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On Area-Universal Quadrangulations

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Abstract. We study drawings of plane quadrangulations such that every inner face realizes a prescribed area. A plane graph is area-universal if for every assignment of non-negative weights to the inner faces, there exists a straight-line drawing such that the area of each inner face equals the weight of the face. It has been conjectured that all plane quadrangulations are area-universal. We develop methods to prove area-universality via reduction to the area-universality of related graphs. This allows us to establish area-universality for large classes of plane quadrangulations. In particular, our methods are strong enough to prove area-universality of all plane quadrangulations with up to 13 vertices.

1 Introduction

A plane graph is a planar graph together with a crossing-free drawing. Let G be a plane graph and let F' be its set of inner faces. An *area assignment* is an assignment of a non-negative real number to every face $f \in F'$, i.e., a function $\mathcal{A}\colon F' \to \mathbb{R}_0^+$. A (potentially degenerate) planar straight-line drawing D of G *realizes* the area assignment $\mathcal{A}$ if for every $f \in F'$ the area of f in D is $\mathcal{A}(f)$. A plane graph G is *area-universal* if it has a realizing drawing for every area assignment $\mathcal{A}$.

Ringel [18] considered straight-line drawings of plane graphs such that all faces have the same area. He gave an example of a plane triangulation that has no equiareal drawing, hence, a triangulation which is not area-universal. Thomassen [19] proved that plane cubic graphs are area-universal. Biedl and Velázquez [3] showed area-universality for the class of plane 3-trees, also known as *stacked triangulations* and *Apollonian networks*. Concerning counter examples, Kleist [12, 13] introduced

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a simple counting argument to show that no Eulerian triangulation is area-universal. Moreover, she showed that every plane graph is area-universal in the class of drawings where one bend per edge is allowed. For triangulations with special vertex orders, Kleist [14] presented a sufficient criterion for their area-universality that only requires the investigation of one area assignment. Interestingly, if the sufficient criterion applies to one plane triangulation, then all embeddings of the underlying planar graph are also area-universal. Dobbins et al. [6] investigated the complexity of deciding whether a given graph is area-universal and several related problems. They conjecture that the problem is complete for the complexity class $\forall\exists\mathbb{R}$.

In this paper, we focus on plane bipartite graphs. Because the property of area-universality is preserved under edge-deletions (see also Observation 1), we consider edge-maximal plane bipartite graphs known as *quadrangulations*. Regarding the area-universality of quadrangulations little is known. Evans et al. [9] showed that the $m \times n$ grid is area-universal for all $m, n \geq 2$, even with the additional requirement that the outer face of the drawing is a rectangle. Kleist [12] showed that 2-degenerate quadrangulations are area-universal and that in the class of drawings where one bend per edge is allowed all quadrangulations have realizing drawings for all area assignments where only half of the edges have a bend.

The study of drawings in various drawing modes with prescribed face areas is summarized under the name *cartograms*. Cartograms date back to at least 1934 when Raisz [17] studied rectangular population cartograms, where the US population was visualized by representing the states with areas proportional to their population. This kind of visualization is particularly useful when showing geo-referenced statistical data in order to provide insight into patterns, trends and outliers in the world around us [22]. Cartograms have been intensely studied for duals of triangulations and rectilinear drawings with bends. The number of sides of the polygons representing a face has been improved in a series of papers from 40 sides [5], to 34 sides [11], to 12 sides [2]. Finally, Alam et al. [1] showed how to construct drawings with 8-sided faces, which is known to be optimal. Chang and Yen [4] studied contact representations of 2-connected outerplaner graphs and construct contact representations with 4-gons of prescribed area. Note that in the cartogram literature the problem is usually treated in the dual setting, i.e., weights are assigned to the vertices. We refer to Nusrat and Kobourov [16] for a survey of the cartogram literature.

Area-universality has also been studied in the context of rectangular layouts, these are dissections of a rectangle into rectangles with prescribed contacts between the rectangles of the dissection. Eppstein et al. [8] showed that a rectangular layout is area-universal if and only if it is one-sided. The key ingredient in their proof is that the weak equivalence class of any rectangular layout is area-universal. The weak equivalence class is obtained by prescribing the contacts between the segments. The area-universality of the weak equivalence class has been shown by different techniques [7, 10, 21]. This area-universality result is very special because, up to affine transformations, the rectangular layout realizing a given area assignment is actually unique.

The class of drawings: We study area-universality of plane quadrangulations. To realize non-negative face areas, we extend the set of planar straight-line drawings of a plane quadrangulation by all drawings which can be obtained as the limit of a sequence of planar straight-line drawings (e.g. specified by the coordinates of the vertices). In particular, we allow *degenerate drawings* in which vertices and edges sharing a face may (partially) coincide; if two edges partially coincide their union forms a segment.

In various cases, considering this enriched set of drawings allows for simpler proofs [9, 13, 19]. For example, the counting argument by Kleist [13] greatly benefits from allowing face area 0 and degenerate drawings. In the case of triangulations, degeneracies occur if and only if some face has area 0 and the set of realizable area assignments is closed [13, Lemma 4], i.e., allowing or disallowing face area 0 and degenerate drawings does not influence whether or not all considered area assignments are realizable. For examples of how to obtain non-degenerate realizing drawings from degenerate drawings, we refer to 1-bend drawings of plane graphs [13, Theorem 3 & Theorem 6] and table cartograms [9, Theorem 2].

Outline of this paper: In Section 2, we investigate operations that preserve area-universality. In Section 3, we use one of these operations, the edge contraction, to show area-universality of grids and large classes of angle graphs. In particular, we consider angle graphs of triangulations that are close to being area-universal. In Section 4, we study strong area-universality, i.e., area-universality within a prescribed outer face. Strong area-universal graphs may serve as building blocks for constructing area-universal quadrangulations. We show that not every plane bipartite graph is strongly area-universal and present families of strongly area-universal graphs. Shape restrictions are also the subject of Section 5 where we study convex drawings. We present both a large family of quadrangulations that are not convex area-universal and examples of strongly convex area-universal graphs. In Section 6, we use our tools to show area-universality of all quadrangulations with at most 13 vertices. In some cases the argument relies on the known area-universality of the class of double stacking graphs.

2 Area-Universality Preserving Operations

We begin with an easy observation which can also be found in [3] and [13].

Observation 1 *A subgraph of an area-universal plane graph is area-universal.*

Therefore, a proof for the area-universality of plane quadrangulations, i.e., maximal plane bipartite graphs, would imply area-universality of all plane bipartite graphs. The following lemma extends Observation 1 with a new operation. A set of edge contractions in a plane graph G is *face-maintaining* if the contractions do not change the number of faces in G, i.e., for a face of degree d at most $d-3$ edges are contracted.

Lemma 1 *Let G be a plane graph that can be transformed into an area-universal plane graph G' by inserting vertices, inserting edges, and performing face-maintaining edge contractions. Then G is area-universal.*

Proof: Let $\mathcal{A}$ denote an area assignment of G. A face f in G corresponds to a (non-empty) collection of faces C_f in G'. We define $\mathcal{A}'$ such that for each inner face f of G it holds that $\mathcal{A}(f) = \sum_{f' \in C_f} \mathcal{A}'(f')$. Since G' is area-universal, there exists an $\mathcal{A}'$-realizing drawing D' of G'. Simply deleting all vertices and edges of G' which are not in G yields a (degenerate) drawing D of G. By definition of $\mathcal{A}'$, D is $\mathcal{A}$-realizing. □

There exists a further operation that preserves area-universality and is based on decomposition. For an illustration consider Figure 1. From a plane graph G with a simple cycle C, we obtain two plane graphs G_{I} and G_{E} by *decomposing along* C: G_{I} is the subgraph of G consisting of C and its interior, while G_{E} is the subgraph of G consisting of C and its exterior. Reversely, we obtain G from G_{I} and G_{E} by identifying the outer face of G_{I} with the inner face of G_{E} whose boundary is C.

A plane graph G is *strongly area-universal* if for every area assignment $\mathcal{A}$ of G and every fixed polygonal placement of the outer face of area $\Sigma\mathcal{A}$, there exists a realizing straight-line drawing of G within the prescribed outer face. Here we have used $\Sigma\mathcal{A}$ to denote the sum of all assigned areas,

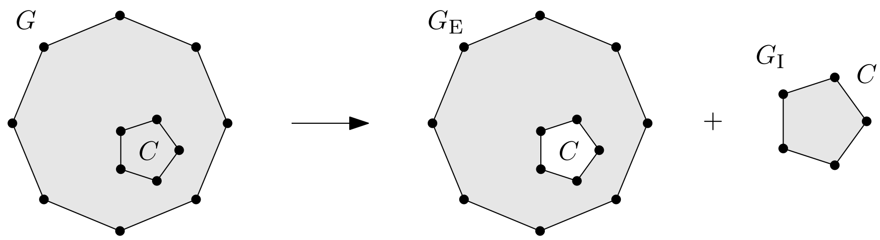


Figure 1: Decomposing G along C yields two plane graphs G_{E} and G_{I}. If G_{E} is area-universal and G_{I} strongly area-universal, then G is area-universal.

i.e., $\Sigma\mathcal{A} := \sum_{f\in F'} \mathcal{A}(f)$. Since all triangles are affine equivalent, a plane graph with a triangular outer face has a realizing drawing (if it exists) within every triangle of correct area [13]. It follows that:

Observation 2 *Plane graphs with triangular outer faces (e.g. triangulations) are area-universal if and only if they are strongly area-universal.*

A similar result for quadrangulations would be a pleasant surprise.

Lemma 2 *Let G be a plane graph with a simple cycle C, and G_{I} and G_{E} obtained by decomposing G along C. If G_{E} is area-universal and G_{I} is strongly area-universal, then G is area-universal. Moreover, if G_{E} is strongly area-universal, then G is also strongly area-universal.*

Proof: Let $\mathcal{A}$ be an area assignment of G. For $i \in \{\text{I},\text{E}\}$, $\mathcal{A}_i$ denotes the induced area assignment of G_i. Note that the interior of C is a face f of G_{E}. Particularly, it holds that $\mathcal{A}_{\text{E}}(f) = \Sigma A_{\text{I}}$. Because G_{E} is area-universal, there exists an $\mathcal{A}_{\text{E}}$-realizing drawing D_{E} of G_{E}. Since G_{I} is strongly area-universal, we find an $\mathcal{A}_{\text{I}}$-realizing drawing D_{I} of G_{I} whose outer face is the polygon representing C in D_{E}. Thus, identifying D_{E} and D_{I} along C yields an $\mathcal{A}$-realizing drawing of G. $\square$

The ideas of this lemma have been used in [13] to show the strong area-universality of 2-degenerate quadrangulations. Recall that a graph is k-degenerate if and only if every subgraph contains a vertex of degree at most k.

Proposition 3 (Kleist [13], Proposition 15) *Every 2-degenerate quadrangulation is strongly area-universal.*

It is easy to see that K_4 is area-universal, i.e., a vertex of degree 3 can be inserted into a triangle so that the three small triangles partition the big triangle in any prescribed ratio. This yields the following lemma.

Lemma 4 *Let T be a plane graph and T^+ the plane graph where a vertex of degree 3 is inserted (stacked) into a triangle of T. Then T is area-universal if and only if T^+ is area-universal.*

Since a plane 3-tree is obtained from a triangle by iteratively stacking vertices into faces, Lemma 4 yields the result from [3]: Plane 3-trees are area-universal.

3 Area-Universality via Edge Contractions

In this section, we discuss some implications of the edge contractions of Lemma 1.

Firstly, we show an alternative proof for the area-universality of grid graphs. The *grid* $G(m, n)$ is the Cartesian product $P_m \times P_n$ of paths P_m and P_n on m and n vertices, respectively. Figure 2(a) illustrates the grid $G(9, 6)$. Area-universality of grid graphs was first proved in the context of *table cartograms* where additionally the outer face is required to be a rectangle [9]. Our new proof does not yield a rectangular outer face; however, it is straight-forward and very simple. The reader is invited to check that Proposition 3 does not imply the area-universality of grid graphs, because $G(m, n)$ is not a subgraph of a 2-degenerate quadrangulation if $n, m \geq 3$.

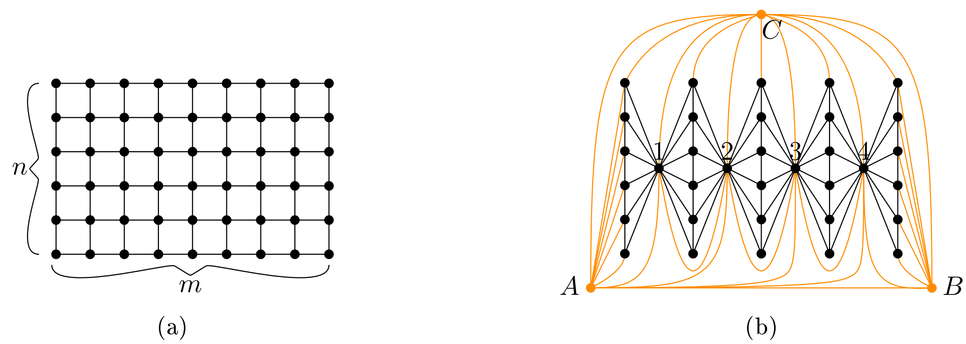


Figure 2: Illustration of Proposition 5 and its proof. (a) The grid graph $G(m, n)$ and (b) an area-universal triangulation 'containing' it.

Proposition 5 *Every grid is area-universal.*

Proof: The idea of this proof is easy to convey by picture; see Figure 2. Contract the edges of every second column of $G(m, n)$ to *super vertices* that are labeled by $1, \dots, k$ from left to right. Then, we add vertices and edges to enhance the resulting graph to a triangulation G as depicted in Figure 2(b). The graph G is a stacked triangulation since the interior of each triangle $(i, i+1, C)$ with $1 \leq i \leq k-1$, and the graph induced by $A, B, C, 1, \dots, k$ is a stacked triangulation. Thus, G is area-universal. Therefore, every grid graph can be transformed into a subgraph of an area-universal graph using face-maintaining edge contractions. Hence, by Lemma 1, grids are area-universal. □

The *angle graph* of a plane graph G is the graph Q_G with vertex set consisting of the vertices and faces of G and edges corresponding to face-vertex incidences. If G is 2-connected, then Q_G is a quadrangulation. Clearly, an angle graph is bipartite where the two bipartition classes are the vertices V and the faces F of G. In the following we consider angle graphs of triangulations. For a plane graph G and its angle graph Q, their *union (graph)* $G + Q$, consists of the union of the vertex and edge sets of G and Q. Note that the union is again a plane graph: Indeed, the vertex set of $G + Q$ coincides with the vertex set of Q. Hence, $G + Q$ can be understood as the quadrangulation Q together with the edges between the vertices of one bipartition class, namely V.

Proposition 6 *The angle graph Q of an area-universal triangulation T is area-universal.*

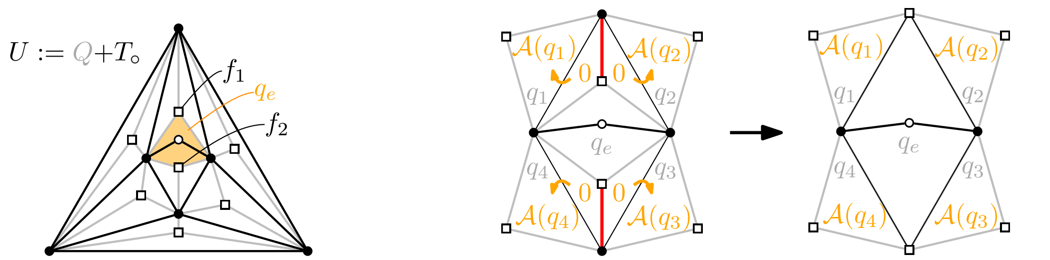
Proof: The graph $T + Q$ can be seen as T where a vertex of degree 3 is inserted in every face. By Lemma 4, $T + Q$ is area-universal. Thus, Observation 1 implies that Q is area-universal. □

Note that the same approach shows that angle graphs of equiareal triangulations are equiareal. Moreover, a straightforward consequence of Proposition 6 is that angle graphs of stacked triangulations are area-universal.

In order for its angle graph to be area-universal, it suffices for a triangulation to be *close to* area-universal. As shown in [12], every plane graph has an area-universal subdivision. The *subdivision number* $s(G)$ of a plane graph G is the minimum number of subdivision vertices to be inserted into G such that it becomes area-universal. If G is area-universal, then clearly $s(G) = 0$. To generalize Proposition 6, we introduce the notion of a *refined area assignment*. If G_1 is a subgraph of G_2, then every face f of G_1 corresponds to a collection of faces C_f in G_2. An area assignment $\mathcal{A}_1$ of G_1 *is refined* by an area assignment $\mathcal{A}_2$ of G_2 if $\mathcal{A}_1(f) = \sum_{s \in C_f} \mathcal{A}_2(s)$. We also say $\mathcal{A}_2$ *refines* $\mathcal{A}_1$.

Theorem 7 *The angle graph Q of a plane triangulation T with $s(T) \leq 1$ is area-universal.*

Proof: Figure 3(a) illustrates the proof. Let e be an edge of T such that subdividing e yields the area-universal graph $T_\circ$. The strategy is as follows: For an area assignment $\mathcal{A}$ of Q, we define a refining area assignment $\mathcal{A}'$ of the union $U := Q + T_\circ$. Let $\mathcal{A}_\circ$ be the unique area assignment of $T_\circ$ such that $\mathcal{A}'$ refines $\mathcal{A}_\circ$. The drawing of $T_\circ$ realizing $\mathcal{A}_\circ$ yields an $\mathcal{A}'$-realizing drawing of U.



(a) The graph $U := Q + T_\circ$ where T is the octahedron graph.

(b) Faces in the neighborhood of the subdivided edge e.

Figure 3: Illustration of Theorem 7 and its proof.

For the definition of $\mathcal{A}'$, note that every face f of Q corresponds to two faces in U. Let q_e denote the face of Q that is split by the subdivided edge e in U. We arbitrarily partition the area $\mathcal{A}(q)$ assigned to face q between the two corresponding faces in U, for all faces q in Q except the four faces q_1, q_2, q_3, and q_4 sharing a boundary edge with q_e. Let f_1 and f_2 denote the two faces adjacent to e in T. For q_i, $1 \leq i \leq 4$, assign area $\mathcal{A}(q_i)$ to the triangular face from U which is neither incident to f_1 nor to f_2 in U; assign an area of 0 to the triangular faces incident to f_1 or f_2. This defines the area assignment $\mathcal{A}'$ and $\mathcal{A}_\circ$ is the area assignment of $T_\circ$ that is refined by $\mathcal{A}'$ of U.

Let $D_\circ$ be an $\mathcal{A}_\circ$-realizing drawing of $T_\circ$. Since each vertex $f \in F - \{f_1, f_2\}$ of U acts as a vertex of degree 3 stacked into a face of $T_\circ$, we can insert f in $D_\circ$ by Lemma 4 such that the areas of $\mathcal{A}'$ are realized. To obtain an $\mathcal{A}'$-realizing drawing of U, it remains to insert f_1 and f_2. We call the highlighted (thick, red) edges in Figure 3(b) incident to f_1 and f_2, the *red edges* of f_1 and f_2, respectively. By definition of A', the red edges must be contracted in every $\mathcal{A}'$-realizing drawing of U. Consequently, given a drawing $T_\circ$, we easily insert f_1 and f_2 at the same location as the already placed vertex of the red edges, respectively. This yields an $\mathcal{A}'$-realizing drawing of U. Since $\mathcal{A}'$ refines $\mathcal{A}$, deleting the edges of $T_\circ$ yields an $\mathcal{A}$-realizing drawing of Q. □

This ties in with a result based on an operation called *diamond addition*. Let G be a plane graph and e an edge incident to two triangular faces consisting of e and the vertices u_1 and u_2, respectively. Applying a diamond addition of order k on edge e of G results in a graph G' in which the edge e is subdivided by vertices $v_1, \dots, v_k$ which are also adjacent to u_1 and u_2, as illustrated in Figure 4.

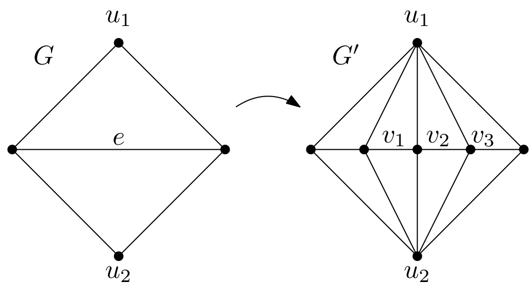


Figure 4: A diamond addition on edge e.

Two diamond additions are *disjoint* if the partitioned triangles are different. For instance, in the left graph of Figure 5, diamond additions on the edges Au and vw are disjoint, while diamond additions on the edges Au and uv are not. Together with Theorem 7, the following theorem implies that the angle graphs of area-universal triangulations on which one diamond addition has been applied are area-universal.

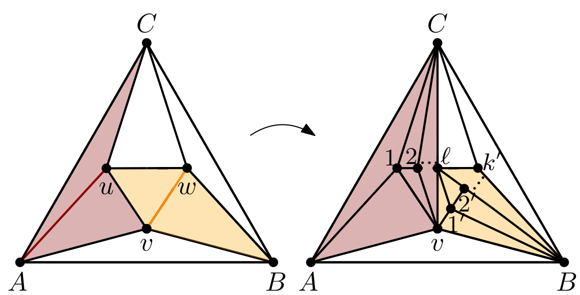


Figure 5: A double stacking graph $H_{\ell,k}$.

Theorem 8 (Kleist [15], Theorem 33) *Let G be a graph obtained from an area-universal graph G' by (multiple disjoint) diamond addition(s) adding k vertices in total. Then, $s(G) \leq k$.*

As a special case of graphs obtained by diamond additions, Kleist [14] studied *double stacking graphs* $H_{\ell,k}$ that can be obtained from the plane octahedron graph. Labeling the octahedron as in Figure 5, $H_{\ell,k}$ is obtained by applying one diamond addition of order $\ell - 1$ on Au and one diamond addition of order $k - 1$ on vw. The octahedron graph is the smallest graph of this class and has parameters $\ell = k = 1$.

In the following we use ideas similar to those used in the proof of Theorem 7, to show the area-universality of angle graphs obtained from area-universal triangulations on which several diamond additions have been applied. We start by considering a special configuration that appears in the neighborhood of an edge on which a diamond addition has been performed. A tent graph T_k is a plane graph with the outer face $v, x_0, x_1, x_2, \dots, x_{k+1}$ and inner vertices $y_0, y_1, \dots, y_k$ where y_i is incident to x_i, x_{i+1} and v. Figure 6(a) depicts the tent graph T_3.

(a) A tent graph T_k. (b) An $\mathcal{A}$-realizing drawing of T_k.

Figure 6: Illustration of Lemma 9 and its proof.

Lemma 9 *Every area assignment $\mathcal{A}$ of a tent graph T_k has an $\mathcal{A}$-realizing drawing within each triangle that has area $\Sigma\mathcal{A}$ and corners v, x_0, x_{k+1}. Moreover, the length of every segment $x_i x_{i+1}$ can be made proportional to the area of the incident triangle.*

Proof: We denoted the assigned areas of T_k by a_i and b_i as depicted in Figure 6(b). We position x_i on the segment $x_0 x_{k+1}$ such that

$$\|x_{i+1} - x_i\| = \frac{b_i}{\sum_j b_j} \|x_{k+1} - x_0\|.$$

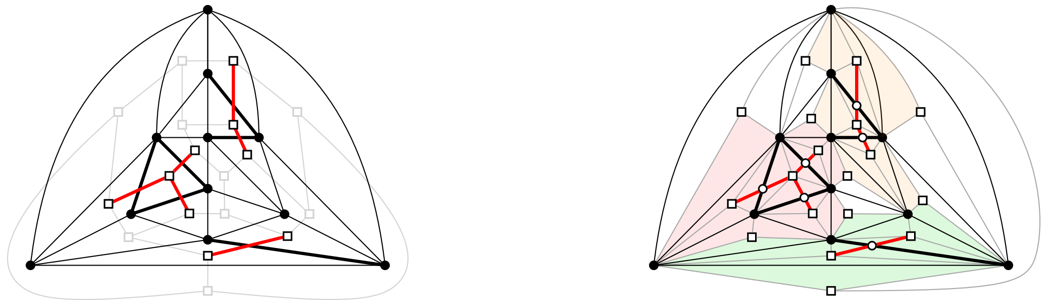
Then, in a realizing drawing the vertices y_i lie on a line ℓ parallel to the segment $x_0 x_{k+1}$ such that for each point x on ℓ the triangle $x_0 x_{k+1} x$ has area $\sum_i b_i$. Note that, by the placement of x_i, each position of y_i on ℓ realizes b_i. Thus, we may use the freedom to realizing a_i when placing y_i on ℓ with the following procedure: Defining y_{-1} as the intersection of the segment vx_0 with the line ℓ, we suppose that y_{i-1} is placed already when we consider y_i for $i \geq 0$. Move y_i rightwards on the line ℓ starting at y_{i-1} and observe the area of the face $vy_{i-1}x_iy_i$. Clearly, it starts at 0 and increases continuously. The intermediate value theorem guarantees a position, where the area equals a_i. We place y_i at the corresponding position and continue with y_{i+1}. Due to the correct total area, the area of a_{k+1} is realized if all other face areas are correct. Thus, we obtain an $\mathcal{A}$-realizing drawing of T_k. □

In the following theorem, we not only consider several disjoint diamond additions, but also groups of non-disjoint diamond additions that are *far apart*. A set of edges of a plane triangulation T is called *far apart* if the subgraph of the dual graph induced by the duals of these edges and their vertices is a collection of stars; see Figure 7(a) for an example.

Theorem 10 *The angle graph Q of a plane triangulation T is area-universal if one of the following holds:*

(i) T is obtained from an area-universal triangulation T' by several disjoint diamond additions of an arbitrary order.

(ii) T has a set of edges that is far apart *such that subdividing each of them at most once yields an area-universal subdivision $T_\circ$.*



(a) The set of thick edges is far apart.

(b) A set of subdivided edges that is far apart.

Figure 7: Illustration of (subdivided) edges that are far apart.

Proof: To prove (i), we consider a diamond addition of order k applied on an edge (u, w) of T'. Let $T'_\circ$ denote the graph obtained from the triangulation T' by subdividing the edge (u, w) with k additional vertices as in T; in other words the edge (u, w) is replaced by a path P with $k+1$ edges. Let A and B denote the two common neighbors of u and w in T' such that Auw and Buw are faces in T'. Recall that Q is the angle graph of T and consider the union $U := Q + T'_\circ$. Define H as the restriction of U to the interior of $AuBw$. Figure 8 depicts H for a diamond addition of order 3.

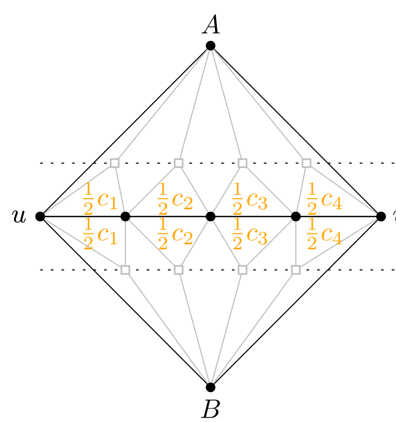


Figure 8: The graph H for a diamond addition of order 3; together with $\mathcal{A}_H$.

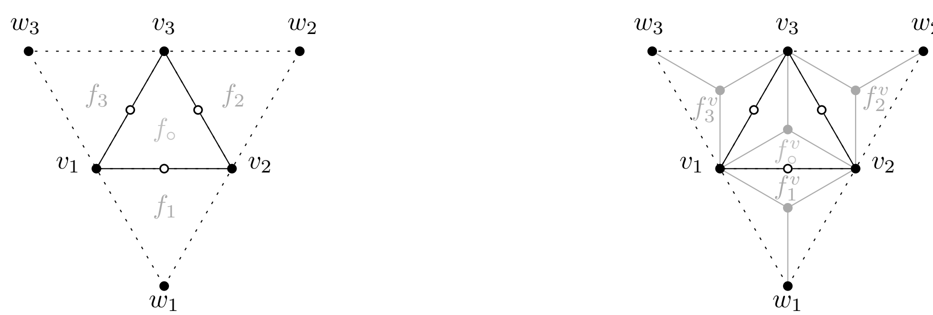
Given an area assignment $\mathcal{A}$ of Q, we construct an area assignment $\mathcal{A}'$ of T' and an area assignment $\mathcal{A}_U$ of U that refines both $\mathcal{A}$ and $\mathcal{A}'$. Observe that every face q of Q is either a face of U or corresponds to two faces of U.

In the latter case, we partition the prescribed area of q equally between its two faces in U and obtain the area assignment $\mathcal{A}_U$ of U. The restriction of $\mathcal{A}_U$ to H is denoted $\mathcal{A}_H$. We define $\mathcal{A}'$ of T' as the area assignment refined by $\mathcal{A}_U$, where we identify the path P and the edge (u, w). From an $\mathcal{A}'$-realizing drawing D' of T', we construct an $\mathcal{A}_U$-realizing drawing of U as follows: First, we add all face vertices not adjacent to P using Lemma 4; recall that they act as vertex of degree 3 in a triangle.

Observe that splitting H along P results in two tent graphs T_k. Consequently, we may use Lemma 9 to reinsert each of the two tent graphs of H. By definition of $\mathcal{A}_H$, the subdivision vertices on (u, w) are placed consistently when applying Lemma 9 to the two tent graphs when splitting H along P. Here we use the fact that the assigned areas of Q were split equally into two when defining $\mathcal{A}_U$. Since A_U refines $\mathcal{A}$, we obtain an $\mathcal{A}$-realizing drawing of Q by deleting the edges of $T'_\circ$. This proves (i).

Now, we show (ii). First, we consider the case when the set of subdivided edges of T forms only one dual star, i.e. T has a face $f_\circ$ such that subdividing each edge incident to $f_\circ$ (at most once) yields an area-universal graph $T_\circ$. Theorem 7 shows the claim if exactly one edge of $f_\circ$ is subdivided. In the following, we show how to deal with the case of three subdivision vertices. The case of two subdivision vertices can be handled by a slight modification which is explained afterwards.

We denote the three faces incident to subdivision vertices in $T_\circ$ by $f_\circ$, f_1, f_2, and f_3 as illustrated in Figure 9; thecorresponding vertices in Q are denoted by $f_\circ^v$, f_1^v, f_2^v, f_3^v. Moreover, we let v_1, v_2, v_3 be the vertices of $f_\circ$ and let w_i be the third vertex of f_i as depicted. Note that two of the w-vertices may coincide implying that a v-vertex has degree 3. However, at most one v-vertex has degree 3; otherwise T is the complete graph on four vertices which is area-universal. Therefore, it remains to consider two cases: One v vertex has degree 3 and none v-vertex has degree 3.



(a) Neighborhood of $f_\circ$ in $T_\circ$.

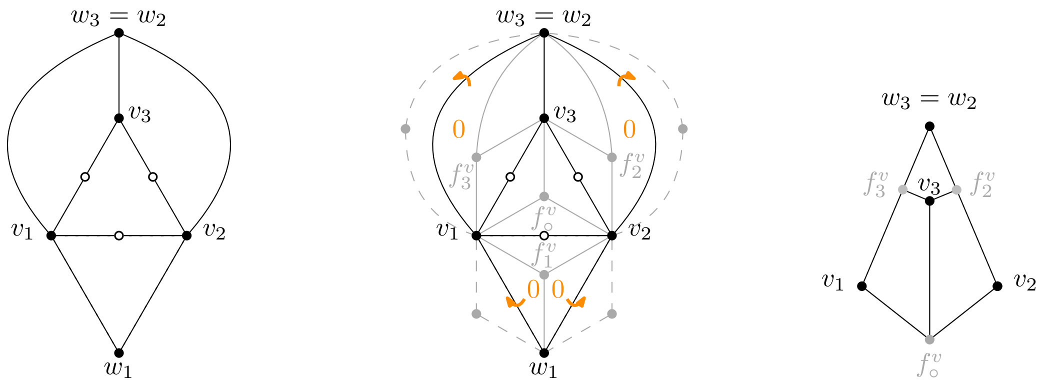
(b) Neighborhood of $f_\circ^v$ in U, where $T_\circ$ is depicted in black and Q in gray.

Figure 9: Illustration of the notation for the proof of Theorem 10(ii).

We first consider the case that v_3 has degree 3. Figure 10(a) illustrates the neighborhood of $f_\circ$ in this case. Our strategy is as follows: For an area assignment $\mathcal{A}$ of Q, we define an area assignment $\mathcal{A}_U$ of the union $U := Q + T_\circ$. This yields a unique area assignment $\mathcal{A}_\circ$ of $T_\circ$ such that $\mathcal{A}_U$ of U refines $\mathcal{A}_\circ$ of $T_\circ$. From an $\mathcal{A}_\circ$-realizing drawing of $T_\circ$, we construct an $\mathcal{A}_U$-realizing drawing of U. Deleting the edges of $T_\circ$ results in an $\mathcal{A}$-realizing drawing of Q.

For a given area assignment $\mathcal{A}$ of Q, we construct an area assignment $\mathcal{A}_U$ of U such that the area of every face q of Q is partitioned between the two faces of $T_\circ$. Let B denote the edges of $T_\circ$ bounding the region formed by the faces $f_1, f_2, f_3, f_\circ$ in $T_\circ$; note that B has four edges since the common vertex of $f_\circ, f_2, f_3$ has degree 3. The edges of B divide four faces of Q in U. For all faces outside B (and not incident to B), the face areas of Q are partitioned arbitrarily between the two faces of U. The area of each of the four faces of Q divided by B is assigned to the subface outside B as indicated in Figure 10(b). We define $\mathcal{A}_\circ$ of $T_\circ$ as the unique area assignment refining $\mathcal{A}_U$.

Given an $\mathcal{A}_\circ$-realizing drawing of $T_\circ$, note that every vertex of Q outside B is of degree 3 and can be inserted with Lemma 4. We will redraw all vertices inside B. By definition of $\mathcal{A}_U$, two incident triangular faces of f_1^v are supposed to be 0. Hence, we must place f_1^v at the same location



(a) Neighborhood of $f_\circ$ in $T_\circ$ when v_3 has degree 3.

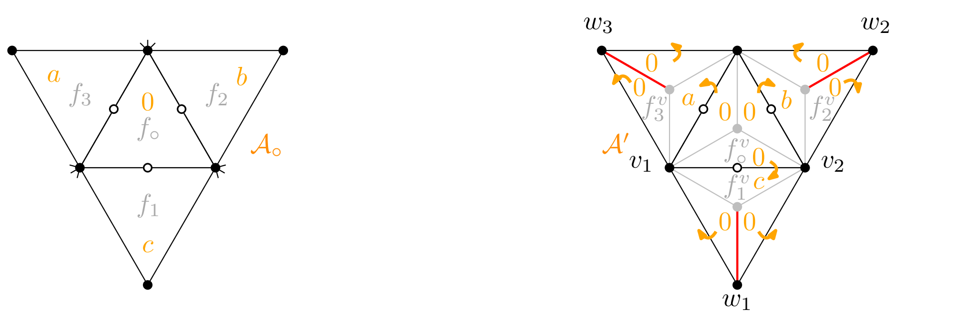
(b) Neighborhood of $f_\circ^v$ in U; $\mathcal{A}_U$ is given in orange.

(c) The graph G.

Figure 10: Illustration of the proof of Theorem 10(ii) for three subdivision vertices if there exists a vertex of degree 3.

as w_1. Then, we place the vertex $f_\circ^v$ as a vertex of degree 2 in the quadrangle $v_1 f_1^v v_2 w_2$ such that the area of the quadrangle $v_1 f_1^v v_2 f_\circ^v$ is correct, this is possible by Proposition 3. It remains to realize the graph G illustrated in Figure 10(c). We later show that every area assignment of G is realizable within any fixed outer face of correct total area: This follows from Lemma 13 and the fact that G is the core of $c(S_3)$.

Now, we turn to the case that no v_i has degree 3. This implies that all w_i are distinct. The resulting neighborhood of $f_\circ^v$ is illustrated in Figure 11(a). Let $\mathcal{A}$ be an area assignment of Q. In a first step, we define an area assignment $\mathcal{A}_U$ of the union $Q + T_\circ =: U$. Note that every face of Q corresponds to two faces in U. Except for the faces incident to $f_\circ^v, f_1^v, f_2^v, f_3^v$, we arbitrarily partition the area $\mathcal{A}(q)$ of a face q of Q between its two faces in U. For the faces incident to $f_\circ^v$, f_1^v, f_2^v, f_3^v, we assign their area as depicted in Figure 11(b).



(a) Neighborhood of $f_\circ$ in $T_\circ$ and $\mathcal{A}_\circ$.

(b) Neighborhood of $f_\circ$ in $Q + T_\circ$ and $\mathcal{A}_U$.

Figure 11: Illustration of the proof of Theorem 10(ii) for three subdivision vertices of high degree.

Let $\mathcal{A}_\circ$ be the area assignment of $T_\circ$ refining $\mathcal{A}_U$. Since $T_\circ$ is area-universal there exists an $\mathcal{A}_\circ$-realizing drawing $D_\circ$ of $T_\circ$. Most face-vertices of Q act as vertices of degree 3 stacked into triangles of $T_\circ$. Hence, by Lemma 4 and we can insert them in $D_\circ$ such that they realize the area of $\mathcal{A}_U$. It remains to insert the vertices f_1^v, f_2^v, f_3^v and $f_\circ^v$. By definition of $\mathcal{A}_U$, we must place f_i^v as to coincide with w_i, i.e., such that the edge $f_i^v w_i$ is contracted. To place $f_\circ^v$ in $D_\circ$, we need some geometric considerations.

The area of $f_\circ$ in $D_\circ$ is 0, therfore at each v_i the two boundary edges of $f_\circ$ are collinear. If the slopes of the 3 supporting lines are pairwise different, then the triangle formed by the three lines is in $f_\circ$, whence the triangle has area 0 which means that the three lines intersect in a point p. This point is the position of the three subdivision vertices in the drawing $D_\circ$ and can be used for $f_\circ^v$. If two of the lines have the same slope, then because they share one of the subdivision vertices they coincide. If the third line has a different slope, then the intersection point of the lines is a good position for $f_\circ^v$. If all three lines coincide there are many different foldings of the boundary of $f_\circ$, we leave it as an exercise to show that in each case there is a position for $f_\circ^v$ such that the edges to v_1, v_2, v_3 can be drawn 'inside' $f_\circ$.

If there are two subdivision vertices on the boundary of $f_\circ$ we use the area assignment $\mathcal{A}_U$ and the corresponding $\mathcal{A}_\circ$ as in the previous case. In the drawing $D_\circ$ we pretend that the third edge of $f_\circ$. The considerations for the case of three subdivided edges show that there is a good position for $f_\circ^v$.

Since the set of subdivided edges is *far apart*, every subdivided edge belongs to a star. We handle each star separately as described above; in particular, the star consists of one, two or three edges since T is a triangulation. By the independence, for every two stars, the edges of T surrounding the regions of the stars are disjoint; these edges form a so-called *boundary cycle* of a star. For an example consider Figure 7(b).

Note that in all cases, when defining $\mathcal{A}_U$ from $\mathcal{A}$, only the areas inside and adjacent to the boundary cycle are affected. Since these sets of faces in U are disjoint, the subdivision vertices can be handled independently. This finishes the proof. □

The results of Proposition 6 and Theorems 7, 8 and 10 imply the area-universality of several classes of angle graphs.

Corollary 11 *The angle graph of a plane triangulation T is area-universal if*

- *T is a stacked triangulation,*
- *T is 4-connected and has at most ten vertices, or*
- *any (possibly a different) embedding of T is a double stacking graph $H_{\ell,k}$.*

Proof: Stacked triangulations are area-universal, hence Proposition 6 implies the area-universality of its angle graphs. Theorem 8 can be used to show that triangulations with at most nine vertices and all embeddings of ℓk-double stacking graphs have subdivision number at most 1 [15]. Consequently, Theorem 7 implies that their angle graphs are area-universal. Moreover, 4-connected plane triangulations on ten vertices can be obtained from area-universal triangulations by at most two disjoint diamond additions. Thus, their area-universality follows from Theorem 10(i). □

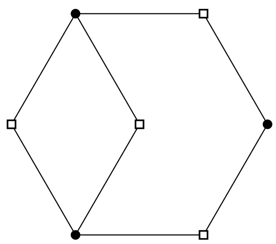
4 Strongly Area-Universal Quadrangulations

In this section, we study strongly area-universal quadrangulations. Recall that a quadrangulation is *strongly area-universal* if it is area-universal within every fixed outer face of the correct total area. A nice property of this class is that we can stack any strongly area-universal into a face of an (strongly) area-universal quadrangulation to obtain an (strongly) area-universal quadrangulation.

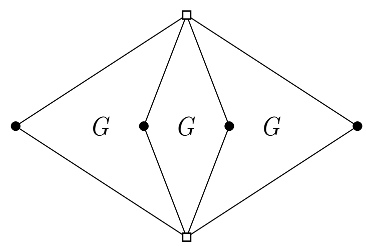
Therefore, few small strongly area-universal quadrangulations can serve as building blocks in order to construct infinite and rich families of area-universal quadrangulations.

Note that for $n > k$, the area of a convex n-gon strictly exceeds the area of any contained k-gon. Therefore, we immediately obtain that the plane bipartite graph depicted in Figure 12(a) is not strongly area-universal: Fixing the outer face as a regular hexagon, there exists no drawing in which the inner 4-face covers more than $^2/_3$ of the area.

Observation 3 *Not all plane bipartite graphs are strongly area-universal.*



(a) A plane bipartite graph that is not strongly area-universal.



(b) Illustration of the proof of Proposition 12.

Figure 12

In contrast, we do not know of a quadrangulation that is not strongly area-universal. Neither do we know of an area assignment that requires a convex outer face. In fact these two questions are closely related.

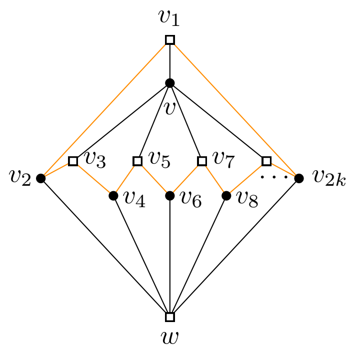
Proposition 12 *If there exists a plane quadrangulation G and an area assignment $\mathcal{A}$ such that every realizing drawing has a convex outer face, then there exists a plane quadrangulation H that is not area-universal. Moreover, if G is 3-connected, we can ensure that H is 3-connected.*

Proof: Suppose we are given a quadrangulation G and an area assignment $\mathcal{A}$ with the described properties. Let H_0 denote the plane graph of $K_{2,4}$, i.e., H_0 has three bounded faces each being a quadrangle. For each bounded face of H_0, we take a copy of G and identify the outer 4-cycle of the copy of G with the boundary of the face. This yields the quadrangulation H as schematically illustrated in Figure 12(b). If G is 3-connected, then H is also 3-connected.

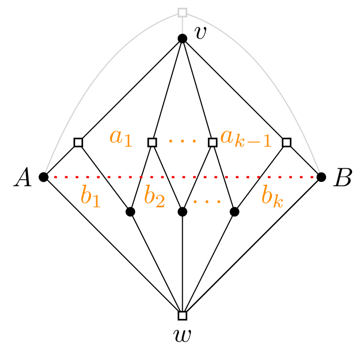
Assigning $\mathcal{A}$ to each copy of G, we claim that H has no realizing drawing. Suppose, by contradiction, that there exists a realizing drawing D. Due to the positive total area of the central copy of G, either the right or the left copy of G has a non-convex boundary cycle in D. Consequently, this copy induces an $\mathcal{A}$-realizing drawing of G where the outer face is not convex and thus, we obtain a contradiction. □

Now, we present a family of plane graphs that have the property of being strongly area-universal. The family consists of the angle graphs of wheels, which are also known as *pseudo-double wheels*. The *pseudo-double wheel* S_k has $2k+2$ vertices and consists of a cycle with vertices $v_1, v_2, \dots, v_{2k}$ and a vertex v adjacent to all vertices on the cycle with odd index and a vertex w adjacent to all vertices on the cycle with even index, see Figure 13(a). Up to the labeling, the plane embedding of S_k is unique.

The smallest pseudo-double wheel S_3 is also known as the *cube graph*. In this section, we show that the cube graph – and more generally, all pseudo-double wheels – are strongly area-universal.



(a) Pseudo-double wheel S_k.



(b) The core $c(S_k)$ of S_k with an area assignment.

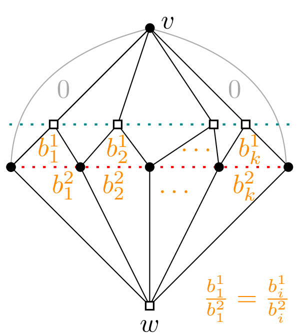
Figure 13

We first study a subgraph of S_k, namely the plane graph $c(S_k)$, called the *core*, which is obtained by deleting v_1. Figure 13(b) illustrates the core of S_5.

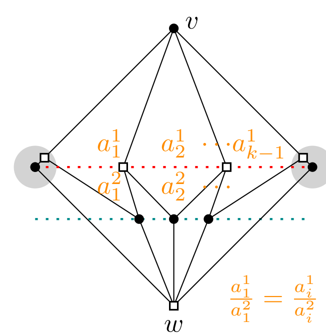
Lemma 13 *Let $c(S_k)$ be the core of a plane pseudo-double wheel with an area assignment $\mathcal{A}$. Let q be a quadrangle of area $\Sigma\mathcal{A}$ containing the diagonal AB and whose corners are identified with the vertices A, w, B, v. Then, $c(S_k)$ has a $\mathcal{A}$-realizing drawing within q.*

Proof: We distinguish two cases. We call the faces of $c(S_k)$ incident to w the *bottom faces* and the faces incident to v the *top faces* of $c(S_k)$. For simplicity, we denote the vertices v_2 and v_{2k} by A and B, respectively, and the face areas by a_i for the top and by b_i for the bottom faces. Consider also Figure 13(b).

Case (i): If $\sum_i b_i > \text{AREA}(\triangle AwB)$, we position the even vertices on the segment AB as illustrated in Figure 14(a). Note that adding the edges of consecutive even vertices and Av and Bv (and deleting w) results in a tent graph. We partition the face area b_i of each bottom face into b_i^1 and b_i^2 such that the ratio b_i^1/b_i^2 coincides for all i and $\sum_i b_i^2 = \text{AREA}(\triangle AwB)$. By Lemma 9, the tent graph has a realizing drawing within the triangle vAB. Due to the same ratio, the vertex



(a) The layout in case (i).



(b) The layout in case (ii).

Figure 14: Illustration of the proof of Lemma 13. The gray disks indicate that the contained vertices are placed at the same position, the center of the disk.

placement on AB also realizes the area for triangles incident to w. Figure 14(a) visualizes the realizing drawing of $c(S_k)$.

Case (ii): If $\sum_i a_i > \text{AREA}(\triangle vAB)$, we position the odd vertices on the segment AB as illustrated in Figure 14(b). Note that the graph in the bottom triangle is a tent graph. Therefore, we partition the area a_i of a top faces into a_i^1 and a_i^2 such that the ratio a_i^1/a_i^2 coincides for all i and $\sum_i a_i^1 = \text{AREA}(\triangle vAB)$. As in case (i), we use Lemma 9 to find a realizing drawing of the tent graph. Figure 14(b) visualizes the realizing drawing of $c(S_k)$. □

This lemma helps us to settle three out of four cases of Theorem 14.

Theorem 14 *The pseudo-double wheel S_k, $k \geq 3$, is strongly area-universal.*

Proof: For an area assignment $\mathcal{A}$ of S_k, we consider an arbitrary but fixed quadrangle q of area $\Sigma\mathcal{A}$ whose corners are identified with $v_1 v_2 w v_{2k}$. We distinguish two cases depending on the shape of q. Note that q can be triangulated by the segment $v_2 v_{2k}$ or $v_1 w$ (or both).

In case 1, the segment $v_2 v_{2k}$ lies inside q. We distinguish two subcases based on the assigned areas a and b of the faces incident to v_1 relative to the area of the triangle $v_1 v_2 v_{2k}$.

Case 1(i): $a + b \leq \text{AREA}(v_1 v_2 v_{2k})$. We can position v such that the triangles $v_1 v_2 v$ and $v_1 v v_{2k}$ realize a and b, respectively. The remaining graph corresponds to the core S_k which we realize in the quadrangle $v v_2 w v_{2k}$ containing the segment $v_2 v_{2k}$ by applying Lemma 13. Figure 15(a) visualizes the resulting layout.

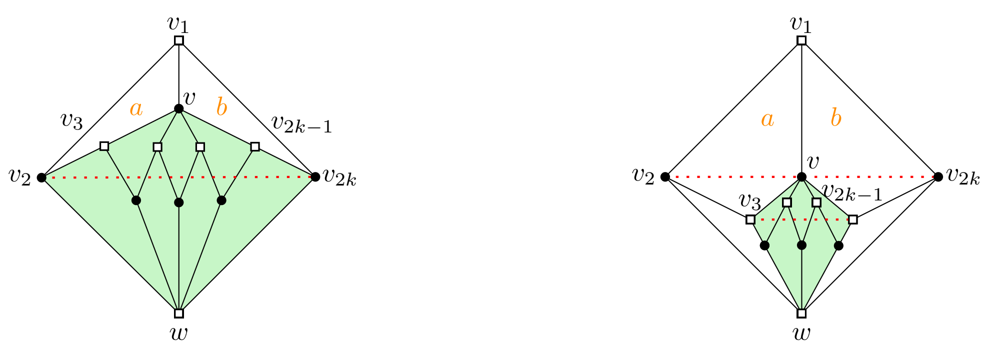
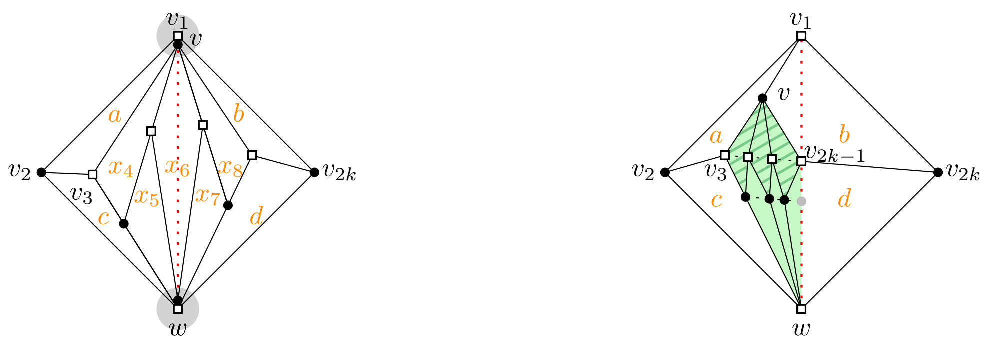


Figure 15: Illustration of case 1 in the proof of Theorem 14 in which the segment $v_2 v_{2k}$ is contained in q. In case 1(i) the face areas incident to v_1 are small; in case 1(ii) the face areas incident to v_1 are big. Both cases reduce to Lemma 13.

Case 1(ii): $a + b > \text{AREA}(v_1 v_2 v_{2k})$. We position v on the segment $v_2 v_{2k}$ such that v_3 and v_{2k-1} are forced to be on a line parallel to the segment $v_2 v_{2k}$, i.e., v partitions $v_2 v_{2k}$ according to the ratio of a and b. The positions of v_3 and v_{2k-1} on the line are such that the areas of the triangles $v_2 w v_3$ and $v_{2k-1} w v_{2k}$ realize b_1 and b_k. The graph induced by the vertices in the interior of $v v_3 w v_{2k-1}$ is the core of a smaller pseudo-double wheel and contains the diagonal v_3, v_{2k-1}. Consequently, Lemma 13 yields a realizing drawing, see Figure 15(b).

In case 2, the segment $v_1 w$ lies inside q. We call the faces incident to v_2 the *left faces* and the faces incident to v_{2k} the *right faces*. We say the left (right) faces are *small* if their assigned area is at most the area of the triangle v_1, v_2, w (v_1, w, v_{2k}). Otherwise, we call the left (right) faces *big*.

Note that the left or right faces must be small. By symmetry, we assume without loss of generality, that the left faces are small. Then we can realize the left faces by triangular faces, by positioning v_3 accordingly.



(a) Case 2(i) (b) Case 2(ii)

Figure 16: Illustration of case 2 in the proof of Theorem 14, in which the segment v_1w is contained in q. In case 2(i), the left and right faces are small; in case 2(ii), the left faces are small and the right faces are big.

Case 2(i): the right faces are small as well; this case is illustrated in Figure 16(a). We reduce this case to subgraphs of two stacked triangulations. To do so, we contract the edge vv_1 and realize the right faces by triangular faces with corner v_{2k-1}. Denote the areas of the inner faces by x_i as illustrated in Figure 16(a). There exists an i with $4 \leq i \leq 2k-1$ such that

$$a + c + \sum_{j=4}^{i-1} x_j \leq \text{AREA}(\triangle v_1 v_2 w) \quad \text{and} \quad \sum_{j=i+1}^{2k-1} x_j + b + d \leq \text{AREA}(\triangle v_1 w v_{2k}).$$

The exact layout depends on whether x_i is the area of a top or bottom face of the core. If it is a top face, then its unique vertex with odd index is placed at w. If x_i belongs to a bottom face, then the unique vertex with even index is placed at v. Afterwards we insert the remaining vertices. For $j < i$, we iteratively insert v_j such that it realizes the face area x_j by triangular face with a flat angle at v_{j+1} from left to right. For $j > i$, we follow the same strategy but in decreasing order.

Case 2(ii): the right faces are big. Figure 16(b) depicts this case. Recall that v_3 has been fixed already. Place v_{2k-1} on v_1w such that the area of the quadrangle $v_1v_3v_{2k-1}v_{2k}$ exceeds b but is not enough to also realize all top faces of the core, i.e., the striped faces incident to v in Figure 16(b). The remaining graph can be handled by Lemma 13: To do so, we choose vv_3wv_{2k-1} as the outer face and insert an artificial vertex on wv_{2k-1}. □

We can combine Lemma 2 and Theorem 14 in order to construct further strongly area-universal graphs. A graph is a *stacked pseudo-double wheel* if there exists a set of cycles such that decomposition along these cycles yields several pseudo-double wheels. A *generalized stacked pseudo-double wheel* can be decomposed into pseudo-double wheels and copies of the unique plane quadrangulation Q_5 on five vertices. Note that Q_5 is the plane graph that can be obtained by starting with a plane C_4, inserting an edge between two non-incident vertices and then subdividing this new edge. It is easy to see that Q_5 is strongly area-universal [12]. Together with Lemma 2 and Theorem 14, it follows that

Corollary 15 *Generalized stacked pseudo-double wheels are strongly area-universal.*

5 Quadrangulations and Convexity

A drawing of a planar graph is *convex* if each face is bounded by the boundary of a convex polygon. Convexity is a visually appealing property of drawings of planar graphs which has therefore been studied extensively in graph drawing. For example, Tutte's spring embeddings [20] guarantee convex drawings for every 3-connected planar graph. In this section, we aim for convex realizing drawings, i.e., given an area assignment $\mathcal{A}$ of a quadrangulation Q, we want to find an $\mathcal{A}$-realizing drawing of Q which is also convex. A planar graph is *convex area-universal* if for every area assignment there exists a convex realizing drawing. Although convex area-universality seems to be a very strong property, there are examples of convex area-universal graphs, such as the cube graph.

Proposition (Kleist [12], Proposition 2) *The cube graph is convex area-universal.*

Indeed, this result can be generalized in two directions. First, the cube graph is convex area-universal for every convex outer face. Second, this holds not only for the cube graph but also for all pseudo-double wheels. We say a graph is *strongly convex area-universal* if for every area assignment $\mathcal{A}$ and every convex drawing of the outer face with total area $\Sigma\mathcal{A}$, there exists a realizing drawing.

Theorem 16 *The pseudo-double wheel S_k, $k \geq 3$, is strongly convex area-universal.*

In Theorem 14 we have shown that pseudo-double wheels are area-universal. The proof made ample use of Lemma 13. Since drawings obtained by using this lemma may contain non-convex faces we need an independent proof for Theorem 16; consider Figures 14(a) and 14(b) for an illustration of the possible appearance of non-convex faces.

Proof: Let $\mathcal{A}$ be a given area assignment of S_k. We denote the areas assigned to the inner faces adjacent to the outer edges by a, b, c, d and the remaining areas by $x_4, x_5, \ldots, x_{2k-1}$ where $x_{2k-1} = c$ as depicted in Figure 17.

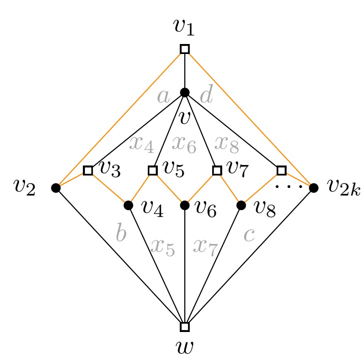


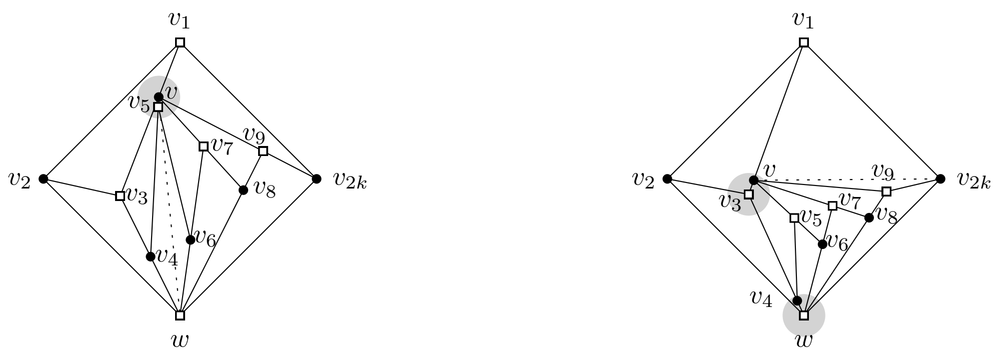
Figure 17: A pseudo-double wheel S_k with a given area assignment.

Let q be a convex quadrangle of area $\Sigma\mathcal{A}$ with corners A, B, C, D which are identified with the outer vertices v_1, v_2, w, v_{2k} of S_k, respectively. Considering the diagonal AC, shows that at least one of the two following inequalities hold: $a + b \leq \text{AREA}(\triangle ABC)$ or $c + d \leq \text{AREA}(\triangle ACD)$. By symmetry, we may assume that the first inequality holds. Hence, we may place v_3 in the triangle ABC such that the areas a and b are realized by the triangular faces ABv_3 and BCv_3 where v and v_4 are placed on Av_3 and Cv_3, respectively. Now we distinguish two cases.

Case 1: $d \leq \text{AREA}(\triangle Av_3D)$. By placing v on the segment Av_3 we realize area d by a triangular face. Split the quadrangle vv_3CD into two parts by the diagonal vC and determine $i \in [4, 2k-1]$ such that

$$\sum_{j=4}^{i-1} x_j \leq \text{AREA}(\triangle v_3Cv) \quad \text{and} \quad \sum_{j=i+1}^{2k-1} x_j \leq \text{AREA}(\triangle vCD).$$

For all $j < i$, we realize the area x_j by a triangle, (namely $uv_{j-1}v_j$ for $u = v$ if j is even and $u = w$ if j is odd), by placing v_j accordingly. Likewise for all $j > i$, we realize the area x_j by a triangle, (namely uv_jv_{j+1} for $u = v$ if j is even and $u = w$ if j is odd), by placing v_j accordingly. Finally, by placing v_i at v if i is odd and at w if i is even we the area x_i with the convex quadrangle $vv_{i-1}wv_{i+1}$. Figure 18(a) visualizes the resulting drawing.



(a) Case 1 (b) Case 2

Figure 18: Illustration of convex realizing drawings in the proof of Theorem 16.

Case 2: $d > \text{AREA}(\triangle Av_3D)$. In this case we place v at v_3 and v_4 at w. We place v_{2k-1} such that the area c and d are realized. In decreasing order, we place v_i such that x_i is realized by the triangle uv_iv_{i+1} with $u = v$ if i is even and $u = w$ if i is odd. This yields a realizing drawing within every convex outer face, see Figure 18(b). □

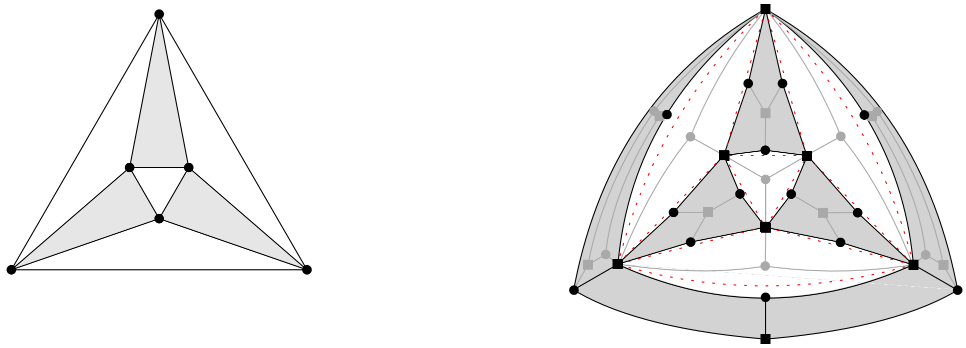
5.1 Not all Quadrangulations are Convex Area-Universal

Plane drawings of $K_{2,n}$ have non-convex faces when $n \geq 4$. Tutte's spring embedding theorem [20], however, warrants that 3-connected quadrangulations have convex drawings. In [12], it was asked whether all 3-connected quadrangulations are convex area-universal. Here we answer this question in the negative.

Theorem 17 *There is a 3-connected quadrangulation that is not convex area-universal.*

Proof: We show that the 3-connected quadrangulation Q, depicted in Figure 19(b), has an area assignment which does not allow for a convex drawing in any (even non-convex) outer face. The construction is based on a non-realizable area assignment $\mathcal{A}$ of the octahedron graph G.

Theorem (Kleist [13], Theorem 1) *For small enough $\varepsilon > 0$, the octahedron graph (with a white/gray-coloring of its faces as illustrated in Figure 19(a)) has no drawing where the white faces have area of at most ε and the gray faces have area of at least 1.*



(a) The octahedron graph G. There exists no drawing of G in which each white face has area 0 and each gray face has area 1 [13, Theorem 1].

(b) The quadrangulation Q and its black subgraph Q'. There exists no convex drawing of Q in which each white face has area 0 and each gray face has area $1/3$.

Figure 19: Illustration of Theorem 17 and its proof.

We show that a convex drawing of Q induces an $\mathcal{A}$-realizing drawing of G, yielding a contradiction. Let Q' be the subgraph of Q which is induced by the black vertices in Figure 19(b). Note that Q' contains a 1-subdivision of the octahedron. We call the two bipartition classes of Q' the *squared* and *circled* vertices.

For the purpose of a contradiction, suppose that for every $\varepsilon > 0$, Q has a convex drawing in which each white face has area $\varepsilon/3$ and each gray face area $1/3$. Then the induced drawing of Q' has the following properties: the white faces have area ε, the gray faces have area 1, and each segment between two squared vertices is contained in some white face. In the remainder, we show that for small enough $\varepsilon > 0$, no such drawing of Q' exists.

Suppose that Q' has such a drawing D. Because each segment between two squared vertices is contained in some white face, these segments together with the squared vertices form a straight-line drawing D' of G where the white faces have area at most ε and the gray faces have area of at least 1. The red dotted graph in Figure 19(b) illustrates D'. The properties of D' contradict the above stated theorem. Consequently, Q is not convex area-universal. □

Remark. Because we did not use the shape of the outer face in the proof of Theorem 17, the quadrangulation Q does not even have an $\mathcal{A}$-realizing drawing where we only require that each inner face is convex.

Remark. The construction of Q in the proof of Theorem 17 is based on a non-realizable area assignment of the octahedron graph. More generally, a white/gray-coloring of the faces of any Eulerian triangulation yields non-realizable area assignment [13, Theorem 1]. This fact allows the construction of a large family of quadrangulations that are not convex area-universal.

6 Small Quadrangulations

In this section we show that our methods are strong enough to prove area-universality for quadrangulations with up to 13 vertices via reductions to know area-universal graphs.

Theorem 18 *Every quadrangulation on at most 13 vertices is area-universal.*

Proof: First, it follows from Lemma 2 and Proposition 3 that a minimal non-area-universal quadrangulation has minimum degree 3. Thus, the smallest quadrangulation of interest is the cube graph on eight vertices. Figure 20 displays all quadrangulations on up to 13 vertices with minimum degree 3. We denote them by $Q_1, \ldots, Q_9$ as illustrated.

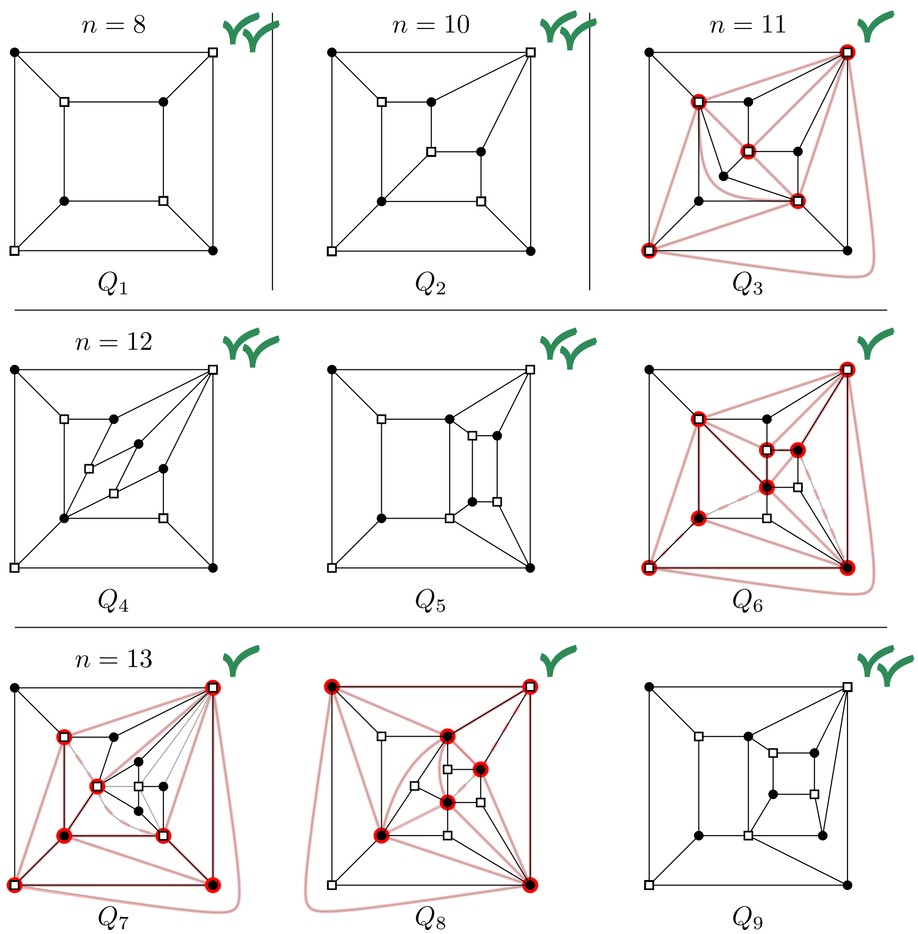


Figure 20: The planar quadrangulations on up to 13 vertices with min-degree 3. A single (double) checkmark indicates (strong) area-universality for all embeddings. The red edges form auxiliary area-universal triangulations.

Since all embeddings of a pseudo-double wheel S_k are equivalent, Theorem 16 proves the (strong) area-universality of pseudo-double wheels. Specifically, this shows the (strong) area-universality of $Q_1 = S_3$, $Q_2 = S_4$ and $Q_4 = S_5$. Moreover, Q_5 is a stacked cube graph and

Q_9 is obtained from the cube by first adding a subdivided diagonal to a face and then stacking a cube graph in one of the two new faces. Since the cube graph is the double wheel S_3, both quadrangulations are area-universal by Corollary 15.

We reduce all remaining quadrangulations to area-universal triangulations. The quadrangulation Q_3 on eleven vertices is a subgraph of a stacked triangulation T, for which every embedding of T is area-universal. In Figure 20, the vertices and edges of the stacked triangulation are highlighted in red.

The three remaining quadrangulations are subgraphs of an area-universal graph family which was shown to be area-universal by Kleist [14].

Theorem (Kleist [14], Theorem 3) *Any (embedding of a) double stacking graph $H_{\ell,k}$ is area-universal if and only if $\ell \cdot k$ is even.*

More precisely, Q_6, Q_7, Q_8 are subgraphs of an area-universal double stacking graph with some additional vertices of degree 3 stacked into triangular faces. Thus, their area-universality follows from Observation 1 and [14, Theorem 3]. In Figure 20 the vertices which remain after iterative removal of degree-3 vertices are highlighted in red. The quadrangulation Q_6 on twelve vertices reduces to the double stacking graph $H_{2,2}$; the quadrangulations Q_7 and Q_8 on 13 vertices reduce to the double stacking graph $H_{2,1}$. The vertices in the interior of red dotted curves in Figure 20 are added by diamond additions on the respective edge. □

7 Conclusions and Future Work

In this paper we develop several useful tools for the study of area-universality of plane quadrangulations. With the help of these tools we prove the area-universality of several non-trivial graph classes, including grid graphs, tent graphs, some types of angle graphs of plane triangulations, pseudo-double wheels and their generalization. We also prove that all quadrangulations with at most 13 vertices are area-universal. Interestingly, pseudo-double wheels are also strongly area-universal and convex area-universal, i.e., the outer face of the realizing drawings can be prescribed or asked to have convex faces. However, these properties do not hold for all quadrangulations: We present examples of quadrangulations and area assignments that admit no realizing drawings with convex faces or a prescribed outer face, respectively. The natural question, whether all quadrangulations are area-universal remains an interesting open problem.

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