

An Adaptive Version of Brandes’ Algorithm for Betweenness Centrality

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Abstract

Betweenness centrality—measuring how many shortest paths pass through a vertex—is one of the most important network analysis concepts for assessing the relative importance of a vertex. The well-known algorithm of Brandes [J. Math. Sociol. ’01] computes, on an n -vertex and m -edge graph, the betweenness centrality of all vertices in $O(nm)$ worst-case time. In later work, significant empirical speedups were achieved by preprocessing degree-one vertices and by graph partitioning based on cut vertices. We contribute an algorithmic treatment of degree-two vertices, which turns out to be much richer in mathematical structure than the case of degree-one vertices. Based on these three algorithmic ingredients, we provide a strengthened worst-case running time analysis for betweenness centrality algorithms. More specifically, we prove an adaptive running time bound $O(kn)$, where $k < m$ is the size of a minimum feedback edge set of the input graph.

Submitted: May 2020	Reviewed: August 2020	Revised: October 2020	Accepted: October 2020
	Final: October 2020	Published: October 2020	
Article type: Regular Paper		Communicated by: Y. Okamoto	

LK was partially supported by DFG Project FPTinP NI 369/16. An extended abstract of this work appeared in the proceedings of the 29th International Symposium on Algorithms and Computation (ISAAC ’18), held in Jiaoxi, Taiwan, December 16–19, 2018.

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1 Introduction

One of the most important building blocks in network analysis is to determine a vertex's relative importance in the network. A key concept herein is *betweenness centrality* as introduced in 1977 by Freeman [11]; it measures centrality based on shortest paths. Intuitively, for each vertex, betweenness centrality counts the (relative) number of shortest paths that pass through the vertex. A straightforward algorithm for computing the betweenness centrality on undirected (unweighted) n -vertex graphs runs in $O(n^3)$ time. For the weighted case, an improvement of this to $O(n^{3-\varepsilon})$ time for any $\varepsilon > 0$ would break the so-called APSP-conjecture [1]. Further, for unweighted graphs with constant maximum degree, computing the betweenness centrality of a single vertex in $O(n^{2-\varepsilon})$ time would break the strong exponential time hypothesis (SETH) [7]. In 2001, Brandes [5] presented the to date theoretically fastest algorithm, improving the running time to $O(nm)$ for graphs with m edges. As many real-world networks are sparse, this is a far-reaching improvement, having a huge impact also in practice. We remark that Newman [23, 24] presented a high-level description of an algorithm computing a variant of betweenness centrality which also runs in $O(nm)$ time.

Since betweenness centrality is a measure of outstanding importance in network science, it finds numerous applications in diverse areas, e.g. in social network analysis [24, 33] or neuroscience [16, 20]. Provably speeding up betweenness centrality computations is the ultimate goal of our research. To this end, we extend previous work and provide a rigorous mathematical analysis that yields a new (parameterized) running time upper bound of the corresponding algorithm.

Our work is in line with numerous research efforts concerning the development of algorithms for computing betweenness centrality, including approximation algorithms [2, 12, 28], parallel and distributed algorithms [30, 32], streaming and incremental algorithms [14, 22], algorithms for updates [19], and exact [9] and fixed-parameter algorithms [6]. Formally, we study the following problem.

BETWEENNESS CENTRALITY

Input: An undirected graph G .

Task: Compute the *betweenness centrality* $C_B(v) := \sum_{s,t \in V(G)} \sigma_{st}(v) / \sigma_{st}$ for each vertex $v \in V(G)$.

Herein, σ_{st} is the number of shortest paths in G from vertex s to vertex t , and $\sigma_{st}(v)$ is the number of shortest paths from s to t that additionally pass through v .¹

Extending previous, more empirically oriented work of Baglioni et al. [3], Puzis et al. [27], and Sariyüce et al. [29] (see Section 2 for a description of their approaches), our main result is an algorithm for BETWEENNESS CENTRALITY that runs in $O(kn)$ time, where k denotes the feedback edge number of the input graph G . The *feedback edge number* of G , also known as the *cyclomatic number*, is the minimum number of edges one needs to delete from G in order

¹To simplify matters, we set $\sigma_{st}(v) = 0$ if $v = s$ or $v = t$. This is equivalent to the definition used by Brandes [5] but differs from the definition used by Newman [23], where $\sigma_{st}(s) = 1$.

to make it a forest.² Clearly, $k = 0$ holds on trees, and $k = m - n + c$ holds in general, where c is the number of connected components of G . Thus our algorithm is *adaptive*, i.e., it interpolates between linear time for constant k and the running time of the best unparameterized algorithm.³ But as $k \approx m - n$, we do not provide asymptotic improvement over Brandes’ algorithm for most graphs. When the input graph is very tree-like ($m = n + o(n)$), however, our new algorithm theoretically improves on Brandes’ algorithm. Real-world networks showing the relation between PhD candidates and their supervisors [8, 15] or the ownership relation between companies [26] typically have a feedback edge number that is smaller than the number of vertices or edges [25] by orders of magnitude.⁴ Moreover, Baglioni et al. [3], building on Brandes’ algorithm and basically shrinking the input graph by deleting degree-one vertices in a preprocessing step, report on significant speedups in comparison with Brandes’ basic algorithm in empirical tests with real-world social networks. For roughly half of their networks, $m - n$ is smaller than n by at least one order of magnitude.

Our algorithmic contribution is to complement the works of Baglioni et al. [3], Puzis et al. [27], and Sariyüce et al. [29] by, roughly speaking, additionally dealing with degree-two vertices. These vertices are much harder to cope with and to analyze since, other than degree-one vertices, they may lie on shortest paths between two vertices. From a practical point of view, one may expect a significant speedup if one can take care of degree-two vertices more quickly. This is due to the nature of many real-world social networks having a power-law degree distribution [4]; thus a large fraction of the vertices are of degree one or two. On the flip side, our more complicated algorithm incurs higher constants in the running time, thus, definitive statements on the practicality require experimental evaluations. The work of Vella et al. [31] can be seen as a first step in this direction: they used a heuristic approach to process degree-two vertices for improving the performance of their BETWEENNESS CENTRALITY algorithms on several real-world networks.

Our work is purely theoretical in spirit, the most profound contribution being the analysis of the worst-case running time of the proposed betweenness centrality algorithm based on degree-one-vertex processing [3], usage of cut vertices [27, 29], and our degree-two-vertex processing. To the best of our knowledge, this provides the first proven worst-case improvement over Brandes’ upper bound in a relevant special case.

Notation. We use mostly standard graph notation. Given a graph G , $V(G)$ and $E(G)$ denote the vertex respectively edge set of G with $n = |V(G)|$ and $m =$

²Notably, BETWEENNESS CENTRALITY computations have also been studied when the input graph is a tree [32], hinting at the practical relevance of this special case.

³We mention in passing that in recent work [21] we employed the same parameter “feedback edge number” in terms of theoretically analyzing known data reduction rules for computing maximum-cardinality matchings. Recent empirical work with this algorithm demonstrated significant accelerations of the state-of-the-art matching algorithm [18, 17].

⁴The networks are available in the Pajek Dataset of Vladimir Batagelj and Andrej Mrvar (2006) (<http://vlado.fmf.uni-lj.si/pub/networks/data/>).

$|E(G)|$. We denote the vertices of degree one, two, and at least three by $V^=1(G)$, $V^=2(G)$, and $V^{\geq 3}(G)$, respectively. A *cut vertex* or *articulation vertex* is a vertex whose removal disconnects the graph. A connected component of a graph is *biconnected* if it does not contain any cut vertices, and hence, no vertices of degree one. A path $P = v_0 \dots v_q$ is a graph with $V(P) = \{v_0, \dots, v_q\}$ and $E(P) = \{\{v_i, v_{i+1}\} \mid 0 \leq i < q\}$. The length of the path P is $|E(P)|$. We call v_0 and v_q the endpoints and v_1, \dots, v_{q-1} the inner vertices of the path. Adding the edge $\{v_q, v_0\}$ to P gives a cycle $C = v_0 \dots v_q v_0$. The distance $d_G(s, t)$ between vertices $s, t \in V(G)$ is the length of the shortest path between s and t in G . The number of shortest s - t -paths is denoted by σ_{st} . The number of shortest s - t -paths containing some vertex v is denoted by $\sigma_{st}(v)$. We set $\sigma_{st}(v) = 0$ if $s = v$ or $t = v$ (or both).

We set $[j, k] := \{j, j+1, \dots, k\}$ and denote for a set X by $\binom{X}{i}$ the size- i subsets of X .

Lastly, when we talk about the time complexity of algorithms, we refer to the number of arithmetic operations.

Paper outline. The presentation of our algorithm is split into two parts: In Section 2 we present the strategy of our algorithm. Section 3 deals with the main technical challenge of our algorithm, namely how to deal with consecutive degree-two vertices. Some proofs in the latter part are deferred to the appendix.

Finally, we conclude in Section 4.

2 Algorithm Overview

In this section, we review our algorithmic strategy to compute the betweenness centrality of each vertex. Before doing so, since we build on the works of Brandes [5], Baglioni et al. [3], Puzis et al. [27], and Sariyüce et al. [29], we first give the high-level ideas behind their algorithmic approaches. Then, we describe the ideas behind our extension. We assume throughout our paper that the input graph is connected. Otherwise, we can process the connected components one after another.

Existing algorithmic approaches. Brandes [5] developed an $O(nm)$ -time algorithm which essentially runs modified breadth-first searches (BFS) from each vertex of the graph. In each of these modified BFS starting in a vertex s , Brandes' algorithm computes the "effect" that s has on the betweenness centrality values of all other vertices. More formally, the modified BFS starting at vertex s computes for every $v \in V(G)$ the value

$$\sum_{t \in V(G)} \frac{\sigma_{st}(v)}{\sigma_{st}}.$$

Reducing the number of performed modified BFS in Brandes' algorithm is one way to speed up Brandes' algorithm. To this end, a popular approach is to

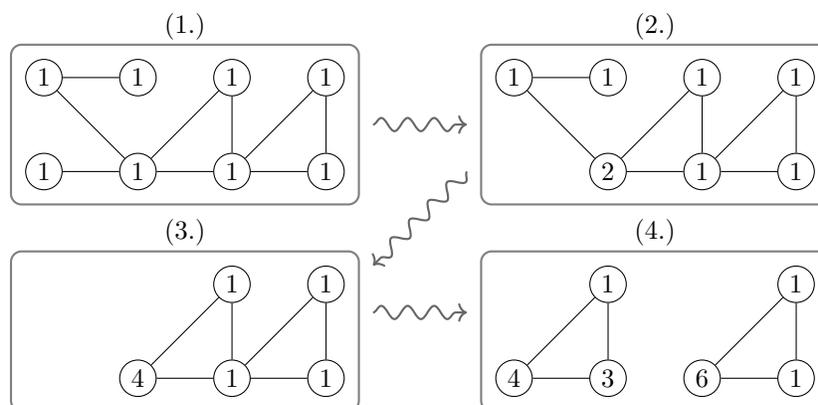


Figure 1: An initial graph where the $\text{Pen}[\cdot]$ -value of each vertex is 1 (top left) and the same graph after deleting one (top right) or both (bottom left) pending trees using Reduction Rule 1. The labels are the respective $\text{Pen}[\cdot]$ -values. Subfigure (4.) shows the graph of (3.) after applying Lemma 1 to the only cut vertex of the graph.

remove in a preprocessing step all degree-one vertices from the graph [3, 27, 29]. By repeatedly removing degree-one vertices, whole “pending trees” (subgraphs that are trees and are connected to the rest of the graph by a single edge) can be deleted. Considering a degree-one vertex v , observe that in *each* shortest path P starting at v , the second vertex in P is the single neighbor u of v . Hence, after deleting v , one needs to store the information that u had a degree-one neighbor. To this end, one uses for each vertex w a counter called $\text{Pen}[w]$ (for *pending*) that stores the number of vertices in the subtree pending on w that were deleted before. In contrast to e. g. Baglioni et al. [3], we initialize for each vertex $w \in V$ the value $\text{Pen}[w]$ with one instead of zero (so we count w as well). This simplifies most of our formulas. See Figure 1 (Parts (1.) to (3.)) for an example of the $\text{Pen}[\cdot]$ -values of the vertices at different points in time. We obtain the following (weighted) problem variant.

WEIGHTED BETWEENNESS CENTRALITY

Input: An undirected graph G and vertex weights $\text{Pen}: V(G) \rightarrow \mathbb{N}$.

Task: Compute for each vertex $v \in V(G)$ the weighted betweenness centrality

$$C_B(v) := \sum_{s,t \in V(G)} \gamma(s, t, v), \tag{1}$$

$$\text{where } \gamma(s, t, v) := \text{Pen}[s] \cdot \text{Pen}[t] \cdot \sigma_{st}(v) / \sigma_{st}.$$

The effect of a degree-one vertex to the betweenness centrality value of its neighbor is captured in the next data reduction rule.

Reduction Rule 1 ([3, 27, 29]) *Let G be a graph, let $s \in V(G)$ be a degree-one vertex, and let $v \in V(G)$ be the neighbor of s . Then increase $\text{Pen}[v]$ by $\text{Pen}[s]$, increase the betweenness centrality of v by $\text{Pen}[s] \cdot \sum_{t \in V(G) \setminus \{s,v\}} \text{Pen}[t]$, and remove s from the graph.*

By Reduction Rule 1 the influence of a degree-one vertex to the betweenness centrality of its neighbor can be computed in constant time. This is since

$$\sum_{t \in V(G) \setminus \{s,v\}} \text{Pen}[t] = \left(\sum_{t \in V(G)} \text{Pen}[t] \right) - \text{Pen}[s] - \text{Pen}[v],$$

and $\sum_{t \in V(G)} \text{Pen}[t]$ can be precomputed in linear time.

A second approach to speed up Brandes' algorithm is to split the input graph G into smaller connected components and process them separately [27, 29]. This approach is a generalization of the ideas behind removing degree-one vertices and works with cut vertices. The basic observation for this approach is as follows. Consider a cut vertex v such that removing v breaks the graph into two connected components C_1 and C_2 (the idea generalizes to more components). Obviously, every shortest path P in G that starts in C_1 and ends in C_2 has to pass through v . For the betweenness centrality values of the vertices inside C_1 (inside C_2) it is not important where exactly P ends (starts). Hence, for computing the betweenness centrality values of the vertices in C_1 , it is sufficient to know which vertices in C_1 are adjacent to v and how many vertices are contained in C_2 . Thus, in a preprocessing step one can just add to C_1 the cut vertex v with $\text{Pen}[v]$ being increased by the sum of $\text{Pen}[\cdot]$ -values of the vertices in C_2 (see Figure 1 (bottom)). Formally, this is done as follows.

Lemma 1 ([27, 29]) *Let G be a connected graph, let v be a cut vertex such that removing v yields $\ell \geq 2$ connected components C_1, \dots, C_ℓ , and let $\xi := \text{Pen}[v]$. Then remove v , add a vertex v_i to every component C_i , make it adjacent to all vertices in the respective component that were adjacent to v , and set*

$$\text{Pen}[v_i] = \xi + \sum_{j \in [1, \ell] \setminus \{i\}} \sum_{w \in V(C_j) \setminus \{v_j\}} \text{Pen}[w].$$

For a vertex v in component C_i denote by $C_B^{C_i}(v)$ the betweenness centrality of v within the component C_i . Computing the betweenness centrality of each connected component independently, increasing the betweenness centrality of v by

$$\sum_{i=1}^{\ell} (C_B^{C_i}(v_i) + (\text{Pen}[v_i] - \xi) \cdot \sum_{s \in V(C_i) \setminus \{v_i\}} \text{Pen}[s]),$$

and ignoring all new vertices v_i is the same as computing the betweenness centrality in G , that is,

$$C_B^G(u) = \begin{cases} C_B^{C_i}(u), & \text{if } u \in V(C_i) \setminus \{v_i\}; \\ \sum_{i=1}^{\ell} (C_B^{C_i}(v_i) + (\text{Pen}[v_i] - \xi) \cdot \sum_{s \in V(C_i) \setminus \{v_i\}} \text{Pen}[s]), & \text{if } u = v. \end{cases}$$

Applying the above procedure as a preprocessing on all cut vertices and degree-one vertices leaves us with biconnected components that we can solve each independently. Here, we split off one special case, namely when a biconnected component consists solely of degree-two vertices, that is, it is a cycle. The reason for this is that our general algorithm requires vertices of degree at least three as a basis. Our algorithm then efficiently processes paths of degree-two vertices that connect these vertices of degree at least three.

We first look at the special case that the biconnected component is a cycle. Then we deal with biconnected components that contain at least two vertices of degree at least three (note that a component with only one vertex of degree at least three cannot be biconnected).

2.1 Dealing with Cycles

We now show how to solve WEIGHTED BETWEENNESS CENTRALITY on cycles with a linear-time dynamic programming algorithm. Note that the vertices in the cycle can have different betweenness centrality values as they can have different $\text{Pen}[\cdot]$ -values.

Proposition 1 *Let $C = x_0 \dots x_q x_0$ be a cycle. Then, one can compute the weighted betweenness centrality of the vertices in C in $O(q)$ time and space.*

Proof: We first introduce some notation needed for the proof. We then show how to compute $BC[v]$ for $v \in V(C)$ efficiently. Finally, we prove the running time.

By $[x_i, x_j]$, $0 \leq i, j \leq q$ we denote the set of vertices $\{x_i, x_{i+1 \bmod (q+1)}, x_{i+2 \bmod (q+1)}, \dots, x_j\}$. For a maximal induced path $P^{\max} = x_0 \dots x_q$, we define

$$W^{\text{left}}[x_i] := \sum_{k=0}^i \text{Pen}[x_k], \text{ and}$$

$$W[x_i, x_j] := \begin{cases} \text{Pen}[x_i], & \text{if } i = j; \\ W^{\text{left}}[x_j] - W^{\text{left}}[x_i] + \text{Pen}[x_i], & \text{if } i < j; \\ W^{\text{left}}[x_q] - W^{\text{left}}[x_i] + W^{\text{left}}[x_j] + \text{Pen}[x_i], & \text{if } i > j. \end{cases}$$

The value $W[x_i, x_j]$ is the sum of the values $\text{Pen}[x_k]$ with $x_k \in [x_i, x_j]$. Further, we denote by $\varphi(i) = (\frac{q+1}{2} + i) \bmod (q + 1)$ the index that is “opposite” to i on the cycle. Note that if $\varphi(i) \in \mathbb{N}$, then $x_{\varphi(i)}$ is the unique vertex in C to which there are two shortest paths from x_i , one visiting $x_{i+1 \bmod (q+1)}$ and one visiting $x_{i-1 \bmod (q+1)}$. Otherwise, if $\varphi(i) \notin \mathbb{N}$, then there is only one shortest path from x_k to any $t \in V(C)$. For the sake of readability, let $\text{Pen}[x_{\varphi(x_i)}] = 0$ if $\varphi(i) \notin \mathbb{N}$. We denote by $\varphi^{\text{left}}(i) = \lceil \varphi(i) \rceil - 1 \bmod (q + 1)$ the index of the vertex to the left of index $\varphi(i)$ and by $\varphi^{\text{right}}(i) = \lfloor \varphi(i) \rfloor + 1 \bmod (q + 1)$ the index of the vertex to the right of index $\varphi(i)$.

We now describe how to compute $BC[x_k]$, $0 \leq k \leq q$, which is the sum of $\gamma(x_i, t, x_k)$ over $i \in [0, q]$ and $t \in V(C)$, with a dynamic programming approach.

Our base case is $BC[x_0]$. Note that the betweenness centrality of a single vertex can in general not be computed in $O(n^{2-\varepsilon})$ time for any $\varepsilon > 0$ unless the SETH fails [7]. We show how to compute the betweenness centrality of x_0 in $O(n)$ time if the input graph is a cycle. Observe that $\gamma(x_i, t, x_0) = 0$ if $x_i = x_0$ or $t = x_0$. Also, for every shortest path starting in $x_{\varphi(0)}$ and ending in some x_j , $1 \leq j \leq q$, it holds that $d_C(x_{\varphi(0)}, x_j) < d_C(x_{\varphi(0)}, x_0)$. Thus there is no shortest path starting in $x_{\varphi(0)}$ that visits x_0 . So we may ignore the cases $i = 0$ and $i = \varphi(0)$ and

$$\begin{aligned} BC[x_0] &= \sum_{\substack{i \in [0, q] \setminus \{\varphi(0)\} \\ t \in V(C)}} \gamma(x_i, t, x_0) = \sum_{\substack{i \in [1, \varphi_0^{\text{left}}] \\ t \in V(C)}} \gamma(x_i, t, x_0) + \sum_{\substack{i \in [\varphi_0^{\text{right}}, q] \\ t \in V(C)}} \gamma(x_i, t, x_0) \\ &= \sum_{\substack{i \in [1, \varphi_0^{\text{left}}] \\ t \in V(C)}} \text{Pen}[x_i] \cdot \text{Pen}[t] \cdot \frac{\sigma_{x_i t}(x_0)}{\sigma_{x_i t}} + \sum_{\substack{i \in [\varphi_0^{\text{right}}, q] \\ t \in V(C)}} \text{Pen}[x_i] \cdot \text{Pen}[t] \cdot \frac{\sigma_{x_i t}(x_0)}{\sigma_{x_i t}}. \end{aligned}$$

By definition of $\varphi(i)$, we have that $d_C(x_i, x_{\varphi^{\text{left}}(i)}) = d_C(x_i, x_{\varphi^{\text{right}}(i)}) < \frac{q+1}{2}$. Hence, there is a unique shortest path from x_i to $x_{\varphi^{\text{left}}(i)}$ visiting $x_{i+1 \bmod (q+1)}$, and there is a unique shortest path from x_i to $x_{\varphi^{\text{right}}(i)}$ visiting $x_{i-1 \bmod (q+1)}$. This gives us that in the equation above, in the first sum, all shortest paths from x_i to $t \in [x_{\varphi^{\text{right}}(i)}, x_q]$ visit x_0 , and in the second sum, all shortest paths from x_i to $t \in [x_1, x_{\varphi^{\text{left}}(i)}]$ visit x_0 . If $\varphi(x_i) \in \mathbb{N}$, then there are two shortest paths from x_i to $x_{\varphi(i)}$, and one of them visits x_0 . With this at hand, we can rewrite the sum as follows:

$$\begin{aligned} BC[x_0] &= \sum_{i=1}^{\varphi^{\text{left}}(0)} \left(\text{Pen}[x_i] \cdot \text{Pen}[x_{\varphi(i)}] \cdot \frac{1}{2} + \sum_{t \in [x_{\varphi^{\text{right}}(i)}, x_q]} \text{Pen}[x_i] \cdot \text{Pen}[t] \right) \\ &\quad + \sum_{i=\varphi^{\text{right}}(0)}^q \left(\text{Pen}[x_i] \cdot \text{Pen}[x_{\varphi(i)}] \cdot \frac{1}{2} + \sum_{t \in [x_1, x_{\varphi^{\text{left}}(i)}]} \text{Pen}[x_i] \cdot \text{Pen}[t] \right) \\ &= \sum_{i=1}^{\varphi^{\text{left}}(0)} \text{Pen}[x_i] \left(\frac{1}{2} \text{Pen}[x_{\varphi(i)}] + W[x_{\varphi^{\text{right}}(i)}, x_q] \right) \\ &\quad + \sum_{i=\varphi^{\text{right}}(0)}^q \text{Pen}[x_i] \left(\frac{1}{2} \text{Pen}[x_{\varphi(i)}] + W[x_1, x_{\varphi^{\text{left}}(i)}] \right). \end{aligned}$$

We can precompute the values $W^{\text{left}}[\cdot]$ in $O(q)$ time. The values $W[\cdot, \cdot]$ and also the values $\varphi(i)$ and its variants can then be computed in constant time. Thus computing $BC[x_0]$ takes $O(q)$ time.

Assume now that we have computed $BC[x_k]$ for some $0 \leq k < q$. We claim that $BC[x_{k+1}]$ can then be computed as follows:

$$\begin{aligned}
 BC[x_{k+1}] = BC[x_k] - \text{Pen}[x_{k+1}] & \left(\text{Pen}[x_{\varphi(k+1)}] \right. \\
 & + 2W[x_{\varphi^{\text{right}}(k+1)}, x_{k-1 \bmod (q+1)}] \\
 & \left. + \text{Pen}[x_k] \left(\text{Pen}[x_{\varphi(k)}] + 2W[x_{k+2 \bmod (q+1)}, x_{\varphi^{\text{left}}(k)}] \right) \right). \tag{2}
 \end{aligned}$$

To this end, observe that all shortest paths in C that contain x_k as an inner vertex also contain x_{k+1} as an inner vertex, except for those paths that start or end in x_{k+1} . Likewise, all shortest paths in C that contain x_{k+1} as an inner vertex also contain x_k as an inner vertex, except for those paths that start or end in x_k . Hence, to compute $BC[x_{k+1}]$ from $BC[x_k]$, we need to subtract the γ -values for shortest paths starting in x_{k+1} and visiting x_k , and we need to add the γ -values for shortest paths starting in x_k and visiting x_{k+1} . Since by Observation 1 each path contributes the same value to the betweenness centrality as its reverse, it holds

$$BC[x_{k+1}] = BC[x_k] + 2 \cdot \sum_{t \in V(C)} \gamma(x_k, t, x_{k+1}) - \gamma(x_{k+1}, t, x_k). \tag{3}$$

With a similar argumentation as above for the computation of $BC[x_0]$, one can show that shortest paths starting in x_k and visiting x_{k+1} must end in a vertex $t \in [x_{k+2}, x_{\varphi^{\text{left}}(k)}]$ or in $x_{\varphi(k+1)}$. Shortest paths starting in x_{k+1} and visiting x_k must end in $t \in [x_{\varphi^{\text{right}}(k+1)}, x_{k-1}]$, or in $x_{\varphi(k)}$. Just as above, for both $i = k$ and $i = k+1$, some fixed vertex x_j is visited by only half of the shortest paths from x_i to $x_{\varphi(i)}$. With the arguments above, we can rewrite Equation (3) to obtain the claimed Equation (2).

After precomputing the values $W^{\text{left}}[\cdot]$ and $BC[x_0]$ in $O(q)$ time and space, we can compute each of the values $BC[x_{k+1}]$ for $0 \leq k < q$ in constant time. Hence, the procedure requires $O(q)$ time. \square

2.2 Dealing with Other Biconnected Graphs

Recall that, after our preprocessing on all cut vertices and degree-one vertices, we obtain a graph consisting of biconnected components, each of which can be solved independently. Also, in the previous subsection, we showed how to solve WEIGHTED BETWEENNESS CENTRALITY on cycles. It remains to show how to solve the problem on biconnected graphs that are not cycles (but contain at least one).

Remark. Henceforth, in this paper, we assume that we are given a vertex-weighted biconnected graph that is not a cycle.

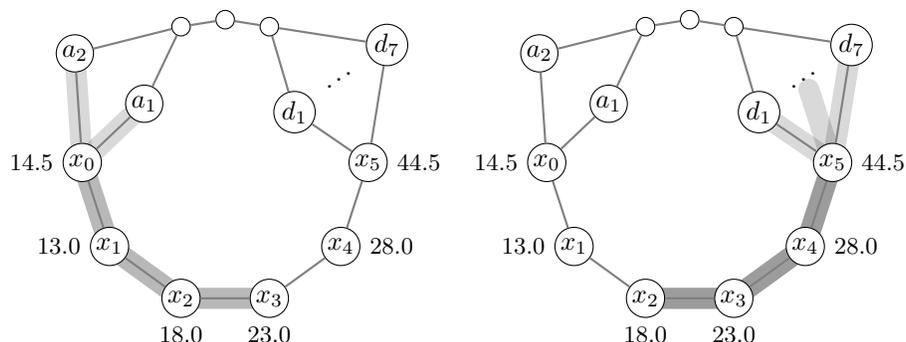


Figure 2: An example graph containing a maximal induced path $x_0 \dots x_5$ (see Definition 1). The labels give the betweenness centrality values of the vertices. Marked are shortest paths from a_i to x_3 (left-hand side) and from d_i to x_2 (right-hand side). The former affect the betweenness centrality value of x_2 , but not of x_3 ; the latter affect the betweenness centrality value of x_3 , but not of x_2 . Hence, most, but not all, of the paths traversing through x_2 also affect the betweenness centrality value of x_3 . Note that this difference cannot be decided locally within the maximal induced path, but can have an arbitrary effect on the difference arbitrarily far away in the graph. In this example graph, one could add more and more “ d -vertices” (the figure shows d_1 – d_7) to further increase the difference in the betweenness centrality values of x_2 and x_3 .

Outline of the algorithmic approach. Starting with a vertex-weighted biconnected graph, our algorithm focuses on degree-two vertices. In contrast to degree-one vertices, degree-two vertices can lie on shortest paths between two other vertices. Moreover, different degree-two vertices on the same shortest path can have different betweenness centrality values (see Figure 2 for an example). This makes degree-two vertices harder to handle: Removing a degree-two vertex v in a similar way as done with degree-one vertices (see Reduction Rule 1) potentially affects many other shortest paths that neither start nor end in v . Thus, we treat degree-two vertices differently: Instead of removing vertices one-by-one, we process multiple degree-two vertices at once and exploit that consecutive degree-two vertices share many shortest paths they lie on, storing information about the shortest paths in a table. To this end we introduce the notion of maximal induced paths.

Definition 1 Let G be a graph. A path $P = v_0 \dots v_\ell$ is a *maximal induced path* in G if $\ell \geq 2$ and the inner vertices $v_1, \dots, v_{\ell-1}$ all have degree two in G , but the endpoints v_0 and v_ℓ do not, that is, $\deg_G(v_1) = \dots = \deg_G(v_{\ell-1}) = 2$, $\deg_G(v_0) \neq 2$, and $\deg_G(v_\ell) \neq 2$. Moreover, \mathcal{P}^{\max} is the set of all maximal induced paths in G .

In a nutshell, our algorithm treats each biconnected component of the input graph in the following three stages (compare with Algorithm 1):

Algorithm 1: Algorithm for computing betweenness centrality of a biconnected graph that is not a cycle.

Input: An undirected biconnected graph G with vertex weights $\text{Pen}: V(G) \rightarrow \mathbb{N}$.

Output: The betweenness centrality values of all vertices.

```

1 foreach  $v \in V(G)$  do  $BC[v] \leftarrow 0$ 
   //  $BC$  will contain the betweenness centrality values
2  $\mathcal{P}^{\max} \leftarrow$  all maximal induced paths of  $G$ 
   // computable in  $O(n + m)$  time, see Lemma 3
3 foreach  $s \in V^{\geq 3}(G)$  do
   // some precomputations taking  $O(kn)$  time, see Lemma 5
4   compute  $d_G(s, t)$  and  $\sigma_{st}$  for each  $t \in V(G) \setminus \{s\}$ 
5    $\text{Inc}[s, t] \leftarrow 2 \cdot \text{Pen}[s] \cdot \text{Pen}[t] / \sigma_{st}$  for each  $t \in V^{\geq 2}(G)$ 
6    $\text{Inc}[s, t] \leftarrow \text{Pen}[s] \cdot \text{Pen}[t] / \sigma_{st}$  for each  $t \in V^{\geq 3}(G) \setminus \{s\}$ 
7 foreach  $x_0 x_1 \dots x_q = P^{\max} \in \mathcal{P}^{\max}$  do
   // initialize  $W^{\text{left}}$  and  $W^{\text{right}}$  in  $O(n)$  time
8    $W^{\text{left}}[x_0] \leftarrow \text{Pen}[x_0]$ ;  $W^{\text{right}}[x_q] \leftarrow \text{Pen}[x_q]$ 
9   for  $i = 1$  to  $q$  do  $W^{\text{left}}[x_i] \leftarrow W^{\text{left}}[x_{i-1}] + \text{Pen}[x_i]$ 
10  for  $i = q - 1$  to  $0$  do  $W^{\text{right}}[x_i] \leftarrow W^{\text{right}}[x_{i+1}] + \text{Pen}[x_i]$ 
11  foreach  $x_0 x_1 \dots x_q = P_1^{\max} \in \mathcal{P}^{\max}$  do
   // case  $s \in V^{\geq 2}(P_1^{\max})$ , see Section 3
   /* deal with the case  $t \in V^{\geq 2}(P_2^{\max})$ , see Section 3.1 */
12  foreach  $y_0 y_1 \dots y_r = P_2^{\max} \in \mathcal{P}^{\max} \setminus \{P_1^{\max}\}$  do
   /* update  $BC$  for the case  $v \in V(P_1^{\max}) \cup V(P_2^{\max})$  */
13  foreach  $v \in V(P_1^{\max}) \cup V(P_2^{\max})$  do
    $BC[v] \leftarrow BC[v] + \gamma(s, t, v)$ 
   /* now deal with the case  $v \notin V(P_1^{\max}) \cup V(P_2^{\max})$  */
14  update  $\text{Inc}[x_0, y_0]$ ,  $\text{Inc}[x_q, y_0]$ ,  $\text{Inc}[x_0, y_r]$ , and  $\text{Inc}[x_q, y_r]$ 
   /* deal with the case that  $t \in V^{\geq 2}(P_1^{\max})$ , see Section 3.2 */
15  foreach  $v \in V(P_1^{\max})$  do  $BC[v] \leftarrow BC[v] + \gamma(s, t, v)$ 
16  update  $\text{Inc}[x_0, x_q]$  // this deals with the case  $v \notin V(P_1^{\max})$ 
17  foreach  $s \in V^{\geq 3}(G)$  do
   // perform modified BFS from  $s$ , see Section 3.3
18  foreach  $t, v \in V(G)$  do  $BC[v] \leftarrow BC[v] + \text{Inc}[s, t] \cdot \sigma_{st}(v)$ 
19 return  $BC$ .

```

1. For all pairs s, t of vertices where s is of degree at least three, precompute $d_G(s, t)$ and σ_{st} , and initialize a table $\text{Inc}[s, t]$ (see Lines 3 to 6).
2. Compute betweenness centrality values for paths starting and ending in maximal induced paths and store them in $\text{Inc}[\cdot, \cdot]$, considering two cases (see Lines 11 to 16):
 - both endpoints of the path are in the same maximal induced path;
 - the endpoints are in two different maximal induced paths.
3. In a postprocessing step, compute the betweenness centrality for all remaining paths (at least one endpoint is of degree at least three) and incorporate the values stored in $\text{Inc}[\cdot, \cdot]$ (see Lines 17 to 18).

Note that in a biconnected graph that is not a cycle, every degree-two vertex is an inner vertex of a maximal induced path. If some degree-two vertex v was not contained in a maximal induced path, then v would be contained in a cycle that contains exactly one vertex u that is of degree at least three. But then u is a cut vertex and the graph would not be biconnected; a contradiction. The remaining part of the algorithm deals with maximal induced paths. Note that if the (biconnected) graph is not a cycle, then all degree-two vertices are contained in maximal induced paths:

Using standard arguments, we can show that the number of maximal induced paths is upper-bounded by the minimum of the feedback edge number k of the input graph and the number n of vertices. Moreover, one can easily compute all maximal induced paths in linear-time (see Line 2 of Algorithm 1).

Lemma 2 *Let G be a graph with feedback edge number k that does not contain degree-one vertices. Then G contains at most $\min\{n, 2k\}$ vertices of degree at least three and at most $\min\{n, 3k\}$ maximal induced paths.*

Proof: Recall that our graph is biconnected. Thus $\sum_{v \in V(G)} \deg(v) = 2m = 2(n - 1 + k)$, and

$$\begin{aligned}
 2(n - 1 + k) &= 2(|V^{=2}(G)| + |V^{\geq 3}(G)| - 1 + k) \\
 &= \sum_{v \in V(G)} \deg(v) = \sum_{v \in V^{=2}(G)} \deg(v) + \sum_{v \in V^{\geq 3}(G)} \deg(v) \\
 &\geq 2 \cdot |V^{=2}(G)| + 3 \cdot |V^{\geq 3}(G)|.
 \end{aligned}$$

Solving for $|V^{\geq 3}(G)|$ gives us that there are at most $2k - 2$ vertices of degree at least three. Then $\sum_{v \in V^{\geq 3}(G)} \deg(v) = 3|V^{\geq 3}(G)| \leq 6k - 6$. It follows that there are at most $3k$ paths whose endpoints are in $V^{\geq 3}(G)$, hence $|\mathcal{P}^{\max}| \leq 3k - 3$. Clearly, for both the number of vertices of degree at least three and number of maximal induced paths, n is also a valid upper bound. \square

Lemma 3 *The set \mathcal{P}^{\max} of all maximal induced paths of a graph with n vertices and m edges can be computed in $O(n)$ time.*

Proof: Iterate through all vertices $v \in V(G)$. If $v \in V^{=2}(G)$, then iteratively traverse the two edges incident to v to discover adjacent degree-two vertices until finding endpoints $v_\ell, v_r \in V^{\geq 3}(G)$. If $v_\ell = v_r$, then we found a cycle which can be ignored. Otherwise, we have a maximal induced path $P^{\max} = v_\ell \dots v_r$, which we add to \mathcal{P}^{\max} .

Note that every degree-two vertex is contained either in exactly one maximal induced path or in exactly one cycle. Hence, we do not need to reconsider any degree-two vertex found in the traversal above and we can find all maximal induced paths in $O(n)$ time. \square

Our algorithm processes the maximal induced paths one by one (see Lines 3 to 18). This part of the algorithm requires pre- and postprocessing (see Lines 3 to 10 and Lines 17 to 18 respectively). In the preprocessing, we initialize tables that are frequently used in the main part (of Section 3). The postprocessing computes the final betweenness centrality values of each vertex as this computation is too time-consuming to be executed for each maximal induced path. When explaining our basic ideas, we will first present the postprocessing as this explains why certain values will be computed during the algorithm.

Recall that we want to compute $\sum_{s,t \in V(G)} \gamma(s, t, v)$ for each $v \in V(G)$ (see Equation (1)). Using the following observations, we split Equation (1) into different parts.

Observation 1 For $s, t, v \in V(G)$ it holds that $\gamma(s, t, v) = \gamma(t, s, v)$.

Observation 2 Let G be a biconnected graph with at least one vertex of degree at least three. Let $v \in V(G)$. Then,

$$\begin{aligned} \sum_{s,t \in V(G)} \gamma(s, t, v) &= \sum_{s \in V^{\geq 3}(G), t \in V(G)} \gamma(s, t, v) + \sum_{s \in V^{=2}(G), t \in V^{\geq 3}(G)} \gamma(t, s, v) \\ &+ \sum_{\substack{s \in V^{=2}(P_1^{\max}), t \in V^{=2}(P_2^{\max}) \\ P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}}} \gamma(s, t, v) + \sum_{\substack{s,t \in V^{=2}(P^{\max}) \\ P^{\max} \in \mathcal{P}^{\max}}} \gamma(s, t, v). \end{aligned}$$

Proof: The first two sums cover all pairs of vertices in which at least one of the two vertices is of degree at least three. The other two sums cover all pairs of vertices which both have degree two. As all vertices of degree two must be part of some maximal induced path, we have $V^{=2}(G) = V^{=2}(\bigcup \mathcal{P}^{\max})$. Two vertices of degree two can thus either be in two different maximal induced paths (third sum) or in the same maximal induced path (fourth sum). \square

In the remaining graph, by Lemma 2, there are at most $O(\min\{k, n\})$ vertices of degree at least three and at most $O(k)$ maximal induced paths. This implies that we can afford to run the modified BFS (similar to Brandes’ algorithm) from each vertex $s \in V^{\geq 3}(G)$ in $O(\min\{k, n\} \cdot (n + k)) = O(kn)$ time. This computes the first summand and, by Observation 1, also the second summand in Observation 2. However, we cannot afford to run such a BFS from every vertex. Thus, we need to compute the third and fourth summands differently.

To this end, note that $\sigma_{st}(v)$ is the only term in $\gamma(s, t, v)$ that depends on v . Our goal is to precompute $\gamma(s, t, v)/\sigma_{st}(v) = \text{Pen}[s] \cdot \text{Pen}[t]/\sigma_{st}$ for as many vertices as possible. Hence, we store precomputed values in a table $\text{Inc}[\cdot, \cdot]$ (see Lines 6, 14 and 16). Then, we plug this factor into the next lemma which provides our postprocessing.

Lemma 4 *Let s be a vertex and let $f: V(G)^2 \rightarrow \mathbb{N}$ be a function such that for each $u, v \in V(G)$ the value $f(u, v)$ can be computed in $O(\tau)$ time. Then, for all $v \in V(G)$ one can compute the value $\sum_{t \in V(G)} f(s, t) \cdot \sigma_{st}(v)$ in $O(n \cdot \tau + m)$ time.*

Proof: This proof generally follows the structure of the proof by Brandes [5, Theorem 6, Corollary 7], the main difference being the generalization of the distance function to an arbitrary function f .

Analogously to Brandes we define $\sigma_{st}(v, w)$ as the number of shortest paths from s to t that contain the edge $\{v, w\}$, and $S_s(v)$ as the set of successors of a vertex v on shortest paths from s , that is, $S_s(v) = \{w \in V(G) \mid \{v, w\} \in E \wedge d_G(s, w) = d_G(s, v) + 1\}$. For the sake of readability we also define $\chi_{sv} = \sum_{t \in V(G)} f(s, t) \cdot \sigma_{st}(v)$. We will first derive a series of equations that show how to compute χ_{sv} . Afterwards we justify Equations (4) and (5).

$$\begin{aligned}
\chi_{sv} &= \sum_{t \in V(G)} f(s, t) \cdot \sigma_{st}(v) \\
&= \sum_{t \in V(G)} f(s, t) \sum_{w \in S_s(v)} \sigma_{st}(v, w) = \sum_{w \in S_s(v)} \sum_{t \in V(G)} f(s, t) \cdot \sigma_{st}(v, w) \quad (4) \\
&= \sum_{w \in S_s(v)} \left(\left(\sum_{t \in V(G) \setminus \{w\}} f(s, t) \cdot \sigma_{st}(v, w) \right) + f(s, w) \cdot \sigma_{sw}(v, w) \right) \\
&= \sum_{w \in S_s(v)} \left(\left(\sum_{t \in V(G) \setminus \{w\}} f(s, t) \cdot \sigma_{st}(w) \cdot \frac{\sigma_{sv}}{\sigma_{sw}} \right) + f(s, w) \cdot \sigma_{sw} \right) \quad (5) \\
&= \sum_{w \in S_s(v)} \left(\chi_{sw} \cdot \frac{\sigma_{sv}}{\sigma_{sw}} + f(s, w) \cdot \sigma_{sw} \right)
\end{aligned}$$

We will now show that Equations (4) and (5) are correct. All other equalities are based on simple arithmetics. To see that Equation (4) is correct, observe that each shortest path from s to any other vertex t that contains v either ends in v , that is, $t = v$, or contains exactly one edge $\{v, w\}$, where $w \in S_s(v)$. If $t = v$, then $\sigma_{st}(v) = 0$ and therefore $\sum_{t \in V} \sigma_{st}(v) = \sum_{t \in V} \sum_{w \in S_s(v)} \sigma_{st}(v, w)$. To see that Equation (5) is correct, observe the following. First, note that the number of shortest paths from s to t that contain a vertex v is

$$\sigma_{st}(v) = \begin{cases} 0, & \text{if } d_G(s, v) + d_G(v, t) > d_G(s, t); \\ \sigma_{sv} \cdot \sigma_{vt}, & \text{otherwise;} \end{cases}$$

second, note that the number of shortest st -paths that contain an edge $\{v, w\}$, $w \in S_s(v)$, is

$$\sigma_{st}(v, w) = \begin{cases} 0, & \text{if } d_G(s, v) + d_G(w, t) + 1 > d_G(s, t); \\ \sigma_{sv} \cdot \sigma_{wt}, & \text{otherwise;} \end{cases}$$

and third, note that the number of shortest sw -paths that contain v is equal to the number of shortest sv -paths. The combination of these three observations yields $\sigma_{st}(v, w) = \sigma_{sv} \cdot \sigma_{wt} = \sigma_{sv} \cdot \sigma_{st}(w) / \sigma_{sw}$.

We next show how to compute χ_{sv} for all $v \in V$ in $O(m + n \cdot \tau)$ time. First, order the vertices in non-increasing distance to s and compute the set of all successors of each vertex in $O(m)$ time using breadth-first search. Note that the number of successors of all vertices is at most m since each edge defines at most one successor-predecessor relation. Then compute χ_{sv} for each vertex by a dynamic program that iterates over the ordered list of vertices and computes

$$\sum_{w \in S_s(v)} \left(\chi_{sw} \cdot \frac{\sigma_{sv}}{\sigma_{sw}} + f(s, w) \cdot \sigma_{sv} \right)$$

in overall $O(m + n \cdot \tau)$ time. This can be done by first computing σ_{st} for all $t \in V$ in overall $O(m)$ time due to Brandes [5, Corollary 4] and $f(s, t)$ for all $t \in V(G)$ in $O(n \cdot \tau)$ time, and then using the already computed values $S_s(v)$ and χ_{sw} to compute

$$\chi_{sv} = \sum_{w \in S_s(v)} \left(\chi_{sw} \cdot \frac{\sigma_{sv}}{\sigma_{sw}} + f(s, w) \cdot \sigma_{sv} \right)$$

in $O(|S_s(v)|)$ time. Note that $\sum_{v \in V} |S_s(v)| \leq O(m)$. This concludes the proof. \square

The proof of Lemma 4 provides us with an algorithm. Our goal is then to only start this algorithm from few vertices, specifically the vertices of degree at least three (see Line 18 of Algorithm 1). Since the term τ in the above lemma will be constant, we obtain a running time of $O(kn)$ for running this postprocessing on all vertices of degree at least three. The most intricate part will be to precompute the factors in $\text{Inc}[\cdot, \cdot]$ (see Lines 14 and 16 of Algorithm 1). We defer the details to Sections 3.1 and 3.2. In these parts, we need the tables W^{left} and W^{right} . These tables store values depending on the maximal induced path a vertex is in. More precisely, for a vertex x_i in a maximal induced path $P^{\text{max}} = x_0 \dots x_q$, we store in $W^{\text{left}}[x_k]$ the sum of the $\text{Pen}[\cdot]$ -values of vertices “left of” x_k in P^{max} ; formally, $W^{\text{left}}[x_k] = \sum_{i=1}^k \text{Pen}[x_i]$. Similarly, we have $W^{\text{right}}[x_k] = \sum_{i=k}^{q-1} \text{Pen}[x_i]$. The reason for having these tables is easy to see: Assume for the vertex $x_k \in P^{\text{max}}$ that the shortest paths to $t \notin V(P^{\text{max}})$ leave P^{max} through x_0 . Then, it is equivalent to just consider the shortest path(s) starting in x_0 and simulate the vertices between x_k and x_0 in P^{max} by “temporarily increasing” $\text{Pen}[x_0]$ by $W^{\text{left}}[x_k]$. This is also the idea behind the argument that we only need to increase the values $\text{Inc}[\cdot, \cdot]$ for the endpoints of the maximal induced paths in Line 14 of Algorithm 1.

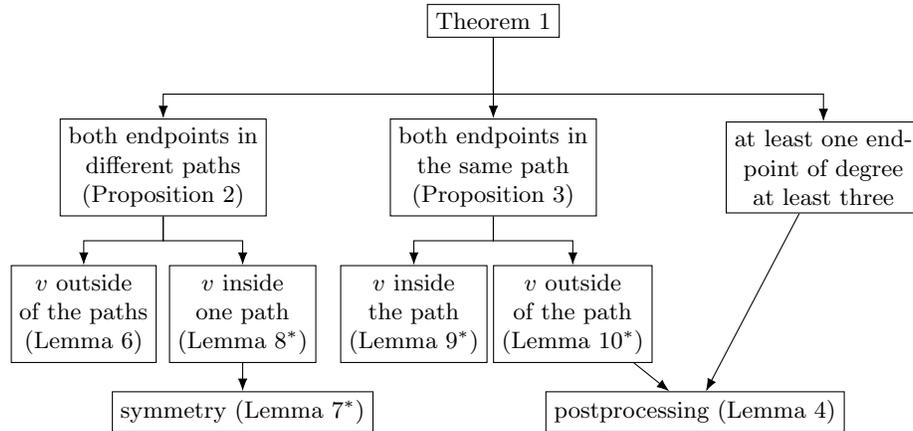


Figure 3: Structure of how the proof of Theorem 1 is split into different cases. By “paths” we mean maximal induced paths. The first layer below the main theorem specifies the positions of the endpoints s and t , whereas the second layer specifies the position of the vertex v , for which the betweenness centrality is computed. The third layer displays further lemmata used to prove the corresponding lemma above. Proofs of lemmata marked with an asterisk are deferred to the appendix.

This leaves us with the remaining part of the preprocessing: the computation of the distances $d_G(s, t)$, the number of shortest paths σ_{st} , and $\text{Inc}[s, t]$ for $s \in V^{\geq 3}(G), t \in V(G)$ (see Lines 3 to 6). This can be done in $O(kn)$ time as well.

Lemma 5 *The initialization in the for-loop in Lines 3 to 6 of Algorithm 1 can be done in $O(kn)$ time.*

Proof: Following Brandes [5, Corollary 4], computing the distances and the number of shortest paths from a fixed vertex s to every $t \in V(G)$ takes $O(m) = O(n + k)$ time. Once these values are computed for a fixed s , computing $\text{Inc}[s, t]$ for $t \in V(G)$ takes $O(n)$ time since the values $\text{Pen}[s]$, $\text{Pen}[t]$, and σ_{st} are known. Since, by Lemma 2, there are $O(\min\{k, n\})$ vertices of degree at least three, it takes $O(\min\{k, n\} \cdot (n + k + n)) = O(kn)$ time to compute Lines 3 to 6. \square

3 Dealing with maximal induced paths

In this section, we focus on degree-two vertices contained in maximal induced paths. Recall that the goal is to compute the betweenness centrality $C_B(v)$ (see Equation (1)) for all $v \in V(G)$ in $O(kn)$ time. In the end of this section, we finally prove our main theorem (Theorem 1).

Figure 3 shows the general proof structure of the main theorem. Based on Observation 2, which we use to split the sum in Equation (1) in the definition of WEIGHTED BETWEENNESS CENTRALITY, we compute $C_B(v)$ in three steps. By

starting a modified BFS from vertices in $V^{\geq 3}(G)$ similarly to Baglioni et al. [3] and Brandes [5], we can compute

$$\sum_{s \in V^{\geq 3}(G), t \in V(G)} \gamma(t, s, v) + \sum_{s \in V^=2(G), t \in V^{\geq 3}(G)} \gamma(s, t, v)$$

for all $v \in V(G)$ in overall $O(kn)$ time. In the next two subsections, we show how to compute the remaining two summands given in Observation 2 (i.e., we prove Propositions 2 and 3). In the last subsection, we prove Theorem 1.

3.1 Paths with endpoints in different maximal induced paths

In this subsection, we look at shortest paths between pairs of maximal induced paths $P_1^{\max} = x_0 \dots x_q$ and $P_2^{\max} = y_0 \dots y_r$, and how to efficiently determine how these paths affect the betweenness centrality of each vertex.

Proposition 2 *In $O(kn)$ time one can compute the following values for all $v \in V(G)$:*

$$\sum_{\substack{s \in V^=2(P_1^{\max}), t \in V^=2(P_2^{\max}) \\ P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}}} \gamma(s, t, v).$$

In the proof of Proposition 2, we consider two cases for every pair $P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}$ of maximal induced paths: First, we look at how the shortest paths between vertices in P_1^{\max} and P_2^{\max} affect the betweenness centrality of those vertices that are not contained in the two maximal induced paths, and second, how they affect the betweenness centrality of those vertices that are contained in the two maximal induced paths. Finally, we prove Proposition 2.

Throughout the following proofs, we will need the following definitions (see Figure 4 for an illustration). Let $t \in P_2^{\max}$. Then we choose vertices $x_t^{\text{left}}, x_t^{\text{right}} \in V^=2(P_1^{\max})$ such that shortest paths from t to $s \in \{x_1, x_2, \dots, x_t^{\text{left}}\} =: X_t^{\text{left}}$ enter P_1^{\max} only via x_0 , and shortest paths from t to $s \in \{x_t^{\text{right}}, \dots, x_{q-2}, x_{q-1}\} =: X_t^{\text{right}}$ enter P_1^{\max} only via x_q . There may exist a vertex x_t^{mid} to which there are shortest paths both via x_0 and via x_q . For computing the indices of these vertices, we determine an index i such that $d_G(x_0, t) + i = d_G(x_q, t) + q - i$ which is equivalent to $i = \frac{1}{2}(q - d_G(x_0, t) + d_G(x_q, t))$. If i is integral, then $x_t^{\text{mid}} = x_i$, $x_t^{\text{left}} = x_{i-1}$ and $x_t^{\text{right}} = x_{i+1}$. Otherwise, x_t^{mid} does not exist, and $x_t^{\text{left}} = x_{i-1/2}$ and $x_t^{\text{right}} = x_{i+1/2}$. For easier argumentation, if x_t^{mid} does not exist, then we say that $\text{Pen}[x_t^{\text{mid}}] = \sigma_{tx_t^{\text{mid}}}(v) / \sigma_{tx_t^{\text{mid}}} = 0$, and hence, $\gamma(x_t^{\text{mid}}, t, v) = 0$.

3.1.1 Vertices outside of the maximal induced paths

We now show how shortest paths between two fixed maximal induced paths P_1^{\max} and P_2^{\max} affect the betweenness centrality of vertices v that are not contained in P_1^{\max} or in P_2^{\max} , that is $v \in V(G) \setminus (V(P_1^{\max}) \cup V(P_2^{\max}))$. Recall that in the

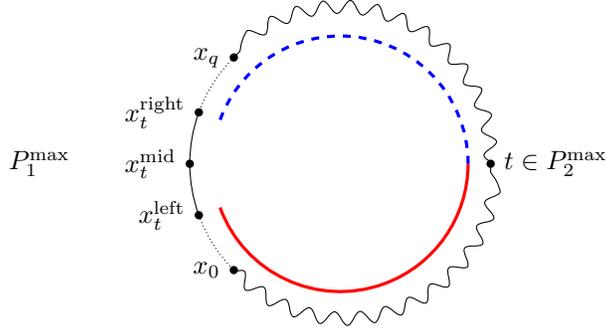


Figure 4: An exemplary graph containing two maximal induced paths $P_1^{\max} = x_0 \dots x_q$ and P_2^{\max} . The curled lines depict shortest paths from t to x_0 and to x_q respectively. We then choose $x_t^{\text{left}}, x_t^{\text{mid}}, x_t^{\text{right}} \in V(P_1^{\max})$ in such a way that the distance from t to x_t^{left} and to x_t^{right} is equal, that is, the red (solid) line and the blue (dashed) line represent shortest paths of same length. Since x_t^{mid} is adjacent to x_t^{left} and x_t^{right} , there are shortest paths from x_t^{mid} to t via both x_0 and x_q , that is, along the blue and the red line.

course of the algorithm we first gather values in $\text{Inc}[s, t]$ for every $s \in V^{\geq 3}(G)$ and $t \in V(G)$. Hereby, $\text{Inc}[s, t] = \gamma(s, t, v) / \sigma_{st}(v)$ measures how much the shortest paths from s to t affect the betweenness centrality of all vertices in the graph. Then, in the final step we compute $\text{Inc}[s, t] \cdot \sigma_{st}(v)$ in $O(kn)$ time.

Lemma 6 *Let $P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}$ and assume that the values $d_G(s, t)$, $W^{\text{left}}[v]$ and $W^{\text{right}}[v]$ are known for $s, t \in V^{\geq 3}(G)$ and $v \in V^=2(G)$, respectively. Then, excluding the running time of the postprocessing (see Lines 17 to 18 in Algorithm 1), one can compute in $O(|V^=2(P_2^{\max})|)$ time the following.*

$$\sum_{s \in V^=2(P_1^{\max}), t \in V^=2(P_2^{\max})} \gamma(s, t, v). \quad (6)$$

Proof: We fix $P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}$ with $P_1^{\max} = x_0 \dots x_q$ and $P_2^{\max} = y_0 \dots y_r$. We show how to compute $\sum_{s \in V^=2(P_1^{\max})} \gamma(s, t, v)$ for a fixed $t \in V^=2(P_2^{\max})$ and $v \in V(G) \setminus (V(P_1^{\max}) \cup V(P_2^{\max}))$. Afterwards, we analyze the running time.

By definition of $x_t^{\text{left}}, x_t^{\text{mid}}$ and x_t^{right} we have

$$\sum_{s \in V^=2(P_1^{\max})} \gamma(s, t, v) = \gamma(x_t^{\text{mid}}, t, v) + \sum_{s \in X_t^{\text{left}}} \gamma(s, t, v) + \sum_{s \in X_t^{\text{right}}} \gamma(s, t, v). \quad (7)$$

By definition of maximal induced paths, every shortest path from $s \in V^=2(P_1^{\max})$ to t visits either y_0 or y_r . For $\psi \in \{x_0, x_q\}$ let $S(t, \psi)$ be a maximal subset of $\{y_0, y_r\}$ such that for each $\varphi \in S(t, \psi)$ there is a shortest st -path via ψ and φ . An example for this notation is given in Figure 5. Then, for $s \in X_t^{\text{left}}$, all st -paths visit x_0 and $\varphi \in S(t, x_0)$. Hence, we have that $\sigma_{st} =$

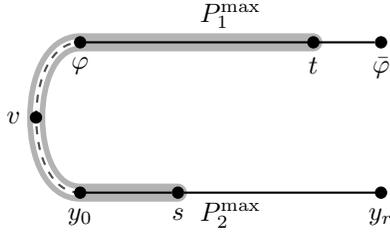


Figure 5: An example for the proof of Lemma 6. The endpoints of P_1^{\max} are φ and $\bar{\varphi}$. In this example we have $s \in X_t^{\text{left}}$, and the set $S(t, x_0) = \{\varphi\}$. Hence, every shortest path from s to t visits y_0 and φ .

$\sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi}$ and $\sigma_{st}(v) = \sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi}(v)$. Analogously, for $s \in X_t^{\text{right}}$ we have that $\sigma_{st} = \sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}$ and $\sigma_{st}(v) = \sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}(v)$. Paths from t to x_t^{mid} may visit x_0 and $\varphi \in S(t, x_0)$ or x_q and $\varphi \in S(t, x_q)$. Hence, $\sigma_{tx_t^{\text{mid}}} = \sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi} + \sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}$. The equality holds analogously for $\sigma_{tx_t^{\text{mid}}}(v)$. With this at hand, we can simplify the computation of the first sum of Equation (7):

$$\begin{aligned} \sum_{s \in X_t^{\text{left}}} \gamma(s, t, v) &= \sum_{s \in X_t^{\text{left}}} \text{Pen}[s] \cdot \text{Pen}[t] \cdot \frac{\sigma_{st}(v)}{\sigma_{st}} \\ &= \left(\sum_{s \in X_t^{\text{left}}} \text{Pen}[s] \right) \cdot \text{Pen}[t] \cdot \frac{\sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi}(v)}{\sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi}} \\ &= W^{\text{left}}[x_t^{\text{left}}] \cdot \text{Pen}[t] \cdot \frac{\sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi}(v)}{\sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi}}. \end{aligned} \tag{8}$$

Analogously,

$$\sum_{s \in X_t^{\text{right}}} \gamma(s, t, v) = W^{\text{right}}[x_t^{\text{right}}] \cdot \text{Pen}[t] \cdot \frac{\sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}(v)}{\sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}}, \tag{9}$$

and

$$\gamma(x_t^{\text{mid}}, t, v) = \text{Pen}[x_t^{\text{mid}}] \cdot \text{Pen}[t] \cdot \frac{\sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi}(v) + \sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}(v)}{\sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi} + \sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}}. \tag{10}$$

With this we can rewrite Equation (7) to

$$\begin{aligned}
& \sum_{s \in V=2(P_1^{\max})} \gamma(s, t, v) \\
(8),(9),(10) \quad & \frac{W^{\text{left}}[x_t^{\text{left}}] \cdot \text{Pen}[t]}{\sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi}} \cdot \sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi}(v) \\
& + \frac{W^{\text{right}}[x_t^{\text{right}}] \cdot \text{Pen}[t]}{\sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}} \cdot \sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}(v) \\
& + \text{Pen}[x_t^{\text{mid}}] \cdot \text{Pen}[t] \cdot \frac{\sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi}(v) + \sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}(v)}{\sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi} + \sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}}.
\end{aligned}$$

By joining values $\sigma_{x_0 \varphi}(v)$ and $\sigma_{x_q \varphi}(v)$ we obtain

$$\begin{aligned}
& \sum_{s \in V=2(P_1^{\max})} \gamma(s, t, v) \\
= & \left(\frac{W^{\text{left}}[x_t^{\text{left}}] \cdot \text{Pen}[t]}{\sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi}} + \frac{\text{Pen}[x_t^{\text{mid}}] \cdot \text{Pen}[t]}{\sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi} + \sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}} \right) \cdot \sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi}(v) \quad (11) \\
& + \left(\frac{W^{\text{right}}[x_t^{\text{right}}] \cdot \text{Pen}[t]}{\sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}} + \frac{\text{Pen}[x_t^{\text{mid}}] \cdot \text{Pen}[t]}{\sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi} + \sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}} \right) \cdot \sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}(v) \quad (12) \\
=: & X_1 \cdot \sum_{\varphi \in S(t, x_0)} \sigma_{x_0 \varphi}(v) + X_2 \cdot \sum_{\varphi \in S(t, x_q)} \sigma_{x_q \varphi}(v).
\end{aligned}$$

Note that we define X_1 and X_2 to be the terms in the parentheses before the two sums.

We need to increase the betweenness centrality of all vertices on shortest paths from s to t via x_0 by the value of Term (11), and those shortest paths via x_q by the value of Term (12). By Lemma 4, increasing $\text{Inc}[s, t]$ by some value A ensures the increment of the betweenness centrality of v by $A \cdot \sigma_{st}(v)$ for all vertices v that are on a shortest path between s and t . Hence, increasing $\text{Inc}[x_0, \varphi]$ for every $\varphi \in S(t, x_0)$ by X_1 is equivalent to increasing the betweenness centrality of v by the value of Term (11). Analogously, increasing $\text{Inc}[x_q, \varphi]$ for every $\varphi \in S(t, x_q)$ by X_2 is equivalent to increasing the betweenness centrality of v by the value of Term (12).

We now have incremented $\text{Inc}[\psi, \varphi]$ for $\psi \in \{x_0, x_q\}$ and $\varphi \in \{y_0, y_r\}$ by certain values, and we have shown that this increment is correct if the shortest $\psi\varphi$ -paths do not visit inner vertices of P_1^{\max} or P_2^{\max} . We still need to show that (1) increasing $\text{Inc}[\psi, \varphi]$ does not affect the betweenness centrality of ψ or φ , and that (2) we increase $\text{Inc}[\psi, \varphi]$ only if no shortest $\psi\varphi$ -path visits inner vertices of P_1^{\max} or P_2^{\max} .

For (1), recall that for each $s, t \in V^{\geq 3}(G)$ the betweenness centrality of $v \in V(G)$ is increased by $\text{Inc}[s, t] \cdot \sigma_{st}(v)$. But since $\sigma_{\psi\varphi}(\psi) = \sigma_{\psi\varphi}(\varphi) = 0$, increments of $\text{Inc}[\psi, \varphi]$ do not affect the betweenness centrality of ψ or φ .

For (2), suppose that there is a shortest $\psi\varphi$ -path that visits inner vertices of P_2^{\max} . Let $\bar{\varphi} \neq \varphi$ be the second endpoint of P_2^{\max} . Then $d_G(\psi, \varphi) = d_G(\psi, \bar{\varphi}) + d_G(\bar{\varphi}, \varphi)$, and for all inner vertices y_i of P_2^{\max} , that is, for all y_i with $1 \leq i < r$, it holds that

$$d_G(\psi, \varphi) + d_G(\varphi, y_i) = d_G(\psi, \bar{\varphi}) + d_G(\bar{\varphi}, \varphi) + d_G(\varphi, y_i) > d_G(\psi, \bar{\varphi}) + d_G(\bar{\varphi}, y_i).$$

Hence, there are no shortest $y_i\psi$ -paths that visit φ , and consequently $\text{Inc}[\psi, \varphi]$ will not be incremented. The same argument holds if there is a shortest $\psi\varphi$ -path that visits inner vertices of P_1^{\max} .

Finally, we analyze the running time. The values $W^{\text{left}}[\cdot]$, $W^{\text{right}}[\cdot]$ and $\text{Pen}[\cdot]$ as well as the distances and number of shortest paths between all pairs of vertices of degree at least three are assumed to be known. With this, $S(t, x_0)$ and $S(t, x_q)$ can be computed in constant time. Hence, the values X_1 and X_2 can be computed in constant time for a fixed $t \in V^{=2}(P_2^{\max})$. Thus, the running time to compute the increments of $\text{Inc}[\cdot, \cdot]$ is upper-bounded by $O(|V(P_2^{\max})|)$. \square

3.1.2 Vertices inside the maximal induced paths

We now consider how shortest paths between pairs of two maximal induced paths $P_1^{\max} \neq P_2^{\max}$ affect the betweenness centrality of their vertices.

When iterating through all pairs $P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}$, one will encounter the pair (P_1^{\max}, P_2^{\max}) and its reverse (P_2^{\max}, P_1^{\max}) . Since our graph is undirected, instead of looking at the betweenness centrality of the vertices in both maximal induced paths, it suffices to consider only the vertices inside the second maximal induced path of the pair. This is shown in the following lemma.

Lemma 7 *Computing for every $P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}$ and for each vertex $v \in V(P_1^{\max}) \cup V(P_2^{\max})$*

$$\sum_{s \in V^{=2}(P_1^{\max}), t \in V^{=2}(P_2^{\max})} \gamma(s, t, v) \tag{13}$$

is equivalent to computing for every $P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}$ and for each $v \in V(P_2^{\max})$

$$X_v = \begin{cases} \sum_{s \in V^{=2}(P_1^{\max}), t \in V^{=2}(P_2^{\max})} \gamma(s, t, v), & \text{if } v \in V(P_1^{\max}) \cap V(P_2^{\max}); \\ 2 \cdot \sum_{s \in V^{=2}(P_1^{\max}), t \in V^{=2}(P_2^{\max})} \gamma(s, t, v), & \text{otherwise.} \end{cases} \tag{14}$$

Proof: We will first assume that $V(P_1^{\max}) \cap V(P_2^{\max}) = \emptyset$ for every $P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}$, and will discuss the special case $V(P_1^{\max}) \cap V(P_2^{\max}) \neq \emptyset$ afterwards.

For every fixed $\{P_1^{\max}, P_2^{\max}\} \in (\mathcal{P}^{\max})_2$ and for every $v \in V(P_2^{\max})$, the betweenness centrality of v is increased by

$$\sum_{s \in V=2(P_1^{\max}), t \in V=2(P_2^{\max})} \gamma(s, t, v) + \sum_{s \in V=2(P_2^{\max}), t \in V=2(P_1^{\max})} \gamma(s, t, v),$$

and by Observation 1 this is equal to

$$2 \cdot \sum_{s \in V=2(P_1^{\max}), t \in V=2(P_2^{\max})} \gamma(s, t, v) \quad (15)$$

Analogously, for every $w \in V(P_1^{\max})$, the betweenness centrality of v is increased by

$$2 \cdot \sum_{s \in V=2(P_1^{\max}), t \in V=2(P_2^{\max})} \gamma(s, t, w).$$

Thus, computing Sum (15) for $v \in V(P_2^{\max})$ for every pair $P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}$ is equivalent to computing Sum (13) for $v \in V(P_1^{\max}) \cup V(P_2^{\max})$ for every pair $P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}$, since when iterating over pairs of maximal induced paths we will encounter both the pairs (P_1^{\max}, P_2^{\max}) and (P_2^{\max}, P_1^{\max}) .

Consider now the special case that there exists a vertex $v \in V(P_1^{\max}) \cap V(P_2^{\max})$. Note that this vertex can only be endpoints of P_1^{\max} and P_2^{\max} , and it is covered once when performing the computations for (P_1^{\max}, P_2^{\max}) , and once when performing the computations for (P_2^{\max}, P_1^{\max}) . Hence, we are doing computations twice. We compensate for this by increasing the betweenness centrality of v only by

$$\sum_{s \in V=2(P_1^{\max}), t \in V=2(P_2^{\max})} \gamma(s, t, v)$$

for all $P_1^{\max} \neq P_2^{\max}$, for vertices $v \in V(P_1^{\max}) \cap V(P_2^{\max})$. \square

With this at hand we can show how to compute X_v for each $v \in V(P_2^{\max})$, for a pair $P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}$ of maximal induced paths. To this end, we show the following lemma.

Lemma 8 *Let $P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}$. Then, given that the values $d_G(s, t)$, σ_{st} , $W^{\text{left}}[v]$ and $W^{\text{right}}[v]$ are known for $s \in V^{\geq 3}(G)$ and $t \in V(G)$, and $v \in V=2(G)$, respectively, one can compute for all $v \in V(P_2^{\max})$ in $O(|V(P_2^{\max})|)$ time:*

$$\sum_{s \in V=2(P_1^{\max}), t \in V=2(P_2^{\max})} \gamma(s, t, v). \quad (16)$$

Since the proof of Lemma 8 is rather tedious and technical, we defer it to Appendix B. The proof consists of two steps. First, we show how to compute the value $\sum_{s \in V=2(P_1^{\max})} \gamma(s, t, v)$ for a fixed $t \in V=2(P_2^{\max})$ and $v \in V(P_2^{\max})$

in constant time; here we use that the values listed in the lemma are known. Second, we use a dynamic program to compute for all $v \in V(P_2^{\max})$ the value of Sum (16) in $O(|V(P_2^{\max})|)$ time, using the fact that the difference between the sums of two adjacent $v, v' \in V(P_2^{\max})$ can be computed in constant time.

We are now ready to combine Lemmata 6 to 8 to prove Proposition 2. As mentioned above, to keep the proposition simple, we assume that the values $\text{Inc}[s, t] \cdot \sigma_{st}(v)$ can be computed in constant time for every $s, t \in V^{\geq 3}(G)$ and $v \in V(G)$. In fact, these values are computed in the last step of the algorithm (see Lines 17 and 18 in Algorithm 1 and Lemma 4).

Proposition 2 (Restated) *In $O(kn)$ time one can compute the following values for all $v \in V(G)$:*

$$\sum_{\substack{s \in V^{=2}(P_1^{\max}), t \in V^{=2}(P_2^{\max}) \\ P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}}} \gamma(s, t, v).$$

Proof: Let $P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}$. Then, for each $v \in V(G) = (V(G) \setminus (V(P_1^{\max}) \cup V(P_2^{\max}))) \cup (V(P_1^{\max}) \cup V(P_2^{\max}))$, we need to compute

$$\sum_{s \in V^{=2}(P_1^{\max}), t \in V^{=2}(P_2^{\max})} \gamma(s, t, v). \tag{17}$$

We first compute in $O(kn)$ time the values $d_G(s, t)$ and σ_{st} for every $s, t \in V^{\geq 3}(G)$, as well as the values $W^{\text{left}}[v]$ and $W^{\text{right}}[v]$ for every $v \in V^{=2}(G)$, see Lines 3 to 10 in Algorithm 1. Combining Lemmata 4 and 6 we can compute Sum (17) in $O(|V(P_2^{\max})|)$ time for $v \in V(G) \setminus (V(P_1^{\max}) \cup V(P_2^{\max}))$, plus $O(kn)$ time for a final postprocessing step. Given the values ρ_i of Lemma 8 we can compute the values X_v defined in Equation (14) for $v = y_i \in V(P_2^{\max})$ as follows:

$$X_v = X_{y_i} = \begin{cases} \rho_i, & \text{if } v \in V(P_1^{\max}) \cap V(P_2^{\max}); \\ 2\rho_i, & \text{otherwise.} \end{cases}$$

This can be done in constant time for a single $v \in V(P_2^{\max})$; thus it can be done in $O(|V(P_2^{\max})|)$ time overall. Hence, by Lemma 7, we can compute Sum (17) for $V(P_1^{\max}) \cup V(P_2^{\max})$ in $O(|V(P_2^{\max})|)$ time.

Sum (17) must be computed for every pair $P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}$. Thus, without the pre- and postprocessing steps, we require

$$\begin{aligned} & O\left(\sum_{P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}} |V(P_2^{\max})|\right) \\ &= O\left(\sum_{P_1^{\max} \in \mathcal{P}^{\max}} \sum_{\substack{P_2^{\max} \in \mathcal{P}^{\max} \\ P_1^{\max} \neq P_2^{\max}}} (|V^{=2}(P_2^{\max})| + |V^{\geq 3}(P_2^{\max})|)\right) \\ &= O\left(\sum_{P_1^{\max} \in \mathcal{P}^{\max}} n\right) = O(kn) \end{aligned} \tag{18}$$

time, since there are at most $O(k)$ maximal induced paths and at most n vertices in all maximal induced paths combined. \square

3.2 Paths with endpoints in the same maximal induced path

We now look at shortest paths starting and ending in a maximal induced path $P^{\max} = x_0 \dots x_q$ and show how to efficiently compute how these paths affect the betweenness centrality of all vertices in the graph. The goal is to prove the following.

Proposition 3 *In $O(kn)$ time one can compute the following for all $v \in V(G)$:*

$$\sum_{\substack{s,t \in V^{=2}(P^{\max}) \\ P^{\max} \in \mathcal{P}^{\max}}} \gamma(s, t, v).$$

We start off by noting the following.

Observation 3 *Let $v \in V(G)$ and let $P^{\max} = x_0 \dots x_q$ be a maximal induced path. Then*

$$\sum_{s,t \in V^{=2}(P^{\max})} \gamma(s, t, v) = \sum_{i,j \in [1, q-1]} \gamma(x_i, x_j, v) = 2 \cdot \sum_{i=1}^{q-1} \sum_{j=i+1}^{q-1} \gamma(x_i, x_j, v).$$

For the sake of readability we set $[x_p, x_r] := \{x_p, x_{p+1}, \dots, x_r\}$, $p < r$. We will distinguish between two different cases that we then treat separately: Either $v \in [x_i, x_j]$ or $v \in V(G) \setminus [x_i, x_j]$. We will show that both cases can be solved in overall $O(|V(P^{\max})|)$ time for P^{\max} . Doing this for all maximal induced paths results in a time of $O(\sum_{P^{\max} \in \mathcal{P}^{\max}} |V^{=2}(P^{\max})|) = O(n)$. In the calculations we will distinguish between the two main cases—all shortest $x_i x_j$ -paths are fully contained in P^{\max} , or all shortest $x_i x_j$ -paths leave P^{\max} —and the corner case that there are some shortest paths inside P^{\max} and some that partially leave it.

We will now compute the value for all paths that only consist of vertices in P^{\max} , that is, we will compute for each x_k with $i < k < j$ the term

$$2 \cdot \sum_{i=1}^{q-1} \sum_{j=i+1}^{q-1} \gamma(x_i, x_j, x_k)$$

with a dynamic program in $O(|V(P^{\max})|)$ time. Since $i < k < j$, by Observation 1, this can be simplified to

$$2 \cdot \sum_{\substack{i \in [1, q-1] \\ i < k}} \sum_{\substack{j \in [i+1, q-1] \\ k < j}} \gamma(x_i, x_j, x_k) = 2 \cdot \sum_{i \in [1, k-1]} \sum_{j \in [k+1, q-1]} \gamma(x_i, x_j, x_k).$$

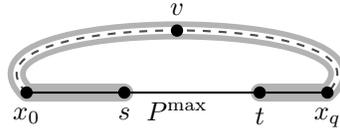


Figure 6: A maximal induced path that affects the betweenness centralities of vertices outside of P^{\max} , such as v . Clearly, if there is a shortest st -path visiting v (thick edge), then there exists a shortest x_0x_q -path visiting v (dashed edge). On an intuitive level, we store the information of the vertices inside of P^{\max} in the table entry $\text{Inc}[x_0, x_q]$.

Lemma 9 *Let $P^{\max} = x_0 \dots x_q$ be a maximal induced path and assume that the values $d_G(s, t)$, σ_{st} , $W^{\text{left}}[v]$ and $W^{\text{right}}[v]$ were precomputed for $s, t \in V^{\geq 3}(G)$ and $v \in V^{=2}(G)$, respectively. Then, in $O(|V(P^{\max})|)$ time, one can compute the following for all x_k with $0 \leq k \leq q$:*

$$\alpha_{x_k} := 2 \cdot \sum_{i \in [1, k-1]} \sum_{j \in [k+1, q-1]} \gamma(x_i, x_j, x_k).$$

The main idea of the dynamic program is the following. Given the value of α_{x_k} , one can compute its difference to $\alpha_{x_{k+1}}$ in constant time, once W^{left} , W^{right} are precomputed (see Lines 7 to 10 in Algorithm 1). These tables can be computed in $O(|V(P^{\max})|)$ time as well. The proof of Lemma 9 is deferred to Appendix C.1.

Now we need to show how to compute the value for all paths that (partially) leave P^{\max} . See Figure 6 for an example of such a path.

Lemma 10 *Let $P^{\max} = x_0 \dots x_q$ be a maximal induced path. Then, excluding the running time of the postprocessing (see Lines 17 and 18 in Algorithm 1), one can compute in $O(|V(P^{\max})|)$ time the following for all $v \in V(G) \setminus [x_i, x_j]$:*

$$\beta_v := \sum_{i \in [1, q-1]} \sum_{j \in [i+1, q-1]} \gamma(x_i, x_j, v) \beta_v.$$

The proof of Lemma 10 is split into two cases: Either $v \in V(G) \setminus V(P^{\max})$, or $v \in V(P^{\max}) \setminus [x_i, x_j]$ (the case that $v \in [x_i, x_j]$ is covered by Lemma 9). The first case makes use of the postprocessing step (see Lines 17 to 18 in Algorithm 1) which was used in an analogous way in the proof of Lemma 6, while the second case uses a dynamic programming approach similar to the one used in the proof of Lemma 9. The proof details can be found in Appendix C.2.

3.3 Postprocessing and algorithm summary

We are now ready to combine all parts and prove our main theorem.

Theorem 1 BETWEENNESS CENTRALITY can be solved in $O(kn)$ time and space, where k is the feedback edge number of the input graph.

Proof: As shown in Proposition 1, if the input graph G is a cycle, then we are done.

We show that Algorithm 1 computes the value

$$C_B(v) = \sum_{s,t \in V(G)} \text{Pen}[s] \cdot \text{Pen}[t] \cdot \frac{\sigma_{st}(v)}{\sigma_{st}} = \sum_{s,t \in V(G)} \gamma(s, t, v)$$

for all $v \in V(G)$ in $O(kn)$ time and space. We use Observation 2 to split the sum as follows.

$$\begin{aligned} \sum_{s,t \in V(G)} \gamma(s, t, v) &= \sum_{s \in V^{\geq 3}(G), t \in V(G)} \gamma(s, t, v) + \sum_{s \in V^=2(G), t \in V^{\geq 3}(G)} \gamma(t, s, v) \\ &+ \sum_{\substack{s \in V^=2(P_1^{\max}), t \in V^=2(P_2^{\max}) \\ P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}}} \gamma(s, t, v) + \sum_{\substack{s,t \in V^=2(P^{\max}) \\ P^{\max} \in \mathcal{P}^{\max}}} \gamma(s, t, v). \end{aligned}$$

By Propositions 2 and 3, we can compute the third and fourth summand in $O(kn)$ time provided that $\text{Inc}[s, t] \cdot \sigma_{st}(v)$ is computed for every $s, t \in V^{\geq 3}(G)$ and every $v \in V(G)$ in a postprocessing step (see Lines 11 to 16). We incorporate this postprocessing into the computation of the first two summands in the equation, that is, we next show that for all $v \in V(G)$ the following value can be computed in $O(kn)$ time:

$$\sum_{\substack{s \in V^{\geq 3}(G) \\ t \in V(G)}} \gamma(s, t, v) + \sum_{\substack{s \in V^=2(G) \\ t \in V^{\geq 3}(G)}} \gamma(s, t, v) + \sum_{\substack{s \in V^{\geq 3}(G) \\ t \in V^{\geq 3}(G)}} \text{Inc}[s, t] \cdot \sigma_{st}(v).$$

To this end, observe that the above is equal to

$$\begin{aligned} &\sum_{\substack{s \in V^{\geq 3}(G) \\ t \in V(G)}} \text{Pen}[s] \text{Pen}[t] \frac{\sigma_{st}(v)}{\sigma_{st}} + \sum_{\substack{s \in V^{\geq 3}(G) \\ t \in V^=2(G)}} \text{Pen}[s] \text{Pen}[t] \frac{\sigma_{st}(v)}{\sigma_{st}} + \sum_{\substack{s \in V^{\geq 3}(G) \\ t \in V^{\geq 3}(G)}} \text{Inc}[s, t] \sigma_{st}(v) \\ &= \sum_{s \in V^{\geq 3}(G)} \left(2 \cdot \sum_{t \in V^=2(G)} \text{Pen}[s] \text{Pen}[t] \frac{\sigma_{st}(v)}{\sigma_{st}} + \sum_{t \in V^{\geq 3}} \sigma_{st}(v) \left(\frac{\text{Pen}[s] \text{Pen}[t]}{\sigma_{st}} + \text{Inc}[s, t] \right) \right). \end{aligned}$$

Note that we initialize $\text{Inc}[s, t]$ in Lines 5 and 6 in Algorithm 1 with $2 \cdot \text{Pen}[s] \text{Pen}[t] / \sigma_{st}$ and $\text{Pen}[s] \text{Pen}[t] / \sigma_{st}$ respectively. Thus we can use the algorithm described in Lemma 4 for each vertex $s \in V^{\geq 3}(G)$ with $f(s, t) = \text{Inc}[s, t]$.

Since $\text{Pen}[s]$, $\text{Pen}[t]$, σ_{st} and $\text{Inc}[s, t]$ can all be looked up in constant time, the algorithm only takes $O(n + m)$ time for each vertex s (see Lines 17 and 18). By Lemma 2 there are $O(\min\{k, n\})$ vertices of degree at least three. Thus, altogether, the algorithm needs $O(\min\{n, k\} \cdot m) = O(\min\{n, k\} \cdot (n+k)) = O(kn)$ time. The precomputations in Lines 3 to 6 require $\Theta(kn)$ space. As the running time is an upper bound on the space complexity, Algorithm 1 requires $\Theta(kn)$ space overall. \square

4 Conclusion

Lifting the processing of degree-one vertices due to Baglioni et al. [3] to a technically much more involved processing of degree-two vertices, we derived a new algorithm for BETWEENNESS CENTRALITY running in $O(kn)$ worst-case time (k is the feedback edge number of the input graph). Our work focuses on algorithm theory and contributes to the field of adaptive algorithm design [10] as well as to the recent “FPT in P” field [13]. It would be of high interest to identify structural parameterizations “beyond” the feedback edge number that might help to get more results in the spirit of our work. In particular, extending our algorithmic approach and mathematical analysis with respect to the treatment of twin vertices [27, 29] might help to get a running time bound involving the modular width or vertex cover number of the input graph. Indeed, Coudert et al. [6] provided first results in this direction; their algorithms however are not adaptive. We believe that improving the dependency on the parameter in the running time is a challenge for future work. As for practical relevance, we firmly believe that a running time of $O(kn)$ as we proved can yield improved performance for some real-world networks. What remains unclear, however, is whether the constants hidden in the O -notation or the non-linear space requirements of our approach can be avoided.

Acknowledgement. We thank the reviewers for their detailed and thorough evaluation of our work which helped to significantly improve the presentation.

References

- [1] Amir Abboud, Fabrizio Grandoni, and Virginia Vassilevska Williams. Subcubic equivalences between graph centrality problems, APSP and diameter. In *Proceedings of the 26th ACM-SIAM Symposium on Discrete Algorithms (SODA '15)*, pages 1681–1697. SIAM, 2015. doi: 10.1137/1.9781611973730.112. 484
- [2] David A. Bader, Shiva Kintali, Kamesh Madduri, and Milena Mihail. Approximating betweenness centrality. In *Proceedings of the 5th International Workshop on Algorithms and Models for the Web-Graph (WAW '07)*, pages 124–137. Springer, 2007. doi: 10.1007/978-3-540-77004-6_10. 484
- [3] Miriam Baglioni, Filippo Geraci, Marco Pellegrini, and Ernesto Lastres. Fast exact computation of betweenness centrality in social networks. In *Proceedings of the 4th International Conference on Advances in Social Networks Analysis and Mining (ASONAM '12)*, pages 450–456. IEEE Computer Society, 2012. doi: 10.1109/ASONAM.2012.79. 484, 485, 486, 487, 488, 499, 509
- [4] Albert-László Barabási and Réka Albert. Emergence of scaling in random networks. *Science*, 286(5439):509–512, 1999. doi: 10.1126/science.286.5439.509. 485

- [5] Ulrik Brandes. A faster algorithm for betweenness centrality. *Journal of Mathematical Sociology*, 25(2):163–177, 2001. doi: 10.1080/0022250X.2001.9990249. 484, 486, 496, 497, 498, 499
- [6] David Coudert, Guillaume Ducoffe, and Alexandru Popa. Fully polynomial FPT algorithms for some classes of bounded clique-width graphs. *ACM Transactions on Algorithms*, 15(3):33:1–33:57, 2019. doi: 10.1145/3310228. 484, 509
- [7] Søren Dahlgaard and Jacob Evald. Tight hardness results for distance and centrality problems in constant degree graphs. *CoRR*, abs/1609.08403, 2016. 484, 490
- [8] Wouter De Nooy, Andrej Mrvar, and Vladimir Batagelj. *Exploratory Social Network Analysis with Pajek*. Cambridge University Press, 3rd edition, 2018. doi: 10.1017/9781108565691. 485
- [9] Dóra Erdős, Vatche Ishakian, Azer Bestavros, and Evimaria Terzi. A divide-and-conquer algorithm for betweenness centrality. In *Proceedings of the 2015 SIAM International Conference on Data Mining (SDM '15)*, pages 433–441. SIAM, 2015. doi: 10.1137/1.9781611974010.49. 484
- [10] Vladimir Estivill-Castro and Derick Wood. A survey of adaptive sorting algorithms. *ACM Computing Surveys*, 24(4):441–476, 1992. doi: 10.1145/146370.146381. 509
- [11] Linton Freeman. A set of measures of centrality based on betweenness. *Sociometry*, 40:35–41, 1977. 484
- [12] Robert Geisberger, Peter Sanders, and Dominik Schultes. Better approximation of betweenness centrality. In *Proceedings of the 10th Meeting on Algorithm Engineering & Experiments (ALENEX '08)*, pages 90–100. SIAM, 2008. 484
- [13] Archontia C. Giannopoulou, George B. Mertzios, and Rolf Niedermeier. Polynomial fixed-parameter algorithms: A case study for longest path on interval graphs. *Theoretical Computer Science*, 689:67–95, 2017. doi: 10.1137/1.9781611972887.9. 509
- [14] Oded Green, Robert McColl, and David A. Bader. A fast algorithm for streaming betweenness centrality. In *Proceedings of the International Conference on Privacy, Security, Risk and Trust (PASSAT '12)*, pages 11–20. IEEE, 2012. doi: 10.1109/SocialCom-PASSAT.2012.37. 484
- [15] David S. Johnson. The genealogy of theoretical computer science: A preliminary report. *ACM SIGACT News*, 16(2):36–49, 1984. doi: 10.1145/1008959.1008960. 485

- [16] Ali Khazaee, Ata Ebrahimzadeh, and Abbas Babajani-Feremi. Identifying patients with Alzheimer’s disease using resting-state fMRI and graph theory. *Clinical Neurophysiology*, 126(11):2132–2141, 2015. doi: 10.1016/j.clinph.2015.02.060. 484
- [17] Tomohiro Koana, Viatcheslav Korenwein, André Nichterlein, Rolf Niedermeier, and Philipp Zschoche. Data reduction for maximum matching on real-world graphs: Theory and experiments. *coRR*, abs/1806.09683, 2019. 485
- [18] Viatcheslav Korenwein, André Nichterlein, Rolf Niedermeier, and Philipp Zschoche. Data reduction for maximum matching on real-world graphs: Theory and experiments. In *Proceedings of the 26th European Symposium on Algorithms (ESA ’18)*, volume 112 of *LIPICs*, pages 53:1–53:13. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018. doi: 10.4230/LIPICs.ESA.2018.53. 485
- [19] Min-Joong Lee, Jungmin Lee, Jaimie Yejean Park, Ryan Hyun Choi, and Chin-Wan Chung. Qube: A quick algorithm for updating betweenness centrality. In *Proceedings of the 21st International Conference on World Wide Web (WWW ’12)*, page 351–360, 2012. doi: 10.1145/2187836.2187884. 484
- [20] John D. Medaglia. Graph theoretic analysis of resting state functional MR imaging. *Neuroimaging Clinics*, 27(4):593–607, 2017. doi: 10.1016/j.nic.2017.06.008. 484
- [21] George B. Mertzios, André Nichterlein, and Rolf Niedermeier. The power of linear-time data reduction for maximum matching. *Algorithmica*, 2020. doi: 10.1007/s00453-020-00736-0. 485
- [22] Meghana Nasre, Matteo Pontecorvi, and Vijaya Ramachandran. Betweenness centrality – incremental and faster. In *Proceedings of the 39th International Conference on Mathematical Foundations of Computer Science (MFCS ’14)*, volume 8634 of *LNCS*, pages 577–588. Springer, 2014. doi: https://doi.org/10.1007/978-3-662-44465-8_49. 484
- [23] Mark E. J. Newman. Who is the best connected scientist? A study of scientific coauthorship networks. In *Proceedings on the 23rd Conference of the Center of Nonlinear Studies (CNLS ’04)*, pages 337–370. Springer, 2004. doi: 10.1007/978-3-540-44485-5_16. 484
- [24] Mark E. J. Newman. *Networks: An Introduction*. Oxford University Press, 2010. doi: 10.1093/acprof:oso/9780199206650.001.0001. 484
- [25] André Nichterlein, Rolf Niedermeier, Johannes Uhlmann, and Mathias Weller. On tractable cases of target set selection. *Social Network Analysis and Mining*, 3(2):233–256, 2013. doi: 10.1007/s13278-012-0067-7. 485

- [26] Kim Norlen, Gabriel Lucas, Michael Gebbie, and John Chuang. Eva: Extraction, visualization and analysis of the telecommunications and media ownership network. In *Proceedings of the 14th International Telecommunications Society Conference (ITS '02)*, 2002. doi: 10.1109/WISEW.2002.1177855. 485
- [27] Rami Puzis, Yuval Elovici, Polina Zilberman, Shlomi Dolev, and Ulrik Brandes. Topology manipulations for speeding betweenness centrality computation. *Journal of Complex Networks*, 3(1):84–112, 2015. doi: 10.1093/com-net/cnu015. 484, 485, 486, 487, 488, 509
- [28] Matteo Riondato and Evgenios M Kornaropoulos. Fast approximation of betweenness centrality through sampling. *Data Mining and Knowledge Discovery*, 30(2):438–475, 2016. doi: 10.1007/s10618-015-0423-0. 484
- [29] Ahmet Erdem Sariyüce, Kamer Kaya, Erik Saule, and Ümit V. Çatalyürek. Graph manipulations for fast centrality computation. *ACM Transactions on Knowledge Discovery from Data*, 11(3):26:1–26:25, 2017. doi: 10.1145/3022668. 484, 485, 486, 487, 488, 509
- [30] Guangming Tan, Dengbiao Tu, and Ninghui Sun. A parallel algorithm for computing betweenness centrality. In *Proceedings of the 38th International Conference on Parallel Processing (ICPP '09)*, pages 340–347. IEEE Computer Society, 2009. doi: 10.1109/ICPP.2009.53. 484
- [31] Flavio Vella, Massimo Bernaschi, and Giancarlo Carbone. Dynamic merging of frontiers for accelerating the evaluation of betweenness centrality. *ACM Journal of Experimental Algorithms*, 23(1):1.4:1–1.4:19, 2018. doi: 10.1145/3182656. 485
- [32] Wei Wang and Choon Yik Tang. Distributed computation of node and edge betweenness on tree graphs. In *Proceedings of the 52nd IEEE Conference on Decision and Control (CDC '13)*, pages 43–48. IEEE, 2013. doi: 10.1109/CDC.2013.6759856. 484, 485
- [33] Stanley Wasserman and Katherine Faust. *Social Network Analysis: Methods and Applications*. Cambridge University Press, 1994. doi: 10.1017/CBO9780511815478. 484

A Notation for proofs in appendix

For the following proofs we will introduce a lot of auxiliary notation. We provide Table 1 as a reference to the definitions of the notations.

Table 1: A reference to the notation used in the Appendix.

We assume $P^{\max} = P_1^{\max} = x_0 \dots x_q$ and $P_2^{\max} = y_0 \dots y_r$.

Symbol	Definition
$\gamma_{st}(v)$	$= \text{Pen}[s] \cdot \text{Pen}[t] \cdot \sigma_{st}(v) / \sigma_{st}$;
$V^{=2}(G)$	the set of vertices of degree two in G ;
$V^{\geq 3}(G)$	the set of vertices of degree at least three in G ;
$\text{Inc}[\cdot, \cdot]$	a table of size $ V^{\geq 3}(G) \times V(G) $ in which intermediary betweenness centrality values are stored;
\mathcal{P}^{\max}	the set of all maximal induced paths;
x_t^{left}	the rightmost vertex in P_1^{\max} such that all shortest paths from $t \in V(G - P_1^{\max})$ to x_t^{left} visit x_0 ;
x_t^{right}	the leftmost vertex in P_1^{\max} such that all shortest paths from $t \in V(G - P_1^{\max})$ to x_t^{right} visit x_q ;
x_t^{mid}	the vertex in P_1^{\max} such that there are shortest paths from $t \in V(G - P_1^{\max})$ to x_t^{left} via x_0 and x_q respectively;
X_t^{left}	$= \{x_1, x_2, \dots, x_t^{\text{left}}\}$;
X_t^{right}	$= \{x_t^{\text{right}}, \dots, x_{q-2}, x_{q-1}\}$;
$W^{\text{left}}[x_k]$	$= \sum_{i=0}^k \text{Pen}[x_i]$, where $x_i \in P_1^{\max}$;
$W^{\text{right}}[x_k]$	$= \sum_{i=k}^{q-1} \text{Pen}[x_i]$, where $x_i \in P_1^{\max}$;
$S(t, \psi)$	for $\psi \in \{x_0, x_q\} = V^{\geq 3}(P_1^{\max})$, the maximal subset of $\{y_0, y_r\} = V^{\geq 3}(P_2^{\max})$ such that for each $\varphi \in S(t, \psi)$ there is a shortest st -path via ψ and φ ;
X_v	see Equation (14);
$\lambda(y_k, y_i)$	$= \sum_{s \in V^{=2}(P_1^{\max})} \gamma(s, y_k, y_i)$, for $0 \leq i \leq r$, $1 \leq k < r$, and $s \in V^{=2}(P_1^{\max})$;
$\eta(y_k, \varphi, \psi)$	is 1 if there is a shortest path from y_k to $\psi \in \{x_0, x_q\}$ to $\varphi \in \{y_0, y_r\}$, 0 otherwise;
ω_i	for $0 < k, i < r$, y_r if $k < i$, y_0 if $k > i$;
$\kappa(y_k, \omega_i)$	see Equation (25);
ρ_i	$= \sum_{k=1}^{i-1} \kappa(y_k, y_r) + \sum_{k=i+1}^{r-1} \kappa(y_k, y_0)$, for $0 < i < r$;
$[x_i, x_j]$	$= \{x_i, x_{i+1}, \dots, x_j\}$ for $0 \leq i < j \leq q$;
i_{mid}^+	$= i + (d_G(x_0, x_q) + q) / 2$, where $0 < i < q$;
j_{mid}^-	$= j - (d_G(x_0, x_q) + q) / 2$; where $0 < j < q$;
α_k	$= 2 \cdot \sum_{i \in [1, k-1]} \sum_{j \in [k+1, q-1]} \gamma(x_i, x_j, x_k)$, where $0 \leq k \leq q$;
β_v	$= \sum_{i \in [1, q-1]} \sum_{j \in [i+1, q-1]} \gamma(x_i, x_j, v)$, where $v \in V(G) \setminus [x_i, x_j]$.

B Proof of Lemma 8

Lemma 8 (Restated) *Let $P_1^{\max} \neq P_2^{\max} \in \mathcal{P}^{\max}$. Then, given that the values $d_G(s, t)$, σ_{st} , $W^{\text{left}}[v]$ and $W^{\text{right}}[v]$ are known for $s \in V^{\geq 3}(G)$ and $t \in V(G)$, and $v \in V^=2(G)$, respectively, one can compute for all $v \in V(P_2^{\max})$ in $O(|V(P_2^{\max})|)$ time:*

$$\sum_{s \in V^=2(P_1^{\max}), t \in V^=2(P_2^{\max})} \gamma(s, t, v). \quad (19)$$

Proof: We first show how to compute $\sum_{s \in V^=2(P_1^{\max})} \gamma(s, t, v)$ for fixed $t \in V^=2(P_2^{\max})$ and $v \in V(P_2^{\max})$ in constant time when the values listed above are known. Then we present a dynamic program that computes for all $v \in V(P_2^{\max})$ the value of Sum (19) in $O(|V(P_2^{\max})|)$ time.

Let $P_1^{\max} = x_0 \dots x_q$ and let $P_2^{\max} = y_0 \dots y_r$. For $v = y_i$, $0 \leq i \leq r$, we compute

$$\begin{aligned} \sum_{s \in V^=2(P_1^{\max}), t \in V^=2(P_2^{\max})} \gamma(s, t, y_i) &= \sum_{s \in V^=2(P_1^{\max})} \sum_{k=1}^{r-1} \gamma(s, y_k, y_i) \\ &= \sum_{k=1}^{r-1} \sum_{s \in V^=2(P_1^{\max})} \gamma(s, y_k, y_i). \end{aligned} \quad (20)$$

For easier reading, we define for $0 \leq i \leq r$ and for $1 \leq k < r$

$$\lambda(y_k, y_i) = \sum_{s \in V^=2(P_1^{\max})} \gamma(s, y_k, y_i).$$

Recall that all shortest paths from y_k to $s \in X_{y_k}^{\text{left}}$ visit x_0 and all shortest paths from y_k to $s \in X_{y_k}^{\text{right}}$ visit x_q . Recall also that for each y_k there may exist a unique vertex $x_{y_k}^{\text{mid}}$ to which there are shortest paths via x_0 and via x_q .

With this at hand, we have

$$\begin{aligned} \lambda(y_k, y_i) &= \gamma(x_{y_k}^{\text{mid}}, y_k, y_i) + \sum_{s \in X_{y_k}^{\text{left}}} \gamma(s, y_k, y_i) + \sum_{s \in X_{y_k}^{\text{right}}} \gamma(s, y_k, y_i) \\ &= \text{Pen}[x_{y_k}^{\text{mid}}] \cdot \text{Pen}[y_k] \cdot \frac{\sigma_{y_k x_{y_k}^{\text{mid}}}(y_i)}{\sigma_{y_k x_{y_k}^{\text{mid}}}} + \sum_{s \in X_{y_k}^{\text{left}}} \text{Pen}[s] \cdot \text{Pen}[y_k] \cdot \frac{\sigma_{s y_k}(y_i)}{\sigma_{s y_k}} \\ &\quad + \sum_{s \in X_{y_k}^{\text{right}}} \text{Pen}[s] \cdot \text{Pen}[y_k] \cdot \frac{\sigma_{s y_k}(y_i)}{\sigma_{s y_k}}. \end{aligned} \quad (21)$$

Next, we rewrite λ in such a way that we can compute it in constant time. To this end, we need to make the values σ independent of s and y_i . To this end, note that if $k < i$, then y_i is visited only by shortest paths from y_k to $s \in V^{=2}(P_1^{\max})$ that also visit y_r . If $k > i$, then y_i is only visited by paths that also visit y_0 . Hence, we need to know whether there are shortest paths from y_k to some endpoint of P_1^{\max} via either y_0 or y_r . For this we define $\eta(y_k, \varphi, \psi)$, which, informally speaking, tells us whether there is a shortest path from y_k to $\psi \in \{x_0, x_q\}$ via $\varphi \in \{y_0, y_r\}$. Formally,

$$\eta(y_k, \varphi, \psi) = \begin{cases} 1, & \text{if } d_{P_2^{\max}}(y_k, \varphi) + d_G(\varphi, \psi) = d_G(y_k, \psi); \\ 0, & \text{otherwise.} \end{cases}$$

Since $d_G(s, t)$ is given for all $s \in V^{\geq 3}(G)$ and all $t \in V(G)$, the values η can be computed in constant time.

We now show how to compute $\sigma_{sy_k}(y_i)/\sigma_{sy_k}$. Let $\omega_i = y_r$ if $k < i$, and $\omega_i = y_0$ if $k > i$. As stated above, for y_i to be on a shortest path from y_k to $s \in V^{=2}(P_1^{\max})$, the path must visit ω_i . If s is in $X_{y_k}^{\text{left}}$, then the shortest paths enter P_1^{\max} via x_0 , and $\sigma_{sy_k}(y_i)/\sigma_{sy_k} = \sigma_{x_0y_k}(y_i)/\sigma_{x_0y_k}$. Note that there may be shortest sy_k -paths that pass via y_0 and sy_k -paths that pass via y_r . Thus we have

$$\frac{\sigma_{x_0y_k}(y_i)}{\sigma_{x_0y_k}} = \frac{\eta(y_k, \omega_i, x_0)\sigma_{x_0\omega_i}}{\eta(y_k, y_0, x_0)\sigma_{x_0y_0} + \eta(y_k, y_r, x_0)\sigma_{x_0y_r}}. \tag{22}$$

With $\sigma_{x_0y_k}(y_i)$ we count the number of shortest x_0y_k -paths visiting y_i . Note that any such path must visit ω_i . If there is such a shortest path visiting ω_i , then all shortest x_0y_k -paths visit y_i , and since there is only one shortest ω_iy_k -path, the number of shortest x_0y_k -paths visiting ω_i is equal to the number of shortest $x_0\omega_i$ -paths, which is $\sigma_{x_0\omega_i}$.

If $s \in X_{y_k}^{\text{right}}$, then

$$\frac{\sigma_{sy_k}(y_i)}{\sigma_{sy_k}} = \frac{\eta(y_k, \omega_i, x_q)\sigma_{x_q\omega_i}}{\eta(y_k, y_0, x_q)\sigma_{x_qy_0} + \eta(y_k, y_r, x_q)\sigma_{x_qy_r}}. \tag{23}$$

Shortest paths from y_k to $x_{y_k}^{\text{mid}}$ may visit any $\varphi \in \{y_0, y_r\}$ and $\psi \in \{x_0, x_q\}$, and thus

$$\frac{\sigma_{y_kx_{y_k}^{\text{mid}}}(y_i)}{\sigma_{y_kx_{y_k}^{\text{mid}}}} = \frac{\sum_{\psi \in \{x_0, x_q\}} \eta(y_k, \omega_i, \psi)\sigma_{\psi\omega_i}}{\sum_{\varphi \in \{y_0, y_r\}} \sum_{\psi \in \{x_0, x_q\}} \eta(y_k, \varphi, \psi)\sigma_{\psi\varphi}}. \tag{24}$$

Observe that

- (1) the values of Equations (22) to (24) can be computed in constant time, since the values σ_{st} are known for $s, t \in V^{\geq 3}(G)$, and

(2) the values $\sigma_{sy_k}(y_i)$ and σ_{sy_k} are independent of s for $s \in X_{y_k}^{\text{left}}$ and for $s \in X_{y_k}^{\text{right}}$ respectively.

Recalling that $W^{\text{left}}[x_j] = \sum_{i=1}^j \text{Pen}[x_i]$ and $W^{\text{right}}[x_j] = \sum_{i=j}^{q-1} \text{Pen}[x_i]$ for $1 \leq j < r$ we define

$$\begin{aligned}
\kappa(y_k, \omega_i) &= \text{Pen}[y_k] \cdot \left(\text{Pen}[x_{y_k}^{\text{mid}}] \cdot \frac{\sum_{\psi \in \{x_0, x_q\}} \eta(y_k, \omega_i, \psi) \sigma_{\psi \omega_i}}{\sum_{\varphi \in \{y_0, y_r\}} \sum_{\psi \in \{x_0, x_q\}} \eta(y_k, \varphi, \psi) \sigma_{\psi \varphi}} \right. \\
&+ \sum_{s \in X_{y_k}^{\text{left}}} \text{Pen}[s] \cdot \frac{\eta(y_k, \omega_i, x_0) \sigma_{x_0 \omega_i}}{\eta(y_k, y_0, x_0) \sigma_{x_0 y_0} + \eta(y_k, y_r, x_0) \sigma_{x_0 y_r}} \\
&+ \left. \sum_{s \in X_{y_k}^{\text{right}}} \text{Pen}[s] \cdot \frac{\eta(y_k, \omega_i, x_q) \sigma_{x_q \omega_i}}{\eta(y_k, y_0, x_q) \sigma_{x_q y_0} + \eta(y_k, y_r, x_q) \sigma_{x_q y_r}} \right) \quad (25) \\
&= \text{Pen}[y_k] \cdot \left(\text{Pen}[x_{y_k}^{\text{mid}}] \cdot \frac{\sum_{\psi \in \{x_0, x_q\}} \eta(y_k, \omega_i, \psi) \sigma_{\psi \omega_i}}{\sum_{\varphi \in \{y_0, y_r\}} \sum_{\psi \in \{x_0, x_q\}} \eta(y_k, \varphi, \psi) \sigma_{\psi \varphi}} \right. \\
&+ W^{\text{left}}[x_{y_k}^{\text{left}}] \cdot \frac{\eta(y_k, \omega_i, x_0) \sigma_{x_0 \omega_i}}{\eta(y_k, y_0, x_0) \sigma_{x_0 y_0} + \eta(y_k, y_r, x_0) \sigma_{x_0 y_r}} \\
&+ \left. W^{\text{right}}[x_{y_k}^{\text{right}}] \cdot \frac{\eta(y_k, \omega_i, x_q) \sigma_{x_q \omega_i}}{\eta(y_k, y_0, x_q) \sigma_{x_q y_0} + \eta(y_k, y_r, x_q) \sigma_{x_q y_r}} \right).
\end{aligned}$$

Note that since the values of $\text{Pen}[\cdot]$, $W^{\text{left}}[\cdot]$ and of $W^{\text{right}}[\cdot]$ are known, $\kappa(y_k, \omega_i)$ can be computed in constant time.

If $k < i$, then

$$\begin{aligned}
\lambda(y_k, y_i) &= \text{Pen}[y_k] \cdot \left(\text{Pen}[x_{y_k}^{\text{mid}}] \cdot \frac{\sigma_{y_k x_{y_k}^{\text{mid}}}(y_i)}{\sigma_{y_k x_{y_k}^{\text{mid}}}} \right. \\
&+ \sum_{s \in X_{y_k}^{\text{left}}} \text{Pen}[s] \cdot \frac{\sigma_{sy_k}(y_i)}{\sigma_{sy_k}} + \sum_{s \in X_{y_k}^{\text{right}}} \text{Pen}[s] \cdot \frac{\sigma_{sy_k}(y_i)}{\sigma_{sy_k}} \left. \right).
\end{aligned}$$

Equations (22) to (24) then give us

$$\begin{aligned}
\lambda(y_k, y_i) &= \text{Pen}[y_k] \cdot \left(\text{Pen}[x_{y_k}^{\text{mid}}] \cdot \frac{\sum_{\psi \in \{x_0, x_q\}} \eta(y_k, y_r, \psi) \sigma_{\psi y_r}}{\sum_{\varphi \in \{y_0, y_r\}} \sum_{\psi \in \{x_0, x_q\}} \eta(y_k, \varphi, \psi) \sigma_{\psi \varphi}} \right. \\
&+ \sum_{s \in X_{y_k}^{\text{left}}} \text{Pen}[s] \cdot \frac{\eta(y_k, y_r, x_0) \sigma_{x_0 y_r}}{\eta(y_k, y_0, x_0) \sigma_{x_q y_0} + \eta(y_k, y_r, x_0) \sigma_{x_q y_r}} \\
&+ \left. \sum_{s \in X_{y_k}^{\text{right}}} \text{Pen}[s] \cdot \frac{\eta(y_k, y_r, x_q) \sigma_{x_q y_r}}{\eta(y_k, y_0, x_q) \sigma_{x_q y_0} + \eta(y_k, y_r, x_q) \sigma_{x_q y_r}} \right) = \kappa(y_k, y_r).
\end{aligned}$$

If $k > i$, then analogously $\lambda(y_k, y_i) = \kappa(y_k, y_0)$. Lastly, if $k = i$, then $\sigma_{sy_k}(y_i) = 0$; thus $\gamma(s, y_k, y_i) = \lambda(y_k, y_i) = 0$. Hence, we can rewrite Sum (20) as

$$\begin{aligned} \sum_{k=1}^{r-1} \sum_{s \in V^{=2}(P_1^{\max})} \gamma(s, y_k, y_i) &= \sum_{\substack{k=1 \\ k \neq i}}^{r-1} \lambda(y_k, y_i) = \left(\sum_{k=1}^{i-1} \lambda(y_k, y_i) + \sum_{k=i+1}^{r-1} \lambda(y_k, y_i) \right) \\ &= \left(\sum_{k=1}^{i-1} \kappa(y_k, y_r) + \sum_{k=i+1}^{r-1} \kappa(y_k, y_0) \right) =: \rho_i. \end{aligned}$$

Towards showing that Sum (19) can be computed in $O(r)$ time, note that $\rho_0 = \sum_{k=1}^{r-1} \kappa(y_k, y_0)$ can be computed in $O(|V(P_2^{\max})|)$ time. Observe that $\rho_{i+1} = \rho_i - \kappa(y_{i+1}, y_r) + \kappa(y_i, y_0)$. Thus, every ρ_i , $1 \leq i \leq r$, can be computed in constant time. Hence, computing all ρ_i , $0 \leq i \leq r$, and thus computing sum (19) for all $v \in V(P_2^{\max})$ takes $O(|V(P_2^{\max})|)$ time. \square

C Proofs of Lemmata 9 and 10

For the proofs of Lemmata 9 and 10 we first make two auxiliary observations and introduce some additional notation.

Observation 4 Let $P^{\max} = x_0 \dots x_q$ be a maximal induced path and let $0 \leq i < j \leq q$. Then

- (i) $d_G(x_i, x_j) = \min\{d_{P^{\max}}(x_i, x_j), i + d_G(x_0, x_q) + q - j\}$, and
- (ii) if $d_{P^{\max}}(x_i, x_j) = i + d_G(x_0, x_q) + q - j$, then $j = i + \frac{d_G(x_0, x_q) + q}{2}$.

Proof: The correctness of (i) is clear. For (ii), note that the claimed equation is equivalent to $j - i = d_{P^{\max}}(x_i, x_j) = i + d_G(x_0, x_q) + q - j$. \square

Observation 5 Let $P^{\max} = x_0 \dots x_q$ be a maximal induced path, let $0 \leq i < j \leq q$, and let $v \in V(G)$. Then

$$\frac{\sigma_{x_i x_j}(v)}{\sigma_{x_i x_j}} = \begin{cases} 0, & \text{if } d_{out} < d_{in} \wedge v \in [x_i, x_j] \text{ or } d_{in} < d_{out} \wedge v \notin [x_i, x_j]; \\ 1, & \text{if } d_{in} < d_{out} \wedge v \in [x_i, x_j]; \\ 1, & \text{if } d_{out} < d_{in} \wedge v \notin [x_i, x_j] \wedge v \in V(P^{\max}); \\ \frac{\sigma_{x_0 x_q}(v)}{\sigma_{x_0 x_q}}, & \text{if } d_{out} < d_{in} \wedge v \notin V(P^{\max}); \\ \frac{1}{\sigma_{x_0 x_q} + 1}, & \text{if } d_{in} = d_{out} \wedge v \in [x_i, x_j]; \\ \frac{\sigma_{x_0 x_q}}{\sigma_{x_0 x_q} + 1}, & \text{if } d_{in} = d_{out} \wedge v \notin [x_i, x_j] \wedge v \in V(P^{\max}); \\ \frac{\sigma_{x_0 x_q}(v)}{\sigma_{x_0 x_q} + 1}, & \text{if } d_{in} = d_{out} \wedge v \notin V(P^{\max}), \end{cases} \tag{26}$$

where $d_{in} = d_{P^{\max}}(x_i, x_j)$ and $d_{out} = i + d_G(x_0, x_q) + q - j$.

Proof: Most cases are self-explanatory. The denominator $\sigma_{x_0x_q} + 1$ is correct since there are $\sigma_{x_0x_q}$ shortest paths from x_0 to x_q (and therefore $\sigma_{x_0x_q}$ shortest paths from x_i to x_j that leave P^{\max}) and one shortest path from x_i to x_j within P^{\max} . Note that if there are shortest paths that are not contained in P^{\max} , then $d_G(x_0, x_q) < q$ and therefore P^{\max} is not a shortest x_0x_q -path. \square

Definition 2 Let $P^{\max} = x_0 \dots x_q$ be a maximal induced path and let $0 \leq i \leq q$. Then we define

$$i_{\text{mid}}^+ = i + (d_G(x_0, x_q) + q)/2 \quad \text{and} \quad j_{\text{mid}}^- = j - (d_G(x_0, x_q) + q)/2.$$

C.1 Proof of Lemma 9

Lemma 9 (Restated) Let $P^{\max} = x_0 \dots x_q$ be a maximal induced path and assume that the values $d_G(s, t)$, σ_{st} , $W^{\text{left}}[v]$ and $W^{\text{right}}[v]$ were precomputed for $s, t \in V^{\geq 3}(G)$ and $v \in V^{\geq 2}(G)$, respectively. Then, in $O(|V(P^{\max})|)$ time, one can compute the following for all x_k with $0 \leq k \leq q$:

$$\alpha_{x_k} := 2 \cdot \sum_{i \in [1, k-1]} \sum_{j \in [k+1, q-1]} \gamma(x_i, x_j, x_k).$$

Proof: We construct a dynamic program, then we show that it is solvable in $O(|V(P^{\max})|)$ time.

Note that $1 \leq i < k$. Thus for $k = 0$ we have

$$\alpha_{x_0} = 2 \sum_{i \in \emptyset} \sum_{j \in [1, q-1]} \gamma(x_i, x_j, x_0) = 0.$$

This will be the base case of the dynamic program.

For every vertex x_k with $1 \leq k < q$ it holds that

$$\begin{aligned} \alpha_{x_k} &= 2 \cdot \sum_{\substack{i \in [1, k-1] \\ j \in [k+1, q-1]}} \gamma(x_i, x_j, x_k) \\ &= 2 \cdot \sum_{\substack{i \in [1, k-2] \\ j \in [k+1, q-1]}} \gamma(x_i, x_j, x_k) + 2 \cdot \sum_{j \in [k+1, q-1]} \gamma(x_{k-1}, x_j, x_k). \end{aligned}$$

Similarly, for x_{k-1} with $1 < k \leq q$ it holds that

$$\begin{aligned} \alpha_{x_{k-1}} &= 2 \cdot \sum_{\substack{i \in [1, k-2] \\ j \in [k, q-1]}} \gamma(x_i, x_j, x_{k-1}) \\ &= 2 \cdot \sum_{\substack{i \in [1, k-2] \\ j \in [k+1, q-1]}} \gamma(x_i, x_j, x_{k-1}) + 2 \cdot \sum_{i \in [1, k-2]} \gamma(x_i, x_k, x_{k-1}). \end{aligned}$$

Next, observe that any path from x_i to x_j with $i \leq k-2$ and $j \geq k+1$ visiting x_k also visits x_{k-1} and vice versa. Substituting this into the equations above yields

$$\alpha_{x_k} = \alpha_{x_{k-1}} + 2 \cdot \sum_{j \in [k+1, q-1]} \gamma(x_{k-1}, x_j, x_k) - 2 \cdot \sum_{i \in [1, k-2]} \gamma(x_i, x_k, x_{k-1}).$$

Now we prove that $\sum_{j \in [k+1, q-1]} \gamma(x_{k-1}, x_j, x_k)$ and $\sum_{i \in [1, k-2]} \gamma(x_i, x_k, x_{k-1})$ can be computed in constant time once W^{left} and W^{right} are precomputed (see Lines 7 to 10 in Algorithm 1). These tables can be computed in $O(|V(P^{\text{max}})|)$ time as well. For the sake of convenience we say that $\gamma(x_i, x_j, x_k) = 0$ if i or j are not integral or are not in $[1, q-1]$ and define $W[x_i, x_j] = \sum_{\ell=i}^j \text{Pen}[x_\ell] = W^{\text{left}}[x_j] - W^{\text{left}}[x_{i-1}]$. Then we can use Observations 4 and 5 to show that

$$\begin{aligned} \sum_{j \in [k+1, q-1]} \gamma(x_{k-1}, x_j, x_k) &= \sum_{j \in [k+1, q-1]} \text{Pen}[x_{k-1}] \cdot \text{Pen}[x_j] \cdot \frac{\sigma_{x_{k-1}x_j}(x_k)}{\sigma_{x_{k-1}x_j}} \\ &= \gamma(x_{k-1}, x_{(k-1)_{\text{mid}}^+}, x_k) + \sum_{j \in [k+1, \min\{\lceil (k-1)_{\text{mid}}^+ \rceil - 1, q-1\}]} \text{Pen}[x_{k-1}] \cdot \text{Pen}[x_j] \\ &= \begin{cases} \text{Pen}[x_{k-1}] \cdot W[x_{k+1}, x_{q-1}], & \text{if } (k-1)_{\text{mid}}^+ \geq q; \\ \text{Pen}[x_{k-1}] \cdot W[x_{k+1}, x_{\lceil (k-1)_{\text{mid}}^+ \rceil - 1}], & \text{if } (k-1)_{\text{mid}}^+ < q \wedge (k-1)_{\text{mid}}^+ \notin \mathbb{Z}; \\ \text{Pen}[x_{k-1}] \cdot (\text{Pen}[x_{(k-1)_{\text{mid}}^+}] \cdot \frac{1}{\sigma_{x_0 x_q} + 1} + W[x_{k+1}, x_{(k-1)_{\text{mid}}^+ - 1}]), & \text{otherwise.} \end{cases} \end{aligned}$$

Herein we use the notation introduced in Definition 2. By $(k-1)_{\text{mid}}^+ \notin \mathbb{Z}$ we mean to say that $(k-1)_{\text{mid}}^+$ is not integral. Analogously,

$$\begin{aligned} \sum_{i \in [1, k-2]} \gamma(x_i, x_k, x_{k-1}) &= \sum_{i \in [1, k-2]} \text{Pen}[x_i] \cdot \text{Pen}[x_k] \cdot \frac{\sigma_{x_i x_k}(x_{k-1})}{\sigma_{x_i x_k}} \\ &= \gamma(x_{k-1}, x_{k_{\text{mid}}^-}, x_{k-1}) + \sum_{i \in [\max\{1, \lfloor (k-1)_{\text{mid}}^- \rfloor + 1\}, k-2]} \text{Pen}[x_i] \cdot \text{Pen}[x_k] \\ &= \begin{cases} \text{Pen}[x_k] \cdot W[x_1, x_{k-2}], & \text{if } k_{\text{mid}}^- < 1; \\ \text{Pen}[x_k] \cdot W[x_{\lfloor k_{\text{mid}}^- \rfloor + 1}, x_{k-2}], & \text{if } k_{\text{mid}}^- \geq 1 \wedge k_{\text{mid}}^- \notin \mathbb{Z}; \\ \text{Pen}[x_k] \cdot (\text{Pen}[x_{k_{\text{mid}}^-}] \cdot \frac{1}{\sigma_{x_0 x_q} + 1} + W[x_{k_{\text{mid}}^- + 1}, x_{k-2}]), & \text{otherwise.} \end{cases} \end{aligned}$$

This completes the proof since $(k-1)_{\text{mid}}^+$, k_{mid}^- , every entry in $W[\cdot, \cdot]$, and all other variables in the equation above can be computed in constant time once $W^{\text{left}}[\cdot]$ is computed. Thus, computing α_{x_i} for each vertex x_i in P^{max} takes constant time. Hence, the computations for the whole maximal induced path P^{max} take $O(|V(P^{\text{max}})|)$ time. \square

C.2 Proof of Lemma 10

Lemma 10 (Restated) *Let $P^{\text{max}} = x_0 \dots x_q$ be a maximal induced path. Then, excluding the running time of the postprocessing (see Lines 17 and 18*

in Algorithm 1), one can compute in $O(|V(P^{\max})|)$ time the following for all $v \in V(G) \setminus [x_i, x_j]$:

$$\beta_v := \sum_{i \in [1, q-1]} \sum_{j \in [i+1, q-1]} \gamma(x_i, x_j, v) \beta_v.$$

Proof: We first show how to compute β_v for all $v \notin V(P^{\max})$ and then how to compute β_v for all $v \in V(P^{\max}) \setminus [x_i, x_j]$ in the given time.

As stated above, the distance from x_i to $x_{i_{\text{mid}}^+}$ (if existing) is the boundary such that all shortest paths to vertices x_j with $j > i_{\text{mid}}^+$ leave P^{\max} and the unique shortest path to any x_j with $i < j < i_{\text{mid}}^+$ is $x_i x_{i+1} \dots x_j$. Thus we can use Observations 4 and 5 to show that for each $v \notin P^{\max}$ and each fixed $i \in [1, q-1]$ it holds that

$$\begin{aligned} \sum_{j \in [i+1, q-1]} \gamma(x_i, x_j, v) &= \sum_{j \in [i+1, q-1]} \text{Pen}[x_i] \cdot \text{Pen}[x_j] \cdot \frac{\sigma_{x_i x_j}(v)}{\sigma_{x_i x_j}} \\ &= \begin{cases} 0, & \text{if } i_{\text{mid}}^+ > q-1; \\ \sum_{j \in [x_{\lceil i_{\text{mid}}^+ \rceil}, q-1]} \text{Pen}[x_i] \cdot \text{Pen}[x_j] \cdot \frac{\sigma_{x_0 x_q}(v)}{\sigma_{x_0 x_q}}, & \text{if } i_{\text{mid}}^+ \leq q-1 \wedge i_{\text{mid}}^+ \notin \mathbb{Z}; \\ \text{Pen}[x_i] \cdot \left(\text{Pen}[x_{i_{\text{mid}}^+}] \cdot \frac{\sigma_{x_0 x_q}(v)}{\sigma_{x_0 x_q} + 1} + \sum_{j \in [x_{i_{\text{mid}}^+ + 1}, q-1]} \text{Pen}[x_j] \cdot \frac{\sigma_{x_0 x_q}(v)}{\sigma_{x_0 x_q}} \right), & \text{otherwise;} \end{cases} \\ &= \begin{cases} 0, & \text{if } i_{\text{mid}}^+ > q-1; \\ \text{Pen}[x_i] \cdot W^{\text{right}}[x_{\lceil i_{\text{mid}}^+ \rceil}] \cdot \frac{\sigma_{x_0 x_q}(v)}{\sigma_{x_0 x_q}}, & \text{if } i_{\text{mid}}^+ \leq q-1 \wedge i_{\text{mid}}^+ \notin \mathbb{Z}; \\ \text{Pen}[x_i] \cdot \left(\text{Pen}[x_{i_{\text{mid}}^+}] \cdot \frac{\sigma_{x_0 x_q}(v)}{\sigma_{x_0 x_q} + 1} + W^{\text{right}}[x_{i_{\text{mid}}^+ + 1}] \cdot \frac{\sigma_{x_0 x_q}(v)}{\sigma_{x_0 x_q}} \right), & \text{otherwise.} \end{cases} \end{aligned}$$

Herein we use the notation introduced in Definition 2. By $i_{\text{mid}}^+ \notin \mathbb{Z}$ we mean to say that i_{mid}^+ is not integral. All variables except for $\sigma_{x_0 x_q}(v)$ can be computed in constant time once W^{right} and $\sigma_{x_0 x_q}$ are computed. Thus we can compute overall in $O(|V(P^{\max})|)$ time the value

$$\begin{aligned} X &= \frac{2 \cdot \sum_{i \in [1, q-1]} \sum_{j \in [i+1, q-1]} \gamma(x_i, x_j, v)}{\sigma_{x_0 x_q}(v)} \\ &= 2 \cdot \sum_{i \in [1, q-1]} \sum_{j \in [i+1, q-1]} \text{Pen}[x_i] \text{Pen}[x_j] \sigma_{x_i, x_j}. \end{aligned} \tag{27}$$

Due to the postprocessing (see Lines 17 and 18 in Algorithm 1) it is sufficient to add X to $\text{Inc}[x_0, x_q]$. This ensures that $X \cdot \sigma_{x_0 x_q}(v)$ is added to the betweenness centrality of each vertex $v \notin V(P^{\max})$. Note that if $X > 0$, then $d_G(x_0, x_q) < q$ and thus the betweenness centrality of any vertex $v \in V(P^{\max})$ is not affected by $\text{Inc}[x_0, x_q]$.

Next, we will compute β_v for all vertices $v \in V(P^{\max})$ (recall that $v \notin [x_i, x_j]$). We start with the simple observation that all paths that leave P^{\max} at some

point have to contain x_0 . Thus β_{x_0} is equal to X by Equation (27). We will use this as the base case for a dynamic program that iterates through P^{\max} and computes β_{x_k} for each vertex x_k , $k \in [0, q]$, in constant time.

Similarly to the proof of Lemma 9 we observe that

$$\begin{aligned} \beta_{x_k} &= 2 \left(\sum_{i \in [k+1, q-1]} \sum_{j \in [i+1, q-1]} \gamma(x_i, x_j, x_k) + \sum_{i \in [1, k-1]} \sum_{j \in [i+1, k-1]} \gamma(x_i, x_j, x_k) \right) \\ &= 2 \left(\sum_{i \in [k+2, q-1]} \sum_{j \in [i+1, q-1]} \gamma(x_i, x_j, x_k) + \sum_{i \in [1, k-1]} \sum_{j \in [i+1, k-1]} \gamma(x_i, x_j, x_k) \right. \\ &\quad \left. + \sum_{j \in [k+2, q-1]} \gamma(x_{k+1}, x_j, x_k) \right) \end{aligned}$$

and

$$\begin{aligned} \beta_{x_{k+1}} &= 2 \left(\sum_{i \in [k+2, q-1]} \sum_{j \in [i+1, q-1]} \gamma(x_i, x_j, x_{k+1}) + \sum_{i \in [1, k]} \sum_{j \in [i+1, k]} \gamma(x_i, x_j, x_{k+1}) \right) \\ &= 2 \left(\sum_{i \in [k+2, q-1]} \sum_{j \in [i+1, q-1]} \gamma(x_i, x_j, x_{k+1}) \right. \\ &\quad \left. + \sum_{i \in [1, k-1]} \sum_{j \in [i+1, k-1]} \gamma(x_i, x_j, x_{k+1}) + \sum_{i \in [1, k-1]} \gamma(x_i, x_k, x_{k+1}) \right). \end{aligned}$$

Furthermore, observe that every st -path with $s, t \neq x_k, x_{k+1}$ that contains x_k also contains x_{k+1} , and vice versa. Thus we can conclude that

$$\beta_{x_{k+1}} = \beta_{x_k} + 2 \left(\underbrace{\sum_{i \in [1, k-1]} \gamma(x_i, x_k, x_{k+1})}_{(*)} - \underbrace{\sum_{j \in [k+2, q-1]} \gamma(x_{k+1}, x_j, x_k)}_{(**)} \right).$$

It remains to show that the sums (*) and (**) can be computed in constant time once W^{left} and W^{right} are computed. Using Observations 4 and 5 we get that

$$\sum_{i \in [1, k-1]} \gamma(x_i, x_k, x_{k+1}) = \begin{cases} 0, & \text{if } k_{\text{mid}}^- < 1; \\ \text{Pen}[x_k] \cdot W^{\text{left}}[x_{\lfloor k_{\text{mid}}^- \rfloor}], & \text{if } k_{\text{mid}}^- \geq 1 \wedge k_{\text{mid}}^- \notin \mathbb{Z}; \\ \text{Pen}[x_k] \cdot \left(W^{\text{left}}[x_{k_{\text{mid}}^- - 1}] + \text{Pen}[k_{\text{mid}}^-] \cdot \frac{\sigma_{x_0 x_q}}{\sigma_{x_0 x_q} + 1} \right), & \text{otherwise;} \end{cases}$$

and

$$\sum_{j \in [k+2, q-1]} \gamma(x_{k+1}, x_j, x_k) = \begin{cases} 0, & \text{if } k_{\text{mid}}^+ < 1; \\ \text{Pen}[x_{k+1}] \cdot W^{\text{right}}[x_{\lceil (k+1)_{\text{mid}}^+ \rceil}], & \text{if } (k+1)_{\text{mid}}^+ \leq q-1 \wedge (k+1)_{\text{mid}}^+ \notin \mathbb{Z}; \\ \text{Pen}[x_{k+1}] \cdot \left(W^{\text{right}}[x_{(k+1)_{\text{mid}}^+ + 1}] + \text{Pen}[(k+1)_{\text{mid}}^+] \cdot \frac{\sigma_{x_0 x_q}}{\sigma_{x_0 x_q} + 1} \right), & \text{otherwise.} \end{cases}$$

Since all variables in these two equalities can be evaluated in constant time, this concludes the proof. \square