Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 25, no. 1, pp. 97–119 (2021)
DOI: 10.7155/jgaa.00551

Equilateral Spherical Drawings of Planar Cayley Graphs

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Submitted: September 2020	Reviewed: November 2020	Revised: January 2021
Accepted: January 2021	Final: January 2021	Published: January 2021
Article type: Regular Paper		Communicated by: G. Liotta

Abstract. In this paper, we study equilateral spherical drawings of planar Cayley graphs. We focus on the case when the underlying group is generated by two rotations. In this case, the set of equilateral drawings can be parameterized by spherical ellipses on the unit sphere. Besides, we give an explicit formula to describe the shortest equilateral spherical drawing and the longest spherical equilateral drawing. Furthermore, we studied the drawing of Schreier coset graphs arising from these equilateral drawings.

1 Introduction

Cayley graph is a graph encoding the structure of a group, which is a central tool in combinatorial and geometric group theory. Given a group G with a symmetric generating set S, the *Cayley graph* $\mathrm{Cay}(G,S)$ associated to (G,S) is an undirected graph with the vertex set G such that two vertices x and y are adjacent if $x = ys$ for some $s \in S$. The group G acts canonically on the graph $\mathrm{Cay}(G,S)$ by left multiplication and the action is transitive on the set of vertices.

A graph is *planar* if it has an embedding on the plane without edge-crossing. When all the edges of the embedding have the same length, it is called an *equilateral embedding*. Eades and Warmold [5] showed that determining whether a 2-connected planar graph has an equilateral embedding is NP-hard. Markenzon and Paciornik [13] presented a linear time algorithm to determine whether a 2-connected chordal graph has a equilateral embedding without edge-crossing.

One important subject in graph drawing is to find a nice drawing algorithm in 3D. For trees, there are several 3D graph drawing algorithms like [1], [9], [12]. For other particular kinds of graphs, there are also some 3D graph drawing algorithms like [6], [11], [3] and [4].

Maschke [14] classified planar Cayley graphs in 1896. A complete list of planar Cayley graphs can be also be found in [7]. He showed that when $X = \mathrm{Cay}(G,S)$ is a planar Cayley graph,

This research was partially supported by the Ministry of Science and Technology of Taiwan under grants MOST 108-2115-M-009-007 and MOST 108-2115-M-009-006.

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G is isomorphic to a group of isometries of the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. In other words, one can draw X in $\mathbb{R}^3$ such that the vertices lie in $\mathbb{S}^2$, the edges are straight lines, and the action of G on X can be extended to $\mathbb{S}^2$ as isometries. We call such a drawing a *symmetric spherical drawing*. However, Maschke's proof does not give an explicit construction of symmetric spherical drawings. He only drew the figures of all planar Cayley graphs and identify them as the skeletons of uniform polyhedrons.

To construct a symmetric spherical drawing explicitly, first we need find the way to identify G as a group of isometries of $\mathbb{S}^2$. This is equivalent to find a particular three dimensional real representation of G such that the image of G is the symmetric group of the corresponding uniform polyhedron. It is not hard to find such a representation by cases, but in fact there is a unified method using the method of spectral drawing [8, 10], which will be introduced in Section 2.4.

After identifying G as a group of isometries on $\mathbb{S}^2$, one can fix a unit vector $\vec{u}$ as the vector used to draw the identity element of G. Then the coordinate of the vertex g is given by $g\vec{u}$. Denote the resulted drawing by $X_{\vec{u}}$. Note that all $X_{\vec{u}}$'s are highly symmetric, since they all admit G as a group of isometries. When all edges of $X_{\vec{u}}$ are of equal length, $X_{\vec{u}}$ is called an *equilateral spherical drawing*. In this paper, we will like to study all equilateral spherical drawings via studying the following set:

$$\mathfrak{D} = \left\{ \vec{u} \in \mathbb{S}^2 \middle| X_{\vec{u}} \text{ is an equilateral spherical drawing} \right\}.$$

For example, let $G = A_5$, the alternating group of degree 5, and $S = \{(12)(34),\ (12345),\ (15432)\}$. The Cayley graph $X = \mathrm{Cay}(G, S)$ is the truncated icoshedral graph. One can identify G as a group of isometries on $\mathbb{S}^2$ from the spectral drawing. (In this case, G is also known as the icosahedral group. See [2] for more details.)

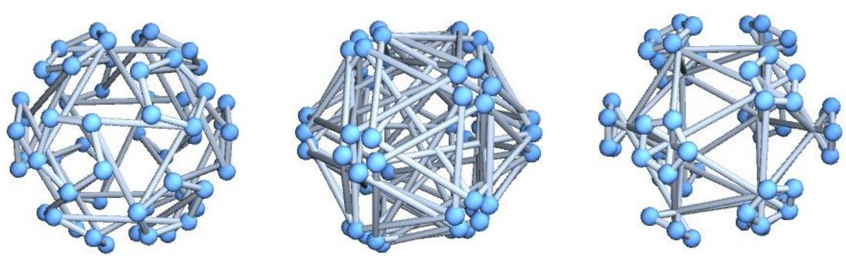


Figure 1: $X_{\vec{u}}$ for some random choices of $\vec{u}$.

The Figure 1 demonstrates $X_{\vec{u}}$ for some random choices of $\vec{u}$. The Figure 2 shows the set $\mathfrak{D}$, which is a union of two so-called spherical ellipses.

In the set $\mathfrak{D}$, the drawing with shortest edge length is indeed the skeleton of a truncated icosahedron; the drawing with longest edge length is the skeleton of an icosahedron. Besides, there exists some drawing equal to the skeleton of a dodecahedron as shown in Figure 3.

Note that the skeleton of a dodecahedron and the skeleton of an icosahedron are the drawings of the Schreier coset graphs X^H associated to $(A_5/H, HSH/H)$ where $H = \langle \rho((123)) \rangle$ and $\langle \rho((12534)) \rangle$, respectively. (See Section 2.1 for the definition of Schreier coset graph.) In general, we would like to know when a Schreier coset graph can be seen by certain $X_{\vec{u}}$.

In this paper, we will only focus on the case that G only consists of rotations as a group of isometries of $\mathbb{S}^2$. In this case, we call $\mathrm{Cay}(G, S)$ a rotational planar graph. The classification of such graphs is given in Section 2.3.

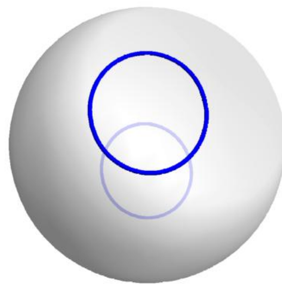


Figure 2: The set $\mathfrak{D}$ for Cay$(A_5, \{(12)(34), (12345), (15432)\})$.

The paper is organized as follows. In Section 2, we recall some background knowledge. In Section 3, we characterize the set $\mathfrak{D}$ and give an explicit form of $\vec{u}$ for the shortest and longest equilateral spherical drawings of $X_{\vec{u}}$. Besides, we study the isomorphism class of $X_{\vec{u}}$. Especially, we show that the shortest and longest drawings are both unique up to isomorphism. Furthermore, we find some particular subgroups H such that there exists some $X_{\vec{u}}$ which is also a graph drawing of the Schreier coset graph X^H. In Section 4, we list the result for all rotational planar graphs.

2 Preliminary

2.1 Schreier coset graphs

For a subgroup H of G, define

$$HSH = \{hsh' | h, h' \in H, s \in S\},$$

which is the H-double coset containing S. It can be also written as a disjoint union of left H-coset, denoted by HSH/H.

The Schreier coset graph X^H associated to $(G/H, HSH/H)$ is the graph in which its vertices are left H-cosets $\{gH | g \in G\}$ in G and two vertices g_1H and g_2H are adjacent if $g_2H = g_1\tilde{s}H$ for some $\tilde{s}H \in HSH/H$.

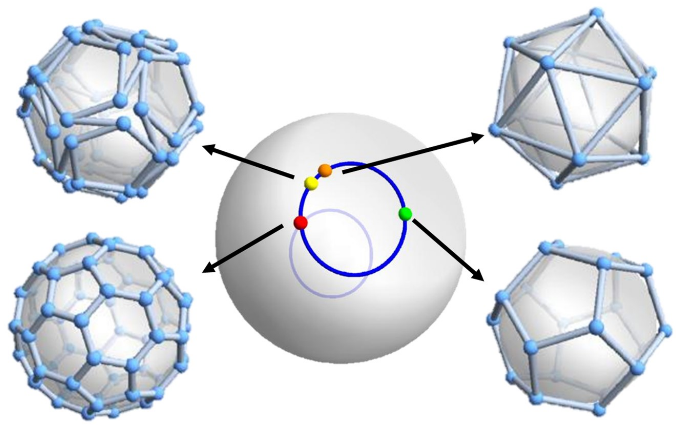
It is well-known that a vertex-transitively graph may not be a Cayley graph, but it is always a Schreier coset graph [15].

2.2 Displacement function

Let $R = R(\theta, \vec{u})$ be a linear rotation around the unit vector $\vec{u}$ in $\mathbb{R}^3$ of degree θ. The square displacement function of a rotation R on $\mathbb{R}^3$ is defined by

$$d_R(\vec{x}) = \|R\vec{x} - \vec{x}\|^2.$$

Denote by $d_{R,\max}$ the maximum value of $d_R(\vec{x})$ on the unit sphere $\mathbb{S}^2$.



(Shortest length drawing) (Longest length drawing)

Figure 3: Some drawings $X_{\vec{u}}$ with $\vec{u} \in \mathfrak{D}$.

Theorem 2.2.1 *For every rotation $R = R(\theta, \vec{u})$ on $\mathbb{R}^3$, we have*

$$d_{R,\max} = 2(1 - \cos\theta) \quad \textit{and} \quad d_R(\vec{x}) = d_{R,\max}(1 - \langle \vec{u}, \vec{x} \rangle^2), \forall\, \vec{x} \in \mathbb{S}^2.$$

Proof: Let $\alpha = \{\vec{v}_1 = \vec{u}, \vec{v}_2, \vec{v}_3\}$ be an orthonormal basis of $\mathbb{R}^3$. The we have

$$[R(\theta, \vec{u})]_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

For $\vec{x} \in \mathbb{S}^2$, let $\vec{x} = x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3}$. Thus we have $x_1^2 + x_2^2 + x_3^2 = 1$ and

$$\begin{aligned} d_R(\vec{x}) = \|R\vec{x} - \vec{x}\|^2 &= (x_2\cos\theta - x_3\sin\theta - x_2)^2 + (x_2\sin\theta + x_3\cos\theta - x_3)^2 \\ &= 2(1-\cos\theta)(x_2^2 + x_3^2) = 2(1-\cos\theta)(1 - \langle \vec{u}, \vec{x} \rangle^2). \end{aligned}$$

Since $1 - \langle \vec{u}, \vec{x} \rangle^2 \leq 1$, we have $d_{R,\max} = 2(1 - \cos\theta)$. □

2.3 Classification of Rotational Planar Groups

A group G is called a planar group if there exists some generating set S such that $\mathrm{Cay}(G, S)$ is a planar graph. Maschke showed that if G can be identified as a group of isometries of $\mathbb{S}^2$. In this case, G contains rotations and reflections. If G only contains rotations, G is a rotation group and its classification can be also found in [7, Theorem 6.3.1]. The following theorem is the result of Maschke's work [14].

Theorem 2.3.1 ([14]) *If $\mathrm{Cay}(G, S)$ is a planar graph and G is a rotation group of S^2. The one of the following holds.*

a) $(G, S) \cong (\mathbb{Z}_n, \{\pm 1\})$ *and* $\mathrm{Cay}(G, S)$ *is a circular graph with n vertices.*

b) $G = \langle x, y | x^2 = y^n = (xy)^2 = 1 \rangle \cong D_n$*, the dihedral group with* $2n$ *elements.*

b1) $S = \{x, y\}$ *and* $\mathrm{Cay}(G, S)$ *is a prism graph with* $2n$ *vertices.*

b2) $S = \{x, xy\}$ *and* $\mathrm{Cay}(G, S)$ *is a circular graph with* $2n$ *vertices.*

c) $G = \langle x, y | x^3 = y^2 = (xy)^3 = 1, \cdots \rangle \cong A_4$*, the alternating group of degree 4.*

c1) $S = \{x, y\}$ *and* $\mathrm{Cay}(G, S)$ *is a truncated tetrahedral graph.*

c2) $S = \{x, xy\}$ *and* $\mathrm{Cay}(G, S)$ *is a cuboctahedral graph.*

d) $G = \langle x, y | x^3 = y^2 = (xy)^4 = 1, \cdots \rangle \cong S_4$*, the symmetric group of degree 4.*

d1) $S = \{x, y\}$ *and* $\mathrm{Cay}(G, S)$ *is a truncated cubical graph.*

d2) $S = \{x, xy\}$ *and* $\mathrm{Cay}(G, S)$ *is a rhombicuboctahedral graph.*

d3) $S = \{y, xy\}$ *and* $\mathrm{Cay}(G, S)$ *is a truncated octahedral graph.*

e) $G = \langle x, y | x^3 = y^2 = (xy)^5 = 1, \cdots \rangle \cong A_5$*, the alternating group of degree 5.*

e1) $S = \{x, y\}$ *and* $\mathrm{Cay}(G, S)$ *is a truncated dodecahedral graph.*

e2) $S = \{x, xy\}$ *and* $\mathrm{Cay}(G, S)$ *is a rhombicosidodecahedral graph.*

e3) $S = \{y, xy\}$ *and* $\mathrm{Cay}(G, S)$ *is a truncated icosahedral graph.*

2.4 Spectral Drawings

Let X be a finite connected undirected graph with the vertex set $V = \{v_1, \cdots, v_n\}$ and the edge set E. Let $\mathbb{R}[V]$ be a real inner product space with an orthonormal basis $\alpha = \{\vec{e}_{v_1}, \cdots, \vec{e}_{v_n}\}$. The Laplacian operator L is a linear transformation on $\mathbb{R}[V]$ characterized by

$$L(\vec{e}_v) = \sum_{(v,u) \in E} \vec{e}_v - \vec{e}_u, \qquad \forall\, v \in V.$$

The Laplacian operator is positive semi-definite with eigenvalues

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n.$$

Let $\left\{ \vec{u}_1 = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \cdots, \vec{u}_n \right\}$ be an orthonormal eigenbasis such that $L(\vec{u}_i) = \lambda_i \vec{u}_i$ for all i.

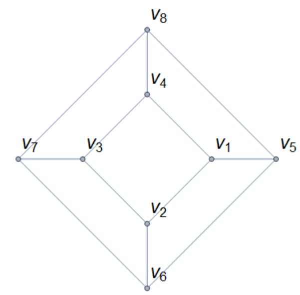
Let W be the k-dimensional subspace of $\mathbb{R}[V]$ with a basis $\beta = \{\vec{u}_2, \cdots, \vec{u}_{k+1}\}$. Then the k-dimensional spectral drawing is a straight-line drawing of X onto $\mathbb{R}^k$ such that the coordinate of the vertex v is given by

$$\mathrm{sp}_k(v) = [\mathrm{proj}_W(\vec{e}_v)]_\beta.$$

Here $\mathrm{proj}_W(\vec{x})$ is the orthogonal projection onto W and $[\vec{x}]_\beta$ is the coordinate vector of $\vec{x}$ under the basis β.

Example 2.4.1 *Let L be the Laplacian matrix of the cubical graph given by*

$$L = \begin{pmatrix} 3 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 3 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & 3 \end{pmatrix}$$



The spectrum of its Laplacian is given by

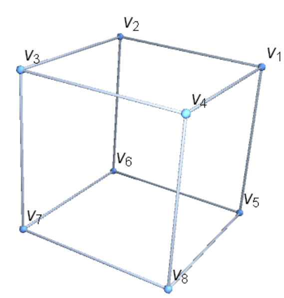
$$0, 2, 2, 2, 4, 4, 4, 6.$$

Let

$$\begin{pmatrix} \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

be the orthonormal basis of the 2-eigenspace. Then we have a three dimensional spectral drawing as follows.

$$\begin{pmatrix} \mathrm{sp}_3(v_1) \\ \mathrm{sp}_3(v_2) \\ \mathrm{sp}_3(v_3) \\ \mathrm{sp}_3(v_4) \\ \mathrm{sp}_3(v_5) \\ \mathrm{sp}_3(v_6) \\ \mathrm{sp}_3(v_7) \\ \mathrm{sp}_3(v_8) \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \Rightarrow$$



2.5 Three Dimensional Real Representations

When X is the underlying graph of the skeleton of a prism, a Platonic solid, or an Archimedean solid $\mathcal{X}$, one can show by cases that the second smallest Laplacian eigenvalues λ_2 is always of multiplicity three. (In other words, we have $\lambda_2 = \lambda_3 = \lambda_4 < \lambda_5$.) In this case, the subspace W spanned by $\vec{u}_2, \vec{u}_3, \vec{u}_4$ is the λ_2-eigenspace. Let G be a group of symmetries of a given solid $\mathcal{X}$, then G has a natural action on $\mathbb{R}[V]$ by

$$g(\vec{e}_v) = \vec{e}_{g(v)}.$$

Theorem 2.5.1 *The subspace W spanned by $\{\vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is G-invariant. In other words, W is g-invariant for all $g \in G$, or equivalently, $g(W) \subseteq W$ for all $g \in G$.*

Proof: For all $g \in G$ and $v \in V$,

$$gL(\vec{e}_v) = g\left(\sum_{(v,u)\in E} \vec{e}_v - \vec{e}_u\right) = \sum_{(v,u)\in E} \vec{g}(e_v) - g(\vec{e}_u) = \sum_{(v,u)\in E} \vec{e}_{g(v)} - \vec{e}_{g(u)} = L(\vec{e}_{g(v)}).$$

Therefore, two linear transformations g and L on $\mathbb{R}[V]$ commute which implies that every eigenspace of L is g-invariant. Especially, the λ_2-eigenspace W is g-invariant. □

Corollary 2.5.2 *The action of G restricted on W is a three dimensional real representation of G.*

By Theorem 2.3.1, when $X = \mathrm{Cay}(G, S)$ is a rotational planar graph and X is not a circular graph, X is the underlying graph of the skeleton of some Platonic or Archimedean solid. Therefore, we can identify G as a group of isometries of $\mathbb{S}^2$ by this manner.

3 Equilateral Drawings

3.1 Setting

Let $\mathrm{Cay}(G, S)$ be a rotational planar graph. Suppose G is not a circular graph. By Section 2.3, we can write $S = \{{s_1}^{\pm 1}, {s_2}^{\pm 1}\}$, where $s_1 = R(\theta_1, \vec{u}_1)$ and $s_2 = R(\theta_2, \vec{u}_2)$ are two rotations on $\mathbb{R}^3$. Let $\delta_i = d_{s_i,\max}$ for $i = 1, 2$. By switching s_1 and s_2, replacing $\vec{u}_i$ by $-\vec{u}_i$, and replacing s_i by ${s_i}^{-1}$ if necessary, we may assume that

- $\theta_1, \theta_2 \in (0, \pi]$.
- $\delta_1 \geq \delta_2$;
- $\cos\psi := \langle \vec{u}_1, \vec{u}_2 \rangle \leq 0$.

Note that by Theorem 2.2.1 ($\delta_i = 2(1-\cos\theta_i)$) and first two assumptions, we always have $\theta_1 \geq \theta_2$. Since G is not a cyclic group, we also have $\vec{u}_1 \neq \pm\vec{u}_2$ and $\sin\psi \neq 0$.

Let $\alpha = \{\vec{u}_1, \vec{u}_2, \vec{u}_1 \times \vec{u}_2\}$, which forms a basis of $\mathbb{R}^3$. For $\vec{x} \in \mathbb{R}^3$, we will denote the coordinate vector (x_1, x_2, x_3) of $\vec{x}$ under the basis α by $[\vec{x}]_\alpha$. Note that under the basis α, the defining equation of $\mathbb{S}^2$ becomes

$$x_1^2 + 2x_1x_2\cos\psi + x_2^2 + x_3^2\sin^2\psi = 1.$$

3.2 Geometric shapes of $\mathfrak{D}$

Recall that

$$\mathfrak{D} = \{\vec{x} \in \mathbb{S}^2 | d_{s_1}(\vec{x}) = d_{s_2}(\vec{x})\}.$$

By Theorem 2.2.1, for $\vec{x} \in \mathbb{S}^2$

$$d_{s_1}(\vec{x}) = \delta_1\big(1 - \langle \vec{u}_1, \vec{x}\rangle^2\big) = \delta_1\big(1 - (x_1 + x_2\cos\psi)^2\big)$$

and

$$d_{s_2}(\vec{x}) = \delta_2\big(1 - \langle \vec{u}_2, \vec{x}\rangle^2\big) = \delta_2\big(1 - (x_1\cos\psi + x_2)^2\big)$$

Let $M_1 = \begin{pmatrix} 1 & \cos\psi \\ \cos\psi & \cos^2\psi \end{pmatrix}$ and $M_2 = \begin{pmatrix} \cos^2\psi & \cos\psi \\ \cos\psi & 1 \end{pmatrix}$, then we can rewrite the difference of the above two equations to obtain

$$d_{s_1}(\vec{x}) - d_{s_2}(\vec{x}) = (\delta_1 - \delta_2) - \begin{pmatrix} x_1 & x_2 \end{pmatrix} \Big(\delta_1 M_1 - \delta_2 M_2\Big) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Therefore, we have

$$\begin{aligned} \mathfrak{D} &= \{\vec{x} \in \mathbb{S}^2 | d_{s_1}(\vec{x}) = d_{s_2}(\vec{x})\} \\ &= \left\{ \vec{x} \in \mathbb{S}^2 \middle| \begin{pmatrix} x_1 & x_2 \end{pmatrix} \Big(\delta_1 M_1 - \delta_2 M_2\Big) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \delta_1 - \delta_2. \right\} \end{aligned}$$

Note that

$$\det \Big(\delta_1 M_1 - \delta_2 M_2\Big) = -\delta_1 \delta_2 \sin\psi^4 < 0,$$

so on the x_1-x_2 plane, the equation $\begin{pmatrix} x_1 & x_2 \end{pmatrix} \Big(\delta_1 M_1 - \delta_2 M_2\Big) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \delta_1 - \delta_2$ defines a hyperbola when $\delta_1 \neq \delta_2$; it defines the union of two lines when $\delta_1 = \delta_2$. In the formal case, the set $\mathfrak{D}$ is the intersection of the unit sphere and a hyperbolic cylinder, which is the disjoint union of two so-called spherical ellipses. In the latter case, the set $\mathfrak{D}$ is the intersection of the unit sphere and the union of two planes, which is the union of two great circles. We summarize the above result as the following theorem.

Theorem 3.2.1 *The following statements hold.*

a) If $\delta_1 = \delta_2$, then $\mathfrak{D}$ is the union of two great circles.

b) If $\delta_1 \neq \delta_2$, then $\mathfrak{D}$ is the disjoint union of two spherical ellipses.

3.3 Maximal and minimal equal displacements

Now we would like to know how does the function

$$d_{s_1}(\vec{x}) = \delta_1\big(1 - (x_1 + x_2 \cos\psi)^2\big)$$

vary on $\mathfrak{D}$. Since the above function only depends on x_1, x_2, it is sufficient to study the two-variables function f defined by

$$f(x_1, x_2) := \delta_1\big(1 - (x_1 + x_2 \cos\psi)^2\big).$$

on the following region:

$$\mathfrak{D}_0 : \begin{cases} x_1^2 + 2x_1x_2\cos\psi + x_2^2 \leq 1 \\ \begin{pmatrix} x_1 & x_2 \end{pmatrix} \Big(\delta_1 M_1 - \delta_2 M_2\Big) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \delta_1 - \delta_2 \end{cases} \tag{1}$$

(Note that $\mathfrak{D}_0$ is the projection of $\mathfrak{D}$ onto the plane spanned by $\vec{u}_1$ and $\vec{u}_2$.) The extreme values of $f(x)$ occur on either the critical points or the boundary points given by

$$\begin{cases} x_1^2 + 2x_1x_2\cos\psi + x_2^2 = 1 \\ \begin{pmatrix} x_1 & x_2 \end{pmatrix} \Big(\delta_1 M_1 - \delta_2 M_2\Big) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \delta_1 - \delta_2 \end{cases} \tag{2}$$

First, let us find the critical points of $f(x)$. Set

$$F(x_1, x_2) := \begin{pmatrix} x_1 & x_2 \end{pmatrix} \Big(\delta_1 M_1 - \delta_2 M_2\Big) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

By the method of Lagrange multiplier, we have $\nabla f - \lambda \nabla F = 0$, where λ is the Lagrange multiplier, which implies that

$$\begin{aligned} -2\delta_1 M_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 2\lambda\Big(\delta_1 M_1 - \delta_2 M_2\Big) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \iff \Big((1+\lambda)\delta_1 M_1 - \lambda\delta_2 M_2)\Big) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

The above equation has non-zero solution only when

$$\det\Big((1+\lambda)\delta_1 M_1 - \lambda\delta_2 M_2)\Big) = -(1+\lambda)\lambda\delta_1\delta_2 \sin^4\psi = 0.$$

Since $\sin\psi \neq 0$, we have $\lambda = -1$ or $\lambda = 0$. When $\lambda = -1$, (x_1, x_2) lies in the kernel of M_2 and it is of the form $a(1, -\cos\psi)$. Plug $(x_1, x_2) = a(1, -\cos\psi)$ into the system of equations (1), we have

$$a^2\delta_1 \sin^4\psi = \delta_1 - \delta_2 \quad \text{and} \quad a^2 \sin^2\psi \leq 1.$$

When $\lambda = 0$, (x_1, x_2) lies in the kernel of M_1 which is the form $b(\cos\psi, -1)$. Plug $(x_1, x_2) = b(\cos\psi, -1)$ into the system of equations (1), we have

$$-b^2\delta_2 \sin^4\psi = \delta_1 - \delta_2 \quad \text{and} \quad b^2 \sin^2\psi \leq 1.$$

Recall that we assume that $\delta_1 \geq \delta_2$. If $\delta_1 = \delta_2$, then there exists a unique solution $(x_1, x_2) = (0, 0)$. If $\delta_1 > \delta_2$, then there exists no solution for b and we have

$$a = \pm\frac{1}{\sin^2\psi}\sqrt{1 - \frac{\delta_2}{\delta_1}}.$$

Moreover, such a satisfies the condition $a^2 \sin^2\psi \leq 1$ if and only if $\delta_1 \cos^2\psi \leq \delta_2$. In this case, the x_3-coordinate of x is given by

$$x_3 = \pm\frac{1}{\sin\psi}\sqrt{1 - x_1^2 - 2x_1x_2\cos\psi - x_2^2} = \pm\frac{1}{\sin\psi}\sqrt{1 - a^2\sin^2\psi}$$

Let us summarize the above computation as the two following theorems.

Theorem 3.3.1 *For the function $f(x_1, x_2)$ on $\mathfrak{D}_0$,*

- *when $\delta_1 = \delta_2$, it has a unique critical points $(0, 0)$. Moreover, $f(0, 0) = \delta_1 = \delta_2$.*
- *when $\delta_1 > \delta_2 \geq \delta_1 \cos^2\psi$, it has two critical points $(a, -a\cos\psi)$, where $a = \pm\frac{1}{\sin^2\psi}\sqrt{1 - \frac{\delta_2}{\delta_1}}$. Moreover, $f(x_1, x_2) = \delta_2$ on these critical points.*
- *when $\delta_1 \cos^2\psi > \delta_2$, it has no critical points.*

Theorem 3.3.2 *For the displacement function $d_{s_1}(\vec{x})$ on $\mathfrak{D}$,*

- *when* $\delta_1 \geq \delta_2 \geq \delta_1 \cos^2\psi$*, it has two (if* $\delta_1 = \delta_2$*) or four (if* $\delta_1 > \delta_2$*) critical points* $(a, -a\cos\psi, \pm\frac{1}{\sin\psi}\sqrt{1-a^2\sin^2\psi})$*, where* $a = \pm\frac{1}{\sin^2\psi}\sqrt{1-\frac{\delta_2}{\delta_1}}$*. Moreover,* $d_{s_1}(\vec{x}) = \delta_2$ *on these critical points.*
- *when* $\delta_1\cos^2\psi > \delta_2$*, it has no critical point.*

Remark. The value δ_2 is the maximum of $d_{s_2}(\vec{x})$, so it is the trivial upper bound of $d_{s_1}(\vec{x})$ on $\mathfrak{D}$.

Next, let us find the values of $f(x_1, x_2)$ at the boundary points. Recall that the boundary points are given by

$$\begin{aligned} x_1^2 + 2x_1x_2\cos\psi + x_2^2 &= 1 \quad (3)\\ \begin{pmatrix} x_1 & x_2 \end{pmatrix} (\delta_1 M_1 - \delta_2 M_2) \begin{pmatrix} x_1 & x_2 \end{pmatrix}^t &= \delta_1 - \delta_2 \quad (4)\end{aligned}$$

By direct computation, Equation (4) becomes

$$(\delta_1 - \delta_2\cos^2\psi)x_1^2 + 2\cos\psi(\delta_1-\delta_2)x_1x_2 + (\cos^2\psi\delta_1 - \delta_2)x_2^2 = \delta_1 - \delta_2.$$

Subtracting $\delta_1 - \delta_2$ times of Equation (3) from Equation (4), we obtain

$$(1-\cos^2\psi)(\delta_2 x_1^2 - \delta_1 x_2^2) = 0,$$

which implies that

$$x_1 = \pm\sqrt{\frac{\delta_1}{\delta_2}}x_2.$$

This is a quite simple characterization of the boundary points.

Plugging the above result into Equation (3), we obtain the following.

Proposition 3.3.3 *The four boundary points of* $\mathfrak{D}_0$ *are*

$$(x_1, x_2) = \pm\left(\sqrt{\frac{\delta_1}{\delta_1 + 2\epsilon\sqrt{\delta_1\delta_2}\cos\psi + \delta_2}}, \epsilon\sqrt{\frac{\delta_2}{\delta_1 + 2\epsilon\sqrt{\delta_1\delta_2}\cos\psi + \delta_2}}\right).$$

where $\epsilon = \pm 1$*.*

In this case,

$$f(x_1, x_2) = \delta_1\big(1 - (x_1 + x_2\cos\psi)^2\big) = \delta_1 x_2^2\sin^2\psi = \frac{\delta_1\delta_2\sin^2\psi}{\delta_1 + 2\epsilon\sqrt{\delta_1\delta_2}\cos\psi + \delta_2}.$$

Combing the result of critical points, we have the following theorem.

Theorem 3.3.4 *For the displacement function* $d_{s_1}(\vec{x})$ *on* $\mathfrak{D}$*, let* $\delta_{\max}$ *and* $\delta_{\min}$ *be the maximal value and the minimal value respectively.*

- *when* $\delta_1 \geq \delta_2 \geq \delta_1\cos^2\psi$*,* $\delta_{\max} = \delta_2$ *and* $\delta_{\min} = \frac{\delta_1\delta_2\sin^2\psi}{\delta_1 - 2\sqrt{\delta_1\delta_2}\cos\psi + \delta_2}$*. Moreover,*
- *when* $\delta_1\cos^2\psi \geq \delta_2$*,* $\delta_{\max} = \frac{\delta_1\delta_2\sin^2\psi}{\delta_1 + 2\sqrt{\delta_1\delta_2}\cos\psi + \delta_2}$ *and* $\delta_{\min} = \frac{\delta_1\delta_2\sin^2\psi}{\delta_1 - 2\sqrt{\delta_1\delta_2}\cos\psi + \delta_2}$*.*

(Here we use the assumption that $\cos\psi \leq 0$*.)*

3.4 Isomorphism classes of $X_{\vec{u}}$

For two drawings $X_{\vec{u}}$ and $X_{\vec{v}}$ of X and $A \in O(3)$, we say A is an isomorphism from $X_{\vec{u}}$ and $X_{\vec{v}}$ if there exists a permutation σ on G such that

a) $Ag\vec{u} = \sigma(g)\vec{v}, \quad \forall\, g \in G$

b) $g_1 \in g_2 S$ if and only if $\sigma(g_1) \in \sigma(g_2)S, \quad \forall\, g_1, g_2 \in G.$

In this case, we say $X_{\vec{u}}$ and $X_{\vec{v}}$ are isomorphic, denoted by $X_{\vec{u}} \cong X_{\vec{v}}$.

Set

$$\mathrm{Aut}(X_{\vec{u}}) = \{A \in O(3) | A \text{ is an isomorphism from } X_{\vec{u}} \text{ to } X_{A\vec{u}}\}$$

which is a subgroup of $O(3)$.

Proposition 3.4.1 *If $X_{\vec{v}} \cong X_{\vec{u}}$, then there exists $A \in \mathrm{Aut}(X_{\vec{u}})$ such that $\vec{v} = A\vec{u}$.*

Proof: Since $X_{\vec{u}} \cong X_{\vec{v}}$, there exist $A' \in O(3)$ and a permutation σ' satisfying the conditions a) and b). Set $A = \sigma'(e)^{-1}A'$ and $\sigma(x) := \sigma'(e)^{-1}\sigma'(x)$. (Here e is the identity of G, which is also equal to the identity matrix.) Then σ is still a permutation with $\sigma(e) = e$ on G and (A, σ) still satisfies the above the conditions a) and b). In this case, we have $A\vec{u} = Ae\vec{u} = \sigma(e)\vec{v} = e\vec{v} = \vec{v}$. Therefore, A is an isomorphism from $X_{\vec{u}}$ to $X_{\vec{v}} = X_{A\vec{u}}$. □

Example 3.4.1 *Let $A = -I$ and σ be the identity map, then for all $g \in G$, $Ag\vec{u} = -g\vec{u} = \sigma(g)(-\vec{u})$. Thus $-I \in \mathrm{Aut}(X_{\vec{u}})$.*

Example 3.4.2 *Let $A = R(\pi, \vec{u}_3/\|\vec{u}_3\|)$ (Recall that $\vec{u}_3 = \vec{u}_1 \times \vec{u}_2$.) Then $A(\vec{u}_1) = -\vec{u}_1$, $A(\vec{u}_2) = -\vec{u}_2$ and $A(\vec{u}_3) = \vec{u}_3$. Moreover, we have $As_1A^{-1} = {s_1}^{-1}$ and $As_2A^{-1} = {s_2}^{-1}$, which implies that $\sigma(g) = AgA^{-1}$ defines a permutation on G. It is easy to see that such $A \in \mathrm{Aut}(X_{\vec{u}})$ and so does $-A$.*

Example 3.4.3 *Suppose $\langle \vec{u}_1, \vec{u}_2 \rangle = 0$. Let $A = R(\pi, \vec{u}_1)$ and $\vec{v} = A\vec{u}$. Then $A(\vec{u_1}) = \vec{u}_1$, $A(\vec{u_2}) = -\vec{u}_2$ and $A(\vec{u}_3) = -\vec{u}_3$. Moreover, we have $As_1A^{-1} = s_1$ and $As_2A^{-1} = {s_2}^{-1}$. By the same token as the previous example, we also have $A \in \mathrm{Aut}(X_{\vec{u}})$. Moreover, the same result also holds for $A = -R(\pi, \vec{u}_1), \pm R(\pi, \vec{u}_2)$.*

For $\vec{u} \in \mathfrak{D}$, set

$$\mathrm{Aut}_0(X) = \begin{cases} \{\pm I, \pm R(\pi, \vec{u}_3/\|\vec{u}_3\|)\} & , \text{ if } \langle \vec{u}_1, \vec{u}_2 \rangle \neq 0; \\ \{\pm I, \pm R(\pi, \vec{u}_1), \pm R(\pi, \vec{u}_2), \pm R(\pi, \vec{u}_3/\|\vec{u}_3\|)\} & , \text{ if } \langle \vec{u}_1, \vec{u}_2 \rangle = 0. \end{cases}$$

From the above examples, we see that $\mathrm{Aut}_0(X)$ is a subgroup of $\mathrm{Aut}(X_{\vec{u}})$ for all $\vec{u} \in \mathbb{S}^2$.

Theorem 3.4.2 *For $\vec{u}, \vec{v} \in \mathfrak{D}$, the following are equivalent.*

a) $X_{\vec{u}} \cong X_{\vec{v}}$.

b) There exists some $A \in \mathrm{Aut}_0(X)$ such that $A\vec{u} = \vec{v}$.

c) $d_{s_1}(\vec{u}) = d_{s_1}(\vec{v})$ and $\langle T\vec{v}, T\vec{v} \rangle = \langle T\vec{u}, T\vec{u} \rangle$ where $T = \sum_{s \in S} s$.

Proof:

b) $\Rightarrow$ a): Follow by the property that $\mathrm{Aut}_0(X)$ is a subgroup of $\mathrm{Aut}(X_{\vec{u}})$.

a) $\Rightarrow$ c): Since $X_{\vec{v}} \cong X_{\vec{u}}$, their edges are all of the same length, which implies that $d_{s_1}(\vec{u}) = d_{s_2}(\vec{v})$. By Proposition 3.4.1, there exists $A \in \mathrm{Aut}(X_{\vec{u}})$ so that $A\vec{u} = \vec{v}$. Since A preserves adjacency relation, we have

$$\{As\vec{u} | s \in S\} = \{s\vec{v} | s \in S\}.$$

c) $\Rightarrow$ b): Suppose $d_{s_1}(\vec{u}) = d_{s_2}(\vec{v}) = \ell$. Let $[\vec{u}]_\alpha = (x_1, x_2, x_3)$, then for $A \in \mathrm{Aut}_0(X)$,

$$[A\vec{u}]_\alpha = \begin{cases} \pm(x_1, x_2, \pm x_3) & , \text{ if } \langle \vec{u}_1, \vec{u}_2 \rangle \neq 0; \\ (\pm x_1, \pm x_2, \pm x_3) & , \text{ if } \langle \vec{u}_1, \vec{u}_2 \rangle = 0. \end{cases}$$

We shall show that $[\vec{v}]_\alpha$ is equal to one of the above. Solving the following equations

$$\ell = d_{s_1}(\vec{u}) = \delta_1\big(1 - (x_1 + x_2 \cos\psi)^2\big) \quad \text{and} \quad \ell = d_{s_2}(\vec{u}) = \delta_2\big(1 - (x_1 \cos\psi + x_2)^2\big),$$

we obtain

$$\begin{cases} x_1 &= k_1 - k_2 \cos\psi \\ x_2 &= -k_1 \cos\psi + k_2 \end{cases}$$

where $k_1 = \pm(\sqrt{1 - \frac{\ell}{\delta_1}})/\sin^2\psi$, and $k_2 = \pm(\sqrt{1 - \frac{\ell}{\delta_2}})/\sin^2\psi$. To solve x_3, recall that the equation of the unit sphere is given by

$$x_1^2 + 2x_1x_2 \cos\psi + x_2^2 + x_3 = 1.$$

Therefore, $x_3 = \pm\sqrt{1 - x_1^2 - 2x_1x_2\cos\psi - x_2^2}$ when $x_1^2 + 2x_1x_2\cos\psi + x_2^2 \leq 1$, or equivalently,

$$1 \geq \sin\psi^2(k_1^2 + k_2^2 - 2k_1k_2\cos\psi).$$

We have the following three cases.

1. When $k_1k_2 = 0$, there are four solutions which are contained in one $\mathrm{Aut}_0(X)$-orbit.
2. When $k_1k_2 \neq 0$ and $\cos\psi = 0$, (or equivalent $\langle \vec{u}_1, \vec{u}_2 \rangle = 0$,) there are eight solutions which are contained in one $\mathrm{Aut}_0(X)$-orbit.
3. When $k_1k_2\cos\psi \neq 0$, there are eight solutions which are contained in two $\mathrm{Aut}_0(X)$-orbits.

It remains to show that when $k_1k_2\cos\psi \neq 0$, then $\vec{u}$ and $\vec{v}$ must be in the same $\mathrm{Aut}_0(X)$-orbit. Suppose not, we may assume that

$$[\vec{u}]_\alpha = (k_1 - k_2\cos\psi, -k_1\cos\psi + k_2, x_3) \text{ and } [\vec{v}]_\alpha = (k_1 + k_2\cos\psi, -k_1\cos\psi - k_2, x_3),$$

which are two solutions of the above equations not contained in the same $\mathrm{Aut}_0(X)$-orbit.

Note that $d_{s_1}(\vec{u}) = d_{s_2}(\vec{v})$ implies that $\langle T\vec{u}, \vec{u} \rangle = \langle T\vec{v}, \vec{v} \rangle$. Together with the assumption $\langle T\vec{u}, T\vec{u} \rangle = \langle T\vec{u}, T\vec{u} \rangle$, we should have

$$\langle T'\vec{v}, T'\vec{v} \rangle = \langle T'\vec{u}, T'\vec{u} \rangle$$

for all T' of the form $T + cI$.

Note that

$$T = \begin{cases} s_1 + {s_1}^{-1} + s_2 + {s_2}^{-1} & \text{, if } s_1 \neq {s_1}^{-1} \text{ and } s_2 \neq {s_2}^{-1}; \\ s_1 + s_2 + {s_2}^{-1} & \text{, if } s_1 = {s_1}^{-1} \text{ and } s_2 \neq {s_2}^{-1}; \\ s_1 + {s_1}^{-1} + s_2 & \text{, if } s_1 \neq {s_1}^{-1} \text{ and } s_2 = {s_2}^{-1}; \\ s_1 + s_2 & \text{, if } s_1 = {s_1}^{-1} \text{ and } s_2 = {s_2}^{-1}. \end{cases}$$

Set

$$T' = T - \begin{cases} (2\cos\theta_1 + 2\cos\theta_2)I & \text{, if } s_1 \neq {s_1}^{-1} \text{ and } s_2 \neq {s_2}^{-1}; \\ (\cos\theta_1 + 2\cos\theta_2)I & \text{, if } s_1 = {s_1}^{-1} \text{ and } s_2 \neq {s_2}^{-1}; \\ (2\cos\theta_1 + \cos\theta_2)I & \text{, if } s_1 \neq {s_1}^{-1} \text{ and } s_2 = {s_2}^{-1}; \\ (\cos\theta_1 + \cos\theta_2)I & \text{, if } s_1 = {s_1}^{-1} \text{ and } s_2 = {s_2}^{-1}. \end{cases}$$

Then by direct computation, we have

$$\langle T'\vec{u}, T'\vec{u}\rangle - \langle T'\vec{v}, T'\vec{v}\rangle = k_1 k_2 \delta_1 \delta_2 \cos\psi \sin^4\psi \cdot \begin{cases} 4 & \text{, if } s_1 \neq {s_1}^{-1} \text{ and } s_2 \neq {s_2}^{-1}; \\ 2 & \text{, if } s_1 = {s_1}^{-1} \text{ and } s_2 \neq {s_2}^{-1}; \\ 2 & \text{, if } s_1 \neq {s_1}^{-1} \text{ and } s_2 = {s_2}^{-1}; \\ 1 & \text{, if } s_1 = {s_1}^{-1} \text{ and } s_2 = {s_2}^{-1}. \end{cases}$$

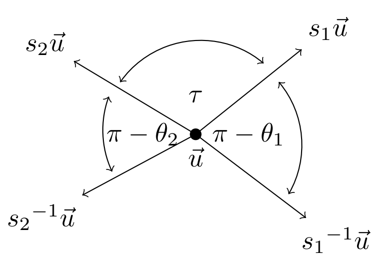
In all cases, $\langle T'\vec{u}, T'\vec{u}\rangle - \langle T'\vec{v}, T'\vec{v}\rangle \neq 0$, which is a contradiction. □

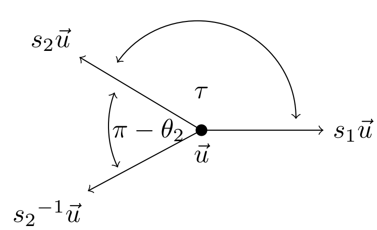
It is easy to see that the four critical points in Theorem 3.3.2 lie in a single $\mathrm{Aut}_0(X)$-orbit and the four boundary points in Proposition 3.3.3 lie in two $\mathrm{Aut}_0(X)$-orbits. Together with Theorem 3.3.4, we conclude that:

Corollary 3.4.3 *For the displacement function $d_{s_1}(\vec{x})$ on $\mathfrak{D}$, let $\delta_{\max}$ and $\delta_{\min}$ be the maximal value and the minimal value respectively. Then for the edge length $\delta_{\max}$ and $\delta_{\min}$, there is a unique equilateral drawing $X_{\vec{u}}$ up to isomorphism.*

3.5 The angle between two edges

To identify the drawing $X_{\vec{u}}$ with the skeleton of some uniform polyhedron, we should find the local configuration of $X_{\vec{u}}$.





s_1 and s_2 are both of order > 2 s_1 is of order 2 and s_2 is of order > 2

Figure 4: Local configurations.

Let us compute the angle τ be between two edges $\vec{u} \to s_1\vec{u}$ and $\vec{u} \to s_2\vec{u}$. Note that

$$\cos\tau = \frac{\langle s_1\vec{u} - \vec{u}, s_2\vec{u} - \vec{u}\rangle}{\|s_1\vec{u} - \vec{u}\|\|s_2\vec{u} - \vec{u}\|}.$$

Besides, under the basis $\alpha = \{\vec{u}_1, \vec{u}_2, \vec{u}_1 \times \vec{u}_2\}$, we have

$$[s_1]_\alpha = [R(\theta_1, \vec{u}_1)]_\alpha = \begin{pmatrix} 1 & \cos\psi(1-\cos\theta_1) & \cos\psi\sin\theta_1 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

and

$$[s_2]_\alpha = [R(\theta_2, \vec{u}_2)]_\alpha = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ \cos\psi(1-\cos\theta_2) & 1 & -\cos\psi\sin\theta_2 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}.$$

By Proposition 3.3.3 and Theorem 3.3.4, for

$$\vec{u} = \frac{1}{\sqrt{\delta_1 + \epsilon 2\sqrt{\delta_1\delta_2}\cos\psi + \delta_2}}(\sqrt{\delta_1}\vec{u}_1 + \epsilon\sqrt{\delta_2}\vec{u}_2),$$

we have

$$\|s_1\vec{u} - \vec{u}\|^2 = \|s_2\vec{u} - \vec{u}\|^2 = \frac{\delta_1\delta_2\sin^2\psi}{\delta_1 + 2\epsilon\sqrt{\delta_1\delta_2}\cos\psi + \delta_2}.$$

By direct computation, we obtain

$$\cos\tau = \frac{\langle s_1\vec{u} - \vec{u}, s_2\vec{u} - \vec{u}\rangle}{\|s_1\vec{u} - \vec{u}\|\|s_2\vec{u} - \vec{u}\|} = -\epsilon\big(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\psi\big).$$

For the drawing $X_{\vec{u}}$ arising from the critical points in Theorem 3.3.2, we have shown that all such drawing are isomorphic and it is sufficient to study one of them. Set $[\vec{u}]_\alpha = (a, -a\cos\psi, \frac{1}{\sin\psi}\sqrt{1 - a^2\sin^2\psi})$, where $a = \frac{1}{\sin^2\psi}\sqrt{1 - \frac{\delta_2}{\delta_1}}$. After a tedious computation, we have

$$\cos\tau = \frac{1}{4\delta_2}\left(\delta_2^2 + 4\cos\psi\sin\theta_1\sin\theta_2 + 2\sin\theta_2\sqrt{(\delta_1 - \delta_2)(\delta_2 - \delta_1\cos^2\psi)}\right).$$

3.6 Drawings of Schreier coset graphs

The drawing $X_{\vec{u}}$ induces a drawing of the Schreier coset graph X^H associated to $(G/H, HSH/H)$ if $\vec{u}$ is fixed by all elements H.

In this case, elements of H are rotations around $\vec{u}$, which implies H is cyclic and it is generated by the rotation of the smallest angle. Therefore, we have the following proposition.

Proposition 3.6.1 *If the Schreier coset graph X^H can be drawn by $X_{\vec{u}}$ for some $\vec{u} \in \mathfrak{D}$, then H is cyclic.*

Suppose $H = \langle h\rangle$. Let $\vec{u} = \vec{u}_h$ be a unit vector fixed by h. Let $\mathrm{Stab}(\vec{u})$ be the stabilizer of $\vec{u}$ in $SO(3)$, which consists all rotations around $\vec{u}$. Note that unit vectors $\vec{x}$ and $\vec{y}$ are in the same $\mathrm{Stab}(\vec{u})$-orbit if $\langle\vec{x}, \vec{u}\rangle = \langle\vec{y}, \vec{u}\rangle$.

Theorem 3.6.2 *For $\vec{u} \in \mathbb{S}^2$, $\vec{u} \in \mathfrak{D}$ if and only if $s_2 \in \mathrm{Stab}(\vec{u})s_1\mathrm{Stab}(\vec{u})$.*

Proof: For $\vec{u} \in \mathbb{S}^2$,

$$d_{s_1}(\vec{u}) = \langle s_1(\vec{u}) - \vec{u}, s_1(\vec{u}) - \vec{u}\rangle = 2 - 2\langle s_1(\vec{u}), \vec{u}\rangle.$$

Therefore, $\vec{u} \in \mathfrak{D}$ if and only if $\langle s_1(\vec{u}), \vec{u}\rangle = \langle s_2(\vec{u}), \vec{u}\rangle$. Suppose $\langle s_1(\vec{u}), \vec{u}\rangle = \langle s_2(\vec{u}), \vec{u}\rangle$, then $s_1(\vec{u})$ and $s_2(\vec{u})$ lie in the same $\mathrm{Stab}(\vec{u})$-orbit. Therefore, $s_2(\vec{u}) = g_1 s_1(\vec{u})$ for some $g_1 \in \mathrm{Stab}(\vec{u})$, which implies that $g_2 := {s_2}^{-1} g_1 s_1$ fixes $\vec{u}$. We conclude that $s_2 = g_1 s_1 {g_2}^{-1} \in \mathrm{Stab}(\vec{u})s_1\mathrm{Stab}(\vec{u})$.

Conversely, suppose $s_2 = g_1 s_1 g_2$ for some $g_1, g_2 \in \mathrm{Stab}(\vec{u})$, then

$$\langle s_2(\vec{u}), \vec{u}\rangle = \langle g_1 s_1 g_2(\vec{u}), \vec{u}\rangle = \langle g_1 s_1(\vec{u}), \vec{u}\rangle = \langle s_1(\vec{u}), {g_1}^{-1}(\vec{u})\rangle = \langle s_1(\vec{u}), \vec{u}\rangle.$$

□

Since we can replace s_2 by ${s_2}^{-1}$ in the above theorem, we also have $\vec{u} \in \mathfrak{D}$ if and only if ${s_2}^{-1} \in \mathrm{Stab}(\vec{u})s_1\mathrm{Stab}(\vec{u})$. On the other hand, when $\vec{u}$ is the rotational axis of h, $\langle h\rangle$ is a subgroup of $\mathrm{Stab}(\vec{u})$. Consequently, we obtain the following simple criterion.

Corollary 3.6.3 *For $h \in G$, if s_2 or ${s_2}^{-1} \in \langle h\rangle s_1 \langle h\rangle$, then $\mathfrak{D}$ contains the fixed vector of h.*

In general, to find all h in Corollary 3.6.3, one can only study by cases. On the other hand, some h exists for all cases.

Example 3.6.1 *Let $h_1 = s_1 s_2$, which is of order m_1, then*

$${s_2}^{-1} = 1 \cdot {s_2}^{-1} = {h_1}^{m_1} {s_2}^{-1} = {h_1}^{m_1 - 1} s_1 \in \langle h_1\rangle s_1 \langle h_1\rangle.$$

Therefore, a drawing of the Schreier coset graph $X^{\langle h_1\rangle}$ can be induced by $X_{\vec{u}_{h_1}}$.

Example 3.6.2 *Let $h_2 = s_1 {s_2}^{-1}$ which is of order m_2, then*

$$s_2 = 1 \cdot s_2 = {h_2}^{m_2} s_2 = {h_2}^{m_2 - 1} s_1 \in \langle h_2\rangle s_1 \langle h_2\rangle.$$

Therefore, a drawing of the Schreier coset graph $X^{\langle h_2\rangle}$ can be induced by $X_{\vec{u}_{h_2}}$.

By the same argument as the above two examples, for $h_3 = {s_1}^{-1} s_2$ and $h_4 = {s_1}^{-1}{s_2}^{-1}$, we obtain drawings of Schreier coset graphs $X^{\langle h_3\rangle}$ and $X^{\langle h_4\rangle}$ respectively, namely $X_{\vec{u}_{h_3}}$ and $X_{\vec{u}_{h_4}}$.

On the other hand, for $A = R(\pi, \vec{u}_3/\|\vec{u}_3\|)$ in Example 3.4.2, we have $A s_1 A^{-1} = {s_1}^{-1}$ and $A s_2 A^{-1} = {s_2}^{-1}$, which implies that

$$A h_1 A^{-1} = A s_1 s_2 A^{-1} = {s_1}^{-1}{s_2}^{-1} = h_3 \quad \text{and } \vec{u}_{h_3} = \pm A \vec{u}_{h_1}.$$

Therefore, $X_{\vec{u}_{h_1}}$ and $X_{\vec{u}_{h_3}}$ are isomorphic. Similarly, $X_{\vec{u}_{h_2}}$ and $X_{\vec{u}_{h_4}}$ are isomorphic.

3.7 Summarization

Let us summarize the results in this section sections. Set

$$\begin{cases} a = \frac{1}{\sin^2\psi}\sqrt{1 - \frac{\delta_2}{\delta_1}} \\ b = (\delta_1 - 2\sqrt{\delta_1\delta_2}\cos\psi + \delta_2)^{-1} \\ c = (\delta_1 + 2\sqrt{\delta_1\delta_2}\cos\psi + \delta_2)^{-1} \end{cases}$$

and

$$\ell = \ell(X_{\vec{u}}) = \sqrt{d_{s_1}(\vec{u})},$$

which is the edge length of $X_{\vec{u}}$. (Note that $b < c$ since $\cos\psi < 0$.) When $[\vec{u}]_\alpha = \left(\sqrt{\delta_1 b}, -\sqrt{\delta_2 b}, 0\right)$, $X_{\vec{u}}$ is the unique shortest equilateral spherical drawing up to isomorphism. Its edge length ℓ is equal to $\sqrt{b\,\delta_1\delta_2}\sin\psi$ and the angle τ between two edges $\vec{u} \to s_1\vec{u}$ and $\vec{u} \to s_2\vec{u}$ satisfies

$$\cos\tau = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\psi.$$

We denote this vector by $\vec{u}_{\min}$. For the longest drawing, we have the following.

Case I: $\delta_1 \geq \delta_2 \geq \delta_1\cos^2\psi$**.**
When $[\vec{u}]_\alpha = (a, -a\cos\psi, \frac{1}{\sin\psi}\sqrt{1 - a^2\sin^2\psi})$, $X_{\vec{u}}$ is the unique longest equilateral drawing up to isomorphism. Its edge length ℓ is equal to $\sqrt{\delta_2}$ and

$$\cos\tau = \frac{1}{4\delta_2}\left(\delta_2^2 + 4\cos\psi\sin\theta_1\sin\theta_2 + 2\sin\theta_2\sqrt{(\delta_1 - \delta_2)(\delta_2 - \delta_1\cos^2\psi)}\right)$$

Case II: $\delta_1\cos^2\psi > \delta_2$**.**
When $[\vec{u}]_\alpha = \left(\sqrt{\delta_1 c}, \sqrt{\delta_2 c}, 0\right)$, $X_{\vec{u}}$ is the unique longest equilateral drawing up to isomorphism. Its edge length ℓ is equal to $\sqrt{c\delta_1\delta_2}\sin\psi$ and

$$\cos\tau = -\big(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\psi\big).$$

In both cases, we denote this vector by $\vec{u}_{\max}$.

Besides, for $h_1 = s_1 s_2$ and $h_2 = s_1 {s_2}^{-1}$, an equilateral drawing of the Schreier coset graph $X^{\langle h_i\rangle}$ can be induced by $X_{\vec{u}_{h_i}}$ for $i = 1, 2$. Here $\vec{u}_{h_i}$ is a unit vector fixed by h_i.

4 Rotational Planar Cayley Graphs

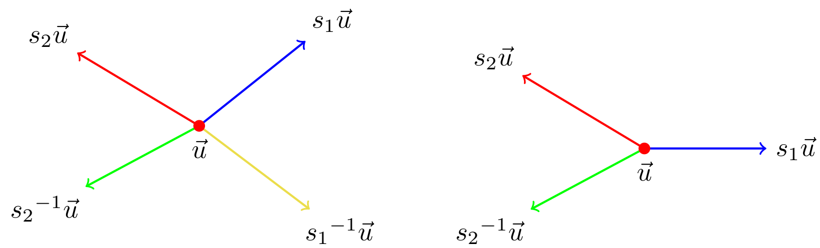
In this section, we study the rotational planar Cayley graphs $\mathrm{Cay}(G, S)$ introduced in Section 2.3, which are not circular. To do so, we will give a permutation representation of the group G so that one can use computer program to compute the real three dimensional representation σ of G introduced in Section 2.5 systematically. In fact, once we know $\cos\psi$, we can set

$$\vec{u}_1 = (1, 0, 0) \quad \text{and} \quad \vec{u}_2 = (\cos\psi, \sin\psi, 0).$$

Then the real three dimensional representation σ can be characterized by

$$\sigma(s_1) = R(2\pi/m_1, \vec{u}_1) \quad \text{and} \quad \sigma(s_2) = R(2\pi/m_2, \vec{u}_2).$$

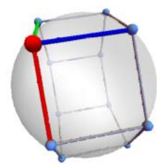
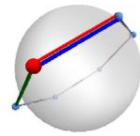
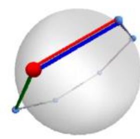
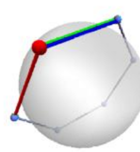
For each Cayley graph, we list the value of $\cos\psi$ and four special equilateral spherical drawings $X_{\vec{u}_{\min}}$, $X_{\vec{u}_{\max}}$, $X_{\vec{u}_{h_1}}$ and $X_{\vec{u}_{h_2}}$. Besides, for each drawing $X_{\vec{u}}$, the value of $\cos\tau$ and the edge length ℓ are listed in the table. We can identify all except two drawings as the skeleton of some uniform polyhedron by computing the local configuration. To make the figures clearly, we add a sphere of suitable radius in the middle of the drawing and color the edges connecting to the point $\vec{u}$ as shown in the following figure.



The configuration when $(s_1)^2 \neq 1$ The configuration when $(s_1)^2 = 1$

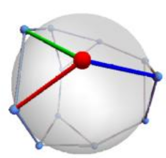
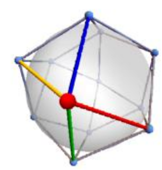
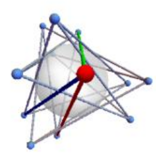
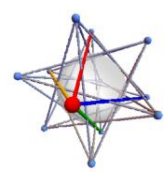
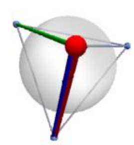
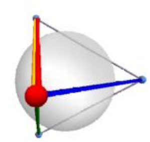
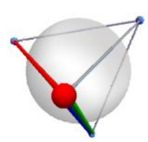
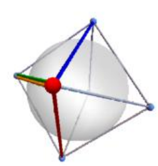
4.1 The dihedral group D_n

Let $G = \langle x, y \rangle$ where $x = \big(1\,n\big)\big(2\,(n-1)\big)\cdots$ and $y = (12\cdots n)$. Then $G \cong D_n$. The following table is the result for $n = 6$.

s_1	$\sigma(y)$
s_2	$\sigma(x)$
$\cos\psi$	0
$X_{\vec{u}_{\min}}$	 $\cos\tau{=}0,\ \ell{=}\frac{2}{\sqrt{5}}$
the skeleton of a	prism
$X_{\vec{u}_{\max}}$	 $\cos\tau{=}1,\ \ell{=}1(=\sqrt{\delta_2})$
the skeleton of a	regular polygon
$X_{\vec{u}_{H_1}}$	 $\cos\tau{=}1,\ \ell{=}1(=\sqrt{\delta_2})$
the skeleton of a	regular polygon
$X_{\vec{u}_{H_2}}$	 $\cos\tau{=}-\frac{1}{2},\ \ell{=}1(=\sqrt{\delta_2})$
the skeleton of a	regular polygon

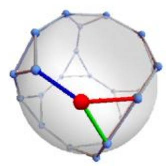
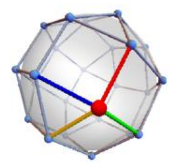
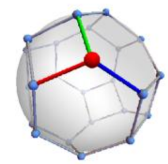
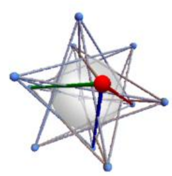
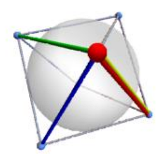
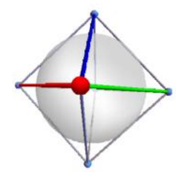
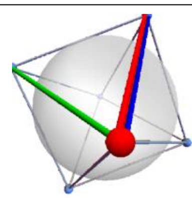
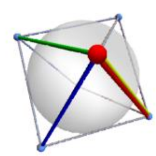
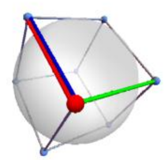
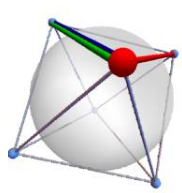
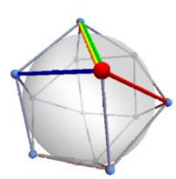
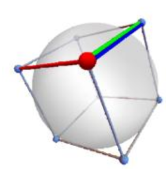
4.2 The tetrahedral group A_4

Let $G = \langle x, y \rangle$ where $x = (123)$ and $y = (12)(34)$. Then $xy = (134)$ and $G \cong A_4$.

s_1	$\sigma(y)$	$\sigma(xy)$
s_2	$\sigma(x)$	$\sigma(x)$
$\cos\psi$	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{3}$
$X_{\vec{u}_{\min}}$	 $\cos\tau = -\frac{1}{2}, \ell = \sqrt{\frac{8}{11}}$	 $\cos\tau = 0, \ell = 1$
the skeleton of a	truncated tetrahedron	cuboctahedron
$X_{\vec{u}_{\max}}$	 $\cos\tau = \frac{1}{12}(9+\sqrt{5}), \ell = \sqrt{3}$	 $\cos\tau = \frac{2}{3}, \ell = \sqrt{3}$
the skeleton of a	unknown	small stellated dodecahedron
$X_{\vec{u}_{H_1}}$	 $\cos\tau = 1, \ell = \sqrt{\frac{8}{3}}$	 $\cos\tau = \frac{1}{2}, \ell = \sqrt{\frac{8}{3}}$
the skeleton of a	tetrahedron	tetrahedron
$X_{\vec{u}_{H_2}}$	 $\cos\tau = \frac{1}{2}, \ell = \sqrt{\frac{8}{3}}$	 $\cos\tau = 0, \ell = \sqrt{2}$
the skeleton of a	tetrahedron	octahedron

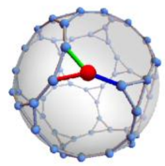
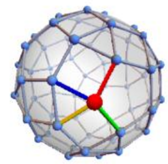
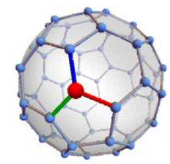
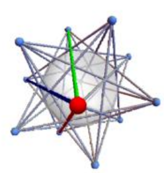
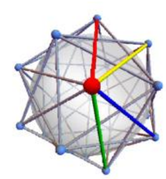
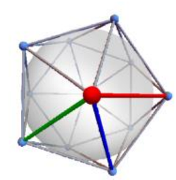
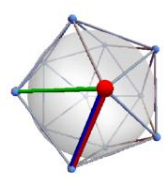
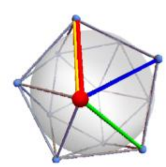
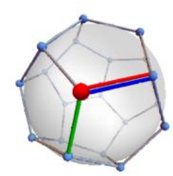
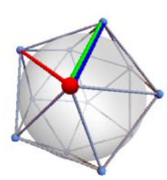
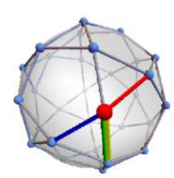
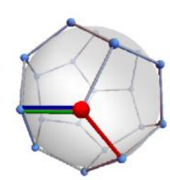
4.3 The octahedral group S_4

Let $G = \langle x, y \rangle$ where $x = (134)$ and $y = (12)$. Then $xy = (1234)$ and $G \cong S_4$.

s_1	$\sigma(y)$	$\sigma(x)$	$\sigma(y)$
s_2	$\sigma(x)$	$\sigma(xy)$	$\sigma(xy)$
$\cos\psi$	$-\sqrt{\frac{2}{3}}$	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{2}}$
$X_{\vec{u}_{\min}}$	 $\cos\tau=-\frac{1}{\sqrt{2}}, \ell=\frac{2}{\sqrt{7+4\sqrt{2}}}$	 $\cos\tau=0, \ell=\frac{2}{\sqrt{5+2\sqrt{2}}}$	 $\cos\tau=-\frac{1}{2}, \ell=\sqrt{\frac{2}{5}}$
the skeleton of a	truncated cube	rhombicuboctahedron	truncated octahedron
$X_{\vec{u}_{\max}}$	 $\cos\tau=\frac{5}{6}, \ell=\sqrt{3}$	 $\cos\tau=\frac{1}{2}, \ell=\sqrt{2}$	 $\cos\tau=\frac{1}{2}, \ell=\sqrt{2}$
the skeleton of a	unknown	octahedron	octahedron
$X_{\vec{u}_{H_1}}$	 $\cos\tau=1, \ell=\sqrt{2}$	 $\cos\tau=\frac{1}{2}, \ell=\sqrt{2}$	 $\cos\tau=1, \ell=\frac{2}{\sqrt{3}}$
the skeleton of a	octahedron	octahedron	cube
$X_{\vec{u}_{H_2}}$	 $\cos\tau=-\frac{1}{2}, \ell=\sqrt{2}$	 $\cos\tau=-\frac{1}{2}, \ell=1$	 $\cos\tau=0, \ell=\frac{2}{\sqrt{3}}$
the skeleton of a	octahedron	cuboctahedron	cube

4.4 The icosahedral group A_5

Let $G = \langle x, y \rangle$ where $x = (124)$ and $y = (23)(45)$. Then $xy = (12345)$ and $G \cong A_5$.

s_1	$\sigma(y)$	$\sigma(x)$	$\sigma(y)$
s_2	$\sigma(x)$	$\sigma(xy)$	$\sigma(xy)$
$\cos\psi$	$-\sqrt{\frac{1}{6}\left(3+\sqrt{5}\right)}$	$-\sqrt{\frac{1}{15}\left(5+2\sqrt{5}\right)}$	$-\sqrt{\frac{1}{10}\left(5+\sqrt{5}\right)}$
$X_{\vec{u}_{\min}}$	 $\cos\tau=\frac{-1}{4}\left(\sqrt{5}+1\right), \ell=\sqrt{\frac{2}{61}\left(37-15\sqrt{5}\right)}$	 $\cos\tau=0, \ell=\sqrt{\frac{1}{41}\left(44-16\sqrt{5}\right)}$	 $\cos\tau=-\frac{1}{2}, \ell=\sqrt{\frac{2}{109}\left(29-9\sqrt{5}\right)}$
the skeleton of a	truncated dodecahedron	rhombicosidodecahedron	truncated icosahedron
$X_{\vec{u}_{\max}}$	 $\cos\tau=\frac{1}{4}\left(\sqrt{5}+1\right), \ell=\sqrt{2+\frac{2}{\sqrt{5}}}$	 $\cos\tau=0, \ell=\frac{2}{\sqrt{3}}$	 $\cos\tau=\frac{1}{2}, \ell=\sqrt{\frac{8}{\sqrt{5}+5}}$
the skeleton of a	small stellated dodecahedron	dodecadodecahedron	icosahedron
$X_{\vec{u}_{H_1}}$	 $\cos\tau=1, \ell=\sqrt{\frac{8}{\sqrt{5}+5}}$	 $\cos\tau=\frac{1}{2}, \ell=\sqrt{\frac{8}{\sqrt{5}+5}}$	 $\cos\tau=1, \ell=\sqrt{\frac{6-2\sqrt{5}}{3}}$
the skeleton of a	icosahedron	icosahedron	dodecahedron
$X_{\vec{u}_{H_2}}$	 $\cos\tau=\frac{1}{2}, \ell=\sqrt{\frac{8}{\sqrt{5}+5}}$	 $\cos\tau=\frac{-1}{4}\left(\sqrt{5}+1\right), \ell=\frac{1}{2}\left(\sqrt{5}-1\right)$	 $\cos\tau=\frac{1}{4}\left(1-\sqrt{5}\right), \ell=\sqrt{\frac{6-2\sqrt{5}}{3}}$
the skeleton of a	icosahedron	icosidodecahedron	dodecahedron

4.5 Conclusion

From the results in Section 4.1 to Section 4.4, we see that the shortest equilateral drawing of a rotational planar Cayley graph gives us the skeleton of the corresponding uniform polyhedron. On the other hand, some longest equilateral drawings can not be identified as the skeleton of any uniform polyhedron and it is interesting to figure out these special structures.

As a final remark, we like to point out all vertex-transitive equilateral polyhedra can be realized by this manner if one consider all possible three representations of the underlying groups of rotational planar Cayley graphs.

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