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Crossing Minimization for 1-page and 2-page Drawings of Graphs with Bounded Treewidth

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Abstract

We investigate crossing minimization for 1-page and 2-page book drawings. We show that computing the 1-page crossing number is fixed-parameter tractable with respect to the number of crossings, that testing 2-page planarity is fixed-parameter tractable with respect to treewidth, and that computing the 2-page crossing number is fixed-parameter tractable with respect to the sum of the number of crossings and the treewidth of the input graph. We prove these results via Courcelle's theorem on the fixed-parameter tractability of properties expressible in monadic second order logic for graphs of bounded treewidth.

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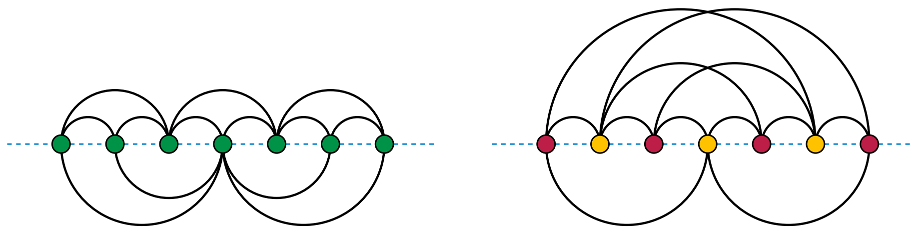

Figure 1: A 2-page book embedding of a planar graph (left) and a 2-page book drawing of the non-planar graph $K_{3,4}$ with two crossings, the minimum possible (right), both drawn as arc diagrams.

1 Introduction

A *k-page book embedding* of a graph G is a drawing that places the vertices of G on a line (the *spine* of the book) and draws each edge, without crossings, inside one of k half-planes bounded by the line (the *pages* of the book) [35, 42]. In one common drawing style, an *arc diagram*, the edges in each page are drawn as circular arcs perpendicular to the spine [49], but the exact shape of the edges is unimportant for the existence of book embeddings. These embeddings can be generalized to *k-page book drawings*: as before, we place each vertex on the spine and each edge within a single page, but with crossings allowed. The *crossing number* of such a drawing is defined to be the sum of the numbers of pairs of edges that cross within each page, and the *k-page crossing number* $\mathrm{cr}_k(G)$ is the minimum crossing number of any k-page book drawing [46]. Figure 1 shows examples of a 2-page book embedding and a minimum-crossing 2-page book drawing. In an optimal drawing, two edges in the same page cross if and only if their endpoints form interleaved intervals on the spine. Therefore, the problem of finding an optimal drawing may be described in purely combinatorial terms as the search for a permutation of the vertices and an assignment of edges to pages that minimizes the number of pairs of edges forming interleaved intervals on the same page.

As with most crossing minimization problems, k-page crossing minimization is NP-hard. Even the simple special case of testing whether the 2-page crossing number is zero is NP-complete [15], as is testing whether the 1-page crossing number is below a given threshold [40]. However, it may still be possible to solve these problems in polynomial time for restricted families of graphs and restricted values of k. For instance, Bannister, Eppstein and Simons [7] showed the computation of $\mathrm{cr}_1(G)$ and $\mathrm{cr}_2(G)$ to be fixed-parameter tractable in the almost-tree parameter. Here, a graph G has almost-tree parameter k if every biconnected component of G can be reduced to a tree by removing at most k edges. In this paper we significantly strengthen these results by finding fixed-parameter tractable algorithms for less-constraining parameters, allowing k-page crossing minimization to be performed in polynomial time for a much wider class of graphs.

1.1 New results

We design fixed-parameter algorithms for the following two problems:

- Computing the minimum number of crossings $\mathrm{cr}_1(G)$ in a 1-page drawing of a graph G.
- Computing the minimum number of crossings $\mathrm{cr}_2(G)$ in a 2-page drawing of G.

Ideally, fixed-parameter algorithms for crossing minimization should be parameterized by their *natural parameter*, which for this problem is the optimal number of crossings. We achieve this ideal bound, for the first time, for $\mathrm{cr}_1(G)$. However, for $\mathrm{cr}_2(G)$, even testing whether a given graph is 2-page planar (that is, whether $\mathrm{cr}_2(G) = 0$) is NP-complete [15]. Therefore, unless P = NP, there can be no fixed-parameter-tractable algorithm parameterized by the crossing number. Instead, we show that $\mathrm{cr}_2(G)$ is fixed-parameter tractable in the sum of the natural parameter and the treewidth of G. One consequence of our result on $\mathrm{cr}_2(G)$ is that it is possible to test whether a given graph has a 2-page book embedding, in time that is fixed-parameter tractable with respect to treewidth.

1.2 Solution technique

We construct these algorithms via Courcelle's theorem [17, 18], which connects the expressibility of graph properties in monadic second order logic with the fixed-parameter tractability of these properties with respect to treewidth. Recall that second order logic extends first order logic by allowing the quantification of k-ary relations in addition to quantification over individual elements. In monadic second order logic we are restricted to quantification over unary relations (equivalently subsets). When applied to the logic of graphs, this means that we are interested in logical formulas whose variables represent vertices, edges, sets of vertices, and sets of edges of the given graph, with predicates for incidence and membership. The property of having a 2-page book embedding is easy to express in (full) second-order logic, via the known characterization that a graph has such an embedding if and only if it is a subgraph of a Hamiltonian planar graph [8]. However, this expression is not allowed in monadic second-order logic because the extra edges needed to make the input graph Hamiltonian cannot be described by a subset of the existing vertices and edges of the graph. Instead, we prove a new structural description of 2-page planarity that is more easily expressed in monadic second order logic.

Like many earlier parameterized algorithms for related problems, our algorithms have a high dependence on their parameter, rendering them impractical. For this reason we have not attempted an exact analysis of their complexity nor have we searched for optimizations to our logical formulas that would improve this complexity.

1.3 Related work

As well as our already-mentioned previous work on crossing minimization for almost-trees [7], related results in fixed-parameter optimization of crossing number include a proof by Grohe, using Courcelle's theorem, that the topological crossing number of a graph is fixed-parameter tractable in its natural parameter [31]. This result was later improved by Kawarabayashi and Reed [36] to be linear in the graph size for any fixed parameter value. Based on these results the crossing number itself was also shown to be fixed-parameter tractable. Pelsmajer et al. showed a similar result for the odd crossing number [43]. Dujmović et al. showed that finding a layered drawing with k crossings and h layers is fixed-parameter tractable in the sum of these two parameters. Their result depends on a bound on the pathwidth of such a drawing, as a function of the two parameters. Here, pathwidth is a parameter closely related to treewidth [23]. We have also used Courcelle's theorem in graph drawing to find the *split thickness* of a graph, the minimum number of vertices into which each vertex should be split in order to produce a planar drawing [27]

Binucci et al. have investigated the *local crossing number* of book drawings [9]. This is a variant of the crossing number in which one counts crossings per edge rather than the total number of crossings of the entire graph. The 1-page graphs of bounded local crossing number can be recognized in quasi-polynomial time [13]. However, without restriction to book drawings, computing local crossing number is NP-hard even for graphs of bounded treewidth [5].

Our research investigates the worst-case parameterized time complexity of exact algorithms for k-page crossing minimization in general graphs, but other approaches to the problem include investigations of the k-page crossing number of special graphs [1, 19, 20, 28, 32]. Many authors have also developed and experimentally compared heuristic approaches to the same problems of minimizing crossings in book drawings of general graphs. For recent work in this area, see [33, 37, 45] and their references.

Subsequently to the appearance of the conference version of this paper [6], Kobayashi et al. [38] found an algorithm for one-page crossing minimization that uses an explicit dynamic program rather than Courcelle's theorem, obtaining running time $O(2^{O(k \log k)}n)$. Despite this improvement, we provide in this work the details for our slower solution to the same problem, as it provides many of the ideas necessary to understand our two-page crossing minimization algorithm.

2 Preliminaries

2.1 Bridges vs flaps and isthmuses

By a *cycle* in a graph we mean a simple cycle: a connected 2-regular subgraph. There is an unfortunate terminological confusion in graph theory: two different concepts, a maximal subgraph that is internally connected by paths that avoid a given cycle, and an edge whose removal disconnects the graph, are both commonly called *bridges*. We need both concepts in our algorithms. To avoid

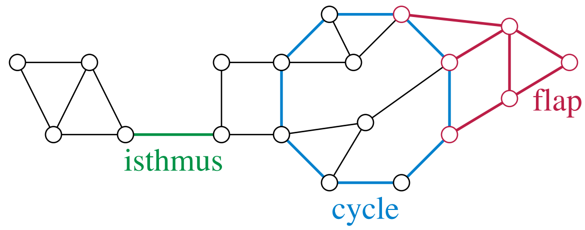

Figure 2: Clarification of our graph-theoretic terminology.

confusion, we call the subgraph-type bridges *flaps* and the edge-type bridges *isthmuses* (Figure 2). The term "flap" has been used with a similar but more general meaning in the theory of graph separators [2]. Although less common than "bridge", the term "isthmus" for a separating edge goes back to Tutte [48] and can still be found in some modern graph theory texts [12, 16].

To be more precise, given a graph G and a cycle C, we define an equivalence relation on the edges of $G \setminus C$ in which two edges are equivalent if they belong to a path that has no interior vertices in C, and we define a *flap* of C to be the subgraph formed by an equivalence class of this relation. (Different cycles may give rise to different flaps.) Given a graph G, we define an *isthmus* of G to be an edge of G that does not belong to any simple cycles in G.

2.2 Treewidth and graph minors

The *treewidth* of G can be defined to be one less than the number of vertices in the largest clique in a chordal supergraph of G that (among possible chordal supergraphs) is chosen to minimize this clique size [11]. Alternatively it can be described in terms of *tree decompositions*. A tree decomposition for a graph G is a tree T whose vertices (called *bags*) are labeled with subsets of vertices of G, such that the bags containing any vertex v of G form a connected subtree of T, and such that the two endpoints of each edge of G both belong to at least one shared bag. The width of a tree decomposition is one less than the largest cardinality of any of its bags, and the width of a graph G is the minimum width of any of its tree decompositions. The problem of computing the treewidth of a general graph is NP-hard [3], but it is fixed-parameter tractable in its natural parameter [10].

A graph H is said to be a *minor* of a graph G if H can be constructed from G via a sequence edge contractions, edge deletions, and vertex deletions. It can be determined whether a graph H is a minor of a graph G, in fixed-parameter tractable time (a polynomial in the size of G multiplied by a computable function of the size of H) [44].

2.3 Logic of graphs

We will be expressing graph properties in *extended monadic second-order logic* (MSO_2). This is a fragment of second-order logic that includes:

- variables for vertices, sets of vertices, edges, and sets of edges;
- binary relations for equality ($=$), inclusion of an element in a set ($\in$) and edge-vertex incidence (I);
- the standard propositional logic operations: $\neg, \wedge, \vee, \rightarrow$;
- the universal quantifier ($\forall$) and the existential quantifier ($\exists$), both which may be applied to variables of any of the four variable types.

To distinguish the variables of different types, we will use $u, v, w, \ldots$ for vertices, $e, f, g, \ldots$ for edges, and capital letters for sets of vertices or edges (with context making clear which type of set). Given a graph G and an MSO_2 formula ϕ we write $G \models \phi$ ("G models ϕ") to express the statement that ϕ is true for the vertices, edges, and sets of vertices and edges in G, with the semantics of this relation defined in the obvious way. MSO_2 differs from full second order logic in that it allows quantification over sets, but not over higher order relations, such as sets of pairs of vertices that are not subsets of the given edges. In Section 3, we provide a brief introduction to MSO_2 logic in which we describe how to express some of the properties we need for our results.

The reason we care about expressing graph properties in MSO_2 is the following powerful algorithmic meta-theorem due to Courcelle.

Lemma 1 (Courcelle's theorem [17, 18]) *Given an integer $k \geq 0$ and an MSO_2-formula ϕ of length ℓ, an algorithm can be constructed that takes as input a graph G of treewidth at most k and decides in $O\big(f(k,\ell)\cdot(n+m)\big)$ time whether $G \models \phi$, where the function f appearing in the time bound is a computable function of the treewidth k and formula length ℓ.*

2.4 Combinatorial enumeration of crossing diagrams

In order to show that the properties we study can be represented by logical formulas of finite length, we need to bound the number of combinatorially distinct ways that a subset of edges in a k-page graph drawing can cross each other.

We define a 1-*page crossing diagram* to be a placement of some points on the circumference of a circle, together with some straight line segments connecting the points such that each point is incident to a segment, no segment is uncrossed and no three segments cross at the same point (Figure 3). Two crossing diagrams are *combinatorially equivalent* if they have the same numbers of points and line segments and there exists a cyclic-order-preserving bijection of their points that takes line segments to line segments. The *crossing number* of a 1-page crossing diagram is the number of pairs of its line segments that cross each other.

We define a 2-*page crossing diagram* to be a 1-page crossing diagram together with a labeling of its line segments by two colors, such that every segment is

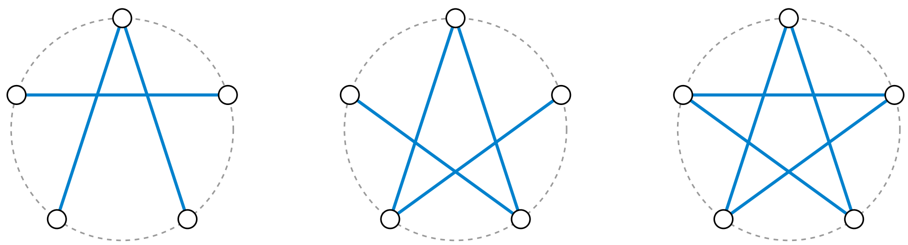

Figure 3: Three inequivalent 1-page crossing diagrams with five points. Every five-point 1-page crossing diagram is equivalent to one of these three diagrams. Their crossing numbers are 2, 3, and 5 respectively.

crossed by another segment of the same color. For a 2-page crossing diagram we define the *crossing number* to be the total number of crossing pairs of line segments that have the same color as each other.

Lemma 2 *There are $2^{O(k^2)}$ 1-page crossing diagrams with k crossings, and there are $2^{O(k^2)}$ 2-page crossing diagrams with k crossings.*

Proof: Place $4k$ points around a circle. Then every 1-page crossing diagram with k or fewer crossings can have at most $2k$ edges and at most $4k$ vertices, so it can be represented by choosing a subset of the points and a set of line segments connecting a subset of pairs of the points. There are $4k$ points and $4k(4k-1)/2$ pairs of points, so $2^{O(k^2)}$ possible subsets to choose.

Similarly, every 2-page crossing diagram with k or fewer crossings can be represented by a subset of the same $4k$ points, and by two disjoint subsets of pairs of points. The number of choices of these subsets can again be bounded by $2^{O(k^2)}$. □

Two combinatorially equivalent crossing diagrams, as defined above, may have a topology that differs from each other, or from combinatorially equivalent diagrams with curved edges (Figure 4). This is because, for an edge with multiple crossings, the order of the crossings along this edge may differ from one diagram to another, but this ordering is not considered as part of our definition of combinatorial equivalence. For our purposes such differences are unimportant, as we are concerned only with the total number of crossings. So we consider two crossing diagrams to be equivalent if they have the same crossing pairs of edges, regardless of whether the crossings occur in the same order.

For a related bound on 1-page crossing diagrams, see Kynčl [39, Prop. 7]. Kynč fixes the set of chords and the ordering of their endpoints (i.e., in our terminology, he fixes a choice of a single 1-page crossing diagram) and proves that, for this choice, there are at most 2^k different ways that this diagram can be realized by choosing the ordering of crossings along each segment. Instead, we consider only which pairs of segments cross (ignoring the order in which they cross along each segment) and bound the number of ways to choose the chords and the endpoint ordering in order to realize a diagram with k crossings.

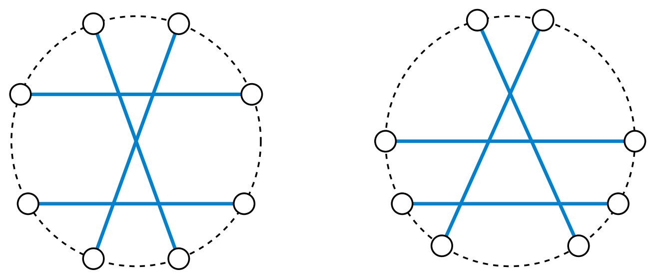

Figure 4: Two combinatorially equivalent 1-page crossing diagrams with different topologies. The set of pairs of segments that cross is the same in each diagram, but the ordering of the crossings along each segment is different.

3 Expressing graph properties in MSO_2

For readers unfamiliar with MSO_2 logic, we provide in this section some standard examples of graph properties that may be expressed in this logic, leading up to the properties that we use in our results. Additional examples may be found in one of the standard introductions to graph logic [18, 22, 29]. The building blocks in this section can be used to construct the formulas that we use throughout our paper.

Because the equal sign (=) is an element that is used within MSO_2 formulas, expressing the equality relation between two vertices, edges, or sets, we instead use the equivalence sign (≡) to express the syntactic equality of two formulas, or the assignment of a name to a formula.

3.1 k-Coloring

The formula $\textsc{color}_k$ that we construct below expresses the k-colorability of a graph. As a step towards the construction of $\textsc{color}_k$, we first construct a formula VERTEX-PARTITION expressing the property that a collection of vertex sets forms a partition of the vertices: the sets are disjoint from each other and their union contains all vertices in the graph.

$$\textsc{vertex-partition}(U_1, \ldots, U_k) \equiv (\forall v)\Bigg[\Big(\bigvee_{i=1}^{k} v \in U_k\Big) \wedge \Big(\bigwedge_{i \neq j} \neg(v \in U_i \wedge v \in U_j)\Big)\Bigg]$$

Although we write the vertex subset U_i using an indexed notation, the allowed operations in MSO do not include this kind of indexing. Instead, when this notation appears in our logical formulas, each U_i should be interpreted as a separate variable name. A formula EDGE-PARTITION expressing the property that a collection of edge sets forms a partition of the edges in the graph may be constructed in the same way by changing vertex variables to edge variables and vertex set variables to edge set variables.

With the ability to partition vertices we can now construct COLOR_k. The construction uses the fact that a k-coloring forms a partition of the vertices with the additional property that, for every color class C, all edges have an endpoint of a different color than C.

$$\begin{aligned}\text{COLOR}_k \equiv (\exists U_1, \ldots, U_k)\Big[&\text{VERTEX-PARTITION}(U_1, \ldots, U_k)\\ &\wedge \bigwedge_{i=1}^{k} (\forall e)(\exists v)[\mathrm{I}(e, v) \wedge v \notin U_i]\Big]\end{aligned}$$

3.2 Minor containment and planarity

Next, we construct a formula MINOR_H expressing the property that a graph has H as a minor. This resembles a coloring problem, where the colors are vertices of H: If we label each of the k vertices in H with a distinct number in the range from 1 to k, then H is a minor of G if and only if there exists a corresponding collection of k connected and disjoint subsets of the vertices of G, say U_1, ..., U_k, such that for each edge (i, j) in H there is an edge from a vertex in U_i to a vertex in U_j.

As part of this construction, we will use a formula CONNECTED expressing the property that a graph is connected. We will construct this formula by first constructing a formula DISCONNECTED expressing the property that a graph is disconnected. This is true if and only if the graph supports a nontrivial cut of the vertices with an empty cut-set.

$$\begin{aligned}\text{DISCONNECTED} \equiv (\exists U)\Big[&(\exists u, v)\big[u \in U \wedge v \notin U\big]\\ &\wedge \neg(\exists e)(\exists u, v)\big[\,\mathrm{I}(e, u) \wedge \mathrm{I}(e, v) \wedge u \in U \wedge v \notin U\big]\Big]\end{aligned}$$

We can now define CONNECTED $\equiv \neg$ DISCONNECTED. A similar construction leads to formulas CONNECTED-VERTICES(V) and CONNECTED-EDGES(E) expressing the properties that vertex set V describes a connected induced subgraph or that edge set E and the endpoints of edges in E describe a connected subgraph.

With the ability to express connectedness we can now construct $\textsc{minor}_H$.

$$\begin{aligned}\textsc{minor}_H \equiv \exists(U_1,\ldots,U_k)\Bigg[&\bigwedge_{i=1}^{k}(\exists u)[u\in U_i] \\ &\wedge \bigwedge_{i=1}^{k}\textsc{connected-vertices}(U_i) \\ &\wedge \bigwedge_{i\neq j}(\forall v)[v\notin U_i \vee v\notin U_j] \\ &\wedge \bigwedge_{(i,j)\in E_H}\textsc{connected-vertices}(U_i\cup U_j)\Bigg]\end{aligned}$$

We can express the existence of this formula as the following result.

Lemma 3 (Corollary 1.15 in [18]) *Given any fixed graph H there exists an MSO_2-formula $\textsc{minor}_H$ such that, for all graphs G, $G \models \phi$ if and only if G contains H as a minor.*

For instance, by Wagner's theorem, the planar graphs are precisely the graphs that have neither K_5 nor $K_{3,3}$ as minors. Therefore we can express the planarity of a graph in MSO_2, in terms of these forbidden minors, as

$$\textsc{planar} \equiv \neg\,\textsc{minor}_{K_5} \wedge \neg\,\textsc{minor}_{K_{3,3}}\,.$$

3.3 Hamiltonicity

Our last example will be a formula expressing the existence of a Hamiltonian cycle in a graph. A set of edges F in a graph is a union of vertex-disjoint cycles if every endpoint of an edge in F is incident to exactly two edges in F.[1] Thus,

$$\textsc{cycle-set}(F) \equiv (\forall e)(\forall v)\Big[\big(e\in F\wedge \mathrm{I}(e,v)\big)\rightarrow(\exists^{=2}f)\big[f\in F\wedge \mathrm{I}(f,v)\big]\Big]$$

expresses the property that F is a disjoint union of cycles. (Here $\exists^{=2}$ is a logical shorthand for the existence of exactly two objects satisfying the given property, i.e. that there exist f_1 and f_2 both satisfying the property, that f_1 and f_2 are unequal, and that there do not exist three unequal edges all satisfying the property.) Then a set of edges is a single cycle if it is a union of cycles and forms a connected subgraph. So we define

$$\textsc{cycle}(F) \equiv \textsc{cycle-set}(F) \wedge \textsc{connected-edges}(F),$$

A set of edges F spans a graph if every vertex is incident to at least one of the edges in F.

$$\textsc{span}(F) \equiv (\forall v)(\exists e)[e\in F\wedge \mathrm{I}(e,v)]$$

[1] An earlier version of this paper used an alternative formulation in which each edge in F is incident to exactly two other edges in F. However, this is also true of a claw $K_{1,3}$ as well as of a cycle.

Finally, a graph is Hamiltonian if it has a spanning cycle.

$$\textsc{hamiltonian} \equiv (\exists F)[\textsc{cycle}(F) \wedge \textsc{span}(F)]$$

4 One-page crossing minimization

In this section we provide the details of our method for one-page crossing minimization. Subsequently to the appearance of the conference version of our work [6], this method has been improved by Kobayashi et al. [38], who provided a faster direct dynamic programming algorithm. Nevertheless, we believe that this material is still relevant as context for our more complex two-page crossing minimization algorithm.

4.1 Outerplanarity

Recall that a graph is *outerplanar* if there exists a placement of its vertices on the circumference of a circle such that when its edges are drawn as straight line segments they do not cross. Topologically, the circle and the half-plane are equivalent, so a graph is outerplanar if and only if it has a crossing-free 1-page drawing. For incorporating a test of outerplanarity into methods using Courcelle's theorem, it is convenient to use a standard characterization of the outerplanar graphs by forbidden minors:

Lemma 4 (Chartrand and Harary [14]) *A graph G is outerplanar (1-page planar) if and only if it contains neither K_4 nor $K_{2,3}$ as a minor.*

Let $\textsc{outerplanar}$ be the formula $\neg\,\textsc{minor}_{K_4} \wedge \neg\,\textsc{minor}_{K_{2,3}}$ combining two minor-containment formulas from Lemma 3. Then Lemma 4 implies that, for all graphs G, $G \models \textsc{outerplanar}$ if and only if G is outerplanar. Because outerplanar graphs have bounded treewidth (at most two), Courcelle's theorem guarantees the existence of a linear time algorithm for testing outerplanarity. There are of course much simpler linear time algorithms for testing outerplanarity [41, 50].

4.2 Crossings vs treewidth

Next, we relate the natural parameter for 1-page crossing minimization (the number of crossings) to the parameter for Courcelle's theorem (the treewidth). This relation will allow us to construct a fixed-parameter-tractable algorithm for the natural parameter.

A *k-clique sum* of two disjoint graphs each containing a k-clique is formed by bijectively identifying each vertex of one k-clique with a vertex of the other k-clique, and then removing one or more of the k-clique edges from the resulting combined graph.

Lemma 5 (Lemma 1 in [21]) *If G_1 and G_2 each have treewidth at most w, then any clique-sum of G_1 and G_2 also has treewidth at most w.*

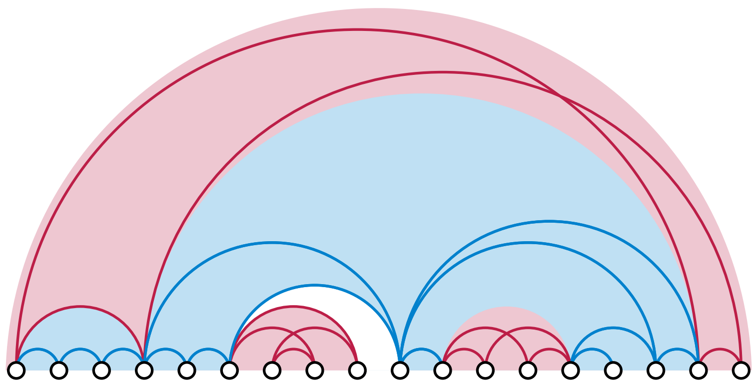

Figure 5: An example of the clique-sum decomposition in Lemma 6. The red regions represent the components with crossings and the blue regions represent outerplanar components. The entire graph may be reconstructed by performing clique-sums on the region boundaries.

Lemma 6 *Every graph G has treewidth $O(\sqrt{\mathrm{cr}_1(G)})$.*

Proof: Let G be a graph with $\mathrm{cr}_1(G) = k$, and D a 1-page drawing of G with k crossings. Then let H be the subgraph of G induced by the endpoints of crossed edges in D. (H is shown as the set of red edges in Figure 5.)

If the edges of H are removed from G, the remaining graph $G \setminus H$ (shown as blue in the figure) has no crossings, so it is outerplanar, and each of its biconnected components is again outerplanar. Because they are outerplanar, their treewidth is at most two.

As can be seen in the figure, G can be decomposed as a clique-sum of the biconnected components of H and of $G \setminus H$, with 1-clique-sums where two components meet at a single articulation vertex of G and 2-clique-sums where a biconnected component of H and a biconnected component of $G \setminus H$ share the same two vertices. Since each clique-sum operation preserves treewidth, and the treewidth of the biconnected components of $G \setminus H$ is at most two, the treewidth of G is bounded by the treewidth of the biconnected components of H.

From each biconnected component C of H we create a planar graph C' by planarizing C with respect to the drawing D. That is, we replace each crossing point of two edges by a new vertex, and we replace each crossed edge by a path through these subdivision vertices. Since C' is a planar graph with $O(k)$ vertices it has treewidth $O(\sqrt{k})$. A tree-decomposition of C' can be transformed into a tree decomposition of C by replacing each subdivision vertex in each bag of the tree decomposition by the four endpoints of its associated two crossing edges, so C also has treewidth $O(\sqrt{k})$, as its treewidth is at most four times that of C'. □

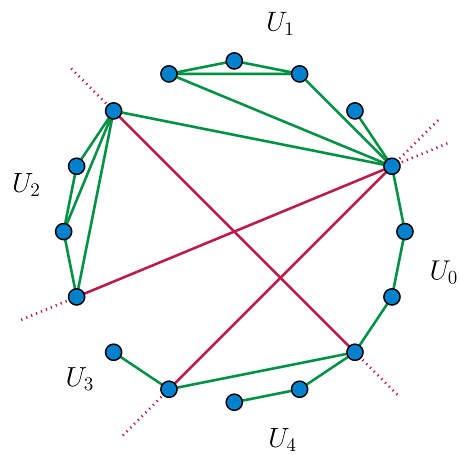

Figure 6: A 1-page drawing of a graph with two crossings and five outerplanar subgraphs, showing the subsets U_i of 1.

4.3 Logical characterization

Let G be a graph with bounded 1-page crossing number, and consider a drawing of G achieving this crossing number. Then the set of crossing edges of the drawing partitions the halfplane into an arrangement of curves, and we can partition G itself into the subgraphs that lie within each face of this arrangement. Each of these subgraphs is itself outerplanar, because it lies within a subset of the halfplane (with its vertices on the boundary of the subset) and has no more crossing edges; see Figure 6. This intuitive idea forms the basis for the following characterization of the 1-page crossing number, which we will use to construct an MSO_2-formula for the property of having a drawing with low crossing number.

Observation 1 *A graph $G = (V, E)$ has $\mathrm{cr}_1(G) \leq k$ if and only if there exist edges $F = \{e_0, \ldots, e_r\}$ with $r = O(k)$, vertices $W = \{v_0, \ldots, v_\ell\}$ with $\ell = O(k)$, and a partition $U_0, \ldots, U_\ell$ of $V \setminus W$ into (possibly empty) subsets, satisfying the following properties:*

1. *W is the set of vertices incident to edges in F.*
2. *F contains all edges in the induced subgraph on W.*
3. *There are no edges between U_i and U_j for $i \neq j$.*
4. *There is an outerplanar embedding of the induced subgraph on $U_i \cup \{v_i, v_{i+1}\}$ with v_i and v_{i+1} consecutive in the spine ordering for all $0 \leq i < \ell$.*
5. *The edges in F produce at most k crossings when their endpoints (the vertices in W) are placed in order according to their indices.*

We now construct a formula $\textsc{onepage}_k$, based on 1, such that $G \models \textsc{onepage}_k$ if and only if $\mathrm{cr}_1(G) \le k$. The formula $\textsc{onepage}_k$ will have the overall form of a disjunction, over all crossing configurations, of a conjunction of sub-formulas representing Properties 1–4 in 1. Property 5 will be represented implicitly, by the enumeration of crossing configurations. The first three properties are easy to express directly: the formulas

$$\begin{aligned} \theta_1(W,F) &\equiv (\forall v)[v \in W \leftrightarrow (\exists e)[e \in F \wedge I(e,v)]] \\ \theta_2(F,W) &\equiv (\forall e)[(\forall v)[I(e,v) \rightarrow v \in W] \rightarrow e \in F] \\ \theta_3(U_i,U_j) &\equiv \neg(\exists e)(\exists u,v)[I(e,u) \wedge I(e,v) \wedge u \in U_i \wedge v \in U_j] \end{aligned}$$

express in MSO_2 Properties 1, 2, and 3 of 1 respectively.

To express Property 4 we use the following characterization of consecutive pairs of vertices in outerplanar embeddings:

Lemma 7 *The following three conditions on an undirected graph G with designated vertices u and v are equivalent to each other:*

1. *G has an outerplanar embedding with u and v consecutive in the spine ordering.*
2. *G is K_4-minor-free, $K_{2,3}$-minor-free, and has no C_4 (four-vertex cycle) minor such that u and v belong to subsets U_i for opposite vertices of the C_4.*
3. *The graph G' formed from G by adding a new vertex w and edges uw and vw is outerplanar.*

Proof: We prove separate implications between these three conditions, as follows.

(1) ⇒ (2)**:**
Because G is assumed outerplanar, it has no K_4 or $K_{2,3}$ minor by Lemma 4. If it had a C_4 minor in which u and v belong to subsets U_i for opposite vertices of the C_4, this minor would necessarily be obtained by a sequence of vertex deletions, edge deletions, and edge contractions (for edge contractions that would not merge u and v into the same supervertex). However, this is impossible, as each of these operations preserves the existence of an outerplanar embedding with u and v consecutive, and they would not be consecutive in the resulting C_4 minor.

(2) ⇒ (3)**:**
We assume the contrary, that Condition 3 fails, and prove that this implies the existence of K_4, $K_{2,3}$, or C_4 (with u and v opposite) as a minor in G. If Condition 3 fails, then G' is not outerplanar and by Lemma 4 it contains a K_4 or $K_{2,3}$ minor H. As discussed in Subsection 3.2, having H as a minor means that the vertices of H can be associated with disjoint connected subsets U_i of vertices of G' in such a way that each edge of

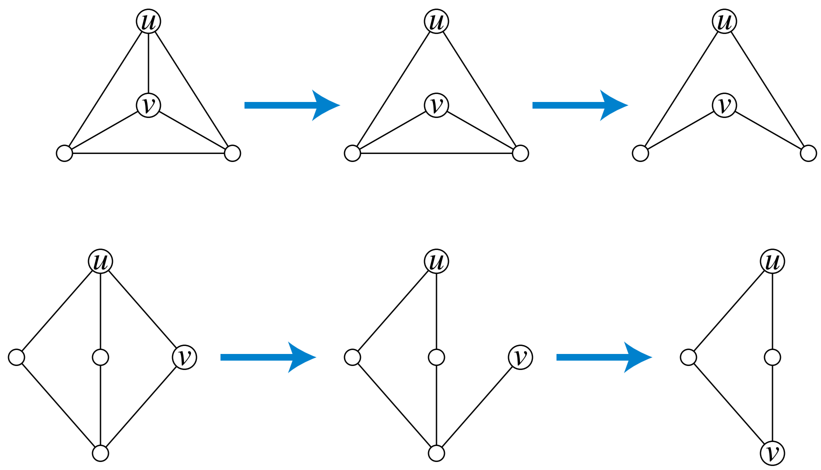

Figure 7: Cases for when $\hat{w} = \hat{u}$ or $\hat{w} = \hat{v}$ (but not both) in Lemma 7. Top: If H is K_4 (top left), then removal of w may eliminate the edge $\hat{u}\hat{v}$ from H (top middle). Removing one more edge leaves a C_4 minor with $\hat{u}$ and $\hat{v}$ opposite (top right). Bottom: If H is $K_{2,3}$ (bottom left), then removing v and eliminating edge $\hat{u}\hat{v}$ leaves a graph with a four-cycle and an extra edge (bottom middle). Contracting the extra edge produces a C_4 minor with $\hat{u}$ and $\hat{v}$ opposite (bottom right).

H can be represented by an edge between the two subsets corresponding to its endpoints. Let $\hat{u}$, $\hat{v}$, and $\hat{w}$ denote the vertices of H (if they exist) whose sets U_i contain u, v, or w. We consider the following cases for how w can participate in this representation.

- If w does not belong to any of the subsets U_i, so $\hat{w}$ does not exist, then H forms a K_4 or $K_{2,3}$ minor in G.
- If w is the only member of its subset U_i, then (as w has only two adjacencies in G') $\hat{w}$ must have degree two in H, and its two neighbors must be two distinct vertices $\hat{u}$ and $\hat{v}$. In this case, H must be $K_{2,3}$ with $\hat{u}$ and $\hat{v}$ as its two degree-three vertices. Removing w from G' and $\hat{w}$ from H leaves a C_4 minor in G in which u and v are opposite.
- In the remaining cases, $\hat{w} = \hat{u}$ or $\hat{w} = \hat{v}$. Suppose first that $\hat{u}$ and $\hat{v}$ are distinct. Then the removal of w from G' cannot disconnect the subset U_i containing w, but it can eliminate the edge between $\hat{u}$ and $\hat{v}$. If H is K_4, the subgraph of H obtained by removing this edge can be transformed into C_4 with u and v opposite by removing one more edge, the one between the other two vertices (Figure 7, top). If H is $K_{2,3}$, then the subgraph obtained by removing edge $\hat{u}\hat{v}$ can be transformed into a C_4 with u and v opposite by the contraction of one more edge, the one that is not in the remaining 4-cycle (Figure 7,

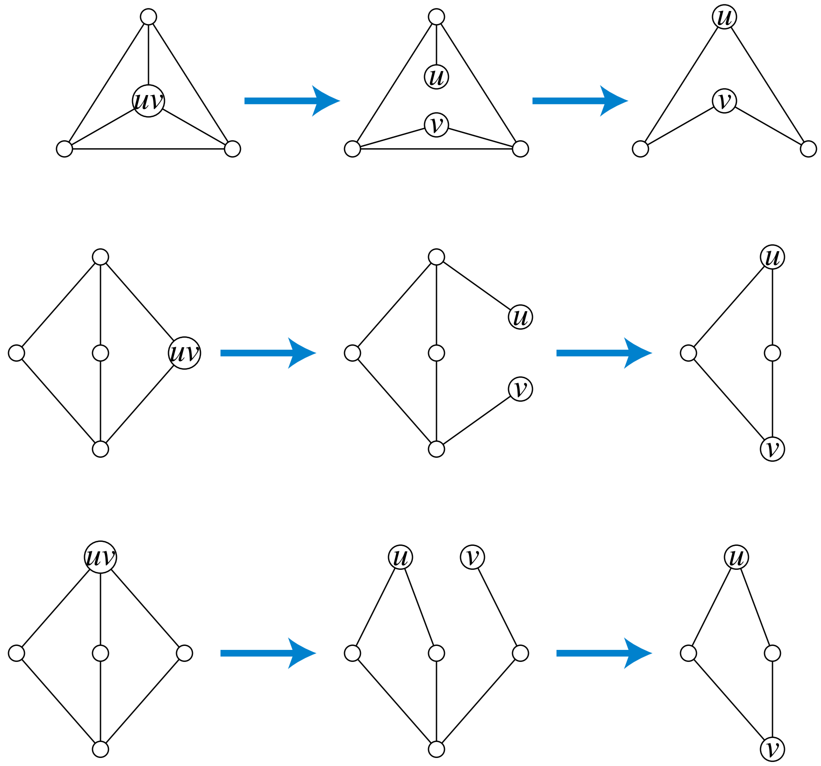

Figure 8: Cases for when $\hat{u} = \hat{v} = \hat{w}$ in Lemma 7, so that removing w may split this vertex of H into two vertices. Top: H is K_4. Middle: H is $K_{2,3}$ and $\hat{w}$ is a degree-two vertex in H. Bottom: H is $K_{2,3}$ and $\hat{w}$ is a degree-three vertex in H. In all cases, the remaining graph after the split has a C_4 minor with $\hat{u}$ and $\hat{v}$ opposite.

bottom).

- Finally, suppose that $\hat{u} = \hat{v} = \hat{w}$. If the removal of w from G' does not disconnect the set U_i containing u, v, and w, then H is a K_4 or $K_{2,3}$ minor of G. If removing w does disconnect this set, it disconnects it into two non-adjacent components, forming a minor of G in which one of the vertices of H has been split into two, and in which the edges incident to the split vertex have been assigned to one of its two copies. Note also that u belongs to one copy, and v belongs to the other copy. We have the following sub-cases:
  - If all of the edges incident to the split vertex are assigned to the same copy of that vertex, then (ignoring the other copy) we have H as a minor of G.
  - If H is K_4, then splitting $\hat{w}$ into the two vertices $\hat{u}$ and $\hat{v}$ leaves a

graph in which contracting one edge (the one incident to whichever of $\hat{u}$ or $\hat{v}$ has degree one) and then deleting one edge (the one incident to neither $\hat{u}$ nor $\hat{v}$) produces a C_4 minor with $\hat{u}$ and $\hat{v}$ opposite (Figure 8, top).

- If H is $K_{2,3}$, and the split vertex has degree two in H, then splitting $\hat{w}$ into the two vertices $\hat{u}$ and $\hat{v}$ leaves a graph in the form of a four-cycle with two additional edges, connecting opposite vertices of the four-cycle to $\hat{u}$ and $\hat{v}$. Contracting these two additional edges produces a C_4 minor with $\hat{u}$ and $\hat{v}$ opposite (Figure 8, middle).
- If H is $K_{2,3}$, and the split vertex has degree three in H, then splitting $\hat{w}$ into the two vertices $\hat{u}$ and $\hat{v}$ leaves a graph in the form of a four-cycle containing one of the two vertices $\hat{u}$ or $\hat{v}$, with the opposite vertex of the four-cycle connected by a two-edge path to the other of $\hat{u}$ or $\hat{v}$. Contracting this two-edge path produces a C_4 minor with $\hat{u}$ and $\hat{v}$ opposite (Figure 8, bottom).

(3) $\Rightarrow$ (1)**:**
By the assumption that Condition 3 holds, G' has an outerplanar drawing. In this drawing, the edges uw and vw partition the bounding disk of the drawing into three regions: a region bounded by edge uw and incident to vertices u and w (but not to vertex v), a second region bounded by edge vw and incident to vertices v and w (but not to u), and a third region bounded by both edges and incident to all three vertices. If the first region is non-empty, the vertices and edges within it touch only each other and u, and can be reflected across u into the space between u and the next vertex on the other side of w, emptying the region without affecting the outerplanarity of the drawing. Similarly, if the second region is non-empty, its vertices and edges touch only each other and v, and can be reflected across v into the space between v and the next vertex on the other side of w, again emptying the region without affecting the outerplanarity of the drawing. Once both of the first two regions have been emptied in this way, w can be removed from the drawing to create an outerplanar drawing of G in which u and v are adjacent.

Thus, each condition implies the other two, so the three conditions are equivalent. □

Corollary 1 *Property 4 can be expressed as an* MSO$_2$*-formula* $\theta_4(U_i, v_i, v_j)$*.*

Proof: We may easily modify Lemma 3 to recognize the three forbidden minors of Lemma 7, by restricting the edges that participate in the minor to the given parameter U_i of θ_4 and by checking that v_i and v_j correspond to opposite vertices of any C_4 minor. □

Lemma 2 tells us that there are $2^{O(k^2)}$ ways of satisfying Property 5 of 1. For each crossing diagram D with k crossings we can construct a formula $\alpha_D(v_0, \ldots, v_\ell, e_0, \ldots, e_r)$ specifying that the vertices $v_0, \ldots, v_\ell$ and edges

$e_0, \ldots, e_r$ are in configuration D. We then construct the formula

$$\begin{aligned}\beta_D \equiv (\exists v_0, \ldots v_\ell)(\exists e_0, \ldots, e_r)(\exists U_0, \ldots, U_\ell) \\ \Big[\alpha_D(v_0, \ldots, v_\ell, e_0, \ldots, e_r) \wedge \Big(\bigcup_{i=0}^{\ell} U_i\Big) = V \setminus \{v_0, \ldots, v_\ell\} \\ \wedge \bigwedge_{i \neq j} \big(U_i \cap U_j = \emptyset\big) \\ \wedge\, \theta_1(v_0, \ldots, v_\ell; e_0, \ldots, e_r) \\ \wedge\, \theta_2(e_0, \ldots, e_r; v_0, \ldots, v_\ell) \\ \wedge \bigwedge_{i \neq j} \theta_3(U_i, U_j) \\ \wedge \bigwedge_{i=0}^{\ell} \theta_4(U_i, v_i, v_{i+1})\Big]\end{aligned}$$

of length $O(k^2)$. This formula expresses the property that, in the given graph G, we can construct a crossing diagram of type D, and a corresponding partition of the vertices into subsets U_i, that obeys Properties 1–4 of 1. By 1, this is equivalent to the property that G has a 1-page drawing with k crossings in configuration D. Finally, we construct ONEPAGE_k by taking the disjunction of the β_D where D ranges over all crossing diagrams with $\leq k$ crossings. Thus, ONEPAGE_k is a formula of length $2^{O(k^2)}$, expressing the property that $\text{cr}_1(G) \leq k$.

Theorem 1 *There exists a computable function f such that $\text{cr}_1(G)$ can be computed in $O(f(k)n)$ time for a graph G with n vertices and with $k = \text{cr}_1(G)$.*

Proof: We have shown the existence of a formula ONEPAGE_k such that a graph $G \models \text{ONEPAGE}_k$ if and only if $\text{cr}_1(G) \leq k$. By Lemma 6, the treewidth of any graph with crossing number k is $O(k)$. Applying Courcelle's theorem with the formula ONEPAGE_k and the $O(k)$ treewidth bound, it follows that computing $\text{cr}_1(G)$ is fixed-parameter tractable in k . □

5 Two-page planarity

A classical characterization of the graphs with planar 2-page drawings is that they are exactly the subhamiltonian planar graphs:

Lemma 8 (Bernhart and Kainen [8]) *A graph is 2-page planar if and only if it is the subgraph of planar Hamiltonian graph.*

However, this characterization does not directly help us to construct an MSO_2-formula expressing the 2-page planarity of a graph, as we do not know how to construct a formula that asserts the existence of a supergraph with the given property. Hamiltonicity and planarity are both straightforward to express in MSO_2, but there is no obvious way to describe a set of edges that may be of more than constant size, is not a subset of the existing edges, and can be used to augment the given graph to form a planar Hamiltonian graph.

For this reason we provide a new characterization, which we model on a standard characterization of planar graphs: a graph is planar if and only if, for every cycle C, the flaps of C can be partitioned into two subsets (the interior and exterior of C) such that no two flaps in the same subset cross each other. For instance, this characterization has been used as the basis for a cubic-time divide and conquer algorithm for planarity testing, which recursively subdivides the graph into cycles and non-crossing subsets of flaps [4, 30, 47]. In our characterization of 2-page graphs, we apply this idea to a special set of cycles, the cycles that lie within one halfplane and are not surrounded by any other cycles. The cycles of this type are edge-disjoint, and if a single cycle of this type has been identified then its interior flaps can also be identified easily: each interior flap is a single edge, and an edge forms an interior flap if and only if it belongs to the same page as the cycle in the book embedding and has both its endpoints on the cycle. As well as identifying which of the two pages each edge of a given graph is assigned to, our MSO_2 formula will partition the edges into three different types of edges: the ones that belong to these special cycles, the ones that form interior flaps of these special cycles, and the remaining *isthmus* edges that, if deleted, would disconnect parts of their page.

Suppose we are given a graph $G = (V, E)$ and a partition of its edges into two subsets A, B, intended to represent the two pages of a 2-page drawing of G. We define the graph separate$(G; A, B)$ that splits each vertex of G into two vertices, one in each page, with a new edge connecting them. Thus, separate$(G; A, B)$ has $2n$ vertices, which can be labeled by pairs of the form (v, X) where v is a vertex in V and X is one of the two sets in A, B. It has an edge between (x, X) and (y, Y) if either of two conditions is met: (1) $x = y$ and $X \neq Y$, or (2) $X = Y$ and there is an edge between x and y in X.

See Figure 10 for an illustration of the separate$(G; A, B)$ construction.

Lemma 9 *A graph $G = (V, E)$ is 2-page planar if and only if there exists a partition A_b, A_c, A_d, B_b, B_c, B_d of the edge set E into six subsets such that, for each of the two choices of $X = A$ and $X = B$, these subsets satisfy the following properties:*

1. *X_c is a union of edge-disjoint cycles.*
2. *$X_c \cup X_b$ does not contain any additional cycles that involve edges in X_b.*
3. *For every edge e in X_d there exists a cycle in X_c containing both endpoints of e.*
4. *The graph formed by the edges $X_d \cup X_c \cup X_b$ is outerplanar.*
5. *For each cycle C in X_c it is not possible to find two vertex-disjoint paths P_1 and P_2 in E such that neither path is a single edge in X_d, all four path endpoints are distinct vertices of C, neither path contains a vertex of C in*

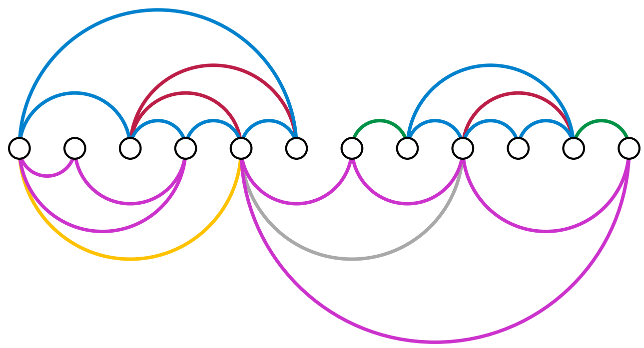

Figure 9: A 2-page planar graph with its edges partitioned into the six sets A_b (green edges), A_c (blue edges), A_d (red edges), B_b (yellow edges), B_c (purple edges), and B_d (gray edges).

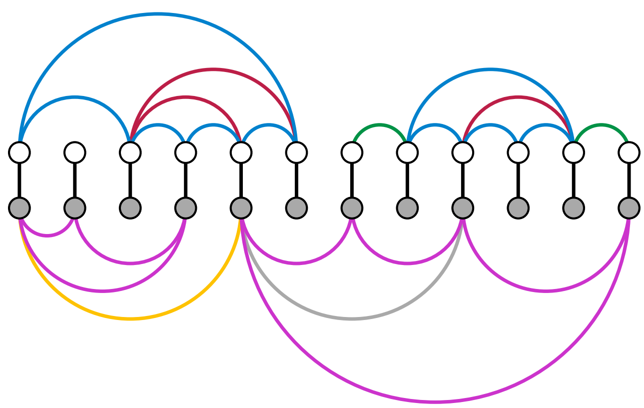

Figure 10: The graph separate$(G; A, B)$ where G is the graph in Figure 9, and A and B are respectively the edges in the first and second page.

its interior, and the two pairs of path endpoints are in crossing position on C.

6. *The subdivision* separate$(G; A_b \cup A_c \cup A_d, B_b \cup B_c \cup B_d)$ *is planar.*

Proof: Suppose G has a 2-page planar drawing. This drawing partitions the edges of G into two sets A and B. For $X = A$ or B, let X_c be the set of edges X forming a union of edge disjoint cycles that surround a maximal subset of their page. Then let X_d be the edges in X drawn in the interior of one of these cycles, and X_b the remaining edges in X. Figure 9 illustrates this division of edges into six subsets. It can be easily verified that the constructed partition satisfies Properties 1 through 6.

Conversely, suppose we have a graph G with a partition of its edges satisfying the properties of the lemma. By Property 6, separate$(G; A_b \cup A_c \cup A_d, B_b \cup B_c \cup B_d)$

has a planar embedding. We may assume without loss of generality that, in this embedding, the cycles of X_c given by Property 1 separate the edges of X_d (interior to the cycles) from the rest of the graph (exterior to the cycles). For, by Property 4, no two interior edges can cross, and by Property 5, no two exterior paths can cross. So, if we have a cycle in X_c that does not properly separate X_d from the rest of the graph, we may modify the embedding to flip the edges of X_d into the interior of the cycle and to flip the components of the rest of the graph to the exterior of the cycle, preserving the (reflected) planar embedding of each flipped component, without introducing any new crossings. By performing this flipping operation to all cycles of A_c and B_c, we obtain an embedding in which the cycles of X_c separate X_d from the rest of the graph, as stated above.

Next, given this embedding of separate$(G; A_b \cup A_c \cup A_d, B_b \cup B_c \cup B_d)$, we contract all of the cycles (X_c) and isthmuses (X_b) in each page ($X = A$ and B), maintaining the orientation and embedding of the edges that were not contracted (Figure 11). As a consequence, the edges in X_d within each cycle of X_c are also contracted. However, in the embedding of separate$(G; A_b \cup A_c \cup A_d, B_b \cup B_c \cup B_d)$, none of the contracted cycles surrounds any part of the graph that is not itself contracted. Because the edges of G are all contracted, the remaining uncontracted edges are only the ones separating A from B, so the contracted graph is bipartite. As a result, we are left with an embedding of a planar embedded bipartite multigraph that has one edge (v, A)–(v, B) for each vertex v in the original graph. Because this multigraph is bipartite, its dual graph has even degree at every vertex, and as the dual graph of a planar graph it is necessarily connected. Thus, the dual of the bipartite multigraph has an Euler tour, and (as with any Eulerian planar graph) this Euler tour can be made non-self-crossing by local uncrossing operations at each vertex. This tour can be represented geometrically as a Jordan curve J that passes through the faces of the embedding of separate$(G; A_b \cup A_c \cup A_d, B_b \cup B_c \cup B_d)$ (in some cases more than once per face) and crosses each edge (v, A)–(v, B) exactly once.

From the embedding of separate$(G; A_b \cup A_c \cup A_d, B_b \cup B_c \cup B_d)$ we can obtain a planar embedding of G itself by contracting all the edges of the form (v, A)–(v, B). If we augment G by adding an edge uv between any two vertices u and v whose edges (u, A)–(u, B) and (v, A)–(v, B) are crossed consecutively by the Jordan curve J, then J can be used to guide a non-crossing placement of these additional edges within the resulting embedding of G. Thus, we have augmented G to a Hamiltonian planar supergraph. The result that G has a 2-page book embedding follows by Lemma 8. □

We construct a formula TWOPAGE based on Lemma 9 with the property that $G \models$ TWOPAGE if and only if G is 2-page planar. First, we construct formulas $\theta_1, \ldots, \theta_5$ expressing Properties 1 through 5 in Lemma 9, as we did for 1-page crossing. Each of these properties has a straightforward expression in MSO_2. To express Property 6 we will need the following technical lemma, which can be proved using the method of syntactic interpretations. (For details on this method see [26, 31].)

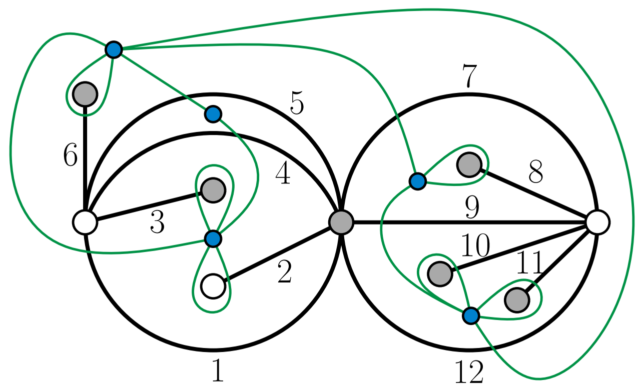

Figure 11: The contraction of the graph in Figure 10 and its planar dual (drawn with blue vertices and green edges). The edge labels correspond to the Hamiltonian cycle ordering of the vertices of G.

Lemma 10 *For every* MSO_2*-formula* ϕ *there exists an* MSO_2*-formula* $\phi^*(A, B)$ *such that* $G \models \phi^*(A, B)$ *if and only if* $\mathrm{separate}(G; A, B) \models \phi$.

Now, we can express Property 6 as an MSO_2-formula θ_6 using Lemma 10, as planarity is expressible by Lemma 3 and the fact that planar graphs are the graph that avoid K_5 and $K_{3,3}$ as minors. Thus, we define TWOPAGE to be the formula expressing the existence of $A_b, A_c, A_d, B_b, B_c, B_d$ satisfying $\theta_1, \ldots \theta_6$.

Theorem 2 *There exists a computable function* f *and an algorithm that can decide whether a given graph with treewidth* k *is* 2*-page planar in* $O(f(k)n)$ *time.*

Proof: The result follows from Courcelle's theorem together with the construction of the MSO_2 formula TWOPAGE representing the existence of a two-page planar embedding. □

6 Two-page crossing minimization

We now extend the results of the previous section from 2-page planarity to 2-page crossing minimization. As in the 1-page case, we will use a formula that involves a disjunction over crossing diagrams. Given a crossing diagram D with k crossings and $r+1$ edges, whose graph is G, we define the *planarization* of G with respect to D to be the graph in which each edge e_i is replaced by a path of degree four vertices, such that two of these replacement paths share a vertex if and only if the original two edges cross in D. As explained earlier, we do not care about the order of crossings along each edge: two crossing diagrams with the same sets of crossing pairs but with different crossing orders are considered equivalent. Nevertheless, we do preserve the order of crossings from (one representative of an equivalence class of) crossing diagrams to their planarizations, in order to ensure that the planarizations form planar graphs.

Lemma 11 *A graph $G = (V, E)$ has $\mathrm{cr}_2(G) \le k$ if and only if there exists edges $e_0, e_1, \cdots, e_r$ with $r < 2k$ and a 2-page crossing diagram D with k crossings on these edges such that when G is planarized with respect to D the resulting graph $G_D = (V_D, E_D)$ has a partition of E_D into $A_b, A_c, A_d, B_b, B_c, B_d$ such that, for $X = A, B$:*

1. *X_c is a union of edge disjoint cycles.*
2. *None of the cycles of $X_c \cup X_b$ contains an edge in X_b.*
3. *If e is an edge introduced in the planarization, then $e \in A_b \cup A_c \cup A_d$ if e is in the first page of D, and $e \in B_b \cup B_c \cup B_d$ if it is in the second page of D.*
4. *Each endpoint of an edge in X_d either belongs to an edge in X_c or is a crossing of D.*
5. *Every path of edges in X_d that starts and ends in vertices of X_c, with no interior points that belong to X_c, starts and ends in vertices of the same cycle in X_c.*
6. *For every cycle C in X_c, let P_C be the subset of X_d consisting of edges that belong to at least one path of edges in X_d that starts and ends at vertices of C and has no interior vertices in C. Let H_C be the graph formed from $C \cup P_C$ by adding a single new vertex incident to all vertices in C. Then $C \cup P_C$ is planar.*
7. *Each edge in X_i belongs to a unique subset P_C.*
8. *For each cycle C in X_c there do not exist two vertex-disjoint paths in E, such that neither path uses edges of $A_d \cup B_d$ nor has any interior vertices on C, with four distinct endpoints on C in crossing position.*
9. *the subdivision $\mathrm{separate}(G; A_b \cup A_c \cup A_d, B_b \cup B_c \cup B_d)$ is planar.*

Proof: We follow the same general steps as the proof of Lemma 9.

If G has $\mathrm{cr}_2(G) \le k$, consider any 2-page drawing with crossing number k, find the diagram D of its crossing edges, and planarize the drawing to produce G_D. Partition G_D into two subgraphs A and B according to the pages of D. For $X = A, B$, let X_c be the graph formed by the cycles in X that are not surrounded in their page by any other cycle, let X_b be the subgraph formed by the edges of X that are not surrounded by cycles of X_c, and let X_d be the edges of X that are surrounded by cycles of X_c. Then the first three items in the lemma follow by construction. Item (4) follows from the fact that all vertices of G belong to the spine of the drawing, so all vertices of G_D that are surrounded by cycles of X_c must correspond to crossings in G. Item (5) follows from the Jordan curve theorem. In item (6), each of the paths defining P_C must lie entirely within P_C in the drawing, for otherwise the path together with an arc of C would form a cycle that surrounds C, contradicting the definition of X_c.

As a subgraph of G_D, $C \cup P_C$ is planar, and because P_C is entirely surrounded by C, adding an extra vertex incident to all vertices of C does not affect its planarity. Item (7) again uses the fact that the subgraphs P_C are surrounded by their cycles, together with the Jordan curve theorem. In item (8), the two paths would both have to be exterior to C in the drawing of G, and would necessarily cross each other. But because the paths avoid $A_d \cup B_d$, they cannot pass through any crossings of the drawing. This contradiction shows that the two paths in question cannot exist. Finally, for item (9), a planar drawing of the subdivision may be obtained from the planar drawing of G_D by replacing the spine of the 2-page drawing by a narrow strip, and replacing each vertex along the spine by two copies of the vertex connected by an edge, as was already depicted (for 2-page embeddings without crossings) in Figure 10.

In the other direction, suppose that the edges e_i, crossing diagram D, and edge partition obeying the conditions of the lemma all exist. By item (9) we can find a planar embedding of separate$(G; A_b \cup A_c \cup A_d, B_b \cup B_c \cup B_d)$. By items (6) and (8) we can modify this drawing (if necessary) by flipping flaps of cycles in X_c so that the flaps in X_d lie inside these cycles and the other flaps lie outside the cycles, without causing any additional crossings with these flips. As in Lemma 9, we then contract all the edges of the embedding except the separation edges to obtain a planar-embedded bipartite multigraph, and use a non-crossing Euler tour of the planar dual of this multigraph to guide a Hamiltonian cycle in an augmentation of the given crossing diagram. This part of the proof is unchanged from Lemma 9, as the parts of G_D that differ from G will all have been contracted. □

The conditions of Lemma 11 do not enforce the condition that each crossing of D remains a crossing in the resulting diagram. Violating this condition this can only reduce the total number of crossings, and does not affect the conclusion of the lemma.

Now, we construct an MSO$_2$-formula ζ_k based on Lemma 11 such that $G \models \zeta_k$ if and only if $\mathrm{cr}_2(G) = k$. To handle the planarization process we use the following lemma. In the lemma, the notation $G^{e_1 \times e_2}$ describes the graph obtained from a graph G by deleting two edges e_1 and e_2 that do not share a common endpoint, and adding a new degree-4 vertex connected to the endpoints of e_1 and e_2.

Lemma 12 (Grohe [31]) *For every* MSO$_2$*-formula* ϕ *there exists an* MSO*-formula* $\phi^*(x_1, x_2)$ *such that* $G \models \phi^*(e_1, e_2)$ *if and only if* $G^{e_1 \times e_2} \models \phi$.

Given any MSO$_2$-formula ϕ and crossing diagram D, we can repeatedly apply the lemma above to construct a formula ϕ^D such that $G \models \phi^D(e_0, \ldots, e_r)$ if and only if $G_D \models \phi$. With this tool in hand it is straightforward to construct a formula γ_D , expressing the property that, in a given graph G we can build a crossing diagram with the structure of D, and partition the planarization G_D into six sets, satisfying Lemma 11. So we can define ζ_k to be the disjunction of the γ_D ranging over all 2-page crossing diagrams with k-crossings.

Theorem 3 *There exists a computable function* f *such that* $\mathrm{cr}_2(G)$ *can be computed in* $O(f(k,t)n)$ *time for a graph* G *with* n *vertices,* $k = \mathrm{cr}_2(G)$*, and* $t = \mathrm{tw}(G)$*.*

7 Conclusion

We have provided new fixed-parameter algorithms for computing the crossing numbers for 1-page and 2-page drawings of graphs with bounded treewidth. The use of monadic second order logic and Courcelle's theorem in our solutions causes the running times of our algorithms to have an impractically high dependence on their parameters. We believe that it should be possible to achieve a better dependence by directly designing dynamic programming algorithms that use tree-decompositions of the given graphs, rather than by relying on Courcelle's theorem to prove the existence of these algorithms. Indeed, Kobayashi et al. have already provided such an algorithm for 1-page crossing minimization [38]. Can this dependency be reduced to the point of producing practical algorithms? For 2-page crossing minimization the runtime is parameterized by both the treewidth and the crossing number. Is 2-page crossing minimization NP-hard for graphs of fixed treewidth? We leave these questions open for future research.

Dujmović and Wood asked [25], "is there a polynomial-time algorithm for computing the book thickness of graphs with bounded treewidth?" Our Theorem 2 provides a partial solution to this question for book thickness 2. Can the graph property of having book thickness k be expressed in MSO_2, answering the question of Dujmović and Wood? The special case of $k = 3$ is of particular interest, to provide a computational attack on the still-open problem of whether there exist planar graphs that require four pages [24, 51]. Heath has shown that every planar graph of treewidth three has a planar 3-page drawing [34], but recognizing three-page graphs of higher treewidth efficiently remains open.

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