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# Lower Bounds for Dynamic Programming on Planar Graphs of Bounded Cutwidth

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**Abstract**

Many combinatorial problems can be solved in time $\mathcal{O}^*(c^{\mathrm{tw}})$ on graphs of treewidth tw, for a problem-specific constant $c$. In several cases, matching upper and lower bounds on $c$ are known based on the Strong Exponential Time Hypothesis (SETH). In this paper we investigate the complexity of solving problems on graphs of bounded cutwidth, a graph parameter that takes larger values than treewidth. We strengthen earlier treewidth-based lower bounds to show that, assuming SETH, Independent Set cannot be solved in $O^*((2-\varepsilon)^{\mathrm{ctw}})$ time, and Dominating Set cannot be solved in $O^*((3-\varepsilon)^{\mathrm{ctw}})$ time. By designing a new crossover gadget, we extend these lower bounds even to planar graphs of bounded cutwidth or treewidth. Hence planarity does not help when solving Independent Set or Dominating Set on graphs of bounded width. This sharply contrasts the fact that in many settings, planarity allows problems to be solved much more efficiently.

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# 1 Introduction

Dynamic programming on graphs of bounded treewidth is a powerful tool in the algorithm designer's toolbox, which has many applications (cf. [5]) and is captured by several meta-theorems [7, 27]. Through clever use of techniques such as Möbius transformation, fast subset convolution [2, 30], cut & count [9], and representative sets [4, 8, 14], algorithms were developed that can solve numerous combinatorial problems on graphs of treewidth tw in $\mathcal{O}^*(c^{\mathrm{tw}})$ time, for a problem-specific constant $c$. In recent work [23], it was shown that under the *Strong Exponential Time Hypothesis* (SETH, see [17, 18]), the base of the exponent $c$ achieved by the best-known algorithm is actually optimal for Dominating Set ($c = 3$) and Independent Set ($c = 2$), amongst others. This prompts the following questions:

1. Do faster algorithms exist for bounded-treewidth graphs that are *planar*?
2. Do faster algorithms exist for a more restrictive graph parameter, such as cutwidth?

It turns out that these questions are related, because the nature of cutwidth allows crossover gadgets to be inserted to planarize a graph without increasing its width significantly.

Before going into our results, we briefly motivate these questions. There is a rich bidimensionality theory (cf. [10]) of how the planarity of a graph can be exploited to obtain better algorithms than in the nonplanar case, leading to what has been called the *square-root phenomenon* [26]: in several settings, parameterized algorithms on planar graphs can be faster by a square-root factor in the exponent, compared to their nonplanar counterparts. Hence it may be tempting to believe that problems on bounded-width graphs can be solved more efficiently when they are planar. Lokshtanov et al. [23, §9] explicitly ask whether their SETH-based lower bounds continue to apply for planar graphs. The same problem is posed by Baste and Sau [1, p. 3] in their investigation on the influence of planarity when solving connectivity problems parameterized by treewidth. This motivates question 1.

When faced with lower bounds for the parameterization by treewidth, it is natural to investigate whether these continue to hold for more restrictive graph parameters. We work with the parameter cutwidth since it is one of the classic graph layout parameters (cf. [11]) which takes larger values than treewidth [22], and has been the subject of frequent study [16, 28, 29]. In their original work, Lokshtanov et al. [23] showed that their lower bounds also hold for pathwidth instead of treewidth. However, the parameterization by *cutwidth* has so far not been considered, which leads us to question 2. (See Section 2 for the definition of cutwidth.)

**Our results** We answer questions 1 and 2 for the problems Independent Set and Dominating Set, which are formally defined in Section 2. Our conceptual contribution towards answering question 1 comes from the following insight: any graph $G$ can be drawn in the plane (generally with crossings) such that the graph $G'$ obtained by replacing each crossing by a vertex of degree four does not have larger cutwidth than $G$. Hence the property of having bounded cutwidth can be preserved while planarizing the graph, which was independently[1] discovered by Eppstein [13]. When we planarize by replacing each crossing by a planar crossover gadget $H$ instead of a single vertex, then we obtain $\mathrm{ctw}(G') \leq \mathrm{ctw}(G) + \mathrm{ctw}(H) + 4$ if the endpoints of the crossing edges each obtain at most one neighbor in the crossover gadget. This gives a means to reduce a problem instance on a general graph of bounded cutwidth to a planar graph of bounded cutwidth, if a suitable crossover gadget is available. The parameter cutwidth is special in this regard: one cannot planarize a drawing of $K_{3,n}$ while keeping the pathwidth or treewidth constant [12, 13].

For the Independent Set problem, the crossover gadget developed by Garey, Johnson, and Stockmeyer [15] can be used in the process described above. Together with the observation that the SETH-based lower bound construction by Lokshtanov et al. [23] for the treewidth parameterization also works for the cutwidth parameterization, this yields our first result.[2]

**Theorem 1** *Assuming SETH, there is no $\varepsilon > 0$ such that* Independent Set *on a planar graph $G$ given along with a linear layout of cutwidth $k$ can be solved in time $\mathcal{O}^*((2-\varepsilon)^k)$.*

For the Dominating Set problem, more work is needed to obtain a lower bound for planar graphs of bounded cutwidth. While the lower bound construction of Lokshtanov et al. [23] also works for the parameter cutwidth after a minor tweak, no crossover gadget for the Dominating Set problem was known. Our main technical contribution therefore consists of the design of a crossover gadget for Dominating Set, which we believe to be of independent interest. Together with the framework above, this gives our second result.

**Theorem 2** *Assuming SETH, there is no $\varepsilon > 0$ such that* Dominating Set *on a planar graph $G$ given along with a linear layout of cutwidth $k$ can be solved in time $\mathcal{O}^*((3-\varepsilon)^k)$.*

Since any linear ordering of cutwidth $k$ can be transformed into a tree decomposition of width at most $k$ in polynomial time (cf. [3, Theorem 47]), the lower bounds of Theorems 1 and 2 also apply to the parameterization by treewidth. Hence our work resolves the question raised by Lokshtanov et al. [23] and by Baste and Sau [1] whether the SETH-lower bounds for Independent Set and Dominating Set parameterized by treewidth also apply for planar graphs.

[1] We learned of Eppstein's result while a previous version of this work was under submission at a different venue; see Footnote 3 in [13]. Our previous manuscript, cited by Eppstein, was later split into two separate parts due to its excessive length. The present paper is one part, and [19, 20] is the other.

[2] The analogous lower bound of $\Omega((2-\varepsilon)^k)$ for solving Independent Set on planar graphs of pathwidth $k$ was already observed by Jansen and Wulms [21], based on an elaborate ad-hoc argument.

**Organization** In Section 2 we provide preliminaries. In Section 3 we present a general theorem for planarizing graphs of bounded cutwidth, using a crossover gadget. It leads to a proof of Theorem 1. In Section 4 we prove Theorem 2. Finally, we provide some conclusions in Section 5. The proofs of two statements marked (★) have been deferred to the appendix. These proofs show how existing lower bounds for treewidth parameterizations can be re-analyzed to give lower bounds for cutwidth parameterizations, and have been placed in the appendix to improve the flow of our contributed argumentation.

## 2 Preliminaries

We use $\mathbb{N}$ to denote the natural numbers, including 0. For a positive integer $n$ and a set $X$ we use $\binom{X}{n}$ to denote the collection of all subsets of $X$ of size $n$. The *power set* of $X$ is denoted $2^X$. The set $\{1, \ldots, n\}$ is abbreviated as $[n]$. The $\mathcal{O}^*$ notation suppresses polynomial factors in the input size $n$, such that $\mathcal{O}^*(f(k))$ is shorthand for $\mathcal{O}(f(k)n^{\mathcal{O}(1)})$. All our logarithms have base two.

We consider finite, simple, and undirected graphs $G$, consisting of a vertex set $V(G)$ and edge set $E(G) \subseteq \binom{V(G)}{2}$. The neighbors of a vertex $v$ in $G$ are denoted $N_G(v)$. The closed neighborhood of $v$ is $N_G[v] := N_G(v) \cup \{v\}$. For a vertex set $S \subseteq V(G)$ the open neighborhood is $N_G(S) := \bigcup_{v \in S} N_G(v) \setminus S$ and the closed neighborhood is $N_G[S] := N_G(S) \cup S$. The subgraph of $G$ induced by a vertex subset $U \subseteq V(G)$ is denoted $G[U]$. The operation of *identifying vertices $u$ and $v$* in a graph $G$ results in the graph $G'$ that is obtained from $G$ by replacing the two vertices $u$ and $v$ by a new vertex $w$ with $N_{G'}(w) = N_G(\{u, v\})$.

An *independent set* is a set of pairwise nonadjacent vertices. A *vertex cover* in a graph $G$ is a set $S \subseteq V(G)$ such that $S$ contains at least one endpoint from every edge. A set $S \subseteq V(G)$ *dominates* the vertices $N_G[S]$. A *dominating set* is a vertex set $S$ such that $N_G[S] = V(G)$. The associated decision problems ask, given a graph $G$ and integer $t$, whether an independent set (dominating set) of size $t$ exists in $G$. The size of a maximum independent set (resp. minimum dominating set) in $G$ is denoted $\textsc{opt}_{\textsc{is}}(G)$ (resp. $\textsc{opt}_{\textsc{ds}}(G)$). The $q$-SAT problem asks whether a given Boolean formula, in conjunctive normal form with clauses of size at most $q$, has a satisfying assignment.

The complexity hypothesis under which our lower bounds hold is formally stated below. Although the truth of this hypothesis is not universally accepted in the community, it is often used [24, 31] to relate the complexity of different problems to each other and to show that further improvements on a problem would imply a breakthrough for CNF-SAT.

**Hypothesis 1 (Strong Exponential Time Hypothesis (SETH) [17, 18])** *For every $\varepsilon > 0$, there is a constant $q$ such that $q$-*SAT *on $n$ variables cannot be solved in time $\mathcal{O}^*((2-\varepsilon)^n)$.*

**Drawings** A *drawing* of a graph $G$ is a function $\psi$ that assigns a unique point $\psi(v) \in \mathbb{R}^2$ to each vertex $v \in V(G)$, and a simple curve $\psi(e) \subseteq \mathbb{R}^2$ to each edge $e \in E(G)$, such that the following four conditions hold. (1) For $e = \{u, v\} \in E(G)$, the endpoints of $\psi(e)$ are exactly $\psi(u)$ and $\psi(v)$. (2) The interior of a curve $\psi(e)$ does not contain the image of any vertex. (3) No three curves representing edges intersect in a common point, except possibly at their endpoints. (4) The interiors of the curves $\psi(e), \psi(e')$ for distinct edges intersect in at most one point. If the interiors of all the curves representing edges are pairwise-disjoint, then we have a *planar drawing*. In this paper we combine (nonplanar) drawings with crossover gadgets to build planar drawings. A graph is *planar* if it admits a planar drawing.

**Cutwidth** For an $n$-vertex graph $G$, a *linear layout* of $G$ is a linear ordering of its vertex set, as given by a bijection $\pi\colon V(G) \to [n]$. The *cutwidth* of $G$ with respect to the layout $\pi$ is:

$$\operatorname{ctw}_\pi(G) = \max_{1 \leq i < n} \left| \left\{ \{u, v\} \in E(G) \;\middle|\; \pi(u) \leq i \wedge \pi(v) > i \right\} \right|.$$

The cutwidth $\operatorname{ctw}(G)$ of a graph $G$ is the minimum cutwidth attained by any linear layout. It is well-known that $\operatorname{ctw}(G) \geq \operatorname{pw}(G) \geq \operatorname{tw}(G)$, where the latter denote the pathwidth and treewidth of $G$, respectively (cf. [3]).

# 3 Planarizing graphs while preserving cutwidth

In this section we show how to planarize a graph without blowing up its cutwidth. An intuitive way to think about cutwidth is to consider the vertices as being placed on a horizontal line in the order dictated by the layout $\pi$, with edges drawn as $x$-monotone curves. For any position $i$ we consider the gap between vertex $\pi^{-1}(i)$ and $\pi^{-1}(i+1)$, and count the edges that *cross* the gap by having one endpoint at position at most $i$ and the other at position after $i$. The cutwidth of a layout is the maximum number of edges crossing any single gap; see Figure 1. The simple but useful fact on which our approach hinges is the following. If we obtain $G'$ by replacing a crossing in the drawing by a new vertex of degree four, and we let $\pi'$ be the left-to-right order of the vertices in the resulting drawing, then $\operatorname{ctw}_\pi(G) = \operatorname{ctw}_{\pi'}(G')$. Hence by repeating this procedure we can eliminate all crossings to obtain a planarized version of $G$ without increasing the cutwidth. To utilize this idea in reductions, we formalize a version of this approach where we planarize the graph by inserting gadgets, rather than simply replacing crossings by degree-four vertices.

**Definition 1** *A* crossover gadget *is a graph $H$ with terminal vertices $u, u', v, v'$ such that:*

1. *there is a planar drawing $\psi$ of $H$ in which all terminals lie on the outer face, and*
2. *there is a closed curve intersecting the drawing $\psi$ only in the terminals, which visits the terminals in the order $u, v, u', v'$.*

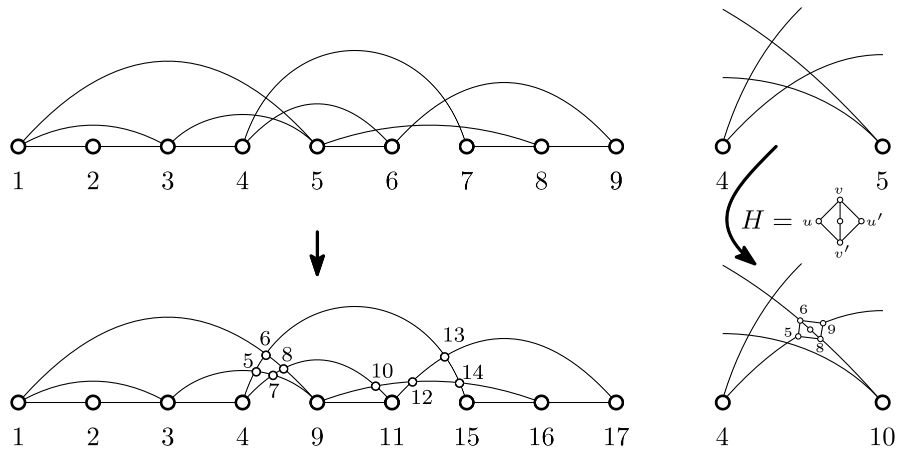


Figure 1: Top-left: a linear layout $\pi$ of a graph $G$ with $\mathrm{ctw}_\pi(G) = 4$. The largest cutsize is attained after vertices 4 and 5. Bottom-left: after inserting vertices at the crossings to obtain $G'$ and extending $\pi$ to $\pi'$ based on the $x$-coordinates of the inserted vertices, we have $\mathrm{ctw}_\pi(G) = \mathrm{ctw}_{\pi'}(G')$. Top-right: enlarged view. Bottom-right: replacing a crossing by gadget $H$.

**Definition 2** *Let $\{a, b\}$ and $\{c, d\}$ be disjoint edges of a graph $G$, and let $H$ be a crossover gadget. The operation of* replacing $\{\{a, b\}, \{c, d\}\}$ by $H$ *removes edges $\{a, b\}$ and $\{c, d\}$, inserts a new copy of the graph $H$, and inserts the edges $\{a, u\}, \{u', b\}, \{c, v\}, \{v', d\}$.*

For a crossover gadget to be useful to planarize instances of a decision problem, a replacement should have a predictable effect on the answer. To formalize this, we say that a *decision problem* $\Pi$ *on graphs* is a decision problem whose input consists of a graph $G$ and integer $t$.

**Definition 3** *A crossover gadget $H$ is* useful for a decision problem $\Pi$ on graphs *if there exists an integer $c_\Pi$ such that the following holds. If $(G, t)$ is an instance of $\Pi$ containing disjoint edges $\{a, b\}, \{c, d\}$, and $G'$ is the result of replacing these edges by $H$, then $(G, t)$ is a* YES*-instance of $\Pi$ if and only if $(G', t + c_\Pi)$ is a* YES*-instance of $\Pi$.*

The following theorem proves that a useful crossover gadget can be used to efficiently planarize instances without blowing up their cutwidth.

**Theorem 3** *If $H$ is a crossover gadget that is useful for decision problem $\Pi$ on graphs, then there is a polynomial-time algorithm that, given an instance $(G, t)$ of $\Pi$ and a linear layout $\pi$ of $G$, outputs an instance $(G', t')$ and a linear layout $\pi'$ of $G'$ such that:*

1. *$G'$ is planar,*

2. $\mathrm{ctw}_{\pi'}(G') \le \mathrm{ctw}_\pi(G) + \mathrm{ctw}(H) + 4$, *and*

3. $(G, t)$ *is a* YES*-instance of* $\Pi$ *if and only if* $(G', t')$ *is a* YES*-instance of* $\Pi$.

**Proof:** Consider the crossover gadget $H$ for $\Pi$ with terminals $u, u, v, v'$. Let $\pi_H$ be a linear layout of $H$ of minimum cutwidth, which is hardcoded into the algorithm along with the integer $c_\Pi$ described in Definition 3.

Let $(G, t)$ be an instance of $\Pi$ with a linear layout $\pi$. We start by constructing a (nonplanar) drawing $\psi$ of $G$ with the following properties.

1. The vertices of $G$ are placed on the $x$-axis, in the order dictated by $\pi$.

2. The image of each edge of $G$ is a strictly $x$-monotone curve.

3. If the drawings of two edges intersect in their interior, then their endpoints are all distinct and the corresponding curves properly cross; they do not only touch.

4. For each pair of edges, their drawings intersect in at most one point.

5. The $x$-coordinates of all crossings are distinct from each other, and from the $x$-coordinates of the vertices.

It is easy to see that such a drawing always exists and can be found in polynomial time; we omit the details as they are not interesting. Properties 1 and 2 together ensure that for any $i \in [|V(G)| - 1]$, the set of edges that cross the gap after vertex $\pi^{-1}(i)$ in the linear layout is exactly the set of edges intersected by a vertical line between $\pi^{-1}(i)$ and $\pi^{-1}(i+1)$, which therefore has size at most $\mathrm{ctw}_\pi(G)$. We will use this property later.

The algorithm replaces the crossings one by one. If two edges $\{a, b\}$ and $\{c, d\}$ intersect in their interior, then their endpoints are all distinct by (3) and they properly cross. Hence we can replace these two edges by a copy of $H$ as in Definition 2. Since there is a planar drawing of $H$ with the terminals alternating along the outer face, after possibly swapping the labels of $a$ and $b$, and of $c$ and $d$, the drawing can be updated so that the crossing between $\{a, b\}$ and $\{c, d\}$ is eliminated. Since each of $a, b, c, d$ is made adjacent to exactly one vertex of $H$, the replacement can be done such that the remaining crossings are in exactly the same locations as before; see the right side of Figure 1. When inserting the crossover gadget, we scale it down sufficiently far that the following holds: all vertices and crossings that were originally on the left of the crossing between $\{a, b\}$ and $\{c, d\}$ lie to the left of all vertices that are inserted to replace this crossing; and all vertices and crossings that were on the right of the $\{\{a, b\}, \{c, d\}\}$ crossing, lie to the right of all vertices inserted for its replacement.

Since each pair of edges intersects at most once by (4), the number of crossings is $\mathcal{O}(|E(G)|^2)$. Hence in polynomial time we can replace all crossings by copies of $H$ to arrive at a graph $G'$. If $\ell$ is the number of replaced crossings, then we set $t' := t + \ell \cdot c_\Pi$. By Definition 3 and transitivity it follows that $(G, t)$ is a

YES-instance of $\Pi$ if and only if $(G', t')$ is a YES-instance. By construction, $G'$ is planar. It remains to define a linear layout of $G'$ and bound its cutwidth.

The layout $\pi'$ of $G'$ is defined as follows. Let the *elements* of the original drawing $\psi$ of $G$ consist of its vertices and its crossings. The elements of $\psi$ are linearly ordered by their $x$-coordinates, by (5). The linear layout $\pi'$ of $G'$ has one block per element of $\psi$, and these blocks are ordered according to the $x$-coordinates of the corresponding element. For elements that consist of a vertex $v$, the block simply consists of $v$. For elements that consist of a crossing $X$, the block consists of the vertices of the copy of $H$ that was inserted to replace $X$, in the order dictated by $\pi_H$. It is easy to see that $\pi'$ can be constructed in polynomial time.

We classify the edges of $G'$ into two types. We have *internal* edges, which are edges within an inserted copy of $H$, and we have *external edges* which connect two different copies of $H$, or which connect a vertex of $V(G) \cap V(G')$ to a copy of $H$. Using this classification we argue that for an arbitrary vertex $v^*$ of $G'$, the cut crossing the gap after vertex $v^*$ in $\pi'$ contains at most $\text{ctw}_\pi(G) + \text{ctw}(H) + 4$ edges. To do so, we distinguish two cases depending on whether $v^*$ is an original vertex from $G$, or was inserted as part of a copy of $H$.

**Claim 4** *If $v^* \in V(G) \cap V(G')$, then the size of the cut after $v^*$ in $\pi'$ is at most* $\text{ctw}_\pi(G)$.

**Proof:** The layout $\pi'$ consists of blocks, and $v^* \in V(G) \cap V(G')$ is a block. So for each copy $C$ of a crossover gadget, the vertices of $C$ all appear on the same side of $v^*$ in the ordering. Hence no internal edge of $C$ crosses the cut after $v^*$, implying that no internal edge is in the cut. Each external edge crossing the cut is (a segment of) an edge of $G$ that is intersected by a vertical line after $v^*$ in the drawing $\psi$; see Figure 1. As such a line intersects at most $\text{ctw}_\pi(G)$ edges as observed above, the cut after $v^*$ has size at most $\text{ctw}_\pi(G)$. ⌟

**Claim 5** *If $v^* \in V(G')$ is a vertex of a copy $C$ of a crossover gadget that was inserted to replace a crossing $X$, then the size of the cut after $v^*$ in $\pi'$ is at most* $\text{ctw}_\pi(G) + \text{ctw}(H) + 4$.

**Proof:** The number of internal edges in the cut after $v^*$ is at most $\text{ctw}(H)$, since the only internal edges in the cut all belong to the same copy $C$ that contains $v^*$ and we ordered them according to an optimal layout $\pi_H$. There are at most four external edges incident on a vertex of $C$, which contribute at most four to the cut. Finally, for each of the remaining external edges in the cut there is a unique edge of $G$ intersected by a vertical line through crossing $X$ in the drawing $\psi$. As at most $\text{ctw}_\pi(G)$ edges are intersected by any vertical line, as observed above, the size of the cut is at most $\text{ctw}_\pi(G) + \text{ctw}(H) + 4$. ⌟

The two claims together show that any gap in the ordering $\pi'$ is crossed by at most $\text{ctw}_\pi(G) + \text{ctw}(H) + 4$ edges, which bounds the cutwidth of $G'$ as required. □

Theorem 3 will be the main ingredient in the proof of Theorem 1. The other ingredients consist of a known crossover gadget along with a lower bound for Independent Set for nonplanar graphs that follows by re-analyzing an earlier construction by Lokshtanov, Marx, and Saurabh [25].

**Theorem 6 (★)** *Assuming SETH, there is no $\varepsilon > 0$ such that* Independent Set *on a (nonplanar) graph $G$ given along with a linear layout of cutwidth $k$ can be solved in time $\mathcal{O}^*((2-\varepsilon)^k)$.*

**Theorem 1** *Assuming SETH, there is no $\varepsilon > 0$ such that* Independent Set *on a planar graph $G$ given along with a linear layout of cutwidth $k$ can be solved in time $\mathcal{O}^*((2-\varepsilon)^k)$.*

**Proof:** First, we observe that the crossover gadget for Vertex Cover due to Garey, Johnson, and Stockmeyer [15, Thm. 2.7] satisfies our conditions of a useful crossover gadget. Since an $n$-vertex graph has a vertex cover of size $k$ if and only if it has an independent set of size $n-k$, it also acts as a useful crossover gadget for Independent Set with $c_\pi = 9$ (cf. [21, Proposition 20]). By Theorem 6, it follows that (assuming SETH) there is no $\varepsilon > 0$ such that Independent Set on a graph $G$ with a linear layout of cutwidth $k$ can be solved in time $\mathcal{O}^*((2-\varepsilon)^k)$. By Theorem 3, if such a runtime could be achieved on *planar* graphs of a given cutwidth, it could be achieved for a general graph as well, since the insertion of crossover gadgets increases the cutwidth by only a constant. Hence Theorem 1 follows. □

# 4 Lower bound for dominating set on planar graphs of bounded cutwidth

In this section we prove a runtime lower bound for solving Dominating Set on planar graphs of bounded cutwidth. Our starting point is the insight that through a minor modification, the lower bound by Lokshtanov et al. [23] for the parameterization by pathwidth can be extended to apply to the parameterization by cutwidth as well.

**Theorem 7 (★)** *Assuming SETH, there is no $\varepsilon > 0$ such that* Dominating Set *on a (nonplanar) graph $G$ given along with a linear layout of cutwidth $k$ can be solved in time $\mathcal{O}^*((3-\varepsilon)^k)$.*

Our contribution is to extend the lower bound of Theorem 7 to apply to *planar* graphs. Following the strategy outlined in Section 3, to achieve this it suffices to develop a useful crossover gadget for Dominating Set as per Definition 3. Since our crossover gadget is fairly complicated (it has more than 100 vertices), we describe its design in steps. The main idea is as follows. We first show that an edge in a Dominating Set instance can be replaced by a longer double-path structure, which contains several triangles. Then we show that when two triangles cross, we can replace their crossing by a suitable adaptation

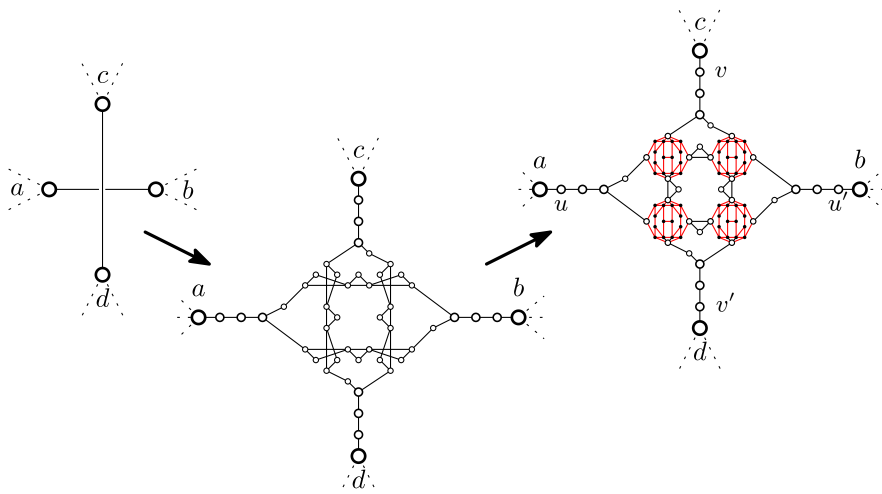


Figure 2: Overview of the method to eliminate an edge crossing in an instance of Dominating Set. Each edge is transformed into a double-path structure, to turn a single crossing edge into four crossing triangles (middle figure). Then each crossing triangle is replaced by a planar gadget (right figure). Some vertices have been omitted for readability. For each red edge $\{u, v\}$, there is a hidden degree-two vertex in the graph that forms a triangle with $u$ and $v$.

of the Vertex Cover crossover gadget due to Garey, Johnson, and Stockmeyer [15, Thm. 2.7]. This two-step approach is illustrated in Figure 2. We follow the same two steps in proving its correctness, starting with the insertion of the double-path structure.

**Lemma 1** *Let $\{x, y\}$ be an edge in a graph $G$. If $G'$ is obtained from $G$ by replacing $\{x, y\}$ by a double-path structure as shown in Figure 3, then* $\text{OPT}_{\text{DS}}(G') = \text{OPT}_{\text{DS}}(G) + 6$.

**Proof:** We prove equality by establishing matching upper and lower bounds.

($\leq$) To show $\text{OPT}_{\text{DS}}(G') \leq \text{OPT}_{\text{DS}}(G) + 6$, consider a minimum dominating set $S \subseteq V(G)$ of $G$. If $S \cap \{x, y\} = \emptyset$, or $S \cap \{x, y\} = \{x, y\}$, then the edge $\{x, y\}$ is not used to dominate vertex $x$ or $y$, and therefore $S \cup \{b_x, b_y, e_x, e_y, g_x, g_y\}$ is a dominating set of size $|S| + 6 = \text{OPT}_{\text{DS}}(G) + 6$ in graph $G'$; see Figure 3. If $S \cap \{x, y\} = \{x\}$, then $S \cup \{c_x, f_x, h_x, e_y, g_y, a_y\}$ is a dominating set of size $\text{OPT}_{\text{DS}}(G) + 6$ in $G'$: the vertex $a_y$ takes over the role of dominating $y$ after the direct edge $\{x, y\}$ is removed, while $a_x$ is dominated from $x$. Symmetrically, if $S \cap \{x, y\} = \{y\}$, then $S \cup \{c_y, f_y, h_y, e_x, g_x, a_x\}$ is a dominating set in $G'$ of size $\text{OPT}_{\text{DS}}(G) + 6$.

($\geq$) To show $\text{OPT}_{\text{DS}}(G') \geq \text{OPT}_{\text{DS}}(G) + 6$, we instead prove $\text{OPT}_{\text{DS}}(G) \leq \text{OPT}_{\text{DS}}(G') - 6$. Consider a minimum dominating set $S' \subseteq V(G')$ of $G'$. Let $B$ be the vertices in the interior of the double-path structure that was inserted into $G'$

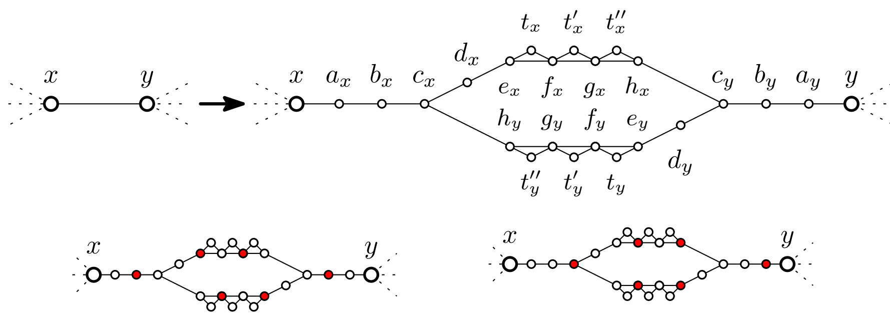


Figure 3: The double-path structure for Dominating Set is the subgraph on the top right minus the vertices $x$ and $y$. Top: an edge $\{x, y\}$ is replaced by a double-path structure. Bottom-left: the interior of the double-path structure can be dominated by six vertices (in red). Bottom-right: there is a set of six vertices that dominates $y$ and all vertices of the double-path structure except $a_x$.

to replace edge $\{x, y\}$. If $|S' \cap B| \geq 7$, then $(S \setminus B) \cup \{x\}$ is a dominating set of size at most $|S'| - 6 \leq \text{OPT}_{\text{DS}}(G') - 6$ in $G$, since $x$ dominates itself and $y$ using the edge $\{x, y\}$. We assume $|S' \cap B| \leq 6$ in the remainder. Then we have $|S' \cap B| = 6$: the closed neighborhoods of the six vertices $\{b_x, b_y, t_x, t_y, t''_x, t''_y\}$ are contained entirely within $B$, and are pairwise disjoint. Hence these six vertices must be dominated by six distinct vertices from $B$. If $S' \cap \{a_x, a_y\} = \emptyset$ then the vertices $x$ and $y$ are not dominated from within the double-path structure, implying that $S' \setminus B$ is a dominating set in $G$ of size $|S'| - 6 \leq \text{OPT}_{\text{DS}}(G') - 6$. It remains to consider the case that $S'$ contains $a_x$, or $a_y$, or both.

**Claim 8** *Let $B' \subseteq B$ be a set of size six that dominates the vertices $B \setminus \{a_x, a_y\}$. If $B'$ contains $a_x$, then $B'$ does not dominate $a_y$. Analogously, if $B'$ contains $a_y$ then it does not dominate $a_x$.*

**Proof:** We prove that if $a_y \in B'$, then $B'$ does not dominate $a_x$. The other statement follows by symmetry. So assume for a contradiction that $B'$ contains $a_y$ and dominates $a_x$, which implies it contains $a_x$ or $b_x$. Since $B'$ dominates the interior of the double-path structure, it contains at least one vertex from the closed neighborhoods of $t_x, t_y, t''_x, t''_y$. Since these are pairwise disjoint, and do not contain $a_y$, $a_x$, or $b_x$, the set $B'$ contains a vertex from the closed neighborhood of each of $\{t_x, t_y, t''_x, t''_y\}$. Since $B'$ has size six, besides the four vertices from these closed neighborhoods, the vertex $a_y$, and the one vertex in $\{a_x, b_x\}$ there can be no further vertices in $B'$. Hence $B'$ does not contain $c_x$ or $d_x$, as these do not occur in the stated closed neighborhoods. This implies that to dominate $d_x$, the set $B'$ contains $e_x$. Then vertex $t'_x$ is not dominated by the vertex from $N_{G'}[t_x]$, and must therefore be dominated by the vertex in $B'$ from $N_{G'}[t''_x]$, implying that $g_x \in B'$. But the vertices mentioned so far do not dominate $c_y$, and regardless of how a vertex is chosen from the closed neighborhoods of $t_y$ and $t''_y$, the resulting choice does not dominate $c_y$ since no vertex from the closed neighborhoods of $t_y, t''_y$ is adjacent to $c_y$. So $c_y$ is not dominated by $B'$; a contradiction. ⌟

Using the claim we finish the proof. The set $B' := S \cap B$ has size six and dominates all of $B \setminus \{a_x, a_y\}$, since those vertices cannot be dominated from elsewhere. If $B'$ contains $a_x$, then $B'$ does not dominate $a_y$. Since $S'$ is a dominating set, and $y$ is the only neighbor of $a_y$ outside of $B$, it follows that $y \in S'$. But then $S' \setminus B$ is a dominating set in $G$ of size $|S'| - 6 = \text{OPT}_{\text{DS}}(G') - 6$ in $G$: the edge $\{x, y\}$ in $G$ ensures that $y$ dominates $x$. If $B'$ contains $a_y$ instead, then the symmetric argument applies. Hence $\text{OPT}_{\text{DS}}(G) \leq \text{OPT}_{\text{DS}}(G') - 6$. □

Using Lemma 1, we can replace a direct edge by a double-path structure while controlling the domination number. This allows two crossing edges to be reduced to four crossing triangles as in Figure 2. Even though more crossings are created in this way, these crossing triangles actually help to planarize the graph. The key point is that crossing triangles enforce a dominating set to locally act like a vertex cover, which allows us to exploit a known gadget for Vertex Cover. The following two statements are useful to formalize these ideas. Recall that a vertex $v$ is *simplicial* in a graph $G$ if $N_G(v)$ forms a clique.

**Observation 9** *If $I$ is an independent set of simplicial degree-two vertices in a graph $G$, then $G$ has a minimum dominating set that contains no vertex of $I$.*

**Proposition 10** *Let $U$ be a set of vertices in a graph $G$, such that for each edge $\{x, y\} \in E(G[U])$ there is a vertex $v \in V(G) \setminus U$ with $N_G(v) = \{x, y\}$. Then there is a minimum dominating set $S$ of $G$ such that $S$ forms a vertex cover of $G[U]$.*

**Proof:** Construct a set $I$ as follows. For each $\{x, y\} \in E(G[U])$, add a vertex $v \in V(G) \setminus U$ with $N_G(v) = \{x, y\}$ to $I$. Then $I$ is an independent set of simplicial degree-two vertices. By Observation 9 there is a minimum dominating set $S$ of $G$ that contains no vertex of $I$. Then $S \cap U$ is a vertex cover of $G[U]$: for an arbitrary edge $\{x, y\} \in E(G[U])$ there is a vertex $v$ in $I$ whose open neighborhood is $\{x, y\}$. Since $I \cap S = \emptyset$ at least one of $x$ and $y$ belongs to $S$ to dominate $v$. Hence the edge $\{x, y\}$ is covered by $S$. □

Proposition 10 relates minimum dominating sets to vertex covers. We therefore use a simplified version of a Vertex Cover crossover gadget in our design. We exploit the graph $H_{\text{VC}}$ with four terminals $\{x, y, p, q\}$ that is shown in Figure 4. It was obtained by applying the "folding" reduction rule for Vertex Cover [6, Lemma 2.3] on the gadget by Garey et al. [15] and omitting two superfluous edges. We use the following property of the graph $H_{\text{VC}}$. It states that in $H_{\text{VC}}$, for every axis from which a vertex cover contains no terminal vertex, the number of non-terminal vertices used in a vertex cover increases.

**Proposition 11** *Let $S$ be a vertex cover of $H_{\text{VC}}$ and let $\ell \in \{0, 1, 2\}$ be the number of pairs among $\{p, q\}$ and $\{x, y\}$ from which $S$ contains no vertices. Then $|S \setminus \{p, q, x, y\}| \geq 9 + \ell$.*

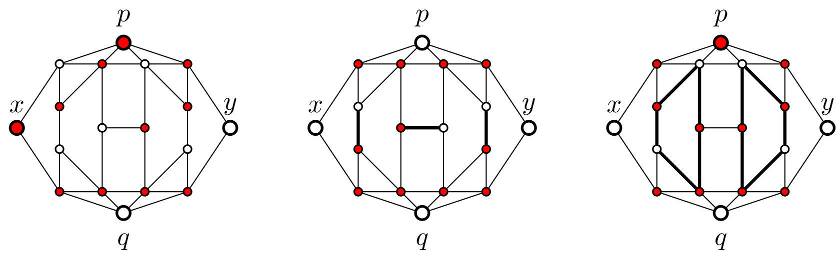


Figure 4: Three copies of the 18-vertex gadget graph $H_{\text{VC}}$, which has four terminals $\{x, y, p, q\}$. Left: A vertex cover for $H_{\text{VC}}$ that contains $p$ and $q$ and has size eleven is shown in red. Middle: Any vertex cover for $H_{\text{VC}}$ that does not contain $p$ or $q$ contains the neighbors of $p$ and $q$ and at least one endpoint of the three thick edges, and contains at least eleven non-terminals. Right: Any vertex cover for $H_{\text{VC}}$ that does not contain $x$ or $y$ contains the four neighbors of $x$ and $y$ and at least three vertices from each of the two highlighted five-cycles, and contains at least ten non-terminals.

**Proof:** We first show $|S \setminus \{p, q, x, y\}| \geq 9$ for any vertex cover $S$ of $H_{\text{VC}}$, proving the claim for $\ell = 0$. The non-terminal vertices of $H_{\text{VC}}$ can be partitioned into four vertex-disjoint triangles and an edge that is vertex-disjoint from the triangles. From any triangle, a vertex cover contains at least two vertices. From the remaining edge, it contains at least one vertex.

If $S$ contains no vertex of $\{p, q\}$, then as illustrated in the middle of Figure 4, $S$ contains at least eleven non-terminals. Hence $|S \setminus \{p, q, x, y\}| \geq 11 \geq 9 + \ell$.

If the previous case does not apply, then $\ell \leq 1$ since $S$ contains a vertex of $\{p, q\}$. If $S$ contains no vertex of $\{x, y\}$, then as illustrated on the right of Figure 4, $S$ contains at least ten non-terminals. Hence $|S \setminus \{p, q, x, y\}| \geq 10 \geq 9 + \ell$. □

Using Proposition 11 we prove that replacing two crossing triangles in a Dominating Set instance by the gadget, increases the optimum by exactly nine.

**Lemma 2** *Let $G$ be a graph, and let $\{x, y, z\}$ and $\{p, q, r\}$ be two vertex-disjoint triangles in $G$ such that $z$ and $r$ have degree two in $G$. Then the graph $G'$ obtained from $G$ by replacing $z$ and $r$ by $H_{\text{VC}}$ as in Figure 5 satisfies $\text{OPT}_{\text{DS}}(G') = \text{OPT}_{\text{DS}}(G) + 9$.*

**Proof:** We prove equality by establishing matching upper and lower bounds.

($\leq$) Consider a minimum dominating set $S$ in $G$ that does not contain $r$ or $z$, which exists by Observation 9. Then $S$ contains at least one of $\{p, q\}$ to dominate $z$, and at least one of $\{x, y\}$ to dominate $r$. We assume without loss of generality, by symmetry, that $p \in S$ and $x \in S$. As shown in Figure 4, there is a vertex cover for $H_{\text{VC}}$ of size 11 that contains $p$ and $x$, and therefore contains nine

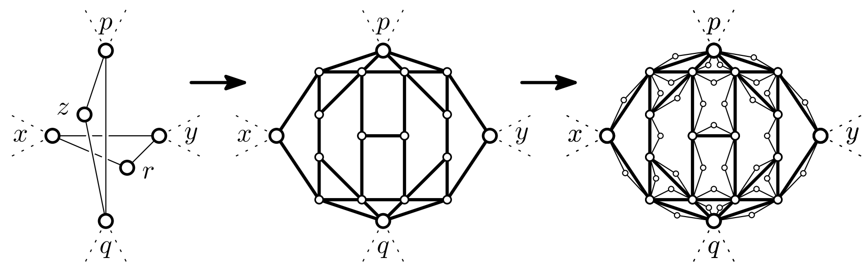


Figure 5: Illustration of how a crossing between two triangles is eliminated in an instance of Dominating Set. A copy of $H_{\text{VC}}$ is inserted, whose four terminals are identified with the four endpoints of the crossing edge. For each edge $\{u, v\}$ of the inserted copy of $H_{\text{VC}}$, an additional degree-two vertex is inserted that forms a triangle with $u$ and $v$.

vertices from the interior of $H_{\text{VC}}$. Let $T$ be this set of nine vertices, and note that $T$ includes a neighbor of $q$ and a neighbor of $y$. We claim that $S' := S \cup T$ is a dominating set for $G'$ of size $|S| + 9 \leq \text{OPT}_{\text{DS}}(G) + 9$. Since $T \cup \{p, x\}$ is a vertex cover of $H_{\text{VC}}$ and $H_{\text{VC}}$ has no isolated vertices, each vertex of $H_{\text{VC}}$ has a neighbor in $S'$ and is dominated. The degree-two vertices that are inserted into $G'$ in the last step are dominated by the vertex that covers the edge with which they form a triangle. Vertices $q$ and $y$ are dominated from their neighbors in $T$. Finally, the remaining vertices of $G'$ are dominated in the same way as in $G$.

($\geq$) To prove $\text{OPT}_{\text{DS}}(G') \geq \text{OPT}_{\text{DS}}(G) + 9$, we instead show $\text{OPT}_{\text{DS}}(G) \leq \text{OPT}_{\text{DS}}(G') - 9$. Let $U \subseteq V(G')$ denote the vertices from the copy of $H_{\text{VC}}$ that was inserted; $U$ contains $p, q, x, y$, but $U$ does not contain the degree-two vertices that were inserted as the last step of the transformation. By Proposition 10, there is a minimum dominating set $S'$ of $G'$ such that $S' \cap U$ is a vertex cover of $G'[U]$. Let $\ell \in \{0, 1, 2\}$ be the number of pairs among $\{p, q\}$ and $\{x, y\}$ from which $S'$ contains no vertices. Since $S' \cap U$ is a vertex cover of $G'[U]$, which is isomorphic to $H_{\text{VC}}$, by Proposition 11 we know that $|(S' \cap U) \setminus \{p, q, x, y\}| \geq 9 + \ell$. Now let $S$ be obtained from $S'$ by removing all vertices of $(S' \cap U) \setminus \{p, q, x, y\}$, adding vertex $x$ if $S' \cap \{x, y\} = \emptyset$, and adding vertex $y$ if $S' \cap \{p, q\} = \emptyset$. Then $|S| \leq |S'| - 9$ since we remove $9 + \ell$ vertices and add $\ell$ new ones. Since $S$ contains at least one vertex from $\{p, q\}$ and at least one vertex from $\{x, y\}$, it dominates the two triangles in $G$. Since it contains a superset of the terminal vertices that $S'$ contains, the remaining vertices of the graph are dominated as before. Hence $S$ is a dominating set in $G$ and $\text{OPT}_{\text{DS}}(G) \leq \text{OPT}_{\text{DS}}(G') - 9$. □

Using the material so far, we can prove that the transformation operation in Figure 2 increases the size of an optimal dominating set by exactly 48.

**Lemma 3** *Let $\{a, b\}$ and $\{c, d\}$ be two disjoint edges of a graph $G$. Let $G'$ be the graph obtained by replacing these two edges as in Figure 2. Then* $\text{OPT}_{\text{DS}}(G') = \text{OPT}_{\text{DS}}(G) + 48$.

**Proof:** The transformation depicted in Figure 2 can be broken down into six steps: transform $\{a, b\}$ into a double-path structure, transform $\{c, d\}$ into a double-path structure, and perform four operations in which crossing triangles are replaced by gadgets. By Lemma 1, the two double-path insertions increase the size of a minimum dominating set by exactly $2 \cdot 6$. By Lemma 2, the four steps in which crossing triangles are eliminated increase the size of a minimum domination set by exactly $4 \cdot 9$. Hence $\textsc{opt}_{\textsc{ds}}(G') = \textsc{opt}_{\textsc{ds}}(G) + 12 + 36$. □

Using Lemma 3 we easily obtain the following.

**Lemma 4** *There is a useful crossover gadget for* Dominating Set*.*

**Proof:** The gadget that is inserted to replace two edges $\{a, b\}$ and $\{c, d\}$ in the procedure of Figure 2 is planar and has its terminals $u, u', v, v'$ on the outer face in the appropriate cyclic ordering. Since Lemma 3 shows that the replacement increases the size of a minimum dominating set by exactly 48, it follows that the structure serves as a useful crossover gadget for Dominating Set as per Definition 3. □

Theorem 2 now follows by combining Lemma 4 with the planarization argument of Theorem 3 and the lower bound for the nonplanar case given by Theorem 7.

**Theorem 2** *Assuming SETH, there is no $\varepsilon > 0$ such that* Dominating Set *on a planar graph $G$ given along with a linear layout of cutwidth $k$ can be solved in time $\mathcal{O}^*((3-\varepsilon)^k)$.*

**Proof:** Suppose Dominating Set on a planar graph with a given linear layout of cutwidth $k$ can be solved in $\mathcal{O}^*((3-\varepsilon)^k)$ time for some $\varepsilon > 0$, by an algorithm called $A$. Then Dominating Set on a nonplanar graph with a given layout of cutwidth $k$ can be solved in time $\mathcal{O}^*((3-\varepsilon)^k)$ by reducing it to a planar graph with a linear layout of cutwidth $k + \mathcal{O}(1)$ (using Theorem 3 and the existence of a useful crossover gadget; this blows up the graph size by at most a polynomial factor) and then running $A$. By Theorem 7, this contradicts SETH. □

# 5 Conclusion

In this work we have investigated whether SETH-based lower bounds for solving problems on graphs of bounded treewidth also apply for (1) planar graphs and (2) graphs of bounded cutwidth. To answer these questions, we showed that the graph parameter cutwidth can be preserved when reducing to a planar instance using suitably restricted crossover gadgets.

For both problems considered in this work, the runtime lower bound for solving the problem on graphs of bounded cutwidth continues to hold for *planar* graphs of bounded cutwidth. Hence planarity seems to offer no algorithmic advantage when working with graphs of bounded cutwidth. Moreover, for both

INDEPENDENT SET and DOMINATING SET the runtime lower bound for the treewidth parameterization also applies for cutwidth.

Future work may explore other combinatorial problems on graphs of bounded cutwidth. For example, what is the optimal running time for FEEDBACK VERTEX SET, ODD CYCLE TRANSVERSAL, or HAMILTONIAN CYCLE on graphs of bounded cutwidth? What is the complexity of the cutwidth parameterization of these problems on planar graphs? For the GRAPH $q$-COLORING problem, these questions are answered in recent work by an overlapping set of authors [20]: planarity offers no advantage, but the parameterization by cutwidth $k$ can be solved in time $\mathcal{O}^*(2^k)$ for all $q$, sharply contrasting that the treewidth parameterization cannot be solved in time $\mathcal{O}^*((q-\varepsilon)^k)$ under SETH.

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# A Lower bound for Independent Set on graphs of bounded cutwidth

**Theorem 6** *Assuming SETH, there is no $\varepsilon > 0$ such that* Independent Set *on a (nonplanar) graph $G$ given along with a linear layout of cutwidth $k$ can be solved in time $\mathcal{O}^*((2-\varepsilon)^k)$.*

**Proof:** This follows from the lower bound of Lokshtanov et al. [23, Thm. 3.1] in terms of pathwidth. It suffices to extend the analysis and provide an analogue of their Lemma 3.3 to bound the cutwidth of the graph $G$ that is constructed from an $n$-variable CNF formula by $n + \mathcal{O}(1)$. Graph $G$ consists of $n+1$ copies of a graph $G_1$; the copies are connected in a path-like fashion. We first recall the structure of $G_1$ and bound its cutwidth.

Graph $G_1$ consists of $n$ paths $P_1, \ldots, P_n$ of $2m$ vertices each, called $p_i^1, \ldots, p_i^{2m}$ for $i \in [n]$, together with $m$ clause gadgets $\hat{C}_j$ for $j \in [m]$. Each clause gadget is easily seen to be a graph of cutwidth $\mathcal{O}(1)$. Between a clause gadget $\hat{C}_j$ and a path $P_i$ there is at most one edge, which connects to either $p_i^{2j-1}$ or $p_i^{2j}$. Each vertex of a clause gadget is adjacent to at most one vertex of one path.

Using this knowledge we describe a linear layout $\pi$ of $G_1$ of cutwidth $n+\mathcal{O}(1)$. It consists of $m$ consecutive blocks $B_1, \ldots, B_m$ of vertices. A block $B_j$ contains the vertices of $\hat{C}_j \cup \{p_i^{2j-1}, p_i^{2j} \mid i \in [n]\}$, and is ordered according to the following process. Start from an optimal ordering for $\hat{C}_j$, of cutwidth $\mathcal{O}(1)$. For every vertex $v$ of $\hat{C}_j$ that is adjacent to a vertex on a path, say to $P_i$, insert vertices $p_i^{2j-1}, p_i^{2j}$ just after $v$ in the ordering. For the paths that are not adjacent to $\hat{C}_j$, put their two vertices $p_i^{2j-1}, p_i^{2j}$ next to each other at the end of the block; the order among these pairs is not important. To see that the resulting ordering $\pi$ has cutwidth $n+\mathcal{O}(1)$, note that from each path $P_i$, the vertices on $P_i$ appear along $\pi$ in their natural order. Hence for any vertex $v \in V(G_1)$, at most one edge from each path $P_i$ crosses the gap after vertex $v$. Additionally, this gap is crossed by at most one clause gadget, whose internal edges contribute $\mathcal{O}(1)$ to the size of the cut. Finally, there is at most one edge from a clause gadget to a path that crosses the cut after $v$: a vertex from $\hat{C}_j$ that has a neighbor on a path $P_i$ is immediately followed by two vertices from $P_i$ that include its neighbor, removing that edge from later cuts. This proves that $\operatorname{ctw}_\pi(G_1) \leq n + \mathcal{O}(1)$.

To see that $\operatorname{ctw}_\pi(G) \leq n + \mathcal{O}(1)$, we note that $G$ is obtained from $n+1$ copies $G_1, G_2, \ldots, G_{n+1}$ of $G_1$ by connecting the last vertex on the $j$-th path in $G_i$, to the first vertex of the $j$-th path in $G_{i+1}$, for all $i \in [n]$ and $j \in [n]$. Hence the number of edges connecting any $G_i$ to vertices in later copies is at most $n$. From this it follows that by simply constructing the order $\pi$ for each graph individually, and concatenating these, we obtain a linear ordering of $G$ of cutwidth $n + \mathcal{O}(1)$. □

# B Lower bound for Dominating Set on graphs of bounded cutwidth

**Theorem 7** *Assuming SETH, there is no $\varepsilon > 0$ such that* Dominating Set *on a (nonplanar) graph $G$ given along with a linear layout of cutwidth $k$ can be solved in time $\mathcal{O}^*((3-\varepsilon)^k)$.*

**Proof:** The proof follows the argumentation of Lokshtanov et al. [23, Thm 4.1] with small modifications along the way. To avoid having to repeat the entire proof, at several steps we only describe how to modify the existing construction.

Suppose that there is an $\varepsilon > 0$ such that Dominating Set on graphs given with a linear layout of cutwidth $k$ can be solved in time $\mathcal{O}^*((3-\varepsilon)^k)$. We will show that this assumption contradicts SETH, by showing that it implies the existence of $\delta > 0$ such that $n$-variable $q$-SAT can be solved in time $\mathcal{O}^*((2-\delta)^n)$ for each fixed $q$.[3] We choose an integer $p$ depending on $\varepsilon$ in a manner that is described at the end of the proof. Consider now an $n$-variable input formula $\phi$ of $q$-SAT for some constant $q$. Let $C_1, \ldots, C_m$ be the clauses of $\phi$. We split the variables of $\phi$ into groups $F_1, \ldots, F_t$, each of size at most $\beta := \lfloor \log 3^p \rfloor = \lfloor p \log 3 \rfloor$ so that $t = \lceil n/\beta \rceil$. (Recall that all logarithms in this paper have base two.)

We now follow the construction of Lokshtanov et al. to produce a graph $G$ that has a dominating set of size $\ell := (p+1)tm(2pt+1)+1$ if and only if $\phi$ is satisfiable ([23, Lemmas 4.1, 4.2]). We then modify the graph $G$ slightly to obtain $G'$ which has a dominating set of size $\ell + 1$ if and only if $G$ has a dominating set of size $\ell$. To describe the modification, we summarize the essential features of the graph $G$ built in the original construction.

The graph $G$ consists of *group gadgets* $\widehat{B}_i^j$ for $i \in [t]$ and $j \in [m(2pt+1)]$, of *clause vertices* $\hat{c}_i^\ell$ for $j \in [m]$ and $0 \leq i < 2pt+1$, and of two special vertices $h$ and $h'$ (see [23, Figure 3]). Each group gadget contains $\mathcal{O}(3^p)$ vertices. It has $p$ *entry vertices* and $p$ *exit vertices*; these $2p$ vertices are all distinct. For each $i \in [t]$ and $j \in [m(2pt+1)-1]$ there is a matching between the exit vertices of $\widehat{B}_i^j$ and the entry vertices of $\widehat{B}_i^{j+1}$. There are no other edges between group gadgets. Each clause vertex $\hat{c}_j^\ell$ is adjacent to at most $q$ different group gadgets, corresponding to the groups that contain the literals in clause $C_j$. The group gadgets to which $\hat{c}_j^\ell$ is adjacent belong to $\{\widehat{B}_i^{m\ell+j} \mid i \in [t]\}$. Finally, vertex $h'$ has degree one and is adjacent to $h$. The other neighbors of $h$ are the entry vertices of $\{\widehat{B}_i^1 \mid i \in [t]\}$, and the exit vertices of $\{\widehat{B}_i^{2pt+1} \mid i \in [t]\}$.

With this summary of $G$, our modification to obtain $G'$ can be easily described. We remove vertices $h$ and $h'$ and replace them by $h_1, h_2, h'_1, h'_2$. Vertices $h'_1$ and $h'_2$ have degree one and are adjacent to $h_1$ and $h_2$, respectively. Vertex $h_1$ is further adjacent to the entry vertices of the group gadgets $\{\widehat{B}_i^1 \mid$

[3] This is a somewhat weaker consequence than used by Lokshtanov et al., who obtain the consequence that CNF-SAT for clauses of *arbitrary* size can be solved by a uniform algorithm in time $\mathcal{O}^*((2-\delta)^n)$ for some $\delta > 0$. By making more significant modifications to the construction we could arrive at the same consequence, but for ease of presentation we will simply show that SETH fails.

$i \in [t]\}$, and vertex $h_2$ is further adjacent to the exit vertices of the group gadgets $\{\widehat{B}_i^{2pt+1} \mid i \in [t]\}$. Essentially, we have split the vertex $h$ into two vertices to reduce the cutwidth of the graph. Note that $\text{OPT}_{\text{DS}}(G') = \text{OPT}_{\text{DS}}(G) + 1$. The presence of the degree-one vertices ensures that there is always a minimum dominating set of $G$ that contains $h$, and of $G'$ that contains both $h_1$ and $h_2$. Such dominating sets may be transformed into one another by exchanging $h$ with $h_1$ and $h_2$, since $h$ dominates the same as $h_1$ and $h_2$ combined. Hence we find that $G'$ has a dominating set of size $\ell + 1$ if and only if $\phi$ is satisfiable.

We proceed to bound the cutwidth of $G'$. For $j \in [2pt+1]$, let the *$j$-th column* of $G'$ consist of the group gadgets $\{\widehat{B}_i^j \mid i \in [t]\}$, together with the unique clause vertex that is adjacent to those group gadgets. The linear layout $\pi'$ of $G'$ starts with vertices $h'_1$ and $h_1$. Then, for each $j \in [2pt + 1]$, it first has the clause vertex of the $j$-th column, followed by the contents of the group gadgets in that column, one gadget at a time. It ends with $h_2$ and finally $h'_2$.

**Claim 12** $\text{ctw}_{\pi'}(G') \leq tp + \mathcal{O}(q \cdot 3^p + (3^p)^2)$.

**Proof:** Consider an arbitrary vertex $v^* \in V(G')$ and the cut consisting of the edges crossing the gap after $v^*$ in layout $\pi'$. The cut after $h'_1$ or $h_2$ has size one. The cut after $h_1$ consists of the $tp$ edges to the entry vertices of the group gadgets in the first column, and the cut after $h'_2$ is empty. It remains to consider the case that $v^*$ belongs to some column $j$.

If $v^*$ is the clause gadget of column $j$, then the cut after $v^*$ consists of the edges from $v^*$ to its neighbors in the group gadgets in that column, together with the $tp$ edges on the $t$ matchings of size $p$ that connect the group gadgets in column $j - 1$ to the group gadgets in column $j$. Since each group gadget has $\mathcal{O}(3^p)$ vertices and a clause vertex is adjacent to at most $q$ different group gadgets, it follows that the size of the cut after $v^*$ is $tp + \mathcal{O}(q \cdot 3^p)$.

If $v^*$ is not the clause gadget of column $j$, then it belongs to some group gadget $\widehat{B}_i^j$ of column $j$. For all other group gadgets, its vertices occur on the same side of $v^*$ in the ordering. Hence the cut after $v^*$ contains edges that are internal to at most one group gadget. Since a group gadget has $\mathcal{O}(3^p)$ vertices, it has $\mathcal{O}((3^p)^2)$ edges that can be contributed to the cut. For all group gadgets in column $j$ that appeared before $\widehat{B}_i^j$ in the ordering, the $p$ matching edges from their exit vertices to the entry vertices of the next column (or to $h_2$) belong to the cut. Similarly, for the group gadgets in column $j$ that appear after $\widehat{B}_i^j$, the $p$ matching edges from their entry vertices to the exits of the previous column belong to the cut. For $\widehat{B}_i^j$ itself, there are at most $2p$ edges connecting to other columns in the cut. The only other edges that can be in the cut are from the group gadgets of column $j$ to its clause gadget, and as argued above there are $\mathcal{O}(q \cdot 3^p)$ of those. It follows that the size of the cut after $v^*$ is at most $tp + \mathcal{O}(q \cdot 3^p + (3^p)^2)$. ⌟

Using this construction of $G'$ and linear layout $\pi'$ we complete the proof. Suppose that Dominating Set on graphs with a given linear layout of cutwidth $k$ can be solved in $\mathcal{O}^*((3 - \varepsilon)^k) = \mathcal{O}^*(3^{\lambda k})$ time, for $\lambda < 1$. We choose $p$ large enough that $\lambda \cdot \frac{p}{\lfloor p \log 3 \rfloor} \leq \frac{\delta}{\log 3}$ for some $\delta < 1$. Choose a function $f(p,q)$ such that the cutwidth in Claim 12 is bounded by $tp + f(p,q)$. Then an instance $\phi$ of $q$-SAT for fixed $q$ can be solved by transforming it into an instance of Dominating Set in time polynomial in $\phi + 3^p$ and applying the assumed algorithm in time

$$\begin{aligned}
\mathcal{O}^*(3^{\lambda(tp+f(p,q))}) &\leq \mathcal{O}^*(3^{\lambda p \lceil n/\beta \rceil}) && \lambda \cdot f(p,q) \in \mathcal{O}(1),\ t = \lceil n/\beta \rceil \\
&= \mathcal{O}^*(3^{\lambda p \lceil \frac{n}{\lfloor p \log 3 \rfloor} \rceil}) && \beta = \lfloor p \log 3 \rfloor \\
&\leq \mathcal{O}^*(3^{\lambda p (\frac{n}{\lfloor p \log 3 \rfloor} + 1)}) && \text{Ceiling adds at most one} \\
&\leq \mathcal{O}^*(3^{\lambda p \frac{n}{\lfloor p \log 3 \rfloor}}) && \lambda p \in \mathcal{O}(1) \\
&\leq \mathcal{O}^*(3^{\frac{\delta n}{\log 3}}) \leq \mathcal{O}^*(2^{\delta n}). && \text{choice of } \delta
\end{aligned}$$

We used the fact that since $p$ and $q$ are constants, their contributions can be absorbed into the $\mathcal{O}^*$ notation. Since this shows that $q$-SAT for any constant $q$ can be solved in time $\mathcal{O}^*(2^{\delta n})) = \mathcal{O}^*((2-\delta')^n)$ for some $\delta' < 1$, this contradicts SETH and concludes the proof of Theorem 7. □