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An Experimental Study on the Ply Number of Straight-line Drawings

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Abstract

The *ply number* of a drawing is a new criterion of interest for graph drawing. Informally, the ply number of a straight-line drawing of a graph is defined as the maximum number of overlapping disks, where each disk is associated with a vertex and has a radius that is half the length of the longest edge incident to that vertex. This paper reports the results of an extensive experimental study that attempts to estimate correlations between the ply number and other aesthetic quality metrics for a graph layout, such as stress, edge-length uniformity, and edge crossings. We also investigate the performance of several graph drawing algorithms in terms of ply number, and provide new insights into the theoretical gap between lower and upper bounds on the ply number of k-ary trees.

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1 Introduction

Graphs occur naturally in many domains: from sociology and biology, to software engineering and transportation. When the vertices and edges of the given graph have no inherent geographical locations, graph layout algorithms are used to capture the underlying relationships in the data (see, e.g., [10, 12, 24, 37]). In order to make the graph layout readable, such algorithms are designed to optimize several quality metrics, such as minimizing the number of edge crossings, striving for uniform edge lengths, or maximizing the vertex angular resolution [10].

Force-directed methods are among the most flexible and popular graph layout algorithms [26]. They tend to compute drawings that are aesthetically pleasing, exhibit symmetries, and contain no, or a few, edge crossings when the graph is planar. Classic examples include the spring layout method of Eades [14] and the algorithm of Fruchterman and Reingold [17], both of which rely on spring forces, similar to those in Hooke's law. In these methods, there are repulsive forces between all nodes and attractive forces between nodes that are adjacent. Alternatively, forces between the nodes can be computed based on their graph theoretic distances, determined by the lengths of the shortest paths between them. For instance, the algorithm of Kamada and Kawai [25] uses spring forces proportional to the graph-theoretic distances. In general, force-directed methods define an objective function which maps each graph layout into a number in $\mathbb{R}^+$ representing the *energy* of the layout. This function is defined in such a way that low energies correspond to layouts in which adjacent nodes are near some pre-specified distance from each other, and in which non-adjacent nodes are well-spaced. The notion of the energy of the system is related to the notion of *stress* in multidimensional scaling [28]. A careful look into stress-based layout algorithms and force-directed algorithms show that despite some similarities, they optimize a spectrum of different functions. For example, methods such as Kamada-Kawai optimize long graph distances, while Noack's LinLog layout [31] optimizes short graph distances. A later study by Chen and Buja explores further the differences between the energy models [6].

Recently, a new parameter, called *ply number*, has been proposed as a quality metric for graph layouts [11], partly inspired by the fact that real road networks, when interpreted as embedded graphs, tend to have a small ply number [15]. We also recall that ideas related to embedding graphs as road networks can be found in [29, 30]. Roughly speaking, a drawing has a small ply number if some, suitably defined, regions of influence of the vertices in the drawing are well spread out. More precisely, let Γ be a straight-line drawing of a graph. For each vertex $v \in \Gamma$, let C_v be the *open* disk centered at v whose radius r_v is half the length of the longest edge incident to v. Denote by S_q the set of disks C_v sharing a point $q \in \mathbb{R}^2$. The *ply number* of Γ is defined as $\mathsf{pn}(\Gamma) = \max_{q \in \mathbb{R}^2} |S_q|$. In other words, the ply number of Γ is the maximum number of disks C_v mutually intersecting in Γ (see, e.g., Figure 1). The ply number $\mathsf{pn}(G)$ of a graph G is the minimum ply number over all straight-line drawings of G. Figure 2 shows two drawings of the same graph. The drawing to the left is more readable than

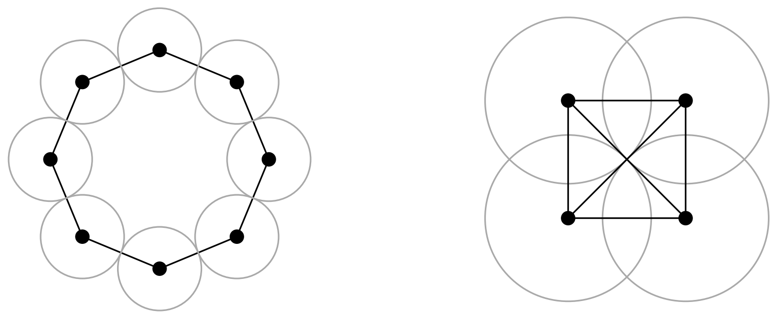


Figure 1: (a) A drawing with ply number 1. (b) A drawing with ply number 2.

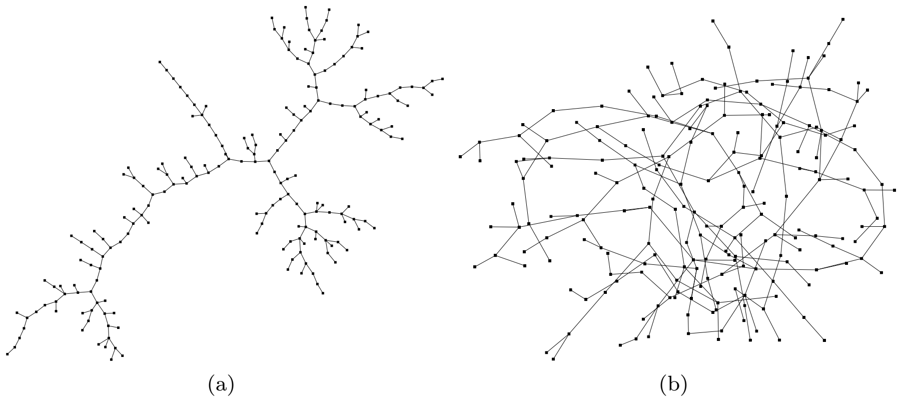


Figure 2: Two drawings of the same graph with ply number 3 (a) and 12 (b).

the drawing to the right, as it clearly conveys the structure of the underlying graph; indeed, the ply number of the left drawing is significantly smaller than the ply number of the right drawing.

Di Giacomo et al. [11] showed that computing the ply number of a given graph is NP-hard. They also proved that binary trees and caterpillars have ply number 2, which is worst-case optimal. The study of the ply number of trees has been continued by Angelini et al. [1] who proved that k-ary trees have non-constant ply number for $k \geq 10$. A new parameter related to the ply number has been recently introduced and studied by Angelini et al. [2]. The *vertex-ply number* of a straight-line drawing is the maximum number of disks in which a vertex is contained, i.e., the vertex-ply number is $\max_{v \in V} |S_v|$. A drawing with vertex-ply number 1 is an *empty-ply drawing*. Angelini et al. [2] show a non-trivial relationship between the ply number and the vertex-ply number and prove various results about empty-ply drawings. A fast algorithm to compute the ply number of a given drawing has been proposed by Heinsohn and Kaufmann [22], who also present methods to reduce the ply number of a drawing.

Our Contribution. While preliminary theoretical results about computing drawings with low ply number have already appeared [1, 2, 11], our work is an experimental study whose main goals are: (*i*) to shed more light on the quality, in terms of ply number, of drawings computed by some of the most popular algorithms, and in particular by different types of force-directed methods; (*ii*) to investigate whether the ply number of a drawing can be actually regarded as a quality metric, which possibly encompasses other popular metrics; (*iii*) to guide further theoretical studies of the combinatorial properties of drawings with low ply number. Specifically, our experiments involve several graph layout algorithms and several graph families, and we establish a correlation between the ply number and some classical quality metrics such as stress and edge length uniformity. Additionally, we give some insights about the theoretical gap between lower and upper bounds for the ply number of k-ary trees.

The paper is structured as follows. Sections 2, 3, and 4 provide details about our experimental questions, setting, and procedures, respectively. The results of our study are presented and discussed in Section 5. Conclusions and future research directions are given in Section 6.

2 Experimental Questions

As mentioned in the introduction, our experiment has the following main objectives: (*i*) to shed more light on the quality, in terms of ply number, of drawings computed by some of the most popular algorithms, with a particular focus on force-directed methods; (*ii*) to investigate whether the ply number of a drawing can be regarded as a quality metric; (*iii*) to guide further theoretical studies of the combinatorial properties of drawings with low ply number. We pose the following specific experimental questions:

Q1. *How good are the layouts computed by different drawing algorithms in terms of ply number?*

Q2. *For trees and cycles, how close is the ply number of drawings produced by existing algorithms to the ply number of the input graph (i.e., to the optimum value)?*

Q3. *Does the ply number correlate with some other commonly used quality metrics?*

Q4. *Can we establish empirical upper bounds on the ply number of k-ary trees?*

Questions Q1 and Q2 are strictly related and are concerned with objective (*i*), which focuses on the performance of existing drawing algorithms in terms of ply number. Question Q3 is concerned with objective (*ii*); in particular, possible correlations of the ply number with commonly accepted quality metrics would suggest that the ply number may be regarded as an alternative quality metric. Question Q4 is mostly concerned with objective (*iii*). Below, we discuss the motivation behind each question in more detail.

The possibility that force-directed algorithms indirectly optimize the ply number has been suggested in [11]: With Q1 we compare force-directed algorithms based on different force models to experimentally investigate this hypothesis. In [11] it is observed that non-planar drawings may have significantly smaller ply number than planar ones. Hence, for planar graphs, we also consider algorithms that compute straight-line planar drawings in comparison with drawings computed by force-directed algorithms.

Besides Q1, Question Q2 aims to estimate the gap between the ply number of drawings computed by existing algorithms and the optimum. In principle, this question may be asked for general input graphs; however, computing the optimum value for the ply number over all drawings of a graph is NP-hard [11], and the (worst-case) optimum value of ply number is known only for cycles and for specific types of trees, namely, paths, binary trees, and caterpillars (whose optimum is either 1 or 2). This is why we restrict Q2 to these families.

Question Q3 is more focused on understanding whether the ply number can be used as a quality metric for graph layouts, which possibly encompasses several other popular quality measures. We are mainly interested in three measures that are among the most used in graph visualization and that we expect to affect (positively or negatively) the ply number more than others: number of crossings, stress, and edge-length uniformity. It is worth remarking that the number of crossings is widely adopted to evaluate the quality of graph layouts, especially for graphs of small and medium size (see, e.g., [23, 33, 34, 39]). Studying the correlation between ply number and crossings, is further motivated by the fact that, as observed above, the ply number is sometimes reduced at the expense of edge crossings. The *stress* of a graph layout captures how well the realized geometric distances between pairs of vertices reflect their graph-theoretic distances in the graph; in the standard formulation, all edges are assumed to have about the same length, thus stress is related to edge-length uniformity (see Section 3). Recent studies give some evidence that reducing the stress of a graph layout is correlated with improved aesthetics [13, 27]. Studying the correlation between ply number and edge-length uniformity is also motivated by the fact that in a drawing with ply number one, all edges have the same length [11].

Question Q4 is motivated by objective (*iii*) and arises from theoretical results on the ply number of k-ary trees. Namely, it is known that every 2-ary (i.e., binary) tree has ply number at most two [11], while the ply number of 10-ary trees is not bounded by a constant [1]. Studying what happens for values of k such that $3 \leq k \leq 9$ is an interesting research direction.

3 Experimental Setting

In order to answer Questions Q1–Q4, we selected different graph datasets, algorithms, and measures. In what follows we describe each of these experimental components.

Graph datasets. To understand whether the experimental results are influenced by the structure of the graph we considered several graph families. All graphs

Table 1: Table summarizing which datasets are used to answer each question.

	Q1	Q2	Q3	Q4
Trees	✓		✓	
Planar	✓		✓	
General	✓		✓	
Scale-free	✓		✓	
Caterpillars		✓		
Paths		✓		
Cycles		✓		
k-ary Trees ($k = 2$)		✓		
k-ary Trees ($k = 3, 6, 9$)				✓

are of small or medium size, expressed as the number of their vertices. In some cases, the size and the number of instances used for each graph family depend on the type of question we want to answer (see Section 5 for details). We used the following datasets:

Trees. Generated with a uniform probability distribution using Prüfer sequences [32].

Planar. Connected simple planar graphs, generated with the OGDF library [7].

General. Connected simple graphs, generated with a uniform probability distribution.

Scale-free. Scale-free graphs, generated according to the Barabási-Albert model [3].

Caterpillars. Each caterpillar of n vertices is generated by first creating a path (spine of the caterpillar) of length $k \in [\frac{n}{4}, \frac{n}{2}]$ (randomly chosen), and attaching each remaining vertex to a randomly selected vertex of the spine.

Paths, Cycles. For a given number of vertices n, there is only one (unlabeled) path and one (unlabeled) cycle on n vertices.

k-ary Trees. Rooted trees where each node has either 0 or k children. Each tree is generated by starting with a single vertex and then creating k children of a randomly selected leaf, until the desired number n of vertices is achieved. When n cannot be obtained, we use a value close to it.

Table 1 shows which datasets are used to answer each question. Note that the datasets Caterpillars, Paths, Cycles, and 2-ary Trees are explicitly used to answer Q2. All these families (except Cycles) are special cases of trees and we do not use them to answer Q1 and Q3. The dataset of k-ary Trees is explicitly designed to answer Q4. See Section 4 for more details.

Algorithms. Among the many force-directed algorithms, we considered some of the most popular ones [26]. We used the following algorithms, available in OGDF:

FR. This algorithm is based on the Fruchterman-Reingold model [17], an improvement of the seminal algorithm by Eades [14]. Vertices are viewed as equally-charged electrical particles and edges act similar to springs; electrical charges cause repulsion between vertices and springs cause attraction. It also introduces a temperature function, which reduces the displacement of the vertices as the layout becomes better.

GEM. This algorithm is proposed by Frick et al. [16]. It is a variant of the FR algorithm, which adds several new heuristics to improve the convergence, including local temperatures, gravitational forces, and the detection of rotations and oscillations.

KK. This algorithm is described by Kamada-Kawai [25]. Unlike FR and GEM, it aims to compute a layout where the geometric distance between two vertices equals their graph-theoretic distance in the graph. The energy function minimized by this algorithm is therefore a type of stress function.

SM. This technique is proposed by Gansner et al. [18]. It minimizes a stress function similar to that proposed by KK, which can be minimized more efficiently via majorization.

FM3. The fast multipole multilevel method of Hachul and Jünger [19] is among the most effective force-directed algorithms in the literature [20].

We also used the following algorithm, available in Gephi [4]:

LL. This is a force-directed algorithm based on the LinLog energy model proposed by Noack [31]. It is specifically conceived to emphasize clusters in the graph.

For instances of Planar, Trees, and 2-ary Trees, we also considered planar straight-line drawing algorithms, still using the implementations in OGDF. The algorithms are:

PL. This is an improved version of the planar straight-line drawing algorithm proposed by Chrobak and Kant [8], based on the shift algorithm of de Fraysseix et al. [9].

TR. The tree layout algorithm of Buchheim et al. [5] is an efficient version of Walker's algorithm [38], which in turn is an extension of the Reingold-Tilford algorithm for rooted binary trees [35].

Measures. We considered four measures: ply number (PN), number of crossings (CR), stress (ST), and edge-length uniformity (EU). Let Γ be a straight-line drawing of a graph $G = (V, E)$. EU corresponds to the normalized standard deviation of the edge length, i.e.:

$$\texttt{EU}(\Gamma) = \sqrt{\sum_{e \in E} \frac{(l_e - l_{avg})^2}{|E| l_{avg}^2}},$$

where l_e is the length of edge e and l_{avg} is the average length of the edges. The stress of Γ is defined as:

$$\texttt{ST}(\Gamma) = \sum_{i,j \in V} w_{ij} (\parallel p_i - p_j \parallel - d_{ij})^2,$$

where $w_{ij} = d_{ij}^{-2}$, p_i and p_j are the positions of i and j in Γ, and d_{ij} is the graph theoretic distance of i and j in G.

4 Experimental Procedures

In the following we describe the different experimental procedures executed to answer each question. In Section 5 we present the results and summarize the main findings.

Procedure for Q1. We drew each instance of each dataset with all the algorithms (clearly, PL has been used only for Planar and TR only for Trees), and measured the ply number of each drawing. Both for Trees and for Planar, we generated 10 instances for each fixed number of vertices $n \in \{50, 100, 150, 200, \dots, 450\}$: the average density of the planar graphs is 1.75. In the General dataset we generated 10 instances for each pair $\langle n, d \rangle$, where $n \in \{50, 100, 150, 200, \dots, 450\}$ is still the number of vertices of the graph and $d \in \{1.5, 2.5\}$ is its edge density. Indeed, we want also understand to which degree the ply number is influenced by the density of the graph. In the Scale-free dataset we generated 10 instances for each pair $\langle n, d \rangle$, where $n \in \{50, 100, 150, 200, \dots, 450\}$ and $d \in \{2, 3\}$ (the graph generator that we used required to specify an integer number as edge density [21]).

Procedure for Q2. To answer Q2 we need to compare the ply number of the drawings computed by the various algorithms with the ply number of the input graph (i.e., the optimum value). Hence, we considered families of graphs for which the worst-case optimal ply number is known. In particular, we considered the Paths and Cycles instances, which have ply number one, and the Caterpillars and 2-ary Trees instances, whose ply number is at most two (see [11]). For each $n \in \{50, 100, 150, 200, \dots, 450\}$, we generated a single instance in Paths and Cycles, and 5 instances in the Caterpillars and 2-ary Trees datasets. For each instance, we computed 10 different drawings with the algorithms KK, FM3, SM (those with better performance based on the results of Q1). We then took the minimum value of ply number over all the drawings of each instance.

Procedure for Q3. We took a representative instance for each sample (i.e., size or size and density) of each dataset. As reported in Table 1, we used the same datasets as for Q1. For each representative instance we computed a series of 60 different drawings and on this series we measured Spearman's rank correlation coefficient ρ [36] between the ply number and all the other quality metrics described in Section 3, i.e., ST, CR, and EU. The series of drawings are produced by running 10 times each of the 6 force-directed algorithms, varying the initial layout every time.

Procedure for Q4. We generated a k-ary tree for each pair $\langle n, k\rangle$, with $k \in \{3, 6, 9\}$ and increasing values of n. More precisely, we considered values of n from 50 to 1000 with steps of 50, from 1000 to 2000 with steps of 100, from 2000 to 7500 with steps of 500. The choice for the values of k is motivated by the fact that we want to experimentally understand if we can empirically establish a constant upper bound to the ply number of k-trees for $2 < k < 10$. For each instance, we measured the ply number of a drawing computed with SM, which turned out to be the best performing algorithm for this measure, according to the experiment for Question Q1.

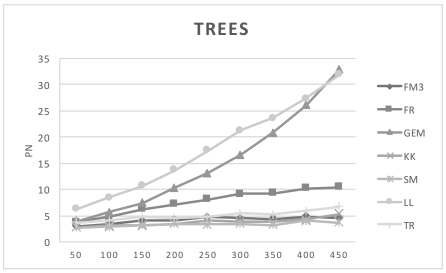
5 Experimental Results and Findings

We first report the experimental data, by presenting tables and charts. Then, we list and discuss the main findings.

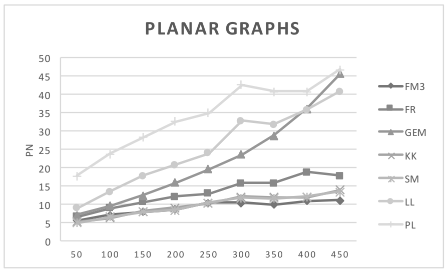
Results for Q1. Figure 3 reports the average ply number for each sample (number of vertices) of drawings computed by the same algorithm. For Trees and Planar we observed a similar trend. The algorithms that give the lowest values of the ply number are SM, KK, and FM3 (with KK and SM consistently producing nearly identical values, and FM3 that is slightly better for larger planar graphs). FR produces drawings with ply number higher than the previous three algorithms, although it has a similar trend. GEM and LL produce drawings with ply number much higher than the other force-directed algorithms, with GEM that becomes worse than LL as n grows. For Trees the TR algorithm performs well (between FM3 and FR), while the algorithm PL (for the Planar dataset) produces the drawings with the worse values of ply, thus highlighting that planar drawings may have higher ply number than non-planar ones, at least when they are computed with approaches similar to that used by PL (which is based on a canonical ordering of the vertex set [8, 9]). For the General and Scale-free datasets, we have a similar situation. The main difference is that the performance of GEM does not worsen as rapidly as in the case of Trees and Planar (in particular, it is always better than LL). For the Scale-free dataset, the values computed by KK and SM increase, as n grows, more than those of FR and FM3. Thus, for larger values of n they are outperformed by FR and FM3, and approach the performance of GEM.

Results for Q2. For paths and cycles, all three algorithms compute drawings with ply number 2, i.e., just one unit larger than the optimal value, for instances up to 250. Figure 4 illustrates two drawing with ply number 2 of two instances (a Binary Tree and a Caterpillar) from the experimental dataset.

For larger sizes, FM3 still computes drawing with ply number 2, while KK and SM produce drawings with ply number 3. It is worth noting that all the drawings computed by FM3 have ply number 2 with the only exception of one



(a)



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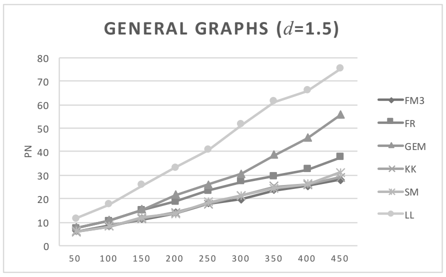

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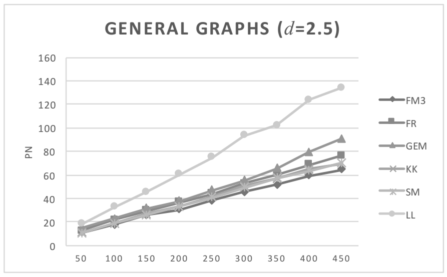

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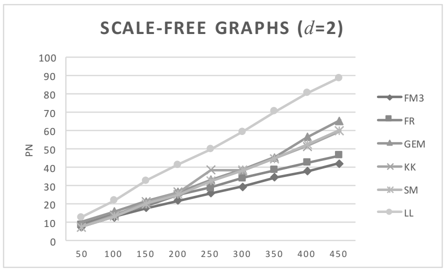

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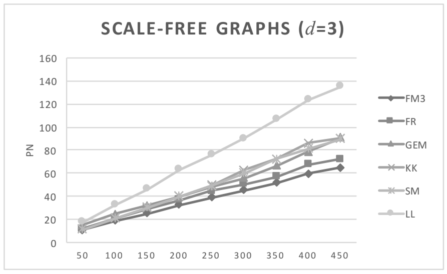
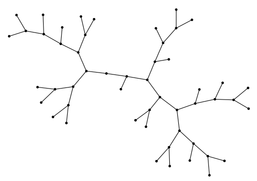


(f)

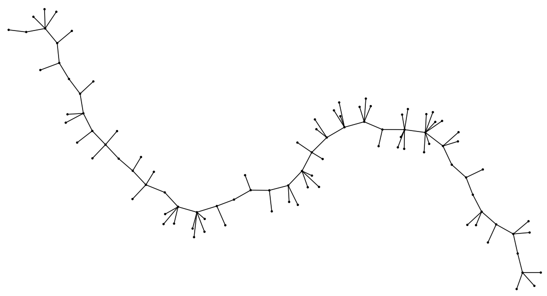
Figure 3: Average ply number for (a) Trees; (b) Planar; (c) General with density $d = 1.5$; (d) General with density $d = 2.5$; (e) Scale-free with density $d = 2$; (f) Scale-free with density $d = 3$. The x-axis reports the number of vertices.

drawing of the cycle of size 450. This is consistent with the fact that FM3 (a multilevel algorithm) is less affected by the initial position of the vertices. The maximum value of ply for drawings produced by KK and SM is 6 and 4, respectively.

Figure 5(a) shows the average ply number of drawings computed by the different algorithms for binary trees of different sizes. In this case, FM3 has the worst performance with an average ply number ranging from 3 to 5.2; the other two algorithms have almost the same values of the average ply number, ranging from 2 to 4.2. In the chart we also show (with error bars) the (average)

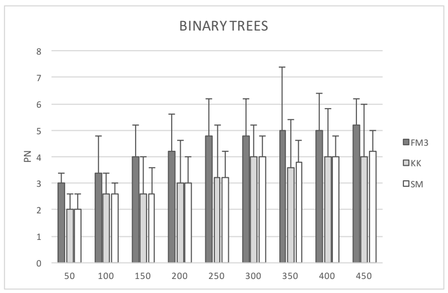


(a)

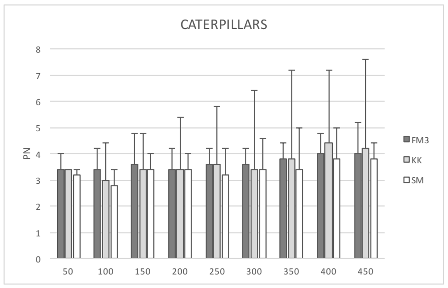


(b)

Figure 4: Drawings from the experimental dataset with ply number 2 (a) A binary tree with $n = 50$ drawn with SM (d) A caterpillar with $n = 100$ drawn with KK.



(a)



(b)

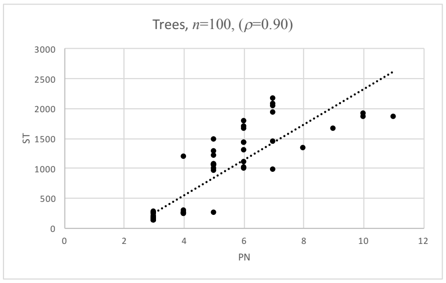
Figure 5: Average ply number for (a) Binary trees; (b) Caterpillars;

maximum ply number of each algorithm. FM3 and KK tend to have larger differences between the (average) maximum and the (average) minimum ply number than SM. Figure 5(b) shows the same data for caterpillars. In this case, the three algorithms perform similarly. The average ply number is around 3.5. Also in this case SM is the algorithm with the smallest difference between maximum and minimum error.

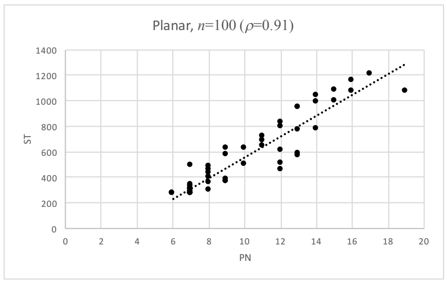
Table 2: Correlation coefficient ρ between ply number (PN) and stress (ST), crossings (CR), edge length uniformity (EU). The values in bold indicate a strong correlation ($\rho \geq 0.7$).

	n	PN, ST	PN, CR	PN, EU		n	PN, ST	PN, CR	PN, EU
Trees	50	**0.92**	**0.88**	**0.86**	General ($d = 2.5$)	50	**0.80**	0.19	**0.89**
	100	**0.90**	**0.84**	**0.89**		100	**0.80**	0.21	**0.72**
	150	**0.94**	**0.92**	**0.78**		150	**0.78**	0.20	0.62
	200	**0.97**	**0.89**	**0.87**		200	**0.79**	0.04	0.54
	250	**0.96**	**0.91**	0.69		250	**0.78**	0.47	0.58
	300	**0.97**	**0.92**	0.65		300	**0.71**	-0.34	0.63
	350	**0.93**	**0.90**	0.62		350	**0.70**	-0.19	0.59
	400	**0.96**	**0.90**	0.61		400	**0.73**	-0.13	0.55
	450	**0.95**	**0.96**	0.51		450	**0.72**	-0.19	0.54
Planar	50	**0.92**	**0.80**	**0.76**	Scale-free ($d = 2$)	50	**0.86**	0.29	**0.74**
	100	**0.91**	**0.83**	**0.86**		100	**0.83**	0.08	**0.80**
	150	**0.90**	0.66	**0.89**		150	0.32	-0.08	**0.82**
	200	**0.77**	**0.87**	**0.71**		200	0.57	0.33	**0.72**
	250	**0.87**	**0.83**	**0.81**		250	0.38	0.27	**0.90**
	300	**0.93**	**0.80**	**0.80**		300	0.21	0.09	**0.90**
	350	**0.80**	**0.91**	**0.74**		350	0.48	0.53	**0.82**
	400	**0.77**	**0.93**	0.67		400	0.45	0.60	**0.84**
	450	**0.76**	**0.84**	0.58		450	0.50	0.54	**0.80**
General ($d = 1.5$)	50	**0.88**	0.39	**0.87**	Scale-free ($d = 3$)	50	**0.72**	0.15	**0.91**
	100	**0.92**	**0.80**	**0.82**		100	0.67	-0.04	**0.88**
	150	**0.89**	**0.81**	**0.86**		150	0.54	-0.32	**0.87**
	200	**0.87**	**0.83**	**0.89**		200	0.47	-0.16	**0.87**
	250	**0.85**	0.60	**0.84**		250	0.30	0.13	**0.90**
	300	**0.85**	**0.73**	**0.73**		300	0.18	0.27	**0.95**
	350	**0.84**	**0.85**	**0.85**		350	0.14	0.36	**0.96**
	400	**0.85**	**0.79**	0.49		400	0.29	0.22	**0.92**
	450	**0.88**	**0.86**	0.59		450	0.30	0.41	**0.90**

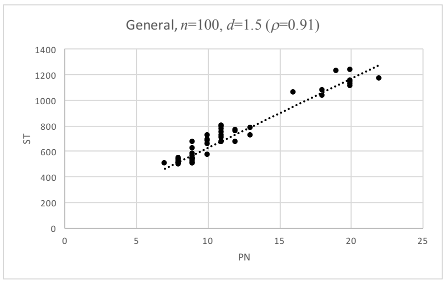
Results for Q3. Table 2 shows, for each instance, the values of the Spearman's rank correlation coefficient ρ. We have high values of correlation (i.e., $\rho \geq 0.7$) between ply number and stress for almost every instance. The exceptions are larger scale-free graphs, for which in most cases we have a moderate correlation (i.e., $0.3 \leq \rho < 0.7$). The correlation between PN and EU is high/moderate in all cases. High values of correlation are obtained for the smaller instances of each dataset, with the only exception of scale-free graphs, where there is a high correlation for all sizes. Concerning PN and CR, we have (high/moderate) correlation only for trees, planar graphs, and for general graphs with density 1.5. Figure 6 shows the relationships between ply number and stress, for the instances with $n = 100$ of each dataset.



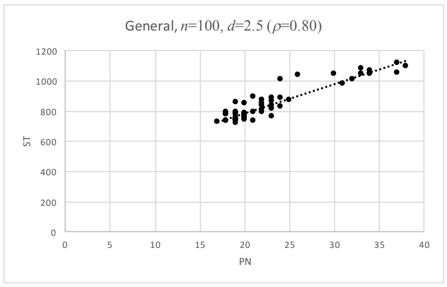
(a)



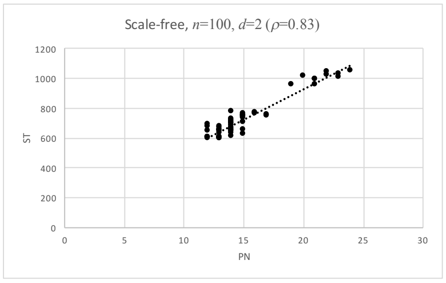
(b)



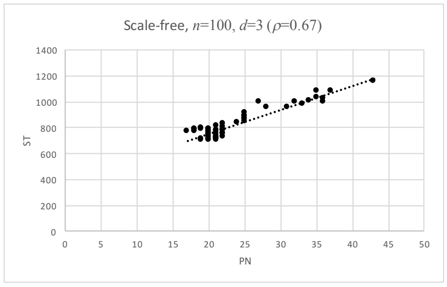
(c)



(d)



(e)



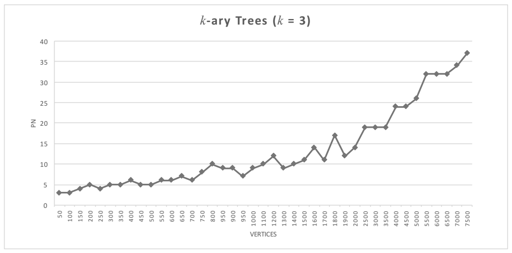
(f)

Figure 6: Relationships between ply number (`PN`) and stress (`ST`), for the instances with $n = 100$ of each dataset.

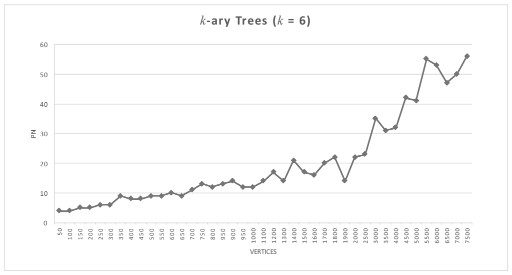
Results for Q4. Figure 7 reports the values of ply number for each k and for increasing values of n. For the sizes of the trees that we considered, we always observed progressively increasing ply numbers, even for $k = 3$, and the ply number function does not exhibit any asymptotic trend towards a constant upper bound.

We now summarize the main findings. We denote by Fi the finding concerned with Question Qi ($i = 1, \ldots, 4$).

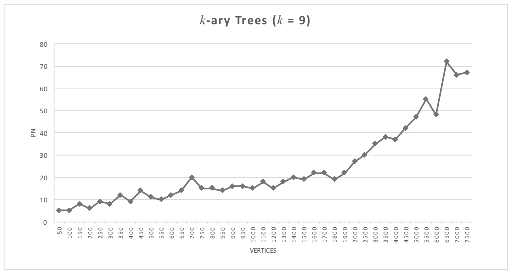
F1. Algorithms designed to minimize stress and edge-length uniformity (such



(a)



(b)



(c)

Figure 7: Trend of the ply number for (a) 3-ary trees, (b) 6-ary trees and (c) 9-ary trees

as SM and KK) compute drawings with smaller values of ply number. This behavior confirms the intuition that low ply number is related to stress and edge uniformity optimization (see also F3). Also multilevel algorithms (like FM3) have good performance and are more stable on denser graphs. We believe this is a consequence of their coarsening phase, which indirectly tends to evenly distribute the vertices in the plane, thus producing drawings with good edge length uniformity, independently of the original placement of the nodes. We also observed a good behavior of FR on denser graphs. Force-directed algorithms whose energy model is conceived to highlight clusters, such as LL, tend to produce drawings with high ply numbers, as they give rise to very different edge lengths in the same drawing.

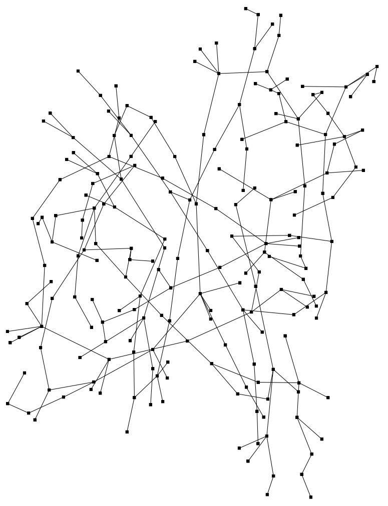
F2. The best performing algorithms in terms of ply number very often generate drawings whose ply number is close to the optimum for graphs like paths, cycles, caterpillars, and binary trees. Hence, they can be considered good heuristics to compute graph layouts with minimum ply number, at least for these simple graph families.

F3. There is a strong correlation between ply and stress, and a strong/moderate correlation between ply and edge-length uniformity. For planar graphs and low density graphs, the correlation between ply and crossings is also observed, while ply is definitely non-correlated with edge crossings on denser graphs and, in particular, on scale-free graphs. Overall, these data indicate that ply number can be often regarded as a unifying quality metric, which encompasses at least stress and edge length uniformity. For very sparse graphs, it also encompasses edge crossings. Note that, the correlation between ply number and stress does not always imply that low ply number equals low stress. For example, Figure 8 shows two drawings of two different graphs with 200 vertices: A binary tree drawn with the GEM algorithm and a Scale-free graph with $d = 3$ drawn with the KK algorithm; the first drawing has low ply and high stress the second one has low stress and high ply number.

F4. We could not observe any asymptotic trend of the ply number towards a constant upper bound for k-ary trees for $k \in \{3, 6, 9\}$; see Fig. 7. In principle, one can interpret this result as an indication that the ply number for such trees is unbounded. However, repeating the same experiment on binary trees (which we know to have bounded ply number), leads to similar results. Thus, we cannot draw any conclusions from this particular experiment.

6 Conclusions and Future Work

Our graph datasets and the data collected in the different experiments are publicly available at `https://www.felicedeluca.com/ply/`. These data answer, or


(a) PN = 8; ST = 9178.

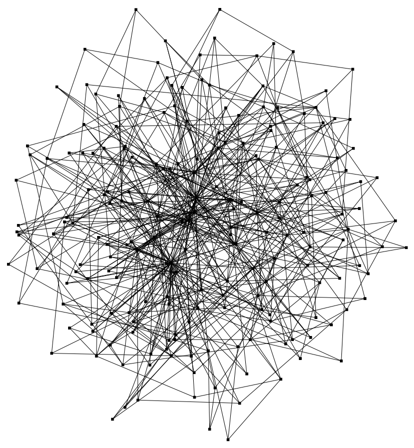

(b) PN = 47; ST = 3297.

Figure 8: Drawings of two graphs with 200 vertices. (a) A Binary tree drawn by GEM algorithm with low ply number and high stress. (b) A Scale-free graph with $d = 3$ drawn by KK with low stress and high ply number.

partially answer, several of our initial questions and raise new interesting questions for further study. We remark that, as in many experimental studies, ours has some limitations and should be interpreted in the context of the specific datasets, layout algorithms, and measurements used. For example, a more complete picture can be obtained with a more diverse set of graphs. In particular, computing the ply number of a drawing in a reliable way requires high arithmetic precision, which is computationally expensive. This limited the sizes of graphs that we could consider. Further, we mostly used algorithms available in OGDF, considering only one other algorithm. Comparing a wider spectrum of algorithms might help to identify what type of stress-minimization and energy minimization functions are best suited to minimize the ply number.

Our study provides answers to several of the questions that we asked and also suggests several natural research directions that remain to be explored:

- Can new layout algorithms be developed that directly optimize ply number? Modifying stress-based methods such as Kamada-Kawai would be difficult, but perhaps force-directed methods such as Fruchterman-Reingold can be augmented with additional forces to separate overlapping disks.

- Considering more carefully the variations in the exact functions used in different stress-based methods and force-directed methods and their impact on optimizing ply might lead to better insights about how to compute low-ply layouts.

- We observed that the algorithm PL produces planar drawings with values of ply number higher than those of non-planar drawings: Is it possible to compute planar drawings with smaller values of ply number?
- There is growing evidence that different types of graph layout algorithms are suited to different types of graphs. A cognitive study could consider the impact of minimizing ply number, compared to the impact of minimizing edge crossings and other aesthetic criteria that are not correlated with the ply number.
- It seems that graph layouts with low ply number increase readability, but perhaps at the cost of increased drawing area. Further experiments can test this hypothesis and look for other possible correlations.
- The results from our experiments on k-ary trees were not sufficient to make a prediction about the theoretical bounds of their ply number. Further experiments might help to provide additional insights in this direction.

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