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Special Issue on Graph Drawing Beyond Planarity Guest Editors' Foreword and Overview

Michael A. Bekos[1] *Michael Kaufmann*[1] *Fabrizio Montecchiani*[2]

[1]Wilhelm-Schickhard-Institut für Informatik, Universität Tübingen, Germany
[2]Dipartimento di Ingegneria, Università degli Studi di Perugia, Italy

E-mail addresses: bekos@informatik.uni-tuebingen.de (Michael A. Bekos) mk@informatik.uni-tuebingen.de (Michael Kaufmann) fabrizio.montecchiani@unipg.it (Fabrizio Montecchiani)

1 Graph Drawing Beyond Planarity

This special issue is devoted to a recent research direction, commonly called *beyond planarity*, which is attracting increasing attention in Graph Drawing [39, 41, 47]. The primary motivation for this research stems from recent cognitive experiments showing that the absence of specific kinds of edge crossing configurations has a positive impact on the human understanding of a graph drawing [42, 43]. Indeed, while minimizing the overall number of edge crossings in a drawing has been the main objective of a large body of literature (see, e.g., [13, 50]), graph drawing beyond planarity is concerned with the study of non-planar graphs that can be drawn by locally avoiding specific edge-crossing configurations or by guaranteeing specific properties for the edge crossings.

Beyond-planar graphs. In this context, several classes of *beyond-planar graphs* have been introduced and studied. Among them: (i) *k-planar graphs* (see, e.g., [45, 49]) which can be drawn in the plane with at most k crossings per edge; (ii) *k-quasi planar graphs* (see, e.g., [4, 35]) which admit drawings without k pairwise crossing edges; (iii) *k-gap-planar graphs* which admit drawings in which each crossing is mapped to one of the two corresponding crossing edges and each edge is assigned with at most k crossings [12]; (iv) *fan-planar graphs* (see, e.g., [15, 20, 44]) which can be drawn in the plane such that no edge crosses two independent edges; (v) *fan-crossing-free graphs* [25] which have drawings where no edge crosses any two edges that are adjacent to each other; (vi) *geometric RAC graphs* (see, e.g., [32]) which admit straight-line drawings such that edge crossings only occur at right angles.

Research topics. At high level, the research on beyond planarity can be mainly classified in four main categories, which we describe in the following. (C.1) The largest body of results is probably the one devoted to the study of the edge density of graphs from a specific family of beyond-planar graphs. (C.2) Studying possible characterizations for beyond-planar graphs and investigating the complexity of the recognition problem are also rather natural problems that received considerable attention. (C.3) Some papers established interesting relationships among different families of beyond-planar graphs. (C.4) Beyond-planar graphs have also been studied in terms of admissible drawing paradigms so to achieve high-readable representations (e.g., polyline drawings with few bends per edge and visibility representations) or to satisfy geometric constraints (e.g., drawings with all vertices on the external boundary or on two parallel lines, or drawings using few slopes for their edge segments).

Some of the main results. We briefly summarize some of the main results with respect to the research categories C.1–C.4 described above.

C.1: For a fixed integer $k > 1$, a k-planar graph with n vertices has at most $4.108\sqrt{k}n$ edges [49]. When $k \leq 4$, improved bounds are known [2, 48, 49]. In particular, when $k = 1$ and $k = 2$, the maximum number of edges is $4n - 8$ and $5n - 10$, respectively, and both bounds are tight [49]. Special bounds are also known for specific subfamilies of 1-planar graphs, e.g., for

IC- and *NIC-planar graphs*, which disallow two pairs of crossing edges to share one [55] and two [54] end-vertices, respectively; see also [23]. We refer the reader to the annotated bibliography by Kobourov et al. [45] for additional references on 1-planar graphs. A long-standing conjecture by Pach, Shahrokhi, and Szegedy affirms that there is a constant c_k such that every k-quasi planar graph on n vertices has at most $c_k n$ edges ($k \geq 2$). This conjecture has been proved for $k = 3$ and $k = 4$, with the best upper bounds currently known being $6.5n - 20$ [4] and $72(n-2)$ [1], respectively. For $k > 4$, the conjecture remains open, and the best upper bound for the maximum number of edges is $c_k n \log n$ [51]. For both fan-planar and 1-gap planar graphs, the maximum number of edges on n vertices is $5n - 10$, and this bound is tight in both cases [12, 44]. An n-vertex fan-crossing free graph has at most $4n - 8$ edges [25]. The maximum number of edges that an n-vertex geometric RAC graph can have is $4n - 10$ [31]. Note that density bounds are also known for RAC drawings with bends [3] as well as for drawings whose crossing angles are less than 90 degrees [9, 28].

C.2: Recognizing 1-planar graphs is NP-complete in general [36, 46], even if the input graph comes with a fixed cyclic ordering of the edges around each vertex [11]. On the positive side, Suzuki [52] proved a characterization of optimal 1-planar graphs (i.e., those with $4n-8$ edges) which has been used to devise a linear-time recognition algorithm [22]. Similar characterizations have been provided for optimal 2-planar and 3-planar graphs [17]. Deciding if a graph is 1-gap planar or fan-planar is also NP-complete [12, 20], even if the graph is given along with a cyclic order of the edges around each vertex [12, 15]. Recognizing geometric RAC graphs is NP-hard [8], as well as the restricted problem of deciding whether a graph admits a straight-line drawing with at most one crossing per edge and right-angle crossings [16].

C.3: All k-planar graphs are $(k+1)$-quasi planar, for every $k \geq 2$ [6, 37]. Furthermore, for every $k \geq 1$, all $2k$-planar graphs are k-gap planar (the converse may not be true), and all k-gap planar graphs are $2k+2$-quasi planar (the converse may not be true) [12]. Moreover, for any $k \geq 2$, there exist fan-planar drawable graphs that are not k-planar, and vice versa [20]. Finally, it is known that there are 1-planar graphs with at most $4n - 10$ edges not admitting straight-line RAC drawings, and on the other hand there exist straight-line RAC graphs that are not 1-planar [33].

C.4: The 1-planar graphs admitting a straight-line (1-planar) drawing have been characterized by Thomassen in 1988 [53]. This characterization also implies the fact that every 1-planar graph can be represented as a polyline drawing with at most one bend per edge. More recently, 1-planar graphs have been studied in terms of visibility representations (see , e.g., [7, 18, 21, 24, 27, 34]) and contact representations [5]. Preliminary results also exist for visibility representations of 2-planar and 3-planar graphs [17]. Constrained drawings in which all vertices must be placed along one line, along the external boundary of the drawing, or along two parallel lines have been studied for example for 1-planar graphs (see, e.g., [10, 14, 29, 38]), 2-planar graphs [40], fan-planar graphs [15, 19], and geometric RAC graphs [26, 30].

2 Contributions

The special issue contains seven papers, out of a total of eight submissions that went through a thorough refereeing and revision process. The papers of this special issue cover a broad range of topics of interest to graph drawing beyond planarity and reflect the state of the art on this topic.

- Ackerman, Keszegh, and Vizer proposed a linear upper bound on the number of edges of beyond-planar topological graphs, called *planarly connected crossing graphs*, in which each pair of crossing edges is independent and there is a crossing-free edge that connects their endpoints. Two notable families belonging to this class are the maximally-dense 1-planar graphs and the maximally-dense fan-planar graphs.

- Bannister, Cabello, and Eppstein studied 1-planarity from the point of view of parameterized complexity and proved that testing whether a graph is 1-planar and finding a corresponding 1-planar drawing is fixed-parameter tractable with respect to the vertex cover, tree-depth, and cyclomatic number. They also proved that testing 1-planarity remains NP-complete for graphs of bounded bandwidth, which implies that it is unlikely that there exists a fixed-parameter tractable algorithm for 1-planarity when parameterized by bandwidth, pathwidth, treewidth, or clique-width.

- Brandenburg expressed the structure of several classes of beyond-planar graphs by simple first-order logic formulas, using two predicates to express a crossing and an adjacency of two edges. He also provided a hierarchical inclusion-relationship diagram for several such graph classes.

- Chimani, Felsner, Kobourov, Ueckerdt, Valtr, and Wolff studied the problem of maximizing the number of crossings over all drawings of a given graph and disproved a recent conjecture of Alpert et al., who claimed that any graph has a straight-line drawing with vertices in convex positions, which maximizes the number of edge crossings. They further investigated the complexity and approximability of the crossing maximization problem and showed that the problem is NP-hard in the topological setting, and is even hard to approximate in the straight-line setting.

- Dujmović and Frati proposed a logarithmic upper bound on the book thickness of k-planar graphs (for fixed values of k), which improves the previously best known bound of $O(\sqrt{n})$ pages for $k > 1$. Notably, their result generalizes to graphs that can be embedded on surfaces of fixed genus with at most k crossings per edge. The main ingredient in their proof is a construction, which guarantees that n-vertex graphs that admit constant layered separators have $O(\log n)$ book thickness.

- Grilli studied the complexity of two problems on simultaneous graph drawing that are closely related to beyond planarity. The first problem, called *GRACSim*, asks for finding planar straight-line embeddings of two planar graphs on the same vertex set, in which a private edge of one graph can cross a private edge of the other graph at a right angle. The second problem, called *k-SEFE*, is a restriction of the simultaneous embedding with fixed edges (SEFE) problem for two planar graphs, in which every private edge may have at most k crossings, for a fixed value of k.

- Hajnal, Igamberdiev, Rote, and Schulz studied simple and 2-simple saturated topological graphs, i.e., drawn graphs in which any two edges share at most one or two points, respectively, and no further edge can be added without destroying simplicity or 2-simplicity, respectively. Improving over previous results, they showed that there exist n-vertex saturated simple and 2-simple topological graphs with only $7n$ edges and $14.5n$ edges, respectively.

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