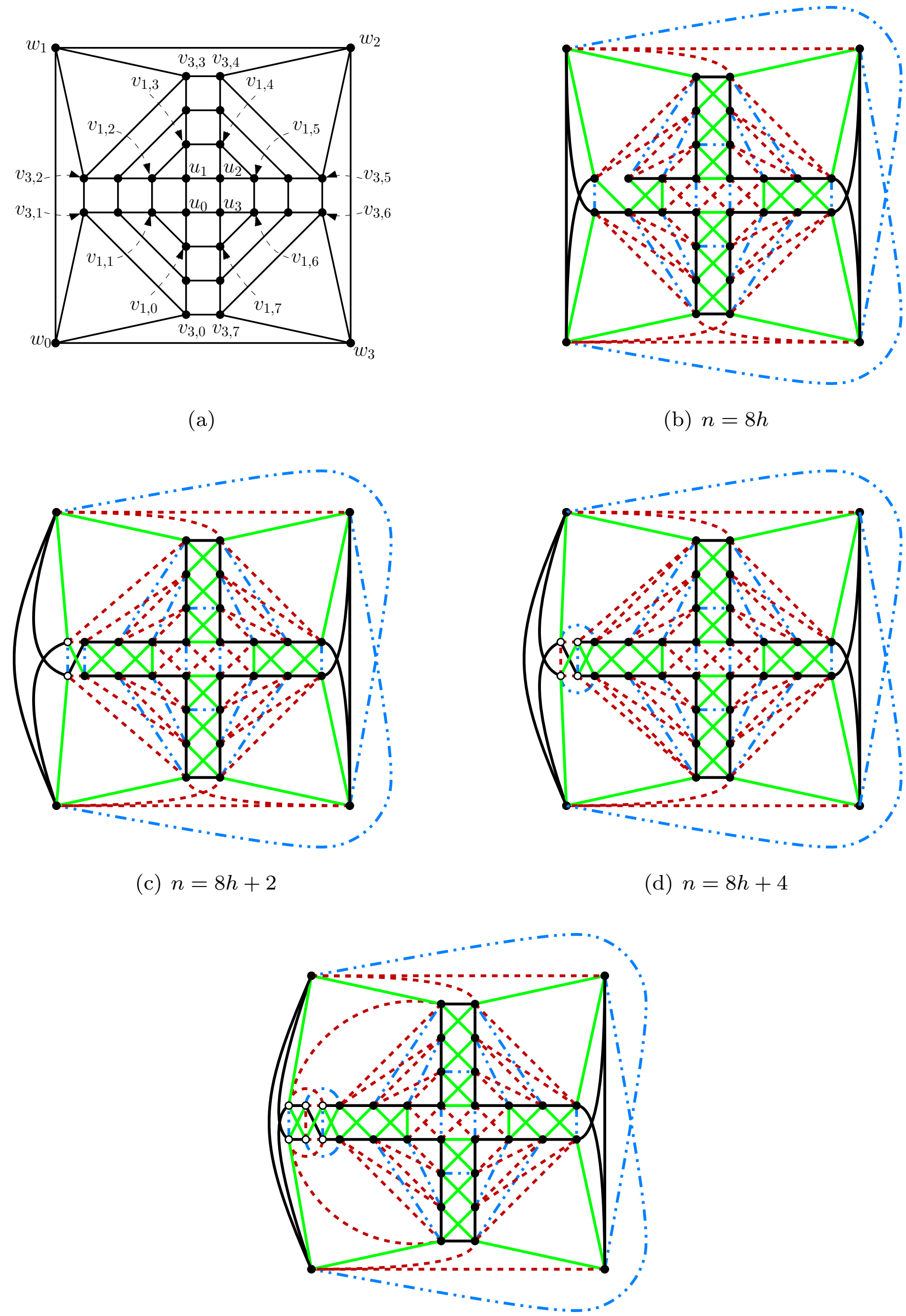


(a) $n = 12$ (b) $n = 14$

(c) $n = 16$ (d) $n = 18$

(e) $n = 20$ (f) $n = 22$

Figure 12: 1-planar packing of three paths and a perfect matching.



(a)

(b) $n = 8h$

(c) $n = 8h + 2$

(d) $n = 8h + 4$

(e) $n = 8h + 6$

Figure 13: (a) Graph G' used in the proof of Theorem 7 ($n = 8h, h = 3$). (b)–(e) 1-planar packings of three paths and a perfect matching obtained starting from G'.

a perfect matching, and the fifth one contains edges that will not be part of the 1-planar packing. For every $n = 8h$ and $h \geq 3$ it is possible to construct a 1-planar graph with n vertices each having degree at least seven as follows. We start with $h-1$ cycles $C_1, C_2, \ldots, C_{h-1}$. Each cycle C_i ($1 \leq i \leq h-1$) has eight vertices $v_{i,j}$ with $0 \leq j \leq 7$. Cycle C_i, for $1 \leq i \leq h-2$, is embedded inside cycle C_{i+1} and is connected to it with edges $(v_{i,j}, v_{i+1,j})$ for each $0 \leq j \leq 7$. We have a cycle with four vertices u_0, u_1, u_2, u_3 embedded inside C_1 and connected to it with edges $(u_j, v_{1,2j})$ and $(u_j, v_{1,2j+1})$. Finally, we have a cycle with four vertices w_0, w_1, w_2, w_3 embedded outside C_{h-1} and connected to it with edges $(w_j, v_{h-1,2j})$ and $(w_j, v_{h-1,2j+1})$. The graph G' described so far has n vertices, is planar, all its vertices have degree four, and each vertex is incident to at most one face of size three (see Figure 13(a)). By adding two crossing edges inside each face of size four, we obtain a 1-planar graph G with n vertices where each vertex has degree at least seven. The graph G and the partition of the edges of G in five sets defining three paths and a matching is shown in Figure 13(b). If n is not a multiple of 8, then it will be $n = 8h + r$, with $0 < r < 8$ and r even (because n is even). In this case we construct G' as explained above and then we extend the paths $u_0, v_{1,1}, \ldots, v_{h-1,1}$ and $u_1, v_{1,2}, \ldots, v_{h-1,2}$ to the left with 1, 2 or 3 vertices each; we then suitably rearrange the edges of G'. The graph G is then obtained, as in the previous case, by adding a pair of crossing edges inside each face of size four. The resulting graph G and a partition of its edges in five sets defining three paths and a matching is shown in Figures 13(c), 13(d), and 13(e), for the cases when $r = 2$, $r = 4$, and $r = 6$, respectively. □

7 Conclusions and Open Problems

We find that the 1-planar packing problem is a fertile and still largely unexplored research subject. We conclude the paper with a list of open problems.

- Theorems 1 and 2 show that not all triples admit a 1-planar packing if at most one of the three trees is a path. This motivated us to study triples when two of the trees are paths. On the other hand, the result of Theorem 2 holds only for $n = 7$. It is natural to ask whether two caterpillars, or even two more complex trees, and a path can always be packed if they have more than 7 vertices.

- In Section 4, we proved that two paths and a 5-legged caterpillar always admit a 1-planar packing (provided that they satisfy Property 1). A natural open problem is to extend Theorem 4 to general caterpillars. As already explained in Section 4, our technique, based on the leaf addition operation, cannot be extended to work with less than five leaves.

- In Section 5, we proved that seven crossings are sufficient for a 1-planar packing of three paths, and that fourteen crossings are sufficient for a 1-planar packing of three cycles. Is it possible to compute a 1-planar packing of three paths or cycles with the minimum number of crossings (three and six, respectively)? Can we compute 1-planar packings with few crossings for triples of other types of trees?

  In Section 6, we studied the 1-planar packing problem by considering quadruples of graphs consisting of three paths and a perfect matching. It would be interesting to investigate what happens if one considers a different number of paths and perfect matchings. In this direction, we report some preliminary observations: (i) Two paths and four perfect matchings do not admit a 1-planar packing, since they have $4n-2$ edges (recall that a 1-planar graph has at most $4n-8$ edges [15]); (ii) two paths and three perfect matchings have $\frac{7}{2}n-2$ edges, hence they may admit a 1-planar packing if $n \geq 12$; (iii) one path and six perfect matchings do not admit a 1-planar packing, since they have $4n-1$ edges; (iv) one path and five perfect matchings have $\frac{7}{2}n-1$ edges, hence they may admit a 1-planar packing if $n \geq 14$.

- We also point at the more general research direction of extending the packing problem to other families of beyond planar graphs [3].

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