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Realization of Posets

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Abstract.

We prove a very general representation theorem for posets and, as a corollary, deduce that any abstract simplicial complex has a geometric realization in the Euclidean space of dimension $\dim P(\Delta) - 1$, where $\dim P(\Delta)$ is the Dushnik-Miller dimension of the face order of Δ.

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1 Introduction

Schnyder proved in [3] that a graph is planar if and only if its incidence poset (that is: the poset where $x < y$ iff x is a vertex, y is an edge and y is incident to x) has dimension at most 3. That an incidence poset has dimension at most 3 implies that the corresponding graph is planar has been extended to abstract simplicial complexes in [2]: if the face order of an abstract simplicial complex Δ is bounded by $d+1$, then Δ has a geometric realization in $\mathbb{R}^d$. We prove here a more general result on poset representation which implies this last result straightforwardly.

We shall first recall some basic definitions from poset theory: A *partially ordered set* (or *poset*) $\mathbf{P}$ is a pair (X, P) where X is a set and P a reflexive, antisymmetric, and transitive binary relation on X. A poset is $\mathbf{P} = (X, P)$ is *finite* if its *ground set* X is finite. We shall write $x \leq y$ in P or $x \leq_P y$ if $(x, y) \in P$. Two elements $x, y \in X$ such that $x \leq y$ in P or $y \leq x$ in P are said to be *comparable*;otherwise, they are said to be *incomparable*.

If P and Q are partial orders on the same set X, Q is said to be an *extension* of P if $x \leq y$ in P implies $x \leq y$ in Q, for all $x, y \in X$. If Q is a *linear order* (that is: a partial order in which every pair of elements are comparable) then it is a *linear extension* of P. The *dimension* $\dim \mathbf{P}$ of $\mathbf{P} = (X, P)$ is the least positive integer t for which there exists a family $\mathcal{R} = (<_1, <_2, \ldots, <_t)$ of linear extensions of P so that $P = \bigcap \mathcal{R} = \bigcap_{i=1}^{t} <_i$. This concept has been introduced by Dushnik and Miller in [1]. A family $\mathcal{R} = (<_1, <_2, \ldots, <_t)$ of linear orders on X is called a *realizer* of P on X if $P = \bigcap \mathcal{R}$.

For an extended study of partially ordered sets, we refer the reader to [4].

We shall further introduce the following notation: the *down-set* (or *filter*) of a poset $\mathbf{P} = (X, P)$ induced by a set $A \subseteq X$ is the set

$$\mathrm{Inf}(A) = \bigcap_{a \in A} \mathrm{Inf}(\{a\}) = \{x \in X, \quad \forall a \in A, x \leq a \text{ in } P\}$$

2 The Poset Representation Theorem

Definition 2.1 *Let $\mathbf{P} = (X, P)$ be a finite poset, n an integer and $f : X \mapsto \mathbb{R}^n$ a mapping from X to the n-dimensional space $\mathbb{R}^n$.*

Then f is said to have the separation property for $\mathbf{P}$ *if, for any $A, B \subseteq X$, there exists a hyperplane of $\mathbb{R}^n$ which separates the points of $f(\mathrm{Inf}(A) \setminus \mathrm{Inf}(B))$ and the ones of $f(\mathrm{Inf}(B) \setminus \mathrm{Inf}(A))$, where $\mathrm{Inf}(Z) = \{x \in X, \forall z \in Z, x \leq_P z\}$ for any $Z \subseteq X$.*

Theorem 2.1 *Let $\mathbf{P} = (X, P)$ be a finite poset and let $d = \dim \mathbf{P}$ be its dimension. Then, there exists a function $f : X \mapsto \mathbb{R}^{d-1}$, which satisfies the separation property for $\mathbf{P}$.*

Proof: Let $\mathcal{R} = \{<_1, \ldots, <_d\}$ be a realizer of $\mathbf{P}$ and denote $\min(X, <_i)$ the minimum element of set X with respect to linear order $<_i$. Let $F_1, \ldots, F_d$ be functions from X to $]1;+\infty[$, each F_i being fast increasing with respect to $<_i$, which means that

$$\forall x <_i y, \quad F_i(x) < d.F_i(y).$$

We define the function $F : X \mapsto \mathbb{R}^d$ by $F(x) = (F_1(x), \dots, F_d(x))$.

For any $A, B \subseteq X$ such that $\mathrm{Inf}(B) \not\subseteq \mathrm{Inf}(A)$, define the linear form $L_{A,B} : \mathbb{R}^{d-1} \mapsto \mathbb{R}$, as:

$$\forall \pi = (\pi_1, \dots, \pi_d) \in \mathbb{R}^d, \quad L_{A,B}(\pi) = \sum_{\substack{1 \leq i \leq d \\ \min(A,<_i) <_i \min(B,<_i)}} \frac{\pi_i}{\min_{a \in A} F_i(a)}.$$

On one hand, for any $z \in \mathrm{Inf}(B) \setminus \mathrm{Inf}(A)$, there exists $a \in A$ and $1 \leq i_0 \leq d$, with $z >_{i_0} a$. Then, we get $F_{i_0}(z) > d.F_{i_0}(a)$. As $\min(B, <_{i_0}) \geq_{i_0} z >_{i_0} \min(A, <_{i_0})$, we obtain: $L_{A,B}(F(z)) > d$.

On the other hand, for any $z \in \mathrm{Inf}(A)$, we have $F_i(z) \leq F_i(a)$ for every $i \in [d]$ and every $a \in A$. Thus, $L_{A,B}(F(z)) \leq d$.

Altogether, for any $A, B \subseteq X$ such that none is included in the other, the hyperplane $H_{A,B}$ with equation $L_{A,B}(\pi) - L_{B,A}(\pi) = 0$ separates the points from $F(\mathrm{Inf}(B) \setminus \mathrm{Inf}(A))$ (for which $L_{A,B}(F(z)) > d \geq L_{B,A}(F(z))$) and those from $F(\mathrm{Inf}(A) \setminus \mathrm{Inf}(B))$ (for which $L_{A,B}(F(z)) \leq d < L_{B,A}(F(z))$). Notice that the origin O belongs to all the so-constructed hyperplanes.

Now, consider a hyperplane H_0 with equation $\sum_{1 \leq i \leq d} \pi_i = 1$, which separates the origin O and the set of the images of X by F. To each element z of X, we associate the point $f(z)$ of H_0 which is the intersection of H_0 with the line $(O, F(z))$.

Now, for any $A, B \subseteq X$ (such that none is included in the other), as $H_{A,B}$ includes O, the hyperplane $H_{A,B} \cap H_0$ of H_0 separates the points from $F(\mathrm{Inf}(B) \setminus \mathrm{Inf}(A))$ and those from $F(\mathrm{Inf}(A) \setminus \mathrm{Inf}(B)$. As $H_0 \simeq \mathbb{R}^{d-1}$ and as the separation property would be obviously true if $A \subseteq B$ or conversely, the theorem follows. □

The preceding theorem is sharp, as proved here using the *standard example* $\mathbf{S}_n$ of poset of dimension n (introduced in [1]):

Theorem 2.2 *For any $n \geq 3$, there exists no function $f : [n] \mapsto \mathbb{R}^{n-2}$ which satisfies the separation property for the* standard example $\mathbf{S}_n$ *of poset of dimension n, which is the height two poset on $\{a_1, \dots, a_n, b_1, \dots, b_n\}$, with minima $\{a_1, \dots, a_n\}$, maxima $\{b_1, \dots, b_n\}$ and such that $\forall i, j, \quad (a_i < b_j) \iff (i \neq j)$.*

Proof: Assume there exists a function $f : \{a_1, \dots, a_n, b_1, \dots, b_n\} \mapsto \mathbb{R}^{n-2}$ having the separation property for $\mathbf{S}_n$.

According to Radon's lemma, for any family of n point in $\mathbb{R}^{n-2}$, there exists a bipartition V, W of them, such that the convex hulls of V and W intersects and thus such that V and W cannot be separated by an hyperplane of $\mathbb{R}^{n-2}$. Let $A = \{b_i, f(a_i) \notin V\}$ and $B = \{b_i, f(a_i) \notin W\}$. Then, $V \subseteq f(\mathrm{Inf}(A))$ and $W \subseteq f(\mathrm{Inf}(B))$. Hence, the separation property fails for A, B. □

From Theorem 2.1, one derives a sufficient condition for a graph to be planar, which is that its incidence poset shall be of dimension at most 3 and this condition is actually also a necessary condition:

Theorem 2.3 (Schnyder [3]) *The incidence poset* $\mathrm{Incid}(G)$ *of a graph* G *has dimension at most* 3 *if and only if* G *is planar, that is: if and only if there exists a mapping* f *from* $V(G) \cup E(G)$ *to* $\mathbb{R}^2$ *having the separation property for* $\mathrm{Incid}(G)$. □

3 Applications

Corollary 3.1 *Let* U *be a finite set, and* $\mathcal{F}$ *a family of subsets of* U *such that:*

$$\forall x, y \in U, \exists X \in \mathcal{F}, \quad x \in X \text{ and } y \notin X. \tag{1}$$

Let d *be the Dushnik-Miller dimension of the inclusion order* $\subset_{\mathcal{F}}$ *on* $\mathcal{F}$.

Then, there exists a function $f : U \mapsto \mathbb{R}^{d-1}$ *such that (denoting* $f(A)$ *the set* $\{f(z), z \in A\}$*, for* $A \subseteq U$*):*

$$\forall X \in \mathcal{F}, \quad \mathrm{Conv}(f(X)) \cap f(U) = f(X), \tag{2}$$

$$\forall X \neq Y \in \mathcal{F}, \quad \mathrm{Conv}(f(X \setminus Y)) \cap \mathrm{Conv}(f(Y \setminus X)) = \emptyset. \tag{3}$$

Proof: Equation (3) is a direct consequence of Theorem 2.1. For (2), consider successively all the elements $z \notin X$: According to (1), the intersection of all the sets in $\mathcal{F}$ including z does not intersect X. Hence, setting $A = \{X\}$ and $B = \{Y \in \mathcal{F}, z \in Y\}$, it follows from Theorem 2.1 that z does not belong to $\mathrm{Conv}(f(X))$. □

An *abstract simplicial complex* Δ is a family of finite sets such that any subset of a set in Δ belongs to Δ: $\forall X \in \Delta, \forall Y \subset X, \quad Y \in \Delta$. The *face order* of Δ is the partial ordering of the elements of Δ by $\subseteq$. A *geometric realization* of Δ is an injective mapping f of the *ground set* $|\Delta| = \bigcup_{X \in \Delta} X$ to some Euclidean space $\mathbb{R}^d$, such that, for any two elements (or *faces*) X, Y of Δ, the convex hulls of the images of X and Y have the convex hull of the image of $X \cap Y$ as their intersection: $\mathrm{Conv}(f(X)) \cap \mathrm{Conv}(f(Y)) = \mathrm{Conv}(f(X \cap Y))$. It is a folklore lemma that a mapping from $|\Delta|$ to $\mathbb{R}^d$ is a geometric realization of Δ if and only if disjoints faces of Δ are mapped to point sets with disjoint convex hulls.

It is well known that an abstract simplicial complex has a geometric realization in $\mathbb{R}^d$ when $d > 2(\max_{X \in \Delta} |X| - 1)$ and that, obviously, it has no geometric realization in $\mathbb{R}^d$ if $d < \max_{X \in \Delta} |X| - 1$.

Theorem 3.2 (Ossona de Mendez [2]) *Let* Δ *be an abstract simplicial complex, and let* d *be the dimension of the face order of* Δ. *Then,* Δ *has a geometric realization in* $\mathbb{R}^{d-1}$.

Proof: Consider the mapping from the ground set $|\Delta|$ of Δ to $\mathbb{R}^{d-1}$, whose existence is ensured by Corollary 3.1. Then, for any disjoint faces F, F' of Δ, we get $\mathrm{Conv}(f(F)) \cap \mathrm{Conv}(f(F')) = \emptyset$, that is: f induces a geometric realization of Δ in $\mathbb{R}^{d-1}$. □

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