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Weighted Upper Edge Cover: Complexity and Approximability

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Abstract

Optimization problems consist of either maximizing or minimizing an objective function. Instead of looking for a maximum solution (resp. minimum solution), one can find a minimum maximal solution (resp. maximum minimal solution). Such "flipping" of the objective function was done for many classical optimization problems. For example, Minimum Vertex Cover becomes Maximum Minimal Vertex Cover, Maximum Independent Set becomes Minimum Maximal Independent Set and so on. In this paper, we propose to study the weighted version of *Maximum Minimal Edge Cover* called Upper Edge Cover, a problem having application in genomic sequence alignment. It is well-known that Minimum Edge Cover is polynomial-time solvable and the "flipped" version is **NP**-hard, but constant approximable. We show that the weighted Upper Edge Cover is much more difficult than Upper Edge Cover because it is not $O(\frac{1}{n^{1/2-\varepsilon}})$ approximable, nor $O(\frac{1}{\Delta^{1-\varepsilon}})$ in edge-weighted graphs of size n and maximum degree Δ respectively. Indeed, we give some hardness of approximation results for some special restricted graph classes such as bipartite graphs, split graphs and k-trees. We counter-balance these negative results by giving some positive approximation results in specific graph classes.

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1 Introduction

Considering a MaxMin or MinMax version of a problem by "flipping" the objective is not a new idea; in fact, such questions have been posed before for many classical optimisation problems. Some of the most well-known examples include the Minimum Maximal Independent Set problem [10] (also known as Minimum Independent Dominating Set), the Maximum Minimal Vertex Cover problem [9, 38], the Lazy Bureaucrat problem [23, 25] (which is a MinMax version of Subset Sum), the Minimum Maximal Matching problem (also known as Minimum Independent Edge Dominating Set) [37], and the Maximum Minimal Dominating Set problem (also called Upper Dominating Set) [1, 6]. However, to the best of our knowledge, weighted MaxMin and MinMax versions have not been considered so far, except for Minimum Independent Dominating Set [14, 31], and weighted upper dominating set problem [11]. MaxMin or MinMax versions of classical problems turn out to be much harder than the originals, especially when one considers complexity and approximation. For example, Maximum Minimal Vertex Cover does not admit any $n^{\frac{1}{2}-\epsilon}$ approximation [9], while Vertex Cover admits a simple 2-approximation. Minimum Maximal Matching is **NP**-hard (but 2-approximable) while Maximum Matching is polynomial.

The focus of this paper is on *edge cover*. An *edge cover* of a graph $G = (V, E)$ without isolated vertices is a subset of edges $S \subseteq E$ which covers all vertices of G. The *edge cover number* of $G = (V, E)$, is the minimum size of an *edge cover* of G. An optimal edge cover can be computed in polynomial time, even for the weighted version where a weight is given for each edge and one wants to minimize the sum of the weight of the edges in the solution (called here the *weighted edge cover number*). An edge cover $S \subseteq E$ is *minimal* (with respect to inclusion) if the deletion of any subset of edges from S destroys the covering property. Minimal edge cover is also known in the literature as an *enclaveless* set [35] or as a *nonblocker* set [20].

In this paper, we study the computational complexity of the *weighted upper edge cover number*, denoted here $\mathrm{uec}(G, w)$, that is the solution with maximum weight among all minimal edge covers. Formally, the associated optimization problem called the Weighted Upper Edge Cover problem asks to find the largest weighted minimal edge cover of an edge-weighted graph.

Weighted Upper Edge Cover
Input: A weighted connected graph $G = (V, E, w)$, where $w(e) \geq 0$ for all $e \in E$.
Solution: Minimal edge cover $S \subseteq E$.
Objective: Maximizing $w(S) = \sum_{e \in S} w(e)$.

Hence, if S^* is an optimal solution of Weighted Upper Edge Cover on (G, w), then $w(S^*) = \mathrm{uec}(G, w)$. The unweighted value of the optimal solution is *uec*(G) (denoted *upper edge cover number*). To the best of our knowledge, the complexity of computing the weighted upper edge cover number has never been studied in the literature, while a lot of results appear for the unweighted case (corresponding to $w(e) = 1$ for all $e \in E$) [33, 4, 16, 27]. The unweighted variant was firstly investigated in [32], where it is proven that the complexity of computing the upper edge cover number is equivalent to solve the dominating set problem because $\text{uec}(G) = |V| - \gamma(G)$ where $\gamma(G)$ is the size of minimum dominating set of graph G. We will consider the implications of this important remark afterwards in the paper.

We will now define a related problem useful in the following because it is proved in [32] that $S \subseteq E$ is a minimal edge cover of $G = (V, E)$ iff S is a spanning star forest of G *without trivial stars* (i.e. without stars consisting of a single vertex). A spanning star forest is a spanning forest of $G = (V, E)$ into stars where a ℓ-star S is a subset of edges such that the partial subgraph induced by S is isomorphic to $K_{1,\ell}$; The center of the star is the vertex different to the leaves (if $\ell \neq 1$) and a 0-star is also called a trivial star.

Maximum Weighted Spanning Star Forest problem (MaxWSSF in short)
Input: An edge-weighted graph (G, w) on n vertices where $G = (V, E)$ and $w(e) \geq 0$ for all $e \in E$.
Solution: Spanning star forest $\mathcal{S} = \{S_1, \ldots, S_p\} \subseteq 2^E$.
Objective: maximizing $w(\mathcal{S}) = \sum_{e \in \mathcal{S}} w(e) = \sum_{i=1}^{p} \sum_{e \in S_i} w(e)$.

Given an instance (G, w) of MaxWSSF, $opt_{MaxWSSF}(G, w)$ denotes the value of an *optimal spanning star forest*. Authors of [33] describe in details how to apply MaxWSSF model to alignment of multiple genomic sequence, a critical task in comparative genomics. They also show that this approach is promising with real data. In this model, taking weights into account is fundamental since it represents alignment score. Also, their model uses each edge of the spanning star forest to output the solution. Therefore, having trivial star is probably undesirable, which enforces the motivation of studying Weighted Upper Edge Cover.

The unweighted version (corresponding to the case $w(e) = 1$ for all edges e) is denoted by MaxSSF. In this case, the optimal value is $opt_{MaxSSF}(G)$. For unweighted graphs without isolated vertices, we have $\text{uec}(G) = opt_{MaxSSF}(G)$ since any spanning star forest (with possible trivial stars) can be (polynomially) converted into a spanning star forest without trivial stars (i.e. a minimal edge cover) with same size [32]. Hence, these two problems are completely equivalent even from an approximation point of view.

Concerning edge-weighted graphs, the relationship between Weighted Upper Edge Cover and MaxWSSF is less obvious. For instance, we only have: $opt_{MaxWSSF}(G, w) \geq \text{uec}(G, w)$ because any minimal edge cover is a particular spanning star forest. However, the difference between these two values can be arbitrarily large as indicated in Figure 1 (in the graph drawn in Figure 1.(b), v_4 is an isolated vertex when ε goes to infinity). This means that isolated vertices play an important role in feasible solutions. Given a spanning star forest $\mathcal{S} = \{S_1, \ldots, S_r\}$ of (G, w), we rename vertices such that there is some $p, 0 \leq p < r$ such that $S_i = \{v_i\}$ are trivial stars for all $1 \leq i \leq p$ (if $p = 0$, then there is no trivial stars), and S_j are non-trivial stars whose c_j is

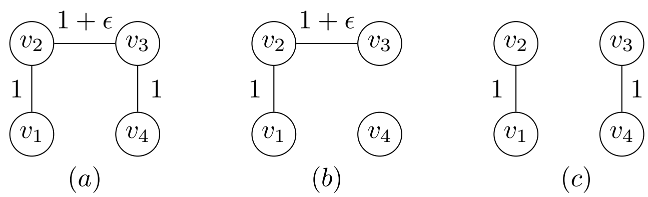

Figure 1: (a) : The weighted graph $G = (V, E, w)$. (b) : Optimal solution of MaxWSSF(G, w). (c) : Optimal solution of Weighted Upper Edge Cover for G with value $\mathrm{uec}(G, w) = 2$.

the center for all $j > p$ (if S_j is a single edge, both endpoints are considered as possible centers). We define $\mathrm{Triv} = \{v_i : i \leq p\}$ as the set of isolated vertices of $(V, E(\mathcal{S}))$ where $E(\mathcal{S}) = \cup_{j>p}^{r} S_j$; moreover, V_l and V_c are respectively the set of leaves and the set of centers of stars in $V \setminus \mathrm{Triv}$. Finally, for $v \in V_l$, $e_v(\mathcal{S}) = c'v \in E(\mathcal{S})$ denotes the edge linking the center c' to the leaf v.

We mainly focus on specific solutions of MaxWSSF called *nice spanning star forests* defined as follows:

Definition 1 *$\mathcal{S}$ is a nice spanning star forest of (G, w) if $\mathrm{Triv} = \{v_i : i \leq p\}$ is an independent set in G and all edges of G starting at Triv are linked to leaves of some ℓ-stars of $\mathcal{S}$ with $\ell \geq 2$. Moreover, $w(uv) \leq w(e_v(\mathcal{S}))$ for $u \in \mathrm{Triv}$, $v \in V_l$.*

Property 2 *Any spanning star forest of (G, w) can be polynomially converted into a nice one with at least the same weight.*

Proof: The weights of (G, w) are non-negative. Thus, if Triv is not an independent set or if some vertex of Triv is linked to some center of $\mathcal{S}$, we could obtain a better spanning star forest with less isolated vertices. In particular, it implies that no vertex of Triv is linked to a 1-star (i.e. a K_2 of $\mathcal{S}$). Finally, if $w(uv) > w(e_v(\mathcal{S}))$, then $\mathcal{S}' = (\mathcal{S} \setminus \{e_v(\mathcal{S})\}) \cup \{uv\}$ is a better spanning star forest. □

It is well known that optimization problems are easier to approximate when the input is a complete weighted graphs satisfying the *triangle inequality*, like for example in the traveling salesman problem. Here, we introduce a generalization of this notion which works to any class of graphs.

Definition 3 *An edge weighted graph (G, w) where $G = (V, E)$ satisfies the* cycle inequality*, if for every cycle C, we have:*

$$\forall e \in C, \quad 2w(e) \leq w(C) = \sum_{e' \in C} w(e')$$

Clearly, for complete graphs, cycle and triangle inequality notions coincide. Definition 3 is interesting when focusing on classes of graphs like split graphs or k-trees. In this article, we are also interested in *bivaluate weights* (resp., *trivalued*) corresponding to the case $w(e) \in \{a, b\}$ with $0 \leq a < b$ (resp., $w(e) \in \{a, b, c\}$ where $0 \leq a < b < c$ are 3 reals). The particular case $a = 0$ and $b = 1$ (called here *binary weights*) is interesting by itself because MaxWSSF with binary weights exactly corresponds to MaxSSF and has been extensively studied in the literature [33, 15, 5]. Moreover for instance, binary weighted Minimum Independent Dominating Set for chordal graphs has been studied in [21], where it is shown that this restriction is polynomial, but bivalued weighted Minimum Independent Dominating Set for chordal graphs with $a > 0$ is **NP**-hard [14].

1.1 Graph terminology and definitions

Throughout this paper, we consider edge-weighted undirected connected graphs $G = (V, E)$ on $n = |V|$ vertices and $m = |E|$ edges. Each edge $e = uv \in E$ between vertices u and v is weighted by a non-negative weight $w(e) \geq 0$. The *degree* $d_G(v)$ of vertex $v \in V$ in G is the number of edges incident to v and $\Delta(G)$ is the *maximum degree* of the graph G.

K_n denotes the *complete graph* on n vertices. A *bipartite graph* (resp., *split graph*) $G = (L \cup R, E)$ is a graph where the vertex set $L \cup R$ can be decomposed into an independent set (resp., a clique) L and an independent set R. A *k-tree* is a graph which can be formed by starting from a k-clique and then repeatedly adding vertices in such a way that each added vertex has exactly k neighbors completely connected together (this neighborhood is a k-clique). For instance, 1-trees are trees and 2-trees are maximal series-parallel graphs. A graph is a *partial k-tree* (or equivalently with *treewidth* at most k) if it is a subgraph of a k-tree.

A *star* $S \subseteq E$ of a graph $G = (V, E)$ is a tree of G where at most one vertex has a degree greater than 1, or, equivalently, it is isomorphic to $K_{1,\ell}$ for some $\ell \geq 0$. The vertices of degree 1 (except the center when $\ell \leq 1$) are called *leaves* of the star while the remaining vertex is called *center* of the star. A ℓ-star is a star of ℓ leaves. If $\ell = 0$, the star is called *trivial* and it is reduced to a single vertex (the center); otherwise, the star is said *non-trivial*.

A *spanning star forest* $\mathcal{S} = \{S_1, \ldots, S_p\} \subseteq 2^E$ of G is a spanning forest into stars, that is, each S_i is a star (possibly trivial), $V(S_i) \cap V(S_j) = \emptyset$ and $\cup_{i=1}^{p} V(S_i) = V$. An *independent set* $S \subseteq V$ of a graph $G = (V, E)$ is a subset of vertices pairwise non-adjacent. The **NP**-hard problem MaxIS seeks an independent set of maximum size. The value of an optimal independent set of G is denoted $\alpha(G)$. A *matching* $M \subseteq E$ is a subset of pairwise non-adjacent edges. A matching M of G is *perfect* if all vertices of G are covered by M. A *dominating set* for a graph G is a subset D of V such that every vertex not in D is adjacent to at least one vertex of D. The *domination number* $\gamma(G)$ is the number of vertices in the smallest dominating set of G.

1.2 Related work

Upper Edge Cover has been investigated intensively during the recent years for unweighted graphs, mainly using the terminologies of *spanning star forests* or *dominating sets*. The *minimum dominating set problem* (denoted MinDS) seeks the smallest dominating set of G of value $\gamma(G)$. As indicated before, we have $\text{uec}(G) = n - \gamma(G)$. Thus, using the complexity results known on MinDS, we deduce that Upper Edge Cover is **NP**-hard in planar graphs of maximum degree 3 [24], chordal graphs [8] (even in *undirected path graphs*, the class of vertex intersection graphs of a collection of paths in a tree), bipartite graphs, split graphs [7] and k-trees with arbitrary k [18], and it is *polynomial* in k-trees with fixed k, convex bipartite graphs [19], strongly chordal graphs [22].

Concerning the approximability, an **APX**-hardness proof with explicit inapproximability bound and a combinatorial 0.6-approximation algorithm is proposed in [33]. Better algorithms with approximation ratio 0.71 and 0.803 are given respectively in [16] and [4]. For any $\varepsilon > 0$, Upper Edge Cover is hard to approximate within a factor of $\frac{259}{260} + \varepsilon$ unless **P**=**NP** [33]. It admits a **PTAS** in k-trees (with arbitrary fixed k). A graph is called *c-dense* if it contains at least $c\frac{n^2}{2}$ edges and it is called *everywhere-c-dense* if the minimum degree is cn; when $c \in (0;1)$ is a constant, we say *dense* and *everywhere-dense* graphs [3]. In [27], it is proved that Upper Edge Cover remains **APX**-complete in c-dense graphs; they also proposed an approximation algorithm with ratio $0.804 + 0.196\sqrt{c}$ for c-dense graphs, while they proved that the problem does not admit a **PTAS** assuming $\mathbf{P} \neq \mathbf{NP}$ for c-dense graphs. He and Liang in [26] claimed that Upper Edge Cover on everywhere-c-dense graphs are easier than c-dense graph; they proved this claim by showing that that for a given graph of order n and minimum degree $\delta(n)$ for every vertex, Upper Edge Cover is **APX**-complete when $1 \leq \delta(n) \leq O(1)$, is **NP**-hard but allows a **PTAS** when $\omega(1) \leq \delta(n) \leq O(n^{1-\epsilon})$ for some constant ϵ, and is not **NP**-hard assuming **ETH** when $\delta(n) \geq \omega(n^{1-\epsilon})$ for every constant $\epsilon > 0$.

In contrast, to the best of our knowledge, for edge weighted graphs with non-negative weights, no result for Weighted Upper Edge Cover is known, although some results are given for Maximum Weighted Spanning Star forest problem: a 0.5-approximation is given in [33] (which is the best ratio obtained so far) and polynomial-time algorithms for special classes of graphs such as trees and cactus graphs are presented in [33, 34]. Negative approximation results are presented in [33, 12, 16]. In particular, MaxWSSF is **NP**-hard to approximate within $\frac{10}{11} + \varepsilon$ [12]. Two generalizations of WSSF, denoted MinExtWSSF and MaxExtWSSF, have been introduced very recently in [29] where the goal consists in *extending* some partial stars into spanning star forests. In this context, a partial feasible solution is given in advance and the goal is to extend this partial solution. Formally, the problem is defined as follow:

Extended weighted spanning star forest problem (ExtWSSF in short)
Input: A weighted graph (G, w) and a packing of stars $\mathcal{U} = \{U_1, \ldots, U_r\}$ where $G = (V, E)$ and $w(e) \geq 0$ for $e \in E$.
Solution: Spanning star forest $\mathcal{S} = \{S_1, \ldots, S_p\} \subseteq E$ containing $\mathcal{U}$.
Output: $w(\mathcal{S}) = \sum_{e \in \mathcal{S}} w(e) = \sum_{i=1}^{p} \sum_{e \in S_i} w(e)$.

In [29], several results have been given for both *minimization*

(MinExtWSSF) and *maximization* (MaxExtWSSF) versions of ExtWSSF (denoted MinExtWSSF and MaxExtWSSF respectively). Dealing with the minimization version for complete graphs: a dichotomy result of the computational complexity is presented depending on parameter c of the (extended) c-relaxed triangle inequality and an FPT algorithm is given. For the maximization version, a positive approximation of 1/2 and a negative approximation result of $\frac{7}{8}$ (even for binary weights) are proposed.

A subset of vertices V' is called *non-blocking* if every vertex in V' has at least one neighbor in $V \setminus V'$. Actually, *non-blocking* is dual of dominating set and vice versa. For a given graph $G = (V, E)$ and a positive integer k, the Non-blocker problem asks if there is a *non-blocking* set $V' \subseteq V$ with $|V'| \geq k$. Hence, for unweighted graphs, optimal value of *non-blocking* number equals the upper edge cover number. In [20] Dehne et al. propose a parameterized perspective of the Non-blocker problem. They give a linear kernel and an **FPT** algorithm running in time $\mathcal{O}^*(2.5154^k)$. They also give faster algorithms for planar and bipartite graphs.

1.3 Contributions

The paper is organized in the following way. We first show in Section 2 that Weighted Upper Edge Cover in complete graphs is equivalent for its approximation to MaxWSSF in general graphs. Then, we study the approximation of Weighted Upper Edge Cover for bipartite graphs, split graphs and k-trees respectively in Sections 3, 4 and 5.

Motivated by the above results mostly negative, we propose a constant approximation ratio algorithm in Section 6 for Weighted Upper Edge Cover in bounded degree graphs.

Note that all results given in this paper are valid if G is isolated vertex free instead of connected.

2 Approximation of Weighted Upper Edge Cover in complete graphs

In this section, we deal with edge-weighted complete graphs. This case seems to be the simplest one because the equivalence between Upper Edge Cover and MaxSSF for the unweighted case proven in [32] remains valid for the weighted case, as proven in the following.

Theorem 4 MaxWSSF *in general graphs is equivalent to approximate* Weighted Upper Edge Cover *in complete graphs.*

Proof: We propose two approximation preserving reductions, one from MaxWSSF in general graphs to Weighted Upper Edge Cover in complete graphs and the other from Weighted Upper Edge Cover to MaxWSSF in complete graphs.

• Reduction from MaxWSSF to Weighted Upper Edge Cover in complete graphs.

Let (G, w) be an instance of MaxWSSF where $G = (V, E)$ is a connected graph with n vertices, edge-weighted using w. We build an instance (K_n, w') of Weighted Upper Edge Cover where K_n is an edge-weighted complete graph $(V, E(K_n))$ over n vertices, edge-weighted with w', such that $\forall u, v \in V$ with $u \neq v$, $w'(uv) = w(uv)$ if $uv \in E$ and $w'(uv) = 0$ otherwise.

Let $S' \subseteq E(K_n)$ be a minimal edge cover of Weighted Upper Edge Cover with weight $w'(S')$. The restriction of S' to G gives a spanning star forest (possibly with trivial stars) $\mathcal{S}$. By construction we have:

$$w(\mathcal{S}) = w'(S') \tag{1}$$

Thus, from equality (1) we deduce $opt_{MaxWSSF}(G, w) \geq \mathrm{uec}(K_n, w')$.

Conversely, let $\mathcal{S}^*$ be an optimal solution of MaxWSSF with value $opt_{MaxWSSF}(G, w)$. By adding some edges from the center of some stars to the isolated vertices of $\mathcal{S}^*$, we obtain a minimal edge cover of K_n of at least same value. Hence, $\mathrm{uec}(K_n, w') \geq opt_{MaxWSSF}(G, w)$. We can deduce,

$$\mathrm{uec}(K_n, w') = opt_{MaxWSSF}(G, w) \tag{2}$$

From equalities (1) and (2), we deduce that any ρ approximation of Weighted Upper Edge Cover for (K_n, w') can be polynomially converted into a ρ approximation of MaxWSSF for (G, w).

• Reduction from Weighted Upper Edge Cover to MaxWSSF in complete graphs.

From an edge-weighted complete graph (K_n, w) instance of Weighted Upper Edge Cover, we set $(G, w') = (K_n, w)$ as an instance of MaxWSSF. Since the graph is complete, the weights are non-negative and the goal is maximization, we can only consider spanning star forests without trivial stars, i.e. minimal edge covers. Hence, Weighted Upper Edge Cover is as a subproblem of MaxWSSF, even from an approximation point of view. □

From Theorem 4 and from known results on MaxWSSF given in [33, 12], we deduce the following:

Corollary 5 *In complete graphs,* Weighted Upper Edge Cover *is* 1/2-*approximable but not approximable within* $\frac{10}{11} + \varepsilon$ *unless* ***P*****=*****NP***.

3 Approximation of Weighted Upper Edge Cover in bipartite graphs

Let us now focus on bipartite graphs. We prove that, even in bipartite graphs with binary weights, Weighted Upper Edge Cover is not $O(n^{\frac{1}{2}-\varepsilon})$ approximable unless $\mathbf{P} = \mathbf{NP}$. Also, we show the problem is **APX**-complete even for bipartite graphs with fixed maximum degree Δ.

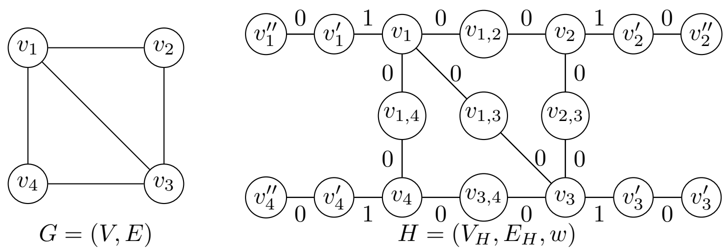

Figure 2: Construction of H from G. The weights are indicated on edges.

Theorem 6 Weighted Upper Edge Cover *in bipartite graphs with binary weights and cycle inequality is as hard as* MaxIS *in general graphs.*

Proof: We propose an approximation preserving **APX**-reduction[1] from Independent Set (denoted by MaxIS) to Weighted Upper Edge Cover.

Given a connected graph $G = (V, E)$ with n vertices and m edges where $V = \{v_1, \ldots, v_n\}$, as an instance of MaxIS, we build a connected bipartite edge-weighted graph $H = (V_H, E_H, w)$ as follows (see also Figure 2):

- For each $v_i \in V$, add a P_3 with edge set $\{v_iv_i', v_i'v_i''\}$.
- For each edge $e = v_iv_j \in E$ where $i < j$, add a middle vertex $v_{i,j}$ on edge e.
- $w(e) := \begin{cases} 1 & \text{if } e = v_iv_i' \text{ for some } v_i \in V \\ 0 & \text{otherwise.} \end{cases}$

Clearly, H is a connected bipartite graph with $|V_H| = 3n + m$ vertices and $|E_H| = 2(m + n)$ edges. Moreover, weights are binary and the instance satisfies cycle inequality.

Let S^* be a maximum independent set of G with size $\alpha(G)$. For each $e \in E$, let $v^e \in V \setminus S^*$ be a vertex which covers e; it is possible since $V \setminus S^*$ is a *vertex cover* of G. Moreover, $\{v^e : e \in E\} = V \setminus S^*$ since S^* is a maximum independent set of G. Clearly, $S' = \{v_{x,y}v^e : e = xy \in E\} \cup \{v_i'v''i : v_i \in V\} \cup \{v_iv_i' : v_i \in S^*\}$ covers all vertices of H and since it does not include any P_3, then S' is a minimal edge cover of H. By construction, $w(S') = |S^*| = \alpha(G)$. Hence, we deduce:

$$\text{uec}(H, w) \geq \alpha(G) \tag{3}$$

Conversely, suppose S' is a minimal edge cover of H with weight $w(S')$. Let us make some simple observations of every minimal edge cover of H. Clearly, $\{v_i'v_i'' : v_i \in V\}$ is part of every feasible solution because v_i'' for $v_i \in V$ are leaves of H. Moreover, for each $e = v_iv_j \in E$ with $i < j$, at least one edge between $v_iv_{i,j}$ or $v_jv_{i,j}$ belongs to any minimal edge cover of H. If $v_iv_{i,j} \notin S'$, it implies

[1]The reduction is actually a Strict-reduction and it is a particular A-reduction which preserves constant approximation.

that $v_j v_{i,j} \in S'$ and $v_j v'_j \notin S'$ is not a part of the feasible solution because of minimality of S'. Hence, $S = \{v_i : v_i v'_i \in S'\}$ is an independent set of G with size $|S| = w(S')$. We deduce:

$$\alpha(G) \geq \text{uec}(H, w) \tag{4}$$

Using inequalities (3) and (4) we deduce:

$$\alpha(G) = \text{uec}(H, w) \tag{5}$$

In conclusion, for each minimal edge cover S' on H, there is an independent set S of G (computed in polynomial-time) such that $|S| \geq w(S')$. □

From Theorem 6, we immediately deduce that WEIGHTED UPPER EDGE COVER in bipartite graphs is not in **APX** unless **P**=**NP**. However, using the results concerning the **APX**-completeness of MAXIS in connected graph G with constant maximum degree $\Delta(G) \geq 3$ or **NP**-hardness of MAXIS in planar graphs [24, 2], we obtain:

Corollary 7 WEIGHTED UPPER EDGE COVER *in bipartite (resp., planar bipartite) graphs of maximum degree* Δ *for any fixed* $\Delta \geq 4$ *and binary weights is* ***APX****-complete (resp.* ***NP****-complete).*

Proof: Let us revisit the construction given in Theorem 6. If the instance of MAXIS has maximum degree 3 (resp. is planar with maximum degree 3), then the constructed instance of WEIGHTED UPPER EDGE COVER is a bipartite (resp., planar bipartite) graph of maximum degree 4. □

Using the strong inapproximability result for MAXIS given in [39], and because the reduction given in the previous theorem is a gap-reduction, we also deduce:

Corollary 8 *For any constant* $\varepsilon > 0$*, and any* $\rho \in \Omega(n^{\varepsilon - \frac{1}{2}})$*,* WEIGHTED UPPER EDGE COVER *does not admit a polynomial* ρ*-approximation algorithm in bipartite graphs of* n *vertices unless* ***P****=****NP****, even for binary weights and cycle inequality.*

Proof: We use the reduction given in Theorem 6 and the inapproximability of MAXIS. MAXIS is known to be hard to approximate [39]. In particular, it is known that, for all $\varepsilon > 0$, it is **NP**-hard to distinguish for an n-vertex graph G between $\alpha(G) > n^{1-\varepsilon}$ and $\alpha(G) < n^{\varepsilon}$.

In the construction of H (see Figure 2), we know that $|V_H| = m + 3n$ and $|E_H| = 2(m + n)$ where m, n are numbers of the edges and vertices of G respectively. Hence, $|V_H| \leq 2n^2$, and the claimed result follows. □

We also deduce one inapproximability result depending on the maximum degree.

Corollary 9 *For any constant* $\varepsilon > 0$*, unless* ***NP****⊆****ZPTIME***$(n^{\text{poly} \log n})$*, it is hard to approximate* WEIGHTED UPPER EDGE COVER *on bipartite graphs of maximum degree* Δ *within a factor of* $\Theta\left(\frac{1}{\Delta^{1-\varepsilon}}\right)$*.*

Proof: We will prove that it is difficult for a graph H (even bipartite with binary weights) of maximum degree Δ to distinguish between the following two cases:

- (Yes-Instance)uec$(H, w) \geq \frac{|V(H)|}{\Delta(G)^{1+\varepsilon}}$,

- (No-Instance) uec$(H, w) \leq \frac{|V(H)|}{\Delta(G)^{2-\varepsilon}}$.

Hence, the result consists of showing that the transformation given in Theorem 6 is a gap reduction. It is proved that: Let $\tau(n)$ be any function from integers to integers. Assuming that $\mathbf{NP}\not\subseteq\mathbf{ZPTIME}(n^{O(\tau(n))})$, there is no polynomial-time algorithm that can solve the following problem [13] (Theorem 5.7, adapted from [36]). For any constant $\varepsilon > 0$ and any integer q, given a regular graph G of size $q^{O(\tau(n))}$ such that all vertices have degree $\Delta = 2^{O(\tau(n))}$, the goal is to distinguish between the following two cases:

- (Yes-Instance) $\alpha(G) \geq \frac{|V(G)|}{\Delta^{\varepsilon}}$,

- (No-Instance) $\alpha(G) \leq \frac{|V(G)|}{\Delta^{1-\varepsilon}}$.

Note that if G is a Δ-regular graph, then graph H resulting of Theorem 6 is a bipartite graph of maximum degree $\Delta + 1 = \Theta(\Delta)$. Thus, since $\alpha(G) =$ uec(H, w) and $|V(H)| = 3|V(G)| + |E(G)| = \Theta(\Delta|V(G)|)$, we get the expected result. □

4 Approximation of Weighted Upper Edge Cover in split graphs

We will now focus on split graphs. Recall that a graph $G = (L \cup R, E)$ is a split graph if the subgraph induced by L and R is a maximum clique and an independent set respectively. It is called *Δ-subregular split graph* if for $v \in L$, $d_G(v) \leq \Delta + |L| - 1$ and for $v \in R$, $d_G(v) \leq \Delta$. This means that the graph induced by crossing edges is of maximum degree at most Δ.

Theorem 10 Weighted Upper Edge Cover *in split graphs with binary weights and cycle inequality is as hard as* MaxIS *in general graphs.*

Proof: The proof is based on a reduction[2] from MaxIS. Given a graph $G = (V, E)$ of n vertices and m edges where $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$, as an instance of MaxIS, we build a split weighted graph $H = (V_H, E_H, w)$ as follows:

- Put two copies of vertices V in H, denoted by $C = \{c_1, \ldots, c_n\}$ and $C' = \{c'_1, \ldots, c'_n\}$ and make them cliques of size n such that all pairs of vertices in C and C' are connected to each other with edges of weight 0.

[2]The reduction is actually a Strict-reduction and it is a particular A-reduction which preserves constant approximation.

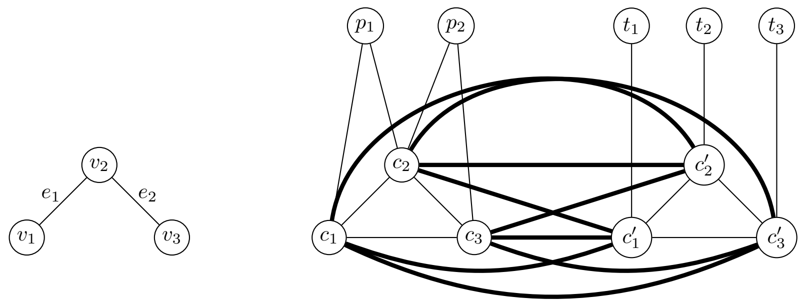

Figure 3: Construction of split graph $H = (V_H, E_H)$ from a P_3. The weights of thick edges in H are 1 and for the others are 0.

- Connect all pairs $c_i c'_j$ for $1 \leq i, j \leq n$ with edges of weight 1 to make a clique of size $2n$.
- Add a set of m new vertices $\{p_1, \ldots, p_m\}$ corresponding to edges of E and connect p_i to c_j, c_k with edges of weight 0 if $e_i = v_j v_k \in E$.
- Add a set of n new vertices $\{t_1, \ldots, t_n\}$ and connect each t_i to c'_i with edges of weight 0.

The built graph H is a weighted split graph with binary weights and cycle inequality, and it contains a clique of size $2n$ and an independent set of size $n + m$. Figure 3 gives an illustration of the construction of H from a P_3.

We claim that G has an independent set of size k iff there exists a minimal edge cover of H with total weight k.

Let S be an independent set of G with size $|S|$. For each $e_i \in E$, there is a vertex $v_{e_i} \notin S$ which covers e_i since S is an independent set of G. Consider the set $\{c_{e_i} : v_{e_i} \notin S\}$ of vertices in C corresponding to vertices of $V \setminus S$, $S' = \{c_{e_i} p_i : e_i \in E\} \cup \{c'_i t_i : v_i \in V\} \cup \{c_i c'_i : v_i \in S\}$ is a minimal edge cover of H. By construction, $w(S') = |S|$. Hence, we deduce:

$$\text{uec}(H, w) \geq \alpha(G) \tag{6}$$

Conversely, let S' be a minimal edge cover of H with weight $w(S')$. Since for $1 \leq i \leq n$, t_i's are leaves in H, $\{t_i c'_i : v_i \in V\}$ is a part of S'. Moreover, for each $e_k = v_i v_j \in E$ with $i < j$, at least one edge among $c_i p_k$ or $c_j p_k$ belongs to S'. W.l.o.g., assume that $c_i p_k \in S'$; this means that $c_i c'_j \notin S'$ for all $1 \leq j \leq n$. Furthermore, for each $c_i \in C$ at most one edge $c_i c'_j \in S'$ for $1 \leq j \leq n$. Hence, $S = \{v_i : c_i c'_j \in S'\}$ is an independent set of G with size $|S| = w(S')$. We deduce,

$$\alpha(G) \geq \text{uec}(H, w) \tag{7}$$

Using inequalities (6) and (7) we deduce $\alpha(G) = \text{uec}(H, w)$. □

Corollary 11 Weighted Upper Edge Cover *in split 3-subregular graphs is* ***APX****-complete and for any constant* $\varepsilon > 0$, *and any* $\rho \in \Omega(n^{\varepsilon - \frac{1}{2}})$, Weighted Upper Edge Cover *does not admit a polynomial* ρ*-approximation algorithm in split graphs of* n *vertices unless* ***P****=****NP****.*

5 Approximation of Weighted Upper Edge Cover in k-trees

Recall that a k-tree is a graph which results from the following inductive definition:

- A K_{k+1} is a k-tree,
- If a graph G is a k-tree, then the addition of a new vertex which has exactly k neighbors in G such that these $k+1$ vertices induce a K_{k+1} forms a k-tree.

As a main result in this section, we prove Weighted Upper Edge Cover is **APX**-complete in weighted-dense k-trees even for binary weights.

5.1 Hardness of approximation

From Corollary 5, we already know that Weighted Upper Edge Cover is **NP**-hard to approximate within a ratio strictly better than $\frac{10}{11}$ because the class of all k-trees contains the class of complete graphs. However, this lower bound needs a non-constant number of distinct values [12]. Moreover, in Theorem 4, we showed that Weighted Upper Edge Cover in weighted complete graphs is equivalent to MaxWSSF in general graphs. In [33, Theorem 3.6], it is proved that MaxSSF is hard to approximate in general graphs within ratio $\frac{259}{260} + \varepsilon$ for any $\varepsilon > 0$, so Weighted Upper Edge Cover in complete graphs and k-trees with binary weights is not strictly approximable within ratio better than $\frac{259}{260} \leq 0.9962$. Here, we propose a new approximation preserving reduction for Weighted Upper Edge Cover in k-trees. Our reduction does not improve the existed bound $\frac{259}{260}$, but help us to find some new upper bounds for weighted Upper edge Cover in weighted-dense k-trees and Upper Edge Cover in dense graphs.

Recall that a graph $G = (V, E)$ with $|V| = n$ is called *c-dense* if $|E| \geq \frac{cn^2}{2}$ [3, 27]. This concept can be adapted to edge-weighted maximization problems as follows. For a non-negative edge-weighted graph (G, w), we assume $w(xy) = 0$ for a non-edge $xy \notin E$ and for all the p distinct weights $w_i > 0$, $i = 1, \ldots, p$ of the instance, the denote by $\bar{w}$ the *average weight* such that $\bar{w} = \frac{\sum_{i=1}^{p} w_i}{p}$.

Definition 12 *An edge weighted graph* $G = (V, E, w)$ *with* $w \geq 0$ *is* c-weighted-dense *if*

$$\sum_{xy \in E} w(xy) \geq cn^2 \times \bar{w}$$

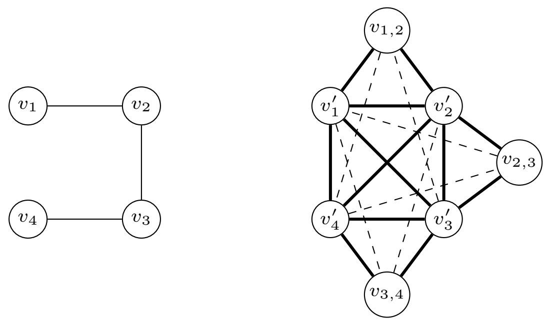

Figure 4: On the right side, the constructed weighted graph $G' = (V', E', w)$ build from a P_4. The edges drawn bold have weight 1 and the weight of all dotted edges is 0.

This notion captures the fact that a huge number of edges with average weight are available in the graph. In particular, using Definition 12, we deduce that a c-dense graph is also weighted c-dense by taking $w(xy) = 1$ if $xy \in E$.

Theorem 13 Weighted Upper Edge Cover *is* ***APX****-hard in the class of weighted c-dense k-trees, even for binary weights and* $c = \frac{4}{25}$.

Proof: We give an approximation preserving reduction from independent set problem. It is known that MaxIS is **APX**-complete in graphs of maximum degree Δ with $\Delta \geq 3$ [2].

Let $G = (V, E)$ be an instance of MaxIS where G is a connected graph of maximum degree Δ and $V = \{v_1, \ldots, v_n\}$ and $|E| = m$. We build a weighted dense k-tree $G' = (V', E', w)$ for Weighted Upper Edge Cover such that $V' = V'_C \cup V'_E$ as follows:

- $V'_C = \{v'_i : v_i \in V\}$ and $V'_E = \{v_{i,j} : e = v_i v_j \in E, 1 \leq i < j \leq n\}$.
- $G'[V'_C]$, the subgraph induced by V'_c is a K_n.
- For each $v_{i,j} \in V'_E$, add n edges $v_k v_{i,j}$ for $1 \leq k \leq n$ to E'.

The weight function w for $xy \in E'$ is defined as follows:

$$w(xy) = \begin{cases} 1 & (x, y \in V'_C) \text{ or } (x = v_{i,j} \text{ and } y \in \{v'_i, v'_j\}), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $|V'| = n + m$ and clearly G' can be constructed from G in polynomial time. G' is a n-tree because initially all $V'_C \cup \{v_{i,j} \in V'_E\}$ is a clique of size $n+1$, and any other vertices in V'_E is connected to all vertices in V'_C which makes a K_{n+1}. Figure 4 proposes an illustration of this construction for a P_4.

We are going to prove that any ρ-approximation for Weighted Upper Edge Cover in k-trees can be polynomially converted into a $(1 + \frac{\Delta^2}{2})\rho - \frac{\Delta^2}{2}$ approximation ratio for MaxIS in graphs of maximum degree Δ.

First, consider an arbitrary independent set S of G. From S, we make a minimal edge cover F of G' of size at least $|S|+m$. For each $e=v_iv_j \in E$, there is a vertex $f(e) \in ((V \setminus S) \cap \{v_i, v_j\})$ because S is an independent set. Let $X=\{f(e) : e \in E\}$, then choose arbitrarily a vertex $r \in X$. We set $F=\{f(e)'v_{i,j} : e=v_iv_j \in E\} \cup \{r'v' : v \in (V \setminus X)\}$. We deduce $\text{uec}(G',w') \geq w(F)=m+|V \setminus X| \geq m+|S|$ and considering S as a maximum independent set induces:

$$\text{uec}(G',w') \geq m+\alpha(G) \tag{8}$$

Conversely, assume that F is a minimal edge cover of G'. We will polynomially modify F into another minimal edge cover F' such that $w(F') \geq w(F)$ which holds the following property.

Property 14 *We can assume that F satisfies the following facts:*

(a) for each $e=v_iv_j \in E$ at least one of v'_i or v'_j is a center of a star in F,

(b) for each $e=v_iv_j \in E$, any vertex of $v_{i,j}$ is a leaf and its center is v_i or v_j.

Remind that for an star $K_{1,1}$, both endpoints can be its center and its leaf.

Proof: For (a), suppose there exist some edges $e=v_iv_j \in E$ such that none of v'_i and v'_j is a center in F. We modify F into F'' by repeating the following process for all of such edges: Let $S'_{i,j}=\{xy \in F : x \in \{v_i, v_j, v_{i,j}\}\}$ and let $S_{i,j}=\{v_iv_j, v_iv_{i,j}\}$, then $F''=(F \setminus S'_{i,j}) \cup S_{i,j}$. Since F is a minimal edge cover of G, F'' is a spanning star forest in G' with possibly some isolated vertices in V'_c. Considering the weight function w, $w(F'')=w(F)$. Easily by connecting all of the trivial stars to one center in F'', we make a new minimal edge cover F' such that $w(F') \geq w(F'') \geq w(F)$.

For (b), suppose there exist some edges $v_iv_j \in E$ such that $v_{i,j}$ is not a leaf in F. We modify F into F'' by repeating the following process for all of such edges: regarding (a), w.l.o.g. suppose v'_i is a center in F. Let $S'_{i,j}=\{xy \in F : x=v_{i,j}\}$ and let $S_{i,j}=v'_iv_{i,j}$, then $F''=(F \setminus S'_{i,j}) \cup S_{i,j}$ is a spanning star forest with possibly trivial stars of G' with $w(F'') \geq w(F)$ which satisfies (b). Notice after these stages, we may create some isolated vertices included in V'_C. However, connecting every isolated vertex in V'_C to an arbitrary center in V'_C induces a minimal edge cover with larger weight. □

Let $X'=\{x : xy \in F, y \in V'_E\}$ and $I'=V'_C \setminus X'$. By considering (a) in Property 14, $I=\{v : v' \in I'\}$ is an independent set of G. Since for each minimal edge cover F, there exist a minimal edge cover F' such that:

$$w(F) \leq w(F')=m+|I| \leq m+\alpha(G) \tag{9}$$

Hence by considering inequality (8) $\text{uec}(G',w')=m+\alpha(G)$.

Let F be a ρ-approximation solution for WEIGHTED UPPER EDGE COVER for (G', w) and I be an independent set of G which made by F' then:

$$\rho \leq \frac{w(F)}{\mathrm{uec}(G', w)} \leq \frac{w(F')}{\mathrm{uec}(G', w)} = \frac{m + |I|}{m + \alpha(G)} \tag{10}$$

since G is connected of maximum degree Δ, so $n \leq \Delta\alpha(G)$ (using Brook's Theorem), and then $m \leq \frac{\Delta^2}{2}\alpha(G)$. Hence, we can deduce that:

$$\Rightarrow 1 - \rho \geq \frac{\alpha(G) - |I|}{m + \alpha(G)} \geq \frac{\alpha(G) - |I|}{(1 + \frac{\Delta^2}{2})\alpha(G)}$$

$$\Rightarrow (1 + \frac{\Delta^2}{2})\rho - \frac{\Delta^2}{2} \leq \frac{|I|}{\alpha(G)}$$

or equivalently $\frac{|I|}{\alpha(G)} \geq (1 + \frac{\Delta^2}{2}) \cdot \frac{w(F)}{\mathrm{uec}(G',w)} - \frac{\Delta^2}{2}$. Remind that G' is a binary weighted graph and $|V'| = n + m$. In the following property, we show that G' is a c-dense graph with $c \geq \frac{4}{(2+\Delta)^2}$.

Property 15 *G' is a c-dense graph with $c \leq \frac{4}{(2+\Delta)^2}$.*

Proof: Regarding the construction of $G' = (V', E', w)$, we have

$$N = |V'| = n + m \leq n + \frac{\Delta}{2}n = (\frac{2 + \Delta}{2})n. \tag{11}$$

On the other hand, since G' is with binary weight function, then $\bar{w} = 1$ and we have $\sum_{u,v \in V'} w(u, v) = 2(\frac{n(n-1)}{2} + 2m) \geq n^2$. Using inequality (11), we have:

$$\sum_{u,v \in V'} w(u, v) \geq \frac{4}{(2 + \Delta)^2} N^2 \tag{12}$$

□

So G' is a c-dense graph with $c = \frac{4}{(2+\Delta)^2}$. By putting $\Delta = 3$, we deduce $c = \frac{4}{25}$. □

He and Liang in [27], studied the MAXSSF (or equivalently UEC) in c-dense graphs where $c \in (0, 1)$. They have shown that MAXSSF in c-dense graphs does not admit a polynomial time approximation scheme (**PTAS**) unless **P** = **NP**. Specifically, they proved that for any $c \in (0, 1)$, there exists $\varepsilon = \varepsilon(c) > 0$ such that approximating MAXSSF in c-dense graphs within a factor $(1 - \varepsilon)$ is **NP**-hard. In the following, we strengthen this result by proving the existence of constant lower bounds for WEIGHTED UPPER EDGE COVER in c-dense graphs and particularly in c-dense k-trees for some different constants c.

Corollary 16 WEIGHTED UPPER EDGE COVER *is not approximable within $\frac{1043}{1045} + \varepsilon$, $\frac{431}{432} + \varepsilon$, $\frac{620}{621} + \varepsilon$ and $\frac{835}{836} + \varepsilon$ for every $\varepsilon > 0$ unless **P=NP** in the class of weighted c-dense k-trees with c equals to $\frac{4}{25}$, $\frac{1}{9}$, $\frac{4}{49}$ and $\frac{1}{16}$ respectively, even for binary weight function.*

Proof: By Property 15, we know that the construction proposed in Theorem 13, is a c-dense graph with $c = \frac{4}{(2+\Delta)^2}$. Thus, by Theorem 13, we deduce that a $(\frac{(1+\Delta^2)}{2}\rho - \frac{\Delta^2}{2})$-approximation for MaxIS in graphs of maximum degree Δ is reachable in polynomial time from a ρ-approximation algorithm of Weighted Upper Edge Cover on c-dense k-trees with $c \leq \frac{4}{(2+\Delta)^2}$. On the other hand, MaxIS is not approximable within ratio $\frac{94}{95} + \varepsilon, \frac{47}{48} + \varepsilon, \frac{45}{46} + \varepsilon$ and $\frac{43}{44} + \varepsilon$ in graphs of maximum degree $3, 4, 5$ and 6 respectively [17]. Hence, by a simple calculation, we complete the proof. □

By deleting all edges of weight 0 in the graph G' given in Theorem 13, we can infer that all the above bounds are valid for Upper edge Cover (or equivalently MaxSSF) in c-dense graphs.

Corollary 17 Upper Edge Cover *is not approximable within* $\frac{1043}{1045} + \varepsilon, \frac{431}{432} + \varepsilon, \frac{620}{621} + \varepsilon$ *and* $\frac{835}{836} + \varepsilon$ *for every* $\varepsilon > 0$ *unless* ***P=NP*** *in* c*-dense graphs with* c *equals to* $\frac{4}{25}, \frac{1}{9}, \frac{4}{49}$ *and* $\frac{1}{16}$ *respectively.*

He and Liang in [27], proposed an approximation algorithm for the unweighted variant of the spanning star forest problem in c-dense graphs with ratio $0.804 + 0.196\sqrt{c}$. Note that this factor is larger than 0.835 when $c = \frac{4}{25}$. However, we showed in Corollary 17 that, for $c = \frac{4}{25}$, it is hard to approximate MaxSSF within ratio $\frac{1043}{1045} + \varepsilon \leq 0.999 + \varepsilon$.

Finally, note that Upper Edge Cover has at least a **PTAS** on everywhere-c-dense graphs using the result given in [26].

5.2 Positive approximation result

Now, we propose a positive approximation result of Weighted Upper Edge Cover via the use of an approximation preserving reduction from MaxWSSF which polynomially transforms any ρ-approximation into a $\frac{k-1}{k+1}\rho$-approximation for weighted upper edge cover.

Theorem 18 *In* k*-trees,* Weighted Upper Edge Cover *is* $\frac{k-1}{2(k+1)}$*-approximable.*

Proof: The proof uses an approximation preserving reduction from MaxWSSF which polynomially transforms any ρ-approximation into a $\frac{k-1}{k+1}\rho$-approximation for weighted upper edge cover. Then, using the 0.5-approximation of MaxWSSF given in [33], we will get the expected result.

Consider an edge-weighted k-tree (G, w) where $G = (V, E)$ and assume G is not complete. Let $\mathcal{S} = \{S_1, \ldots, S_r\} \subseteq E$ be a nice spanning star forest of (G, w) (see Property 2) which is a ρ-approximation of MaxWSSF, that is:

$$w(\mathcal{S}) \geq \rho \cdot opt_{MaxWSSF}(G, w) \tag{13}$$

Now, we show how to modify $\mathcal{S}$ into a minimal edge cover S without loosing "too much".

Before, we need to introduce some definitions and notations. A *vertex-coloring* $\mathcal{C} = (C_1, \ldots, C_q)$ of a graph G is a partition of vertices into independent sets (called *colors*). The *chromatic number* of G, denoted $\chi(G)$, is the minimum number of colors used in a vertex-coloring. If G is a k-tree, it is well known that $\chi(G) = k+1$ and such an optimal vertex-coloring can be done in linear time; hence, consider any optimal vertex-coloring $\mathcal{C} = \{C_1, \ldots, C_{k+1}\}$ of G. Moreover, in k-trees we know that each vertex $u \in C_i$ of color i is adjacent to some vertex $v \in C_j$ of color j for every $j \neq i$.

We color the edges of $E(\mathcal{S})$ incident to every isolated vertex of Triv using the $k+1$ colors where the color of such edge is given by the same color of its leaf. Formally, let $E' = \{uv \in E \colon v \in \text{Triv}\} \subseteq E(\mathcal{S})$ be the subset of edges incident to isolated vertices Triv and let $E_i = \{cv = e_v(\mathcal{S}) \in E(\mathcal{S}) \colon v \in C_i \setminus \text{Triv}\}$ for every $i \leq k+1$ where c is some center of $\mathcal{S}$. The key property is the following:

Property 19 *For any $i < i'$, by deleting some edges of $E_i \cup E_{i'}$ and by adding edges from E' we obtain a minimal edge cover.*

Proof: It is valid because each vertex of color i is adjacent to some vertices of every other colors. Formally, fix two indices $1 \leq i < i' \leq k+1$. Iteratively apply the following procedure: consider $v \in \text{Triv}$; there is $u \in V \setminus \text{Triv}$ such that $u \in C_i \cup C_{i'}$ (say C_i) and $vu \in E$. By hypothesis, u is a leaf of some ℓ-star S_r of $\mathcal{S}$. If at this stage $\ell \geq 2$, then add edge $uv \in E'$ and delete edge $uc \in E_i$ of color i; otherwise $\ell = 1$ and we just add edge $uv \in E'$. At the end, we get a minimal edge cover. □

Now, consider i_1, i_2 with $i_1 < i_2$ such that $w(E_{i_1} \cup E_{i_2}) = \min\{w(E_i \cup E_{i'}) \colon 1 \leq i < i' \leq k+1\}$. Using Property 19 we can polynomially find a minimal edge cover S of (G, w). By construction, $\sum_{i=1}^{k+1} w(E_i) \leq w(E(\mathcal{S}))$ and then:

$$w(E_{i_1} \cup E_{i_2}) \leq \frac{2}{k+1} w(E(\mathcal{S})) \tag{14}$$

Hence using inequalities (13) and (14), we get:

$$w(S') \geq w(E(\mathcal{S})) - w(E_{i_1} \cup E_{i_2}) \geq \frac{k-1}{k+1} w(E(\mathcal{S})) \geq \frac{k-1}{k+1} \rho \cdot opt_{MaxWSSF}(G, w)$$

Finally, since $opt_{MaxWSSF}(G, w) \geq \text{uec}(G, w)$ we get the expected result. □

Remark 20 *Results given in this section implicitly suppose that the parameter k of k-trees is unbounded because using the same approach as given in [30], we can prove that* Weighted Upper Edge Cover *can be solved in $O^*(6^k)$ for graphs of treewidth at most k (containing k-trees).*

6 Approximation of Weighted Upper Edge Cover in bounded degree graphs

In this section, we propose some positive approximation results depending on the maximum degree Δ in complement to the negative result given in Corollary 9.

Theorem 21 *In general graphs with maximum degree Δ, there is an approximation preserving reduction from* Weighted Upper Edge Cover *to* MaxExtWSSF *with expansion*[3] $c(\rho) = \frac{1}{\Delta} \cdot \rho$.

Proof: Consider an edge-weighted graph (G, w) of maximum degree $\Delta(G)$ bounded by Δ as an instance of Weighted Upper Edge Cover. We make an instance (G, w, U) of MaxExtWSSF by putting all pendant edges of G in the forced edge set U. Property 2 also works in this context since U is the set of pendant edges. In particular, we deduce that $opt_{ExtWSSF}(G, w, U) \geq \mathrm{uec}(G, w)$ because U belongs to any minimal edge cover. Let $\mathcal{S} = \{S_1, \dots, S_r\} \subseteq E$ be a nice spanning star forest of (G, w) containing U satisfying:

$$w(\mathcal{S}) \geq \rho \cdot opt_{ExtWSSF}(G, w, U) \geq \rho \cdot \mathrm{uec}(G, w) \tag{15}$$

For each $t \in \mathrm{Triv}$, we choose two edges incident to it with maximum weights $e_1^t = tx_t$ and $e_2^t = ty_t$ in $E \setminus E(\mathcal{S})$ (since by construction $d_G(v) \geq 2$), i.e., $w(e_1^t) \geq w(e_2^t) \geq w(tv)$ for all possible v; let $W = \sum_{t \in \mathrm{Triv}} (w(e_1^t) + w(e_2^t))$ be this global quantity. Also, recall that V_c and V_l are the set of vertices labeled by centers and leaves respectively according to $\mathcal{S}$. We build a new vertex weighted graph $G(\mathcal{S}) = G' = (V', E', w')$ with maximum degree $\Delta(G') \leq \Delta(G) - 1$ as follows:

- $V' = V_l$.
- $uv \in E'$ iff there exists $t \in \mathrm{Triv}$ with $tx_t = tu$ and $ty_t = tv$.
- For $v \in V'$, we set $w'(v) = w(e_v(\mathcal{S}))$ (recall that $e_v(\mathcal{S})$ is the edge of $\mathcal{S}$ linking leaf v to its center).

Clearly, G' is a graph with bounded degree $\Delta - 1$. We mainly prove that from any independent set $I \subseteq V'$ we can polynomially build an upper edge cover S_I of G satisfying:

$$w(S_I) \geq w'(I) + \left(W - \sum_{t \in \mathrm{Triv}} w(e_1^t) \right) \geq w'(I) \tag{16}$$

Let $I \subseteq V'$ be a maximal independent set of G'. This implies that $V' \setminus I$ is a vertex cover of G'. By construction of G', for every $t \in \mathrm{Triv}$, at least one vertex x_t or y_t is not in I (say x_t in the worst case). Recall $e_{x_t}(\mathcal{S})$ is the edge of the

[3] The expansion $c(\rho)$ of a reduction is a mapping transforming a ρ-approximation for the target problem into a $c(\rho)$-approximation for the initial problem.

spanning star forest incident to x_t (since $x_t \in V_l$). We will iteratively apply the following procedure for all $t \in \text{Triv}$ to build S_I:

- If the current ℓ-star S_r of $\mathcal{S}$ containing $e_{x_t}(\mathcal{S})$ satisfies $\ell \geq 2$ (it is true initially by hypothesis), then delete edge $e_{x_t}(\mathcal{S})$ from $\mathcal{S}$, add edge e_1^t and update spanning star forest $\mathcal{S}$.
- Otherwise, $\ell = 1$ and only add e_1^t.

At the end of the procedure, we get a minimal edge cover S_I of G satisfying inequality (16).

Now, apply as solution of I the greedy algorithm of MaxIS for G' taking, at each step, one vertex with maximum weight w' and by removing all the remaining neighbors of it. It is well known that we have:

$$w'(I) \geq \frac{w'(V')}{\Delta(G')+1} \geq \frac{w(\mathcal{S})}{\Delta(G)} \tag{17}$$

Hence, using inequalities (15), (16) and (17), we get the expected result. □

Using the 0.5-approximation algorithm of MaxExtWSSF given in [29], we deduce:

Corollary 22 Weighted Upper Edge Cover *is* $\frac{1}{2\Delta}$*-approximable in graphs with bounded degree* Δ.

7 Conclusion

In this article we gave positive and negative approximability aspects of Weighted Upper Edge Cover for special classes of graphs. We considered different types of weight function w for edges of input graph. Hardness of approximation on complete graphs when w satisfies cycle inequality remains open. Also for graphs with bounded degree Δ, we have shown that our problem is $\frac{1}{2\Delta}$-approximable while we proved it can not be better than $\Theta\left(\frac{1}{\Delta}\right)$. Finding a tighter approximation algorithm depending on Δ or on the average degree can be interesting.

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