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Edge Bounds and Degeneracy of Triangle-Free Penny Graphs and Squaregraphs

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Abstract

We show that triangle-free penny graphs have degeneracy at most two, and that both triangle-free penny graphs and squaregraphs have at most $\min\bigl(2n - \Omega(\sqrt{n}), 2n - D - 2\bigr)$ edges, where n is the number of vertices and D is the diameter of the graph.

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1 Introduction

In this paper we investigate the number of edges and degeneracy of two classes of planar graphs, the triangle-free penny graphs and the squaregraphs.

1.1 Background

Numbers of edges. It is standard that n-vertex planar graphs have at most $3n-6$ edges, and that bipartite planar graphs have at most $2n-4$ edges. The $3n-6$ bound follows by observing that in an embedded graph with n vertices, e edges, and f faces, each face has at least three edges, and by using the corresponding inequality on the number of face-edge incidences, $2e \geq 3f$, to eliminate the number f of faces from Euler's formula $n-e+f=2$. The $2n-4$ bound on bipartite graphs follows in the same way by observing that in these graphs each face has at least four edges, and using the inequality $2e \geq 4f$. Thus, it applies equally well to non-bipartite planar graphs that have no triangular faces, and in particular to triangle-free planar graphs.

In more general terms, planar graphs are sparse: their numbers of edges are within a constant factor of their numbers of vertices. Sparse families of graphs are fundamental to graph theory and graph algorithms [27], and it is of interest to determine bounds on the number of edges for natural graph classes that are as tight as possible. Such studies have been carried out, for instance, on the 1-planar graphs [8] and minor-closed graph families [15]. We continue this line of research on additional planar graph families.

Degeneracy. The degeneracy of an undirected graph G is the minimum number d such that every subgraph of G contains a vertex of degree at most d [25]. Since the induced subgraph on any set of vertices has a superset of the edges of any other subgraph, we may equivalently define d as the minimum number d such that every induced subgraph of G contains a vertex of degree at most d Alternatively, the degeneracy of G is the minimum d such that the edges of G can be oriented to produce a directed acyclic graph with out-degree at most d. Both the degeneracy, and an acyclic orientation with out-degree d, can be computed in linear time by a simple greedy algorithm that repeatedly finds and removes a vertex of minimum degree. Another name for degeneracy is the *coloring number* [17], because graphs of degeneracy d can be colored using $d+1$ colors by choosing colors in the reverse of this removal ordering. When the color for each vertex is chosen, it has at most d already-colored neighbors, so at least one color will be free to choose [34]. The same argument applies equally well to *list coloring*, a version of graph coloring in which the whole graph has more than $d+1$ colors that can be used but each vertex is limited to a smaller list of $d+1$ colors [4].

Penny graphs. Penny graphs are the contact graphs of unit circles. A penny graph may be formed from a set of non-overlapping unit circles by creating a vertex for each circle and an edge for each tangency between two circles [22, 30].

These graphs fit into a long line of graph drawing research on contact graphs of various types of geometric objects [3, 9, 11, 21, 23]. The same graphs (with the exception of the graph with no edges) are also proximity graphs, the graphs determined from a finite set of points in the plane by adding edges between all closest pairs of points. Alternatively, if one adds an edge between all pairs of points that are nearest neighbors of each other, each connected component of the resulting graph can be scaled to become a penny graph. For this reason the same class of graphs has also been called the minimum-distance graphs [10, 33] or the mutual nearest neighbor graphs [14]. A minimum-distance representation can be obtained from a contact representation by choosing a point at the center of each circle, and a contact representation can be obtained from a minimum-distance representation by scaling the points so their minimum distance is two and using each point as the center of a unit circle. However, finding either type of representation given only the graph is NP-hard [14], and remains hard even when the input is a tree [7].

As graph drawings, minimum distance representations are in many ways ideal: they have no crossings, all edges have unit length, all non-adjacent pairs of vertices are at farther than unit distance apart, and the angular resolution is at least $\pi/3$. Every graph that can be drawn with this combination of properties is a penny graph. Moreover, penny graphs have degeneracy at most three, because in every subgraph, every vertex of the convex hull of the subgraph has degree at most three. This bound leads to a linear-time greedy 4-coloring algorithm for penny graphs [20], much simpler than known quadratic-time 4-coloring algorithms for arbitrary planar graphs [31]. Additionally, although planar graphs with n vertices can have $3n-6$ edges, penny graphs have at most $\lfloor 3n-\sqrt{12n-3}\rfloor$ edges [19]. This bound is tight for pennies tightly packed into a hexagon [24], and its lower-order square-root term stands in an intriguing contrast to many similar bounds on the edge numbers of planar graphs, k-planar graphs, quasi-planar graphs, and minor-closed graph families, with constant or unknown lower-order terms [1, 2, 8, 15, 29, 32].

Swanepoel [33] first considered corresponding problems for the *triangle-free* penny graphs. In graph drawing terms, these are the graphs that can be drawn with no crossings, unit-length edges, greater than unit separation for non-adjacent vertices, and angular resolution strictly larger than $\pi/3$. Swanepoel observed that, as with triangle-free planar graphs more generally, an n-vertex triangle-free penny graph can have at most $2n-4$ edges. As a lower bound, the square grids have $\lfloor 2n-2\sqrt{n}\rfloor$ edges, as do some subsets of grids and some pentagonally-symmetric graphs found by Oloff de Wet [33] (Figure 1). Swanepoel conjectured that, of the two bounds, it is the lower bound that is tight. As a subclass of the triangle-free planar graphs, the triangle-free penny graphs are necessarily 3-colorable [18, 35] and can be 3-colored in linear time [12]. However, not every triangle-free planar graph is 3-list-colorable [37] and the known 3-list-colorable subclasses of planar graphs [4, 13, 35] do not include all triangle-free penny graphs.

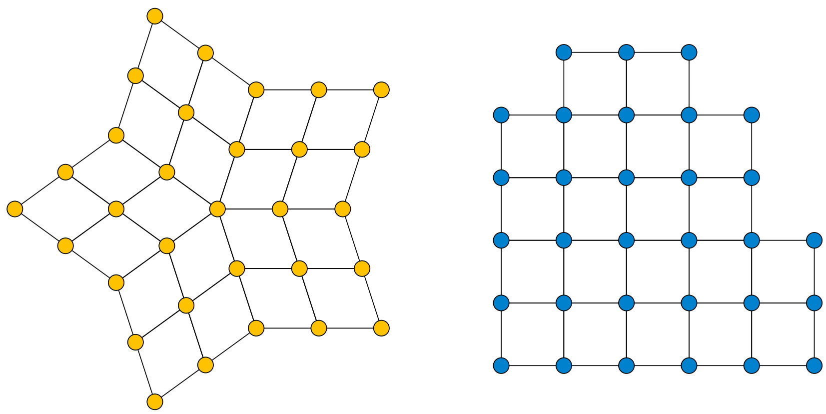

Figure 1: Two graphs that are both triangle-free penny graphs and squaregraphs, with $n = 31$ vertices and $\lfloor 2n - 2\sqrt{n} \rfloor = 50$ edges, Swanepoel's conjectured maximum.

Squaregraphs. These are the graphs that can be embedded in the plane so that every bounded face is a quadrilateral and every vertex that does not belong to the unbounded face has degree at least four. They are also the dual graphs of line arrangements in the hyperbolic plane such that no three lines cross each other. As plane graphs with even-length cycles, they are automatically bipartite, and they form an important subclass of the median graphs [5].

Not every triangle-free penny graph is a squaregraph, and not every squaregraph is a triangle-free penny graph; for instance, Figure 2 depicts a triangle-free penny graph that is not a squaregraph (it is not bipartite) while Figure 6 depicts a squaregraph that is not a bipartite penny graph (it has a vertex of degree greater than five, not possible in bipartite penny graphs). Nevertheless, as we will see, these two classes of graphs behave similarly in many respects.

1.2 New results

We continue these lines of research with the following new results.

- Every triangle-free penny graph with at least one cycle has at least four vertices of degree two or less. Consequently, the triangle-free penny graphs have degeneracy at most two and list chromatic number at most three.

- Every triangle-free penny graph and every squaregraph has at most $2n - D - 2$ edges, where n is the number of vertices in the graph and D is its diameter. Although we base our proof on the existence of degree-two vertices in these graphs, it does not generalize to 2-degenerate triangle-free or bipartite planar graphs more generally: there exist 2-degenerate bipartite graphs with linear diameter and $2n - 4$ edges. Because the diameter of an n-vertex penny graph is $\Omega(n)$, our $2n - D - 2$ edge bound implies that

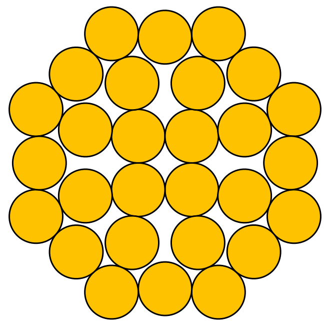

Figure 2: In a triangle-free penny graph, the convex hull vertices may all still have degree three.

every n-vertex triangle-free penny graph has at most $2n - \Omega(\sqrt{n})$ edges. Thus, the form of Swanepoel's conjectured edge bound is correct, although we cannot confirm the conjectured constant factor on the square-root term. However, there exist arbitrarily large squaregraphs of bounded diameter, so we do not obtain a similar bound for squaregraphs in this way.

- We prove more directly that every triangle-free penny graph and every squaregraph has $2n - \Omega(\sqrt{n})$ edges, by combining Euler's formula with the fact that (for both types of graphs) the outer face must have many vertices. For triangle-free penny graphs, the constant in the $\sqrt{n}$ term is better than we would obtain using diameter, but still does not match Swanepoel's conjectured bound. For squaregraphs, we obtain the exact maximum number of edges: it is $\lfloor 2n - 2\sqrt{n} \rfloor$, the same bound conjectured for triangle-free penny graphs by Swanepoel.

2 Degeneracy

We begin by showing that every triangle-free penny graph with at least one cycle has at least four vertices of degree two or less. Unlike the vertices of degree three in arbitrary penny graphs, it is not always possible to find these degree-two vertices on the convex hull (Figure 2). It is convenient to begin with a special case of these graphs, the ones that are biconnected.

Lemma 1 *Every biconnected triangle-free penny graph has at least four vertices of degree two.*

Proof: Given a biconnected triangle-free penny graph G, and its representation as a penny graph, the outer face of the representation (as in any biconnected plane graph) consists of a simple cycle of vertices; in particular each vertex of this face has at least two neighbors. For each vertex v of this simple cycle, let w be the clockwise neighbor of v in the cycle, and let u be the neighbor of v that is

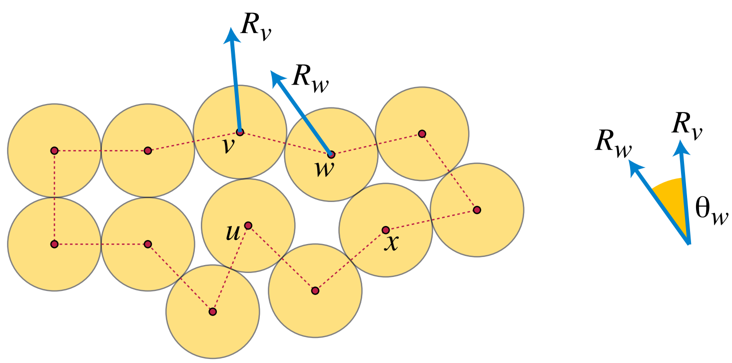

Figure 3: Notation for the proof of Lemma 1. The dashed red cycle is the outer face. In this example, R_w turns counterclockwise with respect to R_v, so θ_w is negative.

next in clockwise order around v from w; define a ray R_v, having the center of the disk of v as its apex, and pointing directly away from the center of u. Given the same boundary vertices v and w in clockwise order, define the angle θ_w to be the angle made by rays R_v and R_w, assigned a sign so that θ_w is positive if R_w turns a clockwise angle (less than $\pi/2$) from R_v, and negative if R_w turns counterclockwise with respect to R_v. If R_v and R_w are parallel, then we define $\theta_w = 0$. See Figure 3 for an illustration of this notation.

Then these rays and their angles have the following properties:

- Each ray R_v stays within an angle of $\pm 2\pi/3$ of the ray directly from v to w along an edge of the outer face of the penny graph. Because the sum of the turning angles of consecutive pairs of edges of any simple polygon is exactly 2π, the same must be true of the sum of the turning angles of the rays R_v. That is, $\sum_v \theta_v = 2\pi$.

- If a boundary vertex w has degree three or more, then $\theta_w \leq 0$. For, if v and w are consecutive on the outer face, with R_v pointing away from a neighbor u of v (as above) and R_w pointing away from a neighbor x of w, then the assumption that w has degree at least three implies that $x \neq v$, and the assumption that G is triangle-free implies that $x \neq u$. If x and u touch, so that $uvwx$ forms a quadrilateral in G, then R_v and R_w are necessarily parallel, so $\theta_w = 0$. In any other case, to prevent x and u from touching, x must be rotated counterclockwise around w from the position where it would touch u, causing angle θ_w to become negative.

- At a boundary vertex w of degree two, $\theta_w < 2\pi/3$. For, in this case, R_w points away from v, the counterclockwise neighbor of w on the outer face. Let u be the neighbor of v such that R_v points away from u; then $w \neq u$. Because both R_v and R_w belong to lines through the center of v, their angle θ_w is complementary to angle wvu, which must be greater than $\pi/3$

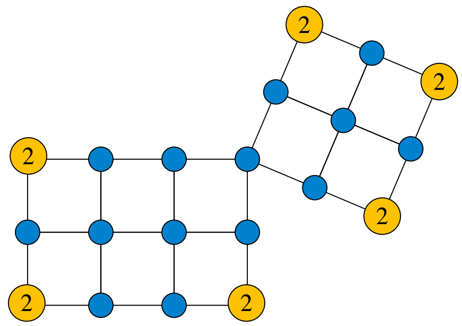

Figure 4: A graph that is both a triangle-free penny-graph and a squaregraph, with two nontrivial biconnected components. Each vertex of degree two in each component C either has degree two in the whole graph, or connects C to other components that include at least one non-articulation vertex of degree two in the whole graph (Lemma 2).

in order to prevent circles u and w from overlapping or touching (and forming a triangle). Therefore, θ_w is less than $2\pi/3$.

For the sequence of angles θ_w, each less than $2\pi/3$, to add to a total angle of 2π, there must be at least four positive angles in the sequence, and therefore there must be at least four degree-two vertices. □

The corresponding result for squaregraphs is known: Every biconnected squaregraph has at least four vertices of degree two [5, Proposition 4.1]. We will extend these results to graphs that are not necessarily biconnected in the following convenient lemma.

Lemma 2 *Every triangle-free penny graph or squaregraph G with at least one cycle has at least four non-articulation vertices of degree two or less.*

Proof: By the assumption that G has at least one cycle, it has at least one nontrivial biconnected component C. By Lemma 1 or its analogue for squaregraphs, C has at least four degree-two vertices, each of which either has degree two in G or forms an articulation point of G. If it forms an articulation point, then the tree of biconnected components connected through it to G has at least one leaf, which must either be a vertex of degree one in G or a nontrivial biconnected component with at least four degree-two vertices, only one of which can be an articulation point. Thus, each of the four degree-two vertices in C is either itself a non-articulation vertex of degree at most two in G or leads to such a vertex (Figure 4). □

The bound on the number of degree-two vertices is tight for square grids.

Theorem 1 *The degeneracy of every triangle-free penny graph or squaregraph is at most two.*

Proof: Every induced subgraph of a triangle-free penny graph is another triangle-free penny graph, so the result follows from Lemma 2 and from the fact that, in a graph with no cycles (a forest) there always exists a vertex of degree one or less (a leaf or an isolated vertex).

For squaregraphs, the same argument does not work directly as it is not true that induced subgraphs of squaregraphs necessarily remain squaregraphs. However, by Lemma 2, every squaregraph has a non-articulation vertex of degree at most two, which by the definition of a squaregraph must belong to the outer face of the squaregraph. Removing this vertex from the squaregraph eliminates any interior face that it belongs to, without changing the number of sides of any other interior face, so it produces another squaregraph. Repeating this process of finding and removing a non-articulation vertex of degree at most two eventually removes all vertices. If each edge is oriented away from the first of its endpoints to be removed, the out-degree of the resulting acyclic orientation is at most 2. Therefore, every squaregraph has degeneracy at most 2. □

This bound is tight as the odd cycles of length ≥ 5 are triangle-free penny graphs with choosability exactly three. (The *choosability* of a graph is the smallest number k such that the graph is k-list-colorable.)

Corollary 1 *Every triangle-free penny graph is 3-list-colorable.*

If a triangle-free penny graph or squaregraph is labeled by a list of three colors for each vertex, then we can find a solution to the list coloring problem for the resulting labeled graph in linear time. The algorithm repeatedly finds and removes a vertex of degree two and then restores the vertices in the reverse of the order they were removed, coloring each vertex differently from its two neighbors when it is restored. It needs as input only the abstract graph, not its representation as a penny graph or squaregraph.

For squaregraphs, 3-list-colorability is known as a special case of the fact that every bipartite planar graph is 3-list-colorable [4, Corollary 3.4]. However, it is unclear how to turn Alon and Tarsi's proof of this more general result into an efficient algorithm.

3 Edges vs diameter

Triangle-free penny graphs and squaregraphs can be distinguished from 2-degenerate triangle-free planar graphs more generally, by a connection between their number of edges and their diameter.

In arbitrary triangle-free planar graphs, or more strongly even in bipartite planar graphs of degeneracy two, having high diameter does not necessarily cause the graph to have fewer edges. Figure 5 shows the construction for a family of 2-degenerate triangle-free planar graphs (actually bipartite planar permutation graphs) with unbounded diameter and $2n - 4$ edges, the maximum possible for any triangle-free planar graph.

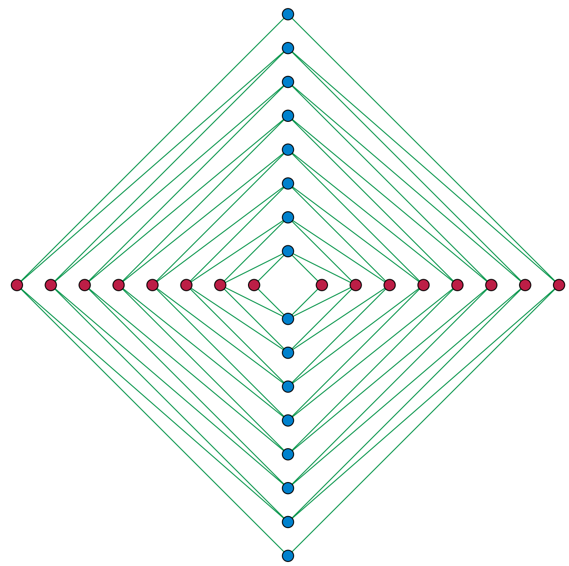

Figure 5: One of a family of 2-degenerate bipartite planar graphs with arbitrarily large diameter and $2n-4$ edges.

In contrast, as we show in this section, both triangle-free penny graphs and squaregraphs obey the following inequality, which we prove by induction using the existence of many degree-two vertices in these graphs:

Theorem 2 *Every connected n-vertex triangle-free penny graph or squaregraph G with diameter D has at most $2n-D-2$ edges.*

Proof: We use induction on n. If G has no cycle, it is a tree, with $n-1$ edges, and the result follows from the fact that $D \leq n-1$. Otherwise, let uw be a diameter pair, and let v be any vertex of degree at most two, whose removal does not disconnect G, distinct from u and w. The existence of v follows from Lemma 2, according to which G has at least four non-articulation vertices of degree at most two, only two of which can be u and w. Then $G-v$ has one less vertex, one or two fewer edges, and diameter at least D. The result follows by applying the induction hypothesis to $G-v$. □

In the case of triangle-free penny graphs, this leads to a $2n-\Omega(\sqrt{n})$ bound on the number of edges, via the following result:

Theorem 3 *Every connected n-vertex penny graph has diameter $\Omega(\sqrt{n})$.*

Proof: By a standard isodiametric inequality [6], for the convex hull of n disjoint unit disks to enclose area $2\pi n$, it must have (geometric) diameter $\Omega(\sqrt{n})$. In order to connect two unit disks at geometric distance $\Omega(\sqrt{n})$ from each other, they must also be at graph-theoretic distance $\Omega(\sqrt{n})$. □

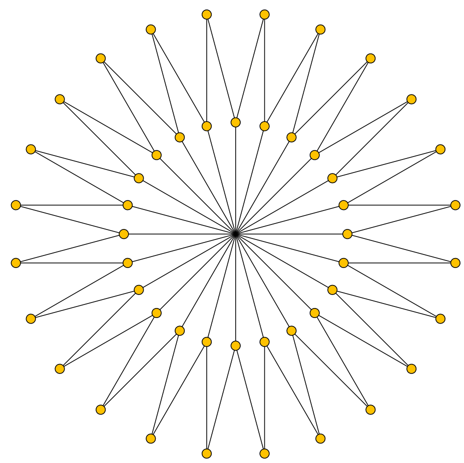

Figure 6: Biconnected squaregraphs of bounded diameter can have arbitrarily many vertices.

In contrast, however, there exist biconnected squaregraphs with arbitrarily many vertices and bounded diameter (Figure 6), so Theorem 2 does not bound the number of edges in these graphs below $2n - O(1)$.

4 Isoperimetry

Next, we derive a bound on the number of edges of a triangle-free penny graph, with a better constant on the $\sqrt{n}$ term than would be given by Theorem 2 and Theorem 3. Our proof uses the isoperimetric theorem to show that the outer face of any representation as a penny graph has many vertices. We then use Euler's formula to show that a planar graph with a large face has few edges. We start with a lemma showing that an n-vertex penny graph must enclose a large area of the plane.

Lemma 3 *Let v be a vertex of a penny graph that (in some representation of the graph as a penny graph) is not on the outer face. Then, in the Voronoi diagram of the centers of the circles in the representation, the Voronoi cell containing v has area at least $2\sqrt{3}$, which is the area of a regular hexagon circumscribed around a unit circle.*

Proof: This is Lemma 5.2 of Pach and Agarwal [28, pp. 48–49]; see there for a proof sketch. □

Lemma 4 *In any penny graph representation of a graph G with n vertices, the number of vertex-face incidences on the outer face of the representation is at least*

$$\sqrt{\pi \cdot 2\sqrt{3} \cdot n} - O(1) \approx 3.3\sqrt{n}.$$

Proof: Unless there are at least this many incidences, by Lemma 3 there must be a total area of at least $2\sqrt{3} \cdot n - O(\sqrt{n})$ enclosed by the outer face, because each Voronoi cell of an inner vertex is enclosed and the Voronoi cells are all disjoint. The result follows from the facts that each vertex-face incidence accounts for 2 units of length of the outer face (the two radii of a single unit circle in the representation, along which the outer face enters and then leaves that circle) and that any curve that encloses area A must have length at least $2\sqrt{\pi A}$ (the isoperimetric theorem, with the shortest enclosing curve being a circle). □

Lemma 5 *Let G be an n-vertex triangle-free plane graph in which one face has k vertex-face incidences. Then G has at most $2n - k/2 - 2$ edges.*

Proof: Vertex-face incidences and edge-face incidences on any face are equal, so the same face of G that has k vertex-face incidences also has k edge-face incidences. We count the number of edge-face incidences in G in two ways: by counting two incidences for each edge, and by summing the lengths of the faces. Each face of G has at least four edges, so if there are e edges and f faces then we have the inequality $2e \geq 4(f-1)+k$, or equivalently $e/2 - k/4 + 1 \geq f$. Using this inequality to replace f in Euler's formula $n - e + f = 2$, we obtain $n - e + e/2 - k/4 + 1 \geq 2$, or equivalently $e \leq 2n - k/2 - 2$ as claimed. □

Theorem 4 *The number of edges in any n-vertex triangle-free penny graph is at most*

$$2n - \frac{1}{2}\sqrt{\pi \cdot 2\sqrt{3} \cdot n} + O(1) \approx 2n - 1.65\sqrt{n}.$$

Proof: Lemma 4 proves the existence of a large face, and plugging the size of this face as the variable k in Lemma 5 gives the stated bound. □

We leave the problem of closing the gap between this upper bound and Swanepoel's $2n - 2\sqrt{n}$ lower bound as open for future research.

5 Tight edge bounds for squaregraphs

The combinatorial structure of squaregraphs makes it easier to get an exact bound on their number of edges which, curiously, has the same formula as the one Swanepoel conjectured for triangle-free penny graphs.

Theorem 5 *The maximum possible number of edges in an n-vertex squaregraph is $\lfloor 2n - 2\sqrt{n} \rfloor$.*

Proof: We use the fact that squaregraphs are dual to hyperbolic line arrangements in which no three lines all cross each other [5, Theorem 6.1] (Figure 7). In a hyperbolic arrangement with ℓ lines and c crossings, the number of squaregraph vertices (dual to cells of the arrangement) is $c + \ell + 1$ and the number of squaregraph edges (dual to the line segments between cells in the arrangement) is $2c + \ell$. Both of these formulas can be seen by induction on the number of lines,

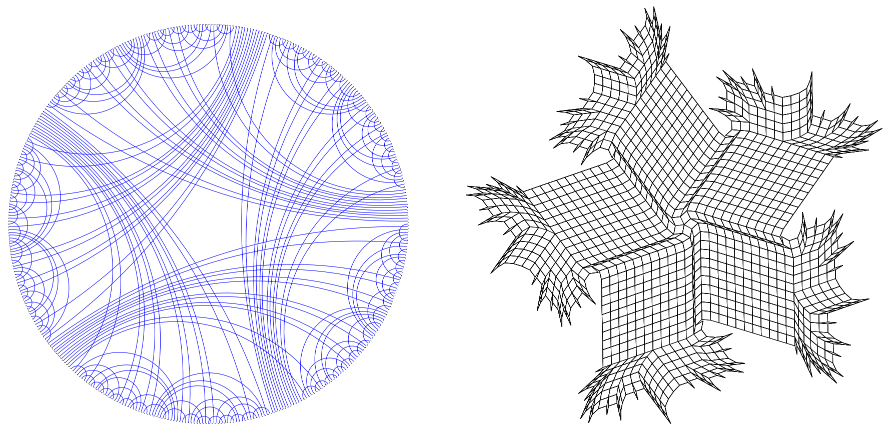

Figure 7: A hyperbolic line arrangement with no three pairwise-crossing lines (left) and its dual squaregraph (right), from [16].

by considering how many new segments and cells are formed by the introduction of a line with a given number of crossings. Therefore, to construct a squaregraph with the maximum number e of edges for a given number $n = c + \ell + 1$ of vertices, we need to maximize c and correspondingly minimize ℓ.

The intersection graph of the lines is triangle-free, and by Mantel's theorem (a special case of Turán's theorem [26, 36]) a triangle-free graph with ℓ vertices has at most $\lfloor \ell/2 \rfloor \cdot \lceil \ell/2 \rceil$ edges (crossings of pairs of lines). That is, $c \leq \lfloor \ell/2 \rfloor \cdot \lceil \ell/2 \rceil$. Combining a simplified and slightly weaker form of this inequality, $c \leq \ell^2/4$, with the formula for the number of vertices in a squaregraph gives

$$\sqrt{n} = \sqrt{c + \ell + 1} \leq \sqrt{\ell^2/4 + \ell + 1} = \ell/2 + 1.$$

Therefore,

$$e = 2c + \ell = 2(c + \ell + 1) - (\ell + 2) \leq 2n - 2\sqrt{n}.$$

The inequality in the statement of the theorem differs from this inequality only in its use of the floor function, and follows from the observation that the number e of edges is an integer.

This bound is tight, as it can be achieved for any n by finding the smallest square grid with at least n vertices and then removing degree-two vertices until the number of remaining vertices is n. See Figure 1 (right) for an example. □

6 Conclusions

We have shown that triangle-free penny graphs are 2-degenerate, and that they have at most $2n - \Omega(\sqrt{n})$ edges. Although we did not obtain Swanepoel's conjectured upper bound of $\lfloor 2n - 2\sqrt{n} \rfloor$ on the number of edges for these graphs, we proved the same bound on the number of edges of squaregraphs. Additionally, we showed that the number of edges in both kinds of graphs is upper bounded by $2n - D - 2$ where D is the diameter of the graph, distinguishing these graphs from more general classes of 2-degenerate triangle-free planar graphs.

We believe that the analogies between triangle-free penny graphs and squaregraphs, opened by this research, are worthy of additional exploration. Proving Swanepoel's conjecture also remains of interest, as does exploring the other graph-theoretic properties of triangle-free penny graphs.

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