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A Direct Proof of the Strong Hanani–Tutte Theorem on the Projective Plane

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Abstract

We reprove the strong Hanani–Tutte theorem on the projective plane. In contrast to the previous proof by Pelsmajer, Schaefer and Stasi, our method is constructive and does not rely on the characterization of forbidden minors, which gives hope to extend it to other surfaces.

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1 Introduction

A drawing of a graph on a surface is a *Hanani–Tutte drawing* if no two vertex-disjoint edges cross an odd number of times.[1] We call vertex-disjoint edges *independent*.

Pelsmajer, Schaefer and Stasi [18] proved the following theorem via consideration of the forbidden minors for the projective plane.

Theorem 1 (Strong Hanani–Tutte for the projective plane, [18]) *A graph G can be embedded into the projective plane if and only if it admits a Hanani–Tutte drawing on the projective plane.*[2]

Our main result is a constructive proof of Theorem 1. The need for a constructive proof is motivated by the question whether the strong Hanani–Tutte theorem is valid on an arbitrary (closed) surface. Currently, this is known to be valid only on the sphere (plane) and on the projective plane. The approach via forbidden minors is relatively simple on the projective plane; however, this approach does not seem applicable to other surfaces, because there is no reasonable characterization of forbidden minors for them. (Already for the torus or the Klein bottle the exact list is not known.)

Given a strong Hanani–Tutte drawing of a graph G on the projective plane, our proof gives an explicit way to transform the drawing into an embedding. In principle, our proof could be transformed into a (relatively efficient) algorithm for this transformation. On the other hand there already exist linear-time algorithms for the deciding embeddability of a graph G on the projective plane [15, 12]. (These algorithms work for any surface but the hidden constant depends exponentially on the genus.)

On the other hand, our approach reveals a number of difficulties that have to be overcome in order to obtain a constructive proof. If the answer to the strong Hanani–Tutte question is affirmative, our approach may serve as a basis for its proof on a general surface. If it is negative, then our approach may perhaps help to reveal appropriate structure needed for a construction of a counterexample.

Unfortunately, our approach needs to build an appropriate toolbox for manipulating with Hanani–Tutte drawings on the projective plane (many tools are actually applicable to a general surface). This significantly prolongs the paper. Therefore, we present the main ideas of our approach in the first four sections of the paper while postponing the technical details to the later sections.

The Hanani–Tutte theorem on the plane and related results. Let us now briefly describe the history of the problem; for complete history and relevant results we refer to a nice survey by Schaefer [21]. Following the work of Hanani [2], Tutte [25] made a remarkable observation now known as the (strong) Hanani–Tutte theorem: a graph is planar if and only if it admits a Hanani–Tutte drawing in the plane. The theorem has also a parallel history

[1] Such a drawing is also called independently even drawing in the literature.
[2] Of course, the "only if" part is trivial.

in algebraic topology, where it follows from the ideas of van Kampen, Flores, Shapiro and Wu [26, 27, 24, 14].

It is a natural question whether the strong Hanani–Tutte theorem can be extended to graphs on other surfaces; as we already said before, it has been confirmed only for the projective plane [18] so far. On general surfaces, only the weak version [1, 20] of the theorem is known to be true: if a graph is drawn on a surface so that every pair of edges crosses an even number of times[3], then the graph can be embedded into the surface while preserving the cyclic order of the edges at all vertices.[4] Note that in the strong version we require that only independent edges cross even number of times, while in the weak version this condition has to hold for all pairs of edges.

We remark that other variants of the Hanani–Tutte theorem generalizing the notion of embedding in the plane have also been considered. For instance, the strong Hanani–Tutte theorem was proved for partially embedded graphs [22] and both weak and strong Hanani–Tutte theorem were proved also for 2-clustered graphs [7].

The strong Hanani–Tutte theorem is important from the algorithmic point of view, since it implies the Trémaux crossing theorem, which is used to prove de Fraysseix-Rosenstiehl's planarity criterion [4]. This criterion has been used to justify the linear time planarity algorithms including the Hopcroft-Tarjan [11] and the Left-Right [3] algorithms. For more details we again refer to [21].

One of the reasons why the strong Hanani–Tutte theorem is so important is that it turns planarity question into a system of linear equations. For general surfaces, the question whether there exists a Hanani–Tutte drawing of G leads to a system of quadratic equations [14] over $\mathbb{Z}_2$. If the strong Hanani–Tutte theorem is true for the surface, any solution to the system then serves as a certificate that G is embeddable. Moreover, if the proof of the Hanani–Tutte theorem is constructive, it gives a recipe how to turn the solution into an actual embedding. Unfortunately, solving systems of quadratic equations is NP-complete.

The original proofs of the strong Hanani–Tutte theorem in the plane used Kuratowski's theorem [13], and therefore are non-constructive. In 2007, Pelsmajer, Schaefer and Štefankovič [19] published a constructive proof. They showed a sequence of moves that change a Hanani–Tutte drawing into an embedding.

A key step in their proof is their Theorem 2.1. We say that an edge is *even* if it crosses every other edge an even number of times (including the adjacent edges).

Theorem 2 (Theorem 2.1 of [19]) *If D is a drawing of a graph G in the plane, and E_0 is the set of even edges in D, then G can be drawn in the plane*

[3] including 0 times

[4] In fact, the embedding preserves the embedding scheme of the graph, where the notion of embedding scheme is a generalization of the rotation systems to arbitrary (even non-orientable) surfaces. For more details on this topic, we refer to [9, Chap. 3.2.3], where embedding schemes are called rotation systems and our rotation systems are called pure.

so that no edge in E_0 is involved in an intersection and there are no new pairs of edges that intersect an odd number of times.

Unfortunately, an analogous result is simply not true on other surfaces, as is shown in [20]. In particular, this is an obstacle for a constructive proof of Theorem 1.

Our approach—replacement of Theorem 2.1 in [19]. The key step of our approach is to provide a suitable replacement of Theorem 2.1 in [19] (Theorem 2); see also Lemma 3 in [8]. For a description of this replacement, let us focus on the following simplified setting.

Let us consider the case that we have a graph G with a Hanani–Tutte drawing D on the sphere S^2. Let Z be a cycle of G which is *simple*, that is, drawn without self-intersections, and such that every edge of Z is even. Theorem 2 then implies that G can be redrawn so that Z is free of crossings without introducing new pairs of edges crossing oddly.

Actually, a detailed inspection of the proof in [19] reveals something slightly stronger in this setting. The drawing of Z splits the plane into two parts that we call the *inside* and the *outside*. This in turn splits G into two parts. The inside part consists of vertices that are inside Z and of the edges that have either at least one endpoint inside Z, or they have both endpoints on Z and they enter the inside of Z next to both endpoints. The outside part is defined analogously. Because we have started with a Hanani–Tutte drawing, it is easy to check that every vertex and every edge is on Z or inside or outside. The proof of Theorem 2 in [19] then implies that the inside and the outside may be fully separated in the drawing; see Fig. 1. Actually, this can be done even by a continuous motion—if the drawing is considered on the sphere (instead of the plane).

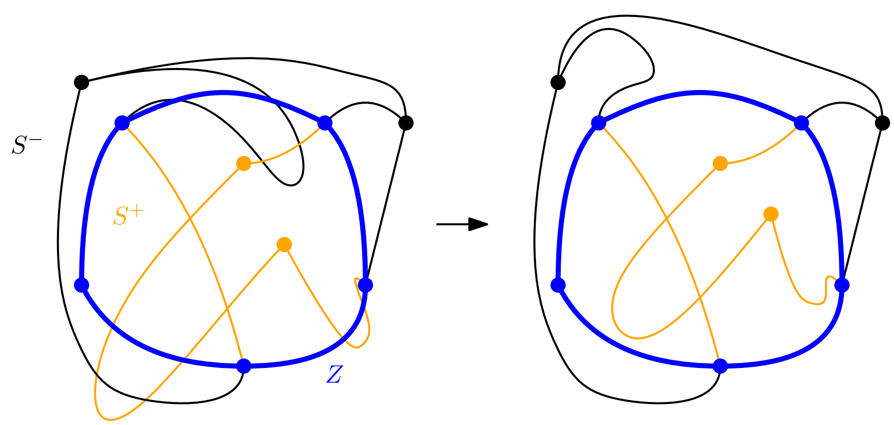

Figure 1: Separating the outside (in black) and the inside (in orange) of a cycle Z (in blue; thick).

The trouble on $\mathbb{R}P^2$ is that it may not be possible to separate the outside and the inside (of a separating cycle) by a continuous motion (of each of the parts separately). This is demonstrated by a projective-planar drawing of K_5 in Fig. 2, left. (The symbol '$\otimes$' stands for the crosscap in the picture.)

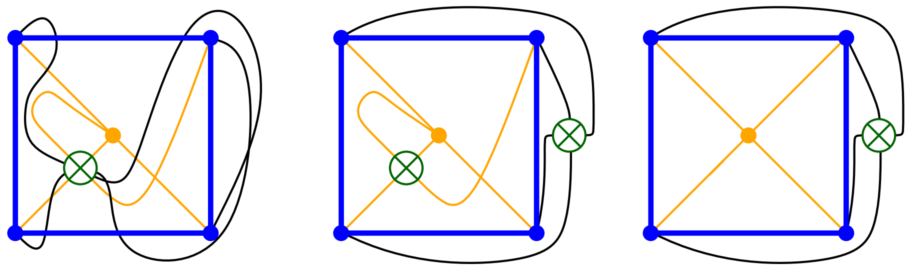

Figure 2: Projective-planar drawing of K_5 where the outside and the inside cannot be separated by a continuous motion (left) and a solution by duplicating the crosscap (middle) and removing one of them (right).

We could easily move part of the graph to the outside as desired if we were allowed to duplicate the crosscap as in Fig. 2, middle. However, the problem is that we cannot afford raising the genus. On the other hand, if we give up on a continuous motion, we may observe that the inside vertices and edges in Fig. 2, middle, may be actually redrawn in a planar way if we remove the 'inside' crosscap. This step changes the homotopy/homology type of many cycles in the drawing.

Our main technical contribution is to show that it is not a coincidence that this simplification of the drawing in Fig. 2 was possible. We will show that it is always possible to redraw one of the sides without using the 'duplicated' crosscap. The precise statement is given by Theorem 10.

The remainder of the proof. As we mentioned above, Theorem 2 is a key ingredient in the proof of the strong Hanani–Tutte theorem in the plane. The rough idea is to find a suitable order on some of the cycles of the graph so that Theorem 2 can be used repeatedly on these cycles eventually obtaining a planar drawing. A detailed proof of Pelsmajer, Schaefer and Štefankovič uses an induction based on this idea.

Similarly, we use Theorem 10 in an inductive proof of Theorem 1. The details in our setting are more complicated, because we have to take care of two types of cycles in the graph based on their homological triviality. We also need to put more effort to set up the induction in a suitable way for using Theorem 10, because our setting for Theorem 10 is slightly more restrictive than the setting of Theorem 2.

Organization of the paper. In Sect. 2 we describe Hanani–Tutte drawings on the projective plane and their properties. There we also set up several tools for modifications of the drawings. In particular, we describe how to transform the Hanani–Tutte drawings on $\mathbb{R}P^2$ into drawings on the sphere satisfying a certain additional condition. This helps significantly in several cases with manipulating these drawings. In Sect. 3 we describe the precise statement of Theorem 10. We also provide a proof of this theorem in that section, however, we postpone the proofs of many auxiliary results to later sections. In Sect. 4 we prove Theorem 1 using Theorem 10 and some of the auxiliary results from Sect. 3. The remaining sections are devoted to the missing proofs of auxiliary results.

2 Hanani–Tutte Drawings

In this section, we consider Hanani–Tutte drawings of graphs on the sphere and on the projective plane. We use the standard notation from graph theory. Namely, if G is a graph, then $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. Given a vertex v or an edge e, by $G-v$ or $G-e$ we denote the graph obtained from G by removing v or e, respectively.

Regarding drawings of graphs, first, let us recall a few standard definitions considered on an arbitrary surface. We put the standard general position assumptions on the drawings. That is, we consider only drawings of graphs on a surface such that no edge contains a vertex in its interior and every pair of edges meets only in a finite number of points, where they *cross* transversally. However, we allow three or more edges meeting in a single point (we do not mind them because we study the pairwise interactions of the edges only). Let us also mention that, in all this paper, we can assume that in every drawing, every edge is free of self-crossings. Indeed, we can remove any self-crossing without changing the image of the edge, except in a small neighborhood of the self-crossing.

We recall from the introduction that two edges are independent if they do not share a vertex. Given a surface S and a graph G, a (strong) *Hanani—Tutte drawing* of G on S is a drawing of G on S such that every pair of independent edges crosses an even number of times. (This is also called *independently even drawing* in the literature.) We will often abbreviate the term (strong) Hanani–Tutte drawing to *HT-drawing*.

Crossing numbers. Let D be a drawing of a graph G on a surface S. Given two distinct edges e and f of G by $\mathrm{cr}(e,f)=\mathrm{cr}_D(e,f)$ we denote the number of crossings between e and f in D modulo 2. We say that an edge e of G is *even* if $\mathrm{cr}(e,f)=0$ for any $f\in E(G)$ distinct from e. We emphasize that we consider the crossing number as an element of $\mathbb{Z}_2$ and all computations throughout the paper involving it are done in $\mathbb{Z}_2$.

HT-drawings on $\mathbb{R}P^2$. It is convenient for us to set up some conventions for working with the HT-drawings on the (real) projective plane, $\mathbb{R}P^2$. There are various ways to represent $\mathbb{R}P^2$. Our convention will be the following: we consider the sphere S^2 and a disk (2-ball) B in it. We remove the interior of B and identify the opposite points on the boundary ∂B. This way, we obtain a representation of $\mathbb{R}P^2$. Let γ be the curve coming from ∂B after the identification. We call this curve a *crosscap*. It is a homologically (homotopically) non-trivial simple cycle (loop) in $\mathbb{R}P^2$, and conversely, any homologically (homotopically) nontrivial simple cycle (loop) may serve as a crosscap up to a self-homeomorphism of $\mathbb{R}P^2$. In drawings, we use the symbol $\otimes$ for the crosscap coming from the removal of the disk 'inside' this symbol. We also use this symbol for ends of proofs.

Given an HT-drawing of a graph on $\mathbb{R}P^2$, it can be slightly shifted so that it meets the crosscap in a finite number of points and only transversally, still keeping the property that we have an HT-drawing. Therefore, we may add to our conventions that this is the case for our HT-drawings on $\mathbb{R}P^2$.

Now, we consider a map $\lambda\colon E(G) \to \mathbb{Z}_2$. For an edge e, we let $\lambda(e)$ be the number of crossings of e and the crosscap γ modulo 2. We emphasize that λ depends on the choice of the crosscap. Afterwards, it will be useful to alter λ via so-called vertex-crosscap switches, which we will explain a bit later.

Given a (graph-theoretic) cycle Z in G, we can distinguish whether Z is drawn as a homologically nontrivial cycle by checking the value of $\lambda(Z) := \sum \lambda(e) \in \mathbb{Z}_2$ where the sum is over all edges of Z. The cycle Z is homologically nontrivial if and only if $\lambda(Z) = 1$. In particular, it follows that $\lambda(Z)$ does not depend on the choice of the crosscap.

Projective HT-drawings on S^2. Let D be an HT-drawing of a graph G on $\mathbb{R}P^2$. It is not hard to derive a drawing D' of the same graph on S^2 such that every pair (e, f) of *independent* edges satisfies $\mathrm{cr}(e, f) = \lambda(e)\lambda(f)$. Indeed, it is sufficient to 'undo' the crosscap, glue back the disk B and then let the edges intersect on B. See the two leftmost pictures in Fig. 3. This motivates the following definition.

Definition 3 *Let D be a drawing of a graph G on S^2 and $\lambda\colon E(G) \to \mathbb{Z}_2$ be a function. Then the pair (D, λ) is a* projective HT-drawing of G on S^2 *if $\mathrm{cr}(e, f) = \lambda(e)\lambda(f)$ for any pair of independent edges e and f of G. (If λ is sufficiently clear from the context, we say that D is a projective HT-drawing of G on S^2.)*

It turns out that a projective HT-drawing on S^2 can also be transformed to an HT-drawing on $\mathbb{R}P^2$.

Lemma 4 *Let (D, λ) be a projective HT-drawing of a graph G on S^2. Then there is an HT-drawing $D_\otimes$ of G on $\mathbb{R}P^2$ such that $\mathrm{cr}_{D_\otimes}(e, f) = \mathrm{cr}_D(e, f) + \lambda(e)\lambda(f)$ for any pair of distinct edges of G, possibly adjacent. In addition, if e and f are arbitrary two edges such that $\lambda(e) = \lambda(f) = 0$ and $D(e)$ and $D(f)$ are disjoint; then $D_\otimes(e)$ and $D_\otimes(f)$ are disjoint as well.*

Proof. It is sufficient to consider a small disk B which does not intersect $D(G)$, replace it with a crosscap and redraw the edges e with $\lambda(e) = 1$ appropriately

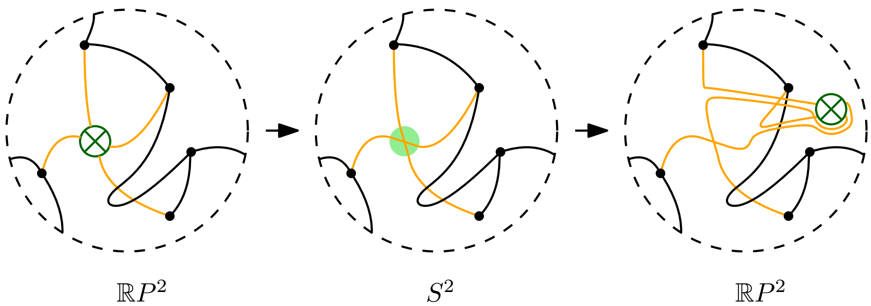

Figure 3: Transformations between HT-drawings on $\mathbb{R}P^2$ and projective HT-drawings on S^2.

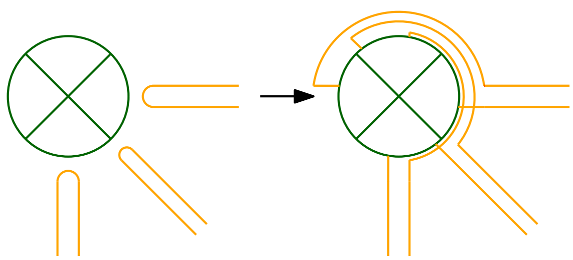

Figure 4: Redrawing the finger-moves around the crosscap.

as described below. (Follow the two pictures on the right in Fig. 3.) From each edge e with $\lambda(e) = 1$, we pull a thin 'finger-move' towards the crosscap which intersects every other edge in pairs of intersection points. Then we redraw the edge in a close neighbourhood of the crosscap as indicated in Fig. 4. After this redrawing, each edge e such that $\lambda(e) = 1$ passes over the crosscap once and each edge e with $\lambda(e) = 0$ does not pass over it. This agrees with our original definition of λ for HT-drawings on $\mathbb{R}P^2$. In addition, we indeed obtain an HT-drawing on $\mathbb{R}P^2$ with $\mathrm{cr}_{D_\otimes}(e, f) = \mathrm{cr}_D(e, f) + \lambda(e)\lambda(f)$, because in the last step we introduce one more crossing among pairs of edges e, f such that $\lambda(e) = \lambda(f) = 1$. $\otimes$

In summary, Lemma 4 together with the previous discussion provide us with two viewpoints on the HT-drawings.

Corollary 5 *A graph G admits a projective HT-drawing on S^2 (with respect to some function $\lambda\colon E(G) \to \mathbb{Z}_2$) if and only if it admits an HT-drawing on $\mathbb{R}P^2$.*

The main strength of Corollary 5 lies in the fact that in projective HT-drawings on S^2 we can ignore the actual geometric position of the crosscap and work in S^2 instead, which is simpler. This is especially helpful when we need to merge two drawings. On the other hand, it turns out that for our approach it will be easier to perform certain parity counts in the language of HT-drawings on $\mathbb{R}P^2$.

In order to distinguish the usual HT-drawings on S^2 from the projective HT-drawings, we will sometimes refer to the former as to the *ordinary* HT-drawings on S^2.

Nontrivial walks. Let (D, λ) be a projective HT-drawing of a graph G and ω be a walk in G. We define $\lambda(\omega) := \sum_{e \in E(\omega)} \lambda(e)$ where $E(\omega)$ is the multiset of edges appearing in ω. Equivalently, it is sufficient to consider only the edges appearing an odd number of times in ω, because $2\lambda(e) = 0$ for any edge e. We say that ω is *trivial* if $\lambda(\omega) = 0$ and *nontrivial* otherwise (that is, $\lambda(\omega) = 1$).

We often use this terminology in special cases when ω is an edge, a path, or a cycle. In particular, a cycle Z is trivial if and only if it is drawn as a homologically trivial cycle in the corresponding drawing $D_\otimes$ of G on $\mathbb{R}P^2$ from Lemma 4.

Given two homologically nontrivial cycles on $\mathbb{R}P^2$ it is well known that they must cross an odd number of times (assuming they cross at every intersection). This fact is substantiated by Lemma 30 later on. However, we first present a weaker version of this statement in the setting of projective HT-drawings, which we need sooner.

Lemma 6 *Let (D, λ) be a projective HT-drawing of a graph G on S^2. Then G does not contain two vertex-disjoint nontrivial cycles.*

Proof. For contradiction, let Z_1 and Z_2 be two vertex-disjoint nontrivial cycles in G. That is, Z_1 as well as Z_2 contains an odd number of nontrivial edges. Therefore, there is an odd number of pairs (e_1, e_2) of nontrivial edges where $e_1 \in Z_1$ and $e_2 \in Z_2$. According to Definition 3, Z_1 and Z_2 must have an odd number of crossings. But this is impossible for two cycles in the plane which cross at every intersection (in D). $\otimes$

Vertex-edge and vertex-crosscap switches. Let D be a drawing of a graph G on S^2. Let us consider a vertex v and an edge e of G such that v is not incident to e. We modify the drawing D into drawing D' so that we pull a thin finger from the interior of e towards v and we let this finger pass over v. We say that D' is obtained from D by the *vertex-edge switch* (v, e).[5] If we have an edge f incident to v, then the crossing number $\mathrm{cr}(e, f)$ of this pair changes (from 0 to 1 or vice versa), but it does not change for any other pair, because the 'finger' intersects the other edges in pairs.

Now, let (D, λ) be a projective HT-drawing of G on S^2. It is very useful to alter λ at the cost of redrawing G. Given a vertex v, we perform the vertex-edge switches (v, e) for all edges e not incident to v such that $\lambda(e) = 1$ obtaining a drawing D'. We also introduce a new function $\lambda' \colon E(G) \to \mathbb{Z}_2$ derived from λ

[5] Another name for the *vertex-edge switch* is the *finger-move* common mainly in topological context in higher dimensions.

by switching the value of λ on all edges of G incident to v. In this case, we say that D' (and λ') is obtained by the *vertex-crosscap switch* over v.[6] It yields again a projective HT-drawing.

Lemma 7 *Let (D,λ) be a projective HT-drawing of G on S^2. Let D' and λ' be obtained from D and λ by a vertex-crosscap switch. Then (D',λ') is a projective HT-drawing of G on S^2.*

Proof. It is routine to check that $\mathrm{cr}_{D'}(e,f) = \lambda'(e)\lambda'(f)$ for any pair of independent edges e and f.

Indeed, let v be the vertex inducing the switch. If neither e nor f is incident to v, then

$$\mathrm{cr}_{D'}(e,f) = \mathrm{cr}_D(e,f) = \lambda(e)\lambda(f) = \lambda'(e)\lambda'(f).$$

It remains to consider the case that one of the edges, say e, is incident to v. Note that $\lambda(e) = 1 - \lambda'(e)$ and $\lambda(f) = \lambda'(f)$ in this case.

If $\lambda(f) = 0$, then

$$\mathrm{cr}_{D'}(e,f) = \mathrm{cr}_D(e,f) = \lambda(e)\lambda(f) = 0 = \lambda'(e)\lambda'(f).$$

Finally, if $\lambda(f) = 1$, then

$$\mathrm{cr}_{D'}(e,f) = 1 - \mathrm{cr}_D(e,f) = 1 - \lambda(e)\lambda(f) = \lambda(f) - \lambda(e)\lambda(f) = \lambda'(e)\lambda'(f).$$

⊗

We also remark that a vertex-crosscap switch keeps the triviality or nontriviality of cycles. Indeed, let Z be a cycle. If Z avoids v, then $\lambda(Z) = \lambda'(Z)$ since $\lambda(e) = \lambda(e')$ for any edge e of Z. If Z contains v, then $\lambda(Z) = \lambda'(Z)$ as well since $\lambda(e) \neq \lambda'(e)$ for exactly two edges of Z.

Planarization. As usual, let (D,λ) be a projective HT-drawing of G on S^2. Now let us consider a subgraph P of G such that every cycle in P is trivial. Then P essentially behaves as a planar subgraph of G, which we make more precise by the following lemma.

Lemma 8 *Let (D,λ) be a projective HT-drawing of G on S^2 and let P be a subgraph of G such that every cycle in P is trivial. Then there is a set $U \subseteq V(P)$ with the following property. Let (D_U,λ_U) be obtained from (D,λ) by the vertex-crosscap switches over all vertices of U (in any order). Then (D_U,λ_U) is a projective HT-drawing of G on S^2 and $\lambda_U(e) = 0$ for any edge e of $E(P)$.*

Proof. The drawing (D_U,λ_U) is a projective HT-drawing by Lemma 7. Let F be a spanning forest of P, the union of spanning trees of each connected component of P, rooted arbitrarily. We first make $\lambda(e) = 0$ for each edge of F,

[6] In the case of drawings on $\mathbb{R}P^2$, a vertex-crosscap switch corresponds to passing the crosscap over v, which motivates our name. On the other hand, it is beyond our needs to describe this correspondence exactly.

as follows: do a breadth-first search on each tree in F; when an edge $e \in F$ with $\lambda(e) = 1$ is encountered, perform a vertex-crosscap switch on the vertex of e farther from the root of the tree. Let λ_U be the resulting map, which is zero on the edges of F. Each edge e in $E(P) \setminus E(F)$ belongs to a cycle Z such that $Z - e \subseteq F$. Since $\lambda_U(Z) = \lambda(Z) = 0$, we have $\lambda_U(e) = 0$ as well. $\otimes$

3 Separation Theorem

In this section, we state the separation theorem announced in the introduction.

As it was explained in the introduction, a simple cycle Z such that every edge of Z is even (in a plane drawing) splits the graph into the outside and the inside. We first introduce a notation for this splitting.

Definition 9 *Let G be a graph and D be a drawing of G on S^2. Let us assume that Z is a cycle of G such that every edge of Z is even and it is drawn as a simple cycle in D. Let S^+ and S^- be the two components of $S^2 \setminus D(Z)$. We call a vertex $v \in V(G) \setminus V(Z)$ an* inside *vertex if it belongs to S^+ and an* outside *vertex otherwise. Given an edge $e = uv \in E(G) \setminus E(Z)$, we say that e is an* inside *edge if either u is an inside vertex or if $u \in V(Z)$ and $D(e)$ points locally to S^+ next to $D(u)$. Analogously we define an* outside *edge.*[7] *We let V^+ and E^+ be the sets of the inside vertices and the inside edges, respectively. Analogously, we define V^- and E^-. We also define the graphs $G^{+0} := (V^+ \cup V(Z), E^+ \cup E(Z))$ and $G^{-0} := (V^- \cup V(Z), E^- \cup E(Z))$.*

Now, we may formulate our main technical tool—the separation theorem for projective HT-drawings.

Theorem 10 *Let (D, λ) be a projective HT-drawing of a 2-connected graph G on S^2 and Z a cycle of G that is simple in D and such that every edge of Z is even. Moreover, we assume that every edge e of Z is trivial, that is, $\lambda(e) = 0$. Then there is a projective HT-drawing (D', λ') of G on S^2 satisfying the following properties.*

- *The drawings D and D' coincide on Z;*
- *the cycle Z is completely free of crossings and all of its edges are trivial in D';*
- *$D'(G^{+0})$ is contained in $S^+ \cup D'(Z)$;*
- *$D'(G^{-0})$ is contained in $S^- \cup D'(Z)$; and*
- *either all edges of G^{+0} or all edges of G^{-0} are trivial (according to λ'); that is, at least one of the drawings $D'(G^{+0})$ or $D'(G^{-0})$ is an ordinary HT-drawing on S^2.*

[7] It turns out that every edge $e \in E(G) \setminus E(Z)$ is either an outside edge or an inside edge, because every edge of Z is even.

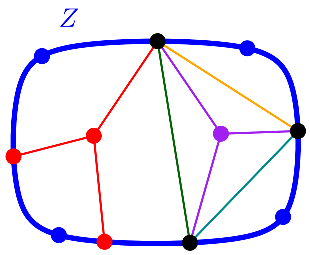

Figure 5: An example of a graph with five inside bridges—marked by different colours. The vertices that belong to several inside bridges are in black.

The assumption that G is 2-connected is not essential for the proof of Theorem 10, but it will slightly simplify some of the steps. (For our application, it will be sufficient to prove the 2-connected case.)

In the remainder of this section, we describe the main ingredients of the proof of Theorem 10 and we also derive this theorem from the ingredients. We will often encounter the setting when G, (D, λ) and Z satisfy the assumptions of Theorem 10. Therefore, we say that G, (D, λ) and Z satisfy the *separation assumptions* if (1) G is a 2-connected graph; (2) (D, λ) is a projective HT-drawing of G; (3) Z is a cycle in G drawn as a simple cycle in D; (4) every edge of Z is even in D and trivial.

Arrow graph. From now on, let us fix G, (D, λ) and Z satisfying the separation assumptions. This also fixes the distinction between the outside and the inside.

Definition 11 *A* bridge *B of G (with respect to Z) is a subgraph of G that is either an edge not in Z but with both endpoints in Z (and its endpoints also belong to B), or a connected component of $G - V(Z)$ together with all edges (and their endpoints in Z) with one endpoint in that component and the other endpoint in Z. (This is a standard definition; see, e.g., Mohar and Thomassen [16, p. 7].)*

We say that B is an inside bridge *if it is a subgraph of G^{+0}, and an* outside bridge *if it is a subgraph of G^{-0} (every bridge is thus either an inside bridge or an outside bridge).*

A walk ω in G is a proper walk *if no vertex in ω belongs to $V(Z)$, except possibly its endpoints, and no edge of ω belongs to $E(Z)$. In particular, each proper walk belongs to a single bridge.*

Since we assume that G is 2-connected, every inside bridge contains at least two vertices of Z. The bridges induce partitions of $E(G) \setminus E(Z)$ and of $V(G) \setminus V(Z)$. See Fig. 5.

We want to record which pairs of vertices on $V(Z)$ are connected with a nontrivial and proper walk inside or outside.[8]

[8] We recall that nontrivial walks are defined in Sect. 2, a bit below Corollary 5.

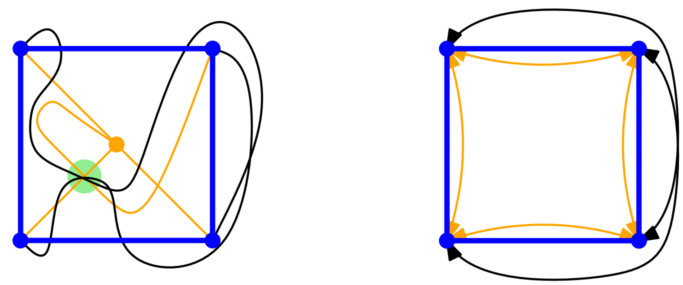

Figure 6: The inside and the outside arrows (right) corresponding to the projective HT-drawing of K_5 (left) derived from its drawing in Fig. 2, left.

For this purpose, we create two new graphs A^+ and A^-, possibly with loops but without multiple edges. In order to distinguish these graphs from G, we draw their edges with double arrows and we call these graphs an *inside arrow graph* and an *outside arrow graph*, respectively. The edges of these graphs are called the *inside/outside arrows*. We set $V(A^+) = V(A^-) = V(Z)$.

Now we describe the *arrows*, that is, $E(A^+)$ and $E(A^-)$. Let u and v be two vertices of $V(Z)$, not necessarily distinct. By W^+_{uv} we denote the set of all proper nontrivial walks in G^{+0} with endpoints u and v. We have an *inside arrow* connecting u and v in $E(A^+)$ if and only if W^+_{uv} is nonempty. In order to distinguish the edges of G from the arrows, we denote an arrow by $\overline{uv} = \overline{vu}$. An arrow which is a loop at a vertex v is denoted by $\overline{vv}$. (This convention will allow us to work with arrows $\overline{uv}$ without a distinction whether $u = v$ or $u \neq v$.) Analogously, we define the set W^-_{uv} and the *outside arrows*.

See Fig. 6 for the arrow graph(s) of the projective HT-drawing of K_5 corresponding to its drawing on $\mathbb{R}P^2$ depicted in Fig. 2, left.

It follows from the definition of the inside bridges that any walk $\omega \in W^+_{uv}$ stays in one inside bridge. Given an inside bridge B, we let $W^+_{uv,B}$ be the set of all walks $\omega \in W^+_{uv}$ which belong to B. In particular, W^+_{uv} decomposes into the disjoint union of the sets $W^+_{uv,B_1}, \dots, W^+_{uv,B_k}$ where $B_1, \dots, B_k$ are all inside bridges. Given an inside arrow $\overline{uv}$ and an inside bridge B, we say that B *induces* $\overline{uv}$ if $W^+_{uv,B}$ is nonempty. (Note that an arrow can be induced by more than one bridge.) An inside bridge B is *nontrivial* if it induces at least one arrow. Given two inside arrows $\overline{uv}$ and $\overline{xy}$ we say that $\overline{uv}$ and $\overline{xy}$ *are induced by different bridges* if there are two different inside bridges B and B' such that B induces $\overline{uv}$ and B' induces $\overline{xy}$. As usual, we define analogous notions for the outside as well. Note that it may happen that there is an inside bridge inducing both $\overline{uv}$ and $\overline{xy}$ even if $\overline{uv}$ and $\overline{xy}$ are induced by different bridges.

Possible configurations of arrows. We plan to utilize the arrow graph in the following way. On the one hand, we will show that certain configurations of arrows are not possible; see Fig. 7. On the other hand, we will show that, since the arrow graph does not contain any of the forbidden configurations,

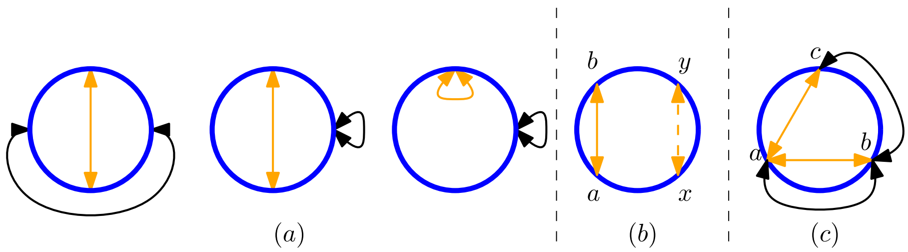

Figure 7: Forbidden configurations of arrows. The cyclic order in (a) may be arbitrary whereas it is important in (b) that the arrows there do not interleave. Different dashing of lines in (b) correspond to arrows induced by different inside bridges. The arrows of the same colour in (c) are induced by the same bridge.

it must contain one of the configurations in Fig. 8 inside or outside. (These configurations are precisely defined in Definition 15.) We will also show that the configurations in Fig. 8 are *redrawable*, that is, they may be appropriately redrawn without the crosscap. The precise statement for redrawings is given by Proposition 17 below.

More concretely, we prove the following three lemmas forbidding the configurations of arrows from Fig. 7. We emphasize that in all three lemmas we assume that the notions used there correspond to a fixed G, (D, λ) and Z satisfying the separation assumptions.

Lemma 12 *Every inside arrow shares a vertex with every outside arrow.*

Lemma 13 *Let $\overline{ab}$ and $\overline{xy}$ be two arrows induced by different inside bridges of G^{+0}. If the two arrows do not share an endpoint, their endpoints have to interleave along Z.*

Lemma 14 *There are no three vertices a, b, c on Z, an inside bridge B^+, and an outside bridge B^- such that B^+ induces the arrows $\overline{ab}$ and $\overline{ac}$ (and no other arrows) and B^- induces the arrows $\overline{ab}$ and $\overline{bc}$ (and no other arrows).*

We prove these three lemmas in Sect. 6. By symmetry, Lemmas 13 and 14 are also valid if we swap the inside and the outside (Lemma 12 as well, but here already the statement of the lemma is symmetric).

Now we describe the redrawable configurations.

Definition 15 *We say that G forms*

(a) *an* inside fan *if there is a vertex common to all inside arrows. (The arrows may come from various inside bridges.)*

(b) *an* inside square *if it contains four vertices a, b, c and d ordered in this cyclic order along Z and the inside arrows are precisely $\overline{ab}$, $\overline{bc}$, $\overline{cd}$ and $\overline{ad}$. In addition, we require that the inside graph G^{+0} has only one nontrivial inside bridge.*

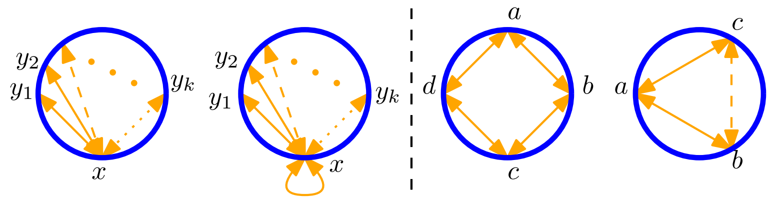

Figure 8: Schematic drawings of the redrawable configurations of arrows from Definition 15. Different dashing of lines correspond to different inside bridges. The loop in the right drawing (a) is an inside loop (drawn outside due to lack of space). The drawing (c) is only one instance of an inside split triangle.

(c) *an* inside split triangle *if there exist three vertices a, b, and c such that the inside arrows of G are $\overline{ab}$, $\overline{ac}$ and $\overline{bc}$. In addition, we require that every nontrivial inside bridge induces either the two arrows $\overline{ab}$ and $\overline{ac}$, or just a single arrow.*

See Fig. 8. We have analogous definitions for an outside fan, outside square *and* outside split triangle*.*

More precisely the notions in Definition 15 depend on G, (D, λ) and Z satisfying the separation assumptions.

A relatively direct case analysis, using Lemmas 12, 13 and 14, reveals the following fact.

Proposition 16 *Let (D, λ) be a projective HT-drawing on S^2 of a graph G and let Z be a cycle in G satisfying the separation assumptions. Then G forms an (inside or outside) fan, square, or split triangle.*

On the other hand, any configuration from Definition 15 can be redrawn without using the crosscap:

Proposition 17 *Let (D, λ) be a projective HT-drawing of G^{+0} on S^2 and Z be a cycle satisfying the separation assumptions. Moreover, let us assume that $D(G^{+0}) \cap S^- = \emptyset$ (that is, G^{+0} is fully drawn on $S^+ \cup D(Z)$). Let us also assume that G^{+0} forms an inside fan, an inside square or an inside split triangle. Then there is an ordinary HT-drawing D' of G^{+0} on S^2 such that D coincides with D' on Z and $D'(G^{+0}) \cap S^- = \emptyset$.*

Proposition 16 is proved in Sect. 5 (assuming there the validity of Lemmas 12, 13 and 14). Proposition 17 is proved in Sect. 7.

Now we are missing only one tool to finish the proof of Theorem 10. This tool is the “redrawing procedure” of Pelsmajer, Schaefer and Štefankovič [19]. More concretely, we need the following variant of Theorem 2. (Note that the theorem below is not in the setting of projective HT-drawings. However, the notions used in the statement are still well defined according to Definition 9.)

Theorem 18 *Let D be a drawing of a graph G on the sphere S^2. Let Z be a cycle in G such that every edge of Z is even and Z is drawn as a simple cycle. Then there is a drawing D'' of G such that*

- *D'' coincides with D on Z;*
- *$D''(G^{+0})$ belongs to $S^+ \cup D(Z)$ and $D''(G^{-0})$ belongs to $S^- \cup D(Z)$;*
- *whenever (e, f) is a pair of edges such that both e and f are inside edges or both e and f are outside edges, then $\mathrm{cr}_{D''}(e, f) = \mathrm{cr}_D(e, f)$.*

It is easy to check that the proof of Theorem 2 in [19] proves Theorem 18 as well. Additionally, we note that an alternative proof of Theorem 2 in [8, Lemma 3] can also be extended to yield Theorem 18. Nevertheless, for completeness, we provide its proof in Sect. 8.

Finally, we prove Theorem 10, assuming the validity of the aforementioned auxiliary results.

Proof of Theorem 10. Let G be the graph, (D, λ) be the drawing and Z be the cycle from the statement.

We use Theorem 18 with G and D to obtain a drawing D'' keeping in mind that all edges of Z are even. See Fig. 9; follow this picture also in the next steps of the proof. We get that Z is drawn on D'' as a simple cycle free of crossings. We also get that $D''(G^{+0})$ is contained in $S^+ \cup D''(Z)$ and $D''(G^{-0})$ is contained in $S^- \cup D''(Z)$. However, there may be no λ'' such that (D'', λ'') is a projective HT-drawing; we still may need to modify it to obtain such a drawing.

By Proposition 16 applied to (D, λ), G forms one of the redrawable configurations on one of the sides; that is, an inside/outside fan, square or split triangle. Without loss of generality, it appears inside. It means that D'' restricted to G^{+0} satisfies the assumptions of Proposition 17. Therefore, there is an ordinary HT-drawing D^+ of G^{+0} satisfying the conclusions of Proposition 17. Finally, we let D' be the drawing of G on S^2 which coincides with D^+ on G^{+0} and with D'' on G^{-0}. Both D'' and D^+ coincide with D on Z; therefore, D' is well defined. We set λ' so that $\lambda'(e) := \lambda(e)$ for an edge $e \in E^-$ and $\lambda'(e) := 0$ for any other edge. Now, we can easily verify that (D', λ') is the required projective HT-drawing.

Indeed, let e and f be independent edges. If both e and f are inside edges, then $\mathrm{cr}_{D'}(e, f) = \mathrm{cr}_{D^+}(e, f) = 0 = \lambda'(e)\lambda'(f)$, since D^+ is an ordinary HT-drawing. If both e and f are outside edges, then $\mathrm{cr}_{D'}(e, f) = \mathrm{cr}_{D''}(e, f) = \mathrm{cr}_D(e, f) = \lambda(e)\lambda(f) = \lambda'(e)\lambda'(f)$. Finally, if one of this edges is an inside edge and the other is an outside edge, then $\mathrm{cr}_{D'}(e, f) = 0 = \lambda'(e)\lambda'(f)$, because $D'(e)$ and $D'(f)$ are separated by $D'(Z)$. ⊗

4 Proof of the Strong Hanani–Tutte Theorem on $\mathbb{R}P^2$

In this section, we prove Theorem 1 assuming validity of Theorem 10 as well as few other auxiliary results from the previous section, which will be proved only

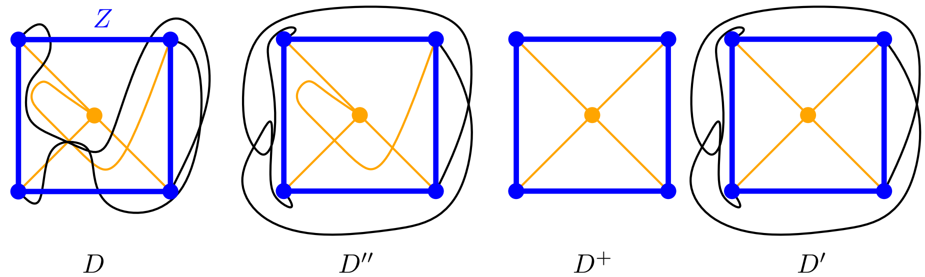

Figure 9: Redrawing a projective HT-drawing of K_5 analogously to the drawing in Fig. 2.

in the later sections.

Given a graph G that admits an HT-drawing on the projective plane, we need to show that G is actually projective-planar. By Corollary 5, we may assume that G admits a projective HT-drawing (D, λ) on S^2. We aim to use Theorem 10. For this, we need that G is 2-connected and contains a suitable trivial cycle Z that may be redrawn so that it satisfies the assumptions of Theorem 10. Therefore, we start with auxiliary claims that will bring us to this setting. Many of them are similar to auxiliary steps in [19] (sometimes they are almost identical, adapted to a new setting).

Before we state the next lemma, we recall the well known fact that any graph admits a (unique) decomposition into *blocks of 2-connectivity* [5, Ch. 3]. Here, we also allow the case that G is disconnected. Each block in this decomposition is either a vertex (this happens only if it is an isolated vertex of G), an edge or a 2-connected graph with at least three vertices. The intersection of two blocks is either empty or it contains a single vertex (which is a cut in the graph). The blocks of the decomposition cover all vertices and edges (a vertex may occur in several blocks whereas any edge belongs to a unique block).

Lemma 19 *If G admits a projective HT-drawing on S^2, then at most one block of 2-connectivity in G is non-planar. Moreover, if all blocks are planar, G is planar as well.*

We note that in [23] it was proved that a minimal counterexample to the strong Hanani–Tutte theorem on any surface is vertex 2-connected. However, for the projective plane the same result can be obtained by much simpler means; therefore, we include its proof here.

Proof. First, for contradiction, let us assume that G contains two distinct non-planar blocks B_1 and B_2. If B_1 and B_2 are disjoint, then Lemma 6 implies that at least one of these blocks, say B_2, does not contain any non-trivial cycle. However, it means that B_2 admits an ordinary HT-drawing on S^2 by Lemma 8. Therefore, B_2 is planar by the strong Hanani–Tutte theorem in the plane [2, 25, 19]. This contradicts our original assumption. It remains to consider the case when B_1 and B_2 share a vertex v (it must be a cut vertex). Let us set

$H := B_1 \cup B_2$. Let P be a spanning tree of H with just two edges e_1, e_2 incident to v and such that $e_1 \in B_1$ and $e_2 \in B_2$. Note that such a tree always exists, because B_1 and B_2 are connected after removing v. By Lemma 8 we may assume that all the edges of P are trivial (after a possible alteration of λ).

Any nontrivial edge e from $E(H) \setminus E(P)$ creates a nontrivial cycle in the corresponding block. If e is not incident to v, then the cycle avoids v by the choice of P. Using Lemma 6 again, we see that at least one of the blocks, say B_2, satisfies that all its nontrivial edges are incident with v. This already implies that B_2 is a planar graph, because D is an HT-drawing of B_2 on S^2 (there are no pairs of nontrivial independent edges in G). This is again a contradiction.

The last item in the statement of this lemma is a well known property of planar graphs. It is sufficient to observe that a disjoint union of two planar graphs is a planar graph, and moreover, that if a graph G contains a cut vertex v and all the components after cutting (and reattaching v) are planar, then G is planar as well. ⊗

Observation 20 *Let (D, λ) be a drawing of a 2-connected graph. If D does not contain any trivial cycle, then G is planar.*

Proof. As G is 2-connected, it is either a cycle or it contains three disjoint paths sharing their endpoints. A cycle is a planar graph as we need. In the latter case, two of the paths are both trivial or both nontrivial. Together, they induce a trivial cycle, therefore this case cannot occur. ⊗

Lemma 21 *Let (D, λ) be a projective HT-drawing on S^2 of a graph G and let Z be a cycle in G. Then G can be redrawn only by local changes next to the vertices of Z to a projective HT-drawing D' on S^2 so that λ remains unchanged and $cr_{D'}(e, f) = \lambda(e)\lambda(f)$ for any pair $(e, f) \in E(Z) \times E(G)$ of distinct (not necessarily independent) edges. In particular, if $\lambda(e) = 0$ for every edge e of Z, then every edge of Z becomes even in D'.*

Proof. Since we have a projective HT-drawing, $cr_D(e, f) = \lambda(e)\lambda(f)$ for every pair of independent edges. To prove the claim it remains to show that local changes allow to change the parity of $cr_D(e, f)$ whenever e is an edge of Z and e and f share a vertex.

This can be done in two steps. First we use local move c) from Fig. 10 to obtain the desired parity of $cr_D(e, f)$, for all pairs of consecutive edges (e, f) on Z. This move may change the parity of crossings between edges on Z and dependent edges not on Z.

Next we use local moves a) and b) from Fig. 10 to obtain the desired parity of crossings between edges on Z and dependent edges not on Z. If v is the vertex common to h, e and f, where e and f are edges on Z, move a) is used when we need to change the parity of $cr_D(e, h)$ and its symmetric version to change the parity of $cr_D(f, h)$. Move b) is used when we need to change the parity for both $cr_D(e, h)$ and $cr_D(f, h)$. Since these moves do not change the parity of $cr_D(e, h')$ or $cr_D(f, h')$ for any other edge h', the claim follows. ⊗

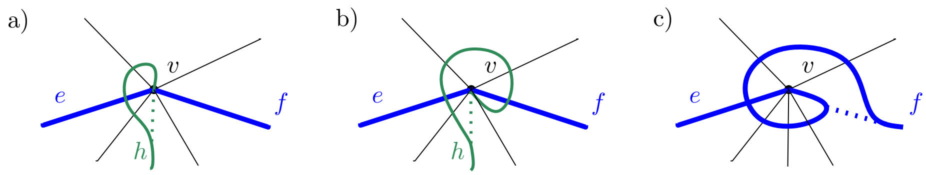

Figure 10: Local changes to make all edges of Z even. The original drawing of the edge near v is dotted.

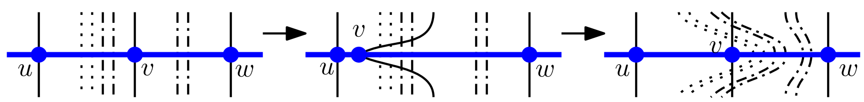

Figure 11: Almost contracting an edge.

Once we know that the edges of a cycle can be made even we also need to know that such a cycle can be made simple.

Lemma 22 *Let (D, λ) be a projective HT-drawing on S^2 of a graph G and let Z be a cycle in G such that each of its edges is even. Then G can be redrawn so that Z becomes a simple cycle, its edges remain even and the resulting drawing is still a projective HT-drawing (with λ unchanged).*

Proof. First, we want to get a drawing such that there is only one edge of Z which may be intersected by other edges. Let us consider three consecutive vertices u, v and w on Z, with $v \notin \{u, w\}$. We *almost-contract* uv so that we move the vertex v (and the edges incident to v) towards u until we remove all intersections between uv and other edges. Note that the image of the cycle Z is not changed; we only slide v towards u along Z. This way, uv is now free of crossings and these crossings appear on vw. See the two leftmost pictures in Fig. 11. (The right picture will be used in the proof of Theorem 18.)

Since uv as well as vw were even edges in the initial drawing, vw remains even after the redrawing. Similarly, the parity of the number of crossings between the edges incident to v and other edges is not affected. If uv and vw intersected, then this step introduces self-intersections of vw.

After performing such redrawing repeatedly, we get that there is only one edge of Z which may be intersected by other edges, as required. We remove possible self-crossings of this edge and the other edges incident with v, as described in Sect. 2, and we are done. ⊗

Apart from lemmas tailored to set up the separation assumptions, we also need one more lemma that will be useful in the inductive proof of Theorem 1.

Lemma 23 *Let (D, λ) be a projective HT-drawing of G and let Z be a cycle satisfying the separation assumptions. Let B be an inside bridge such that any proper path in B with both endpoints on $V(B) \cap V(Z)$ is nontrivial. Then $|V(B) \cap V(Z)| = 2$ and B induces a single arrow and no loop.*

Proof. First, we show that there is no nontrivial cycle in B. For contradiction, there is a nontrivial cycle N in B. By the 2-connectivity of G there exist two vertex disjoint paths p_1 and p_2 (possibly of length zero) that connect Z to N. We consider shortest such paths; thus, each of the paths shares only one vertex with Z and one vertex with N. Let y_1 and y_2 be the endpoints of p_1 and p_2 on N, respectively. Let p_3, p_4 be the two arcs of N between y_1 and y_2. We consider two paths q_1 and q_2 where q_1 is obtained from the concatenation of p_1, p_3 and p_2, while q_2 is obtained from the concatenation of p_1, p_4 and p_2. Since N is non-trivial, one of these paths is trivial, which provides the required contradiction.

Next, we observe that B does not induce any loop in the inside arrow graph. For contradiction, it induces a loop at a vertex x of Z. This means that there is a proper nontrivial walk κ in B with both endpoints x. We set up κ so that it is the shortest such walk. We already know that κ cannot be a cycle, thus it contains a closed nonempty subwalk κ' and we set up κ' so that it is the shortest such subwalk. Therefore, it must be a cycle; by the previous part of this proof, it is trivial. However, it means that κ can be shortened by leaving out κ', which is the required contradiction.

Now, we show that $|V(B) \cap V(Z)| = 2$. By the 2-connectedness of G, we have that $|V(B) \cap V(Z)| \geq 2$. Thus, for contradiction, let a, b, c be three distinct vertices of $V(B) \cap V(Z)$. Let v be one of the inner vertices of B (there must be such a vertex since B cannot be a single edge in this case). By the definition of inside/outside bridges, there exist proper walks p_a, p_b and p_c connecting v to a, b and c, respectively. By the pigeonhole principle, two of the walks have the same value of λ; without loss of generality, let them be p_a and p_b. It follows that the proper walk obtained from the concatenation of p_a and p_b is trivial. Since B does not contain any non-trivial cycle, this walk can be shortened to a trivial proper path between a and b by an analogous argument as in the previous paragraph. A contradiction.

Finally, we know that there are two vertices in $V(B) \cap V(Z)$. Let x and y be these two vertices. Since any path connecting x and y is nontrivial, B induces the arrow $\overline{xy}$ in A^+. No other arrow in A^+ induced by B is possible since there are no loops. $\otimes$

Proposition 24 below is our main tool for deriving Theorem 1 from Theorem 10. It is set up in such a way that it can be inductively proved from Theorem 10. Then it implies Theorem 1, using the auxiliary lemmas from the beginning of this section, relatively easily.

Proposition 24 *Let (D, λ) be a projective HT-drawing of a 2-connected graph G on S^2 and Z a cycle in G that is completely free of crossings in D and such that each of its edges is trivial in D. Assume that (V^+, E^+) or (V^-, E^-) is empty (recall the notation from Definition 9). Then G can be embedded into $\mathbb{R}P^2$ so that Z bounds a face of the resulting embedding homeomorphic to a disk. If, in addition, D is an ordinary HT-drawing on S^2, then G can be embedded into S^2 so that Z bounds a face of the resulting embedding (this face is again*

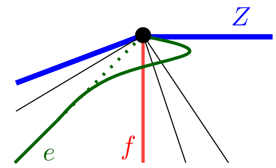

Figure 12: Local changes at u. The original drawing of the edge is dotted, Z is depicted in blue, f (as a part of γ) in red. The changed edge in green.

homeomorphic to a disk—there is in fact no other option on S^2).[9]

Proof. The proof proceeds by induction on the number of edges of G. The base case is when G is a cycle.

Without loss of generality, we assume that (V^-, E^-) is empty. That is, $G = G^{+0}$. If (V^+, E^+) is also empty, G consists only of Z and such a graph can easily be embedded into the plane or projective plane as required. Therefore, we assume that (V^+, E^+) is nonempty.

We find a path γ in $(V(G^{+0}), E(G^{+0}) \setminus E(Z))$ connecting two points x and y lying on Z. We may choose x, y so that $x \neq y$ since G is 2-connected.

Case 1: There exists a trivial γ. First we solve the case that at least one such path γ is trivial. We show that all edges of γ can be made even and simple in the drawing while preserving simplicity of Z, the fact that Z is free of crossings and the projective Hanani–Tutte condition on the whole drawing of G^{+0}.

As the first step, we use Lemma 8 in order to achieve that $\lambda(e) = 0$ for any edge e of Z and γ simultaneously. By inspecting the proof of Lemma 8 we see that we can achieve this by vertex-crosscap switches only over the inner vertices of γ (for this, we set up the root in the proof to be one of the endpoints of γ). In particular we can perform these vertex-crosscap switches inside Z without affecting Z.

Now, we want to make the edges of γ even, again without affecting Z. First, for any pair (e, f) of adjacent edges of γ which intersect oddly, we locally perform the move c) from Fig. 10 similarly as in Lemma 21. Next, we consider any edge $e \notin E(\gamma)$ adjacent to a vertex $u \in V(\gamma) \setminus V(Z)$. For such an edge we perform one of the moves a) or b) from Fig. 10 so that we achieve that e intersects evenly each of the two edges of γ incident with u. Finally, we consider any edge $e \notin E(\gamma) \cup E(Z)$ adjacent to $u \in \{x, y\}$, one of the endpoints of γ on Z. Let f be the edge of γ incident with u. If e and f intersect oddly, we perform the move from Fig. 12. This is possible since Z is free of crossings. This way we achieve that every edge of γ is even.

As the last step of the redrawing of γ, we want to make γ simple (again without affecting Z). This can be done in the same way as in Lemma 22. We

[9]We need to consider the case of ordinary HT-drawings in this proposition for a well working induction.

almost-contract all edges of γ but one so that there is only one edge of γ that intersects with other edges. Then we remove possible self-intersections.

The rest of the argument is easier to explain if we switch inside and outside (this is easily doable by a homeomorphism of S^2) and treat drawings on S^2 as drawings in the plane.

We may assume that after the homeomorphism Z is drawn in the plane as a circle with the inner region empty and with x and y antipodal. The vertices x and y split Z into two paths; we denote by p_1 the 'upper' one and by p_2 the 'lower' one. We may also assume that γ is 'above' p_1 by adapting the initial choice of the correspondence between S^2 and the plane if necessary.

Now we continuously deform the plane so that Z becomes flatter and flatter until it coincides with the line segment connecting x to y, as depicted in Fig. 13 a). We may further require that no inner vertex of p_1 was identified with any inner vertex of p_2.

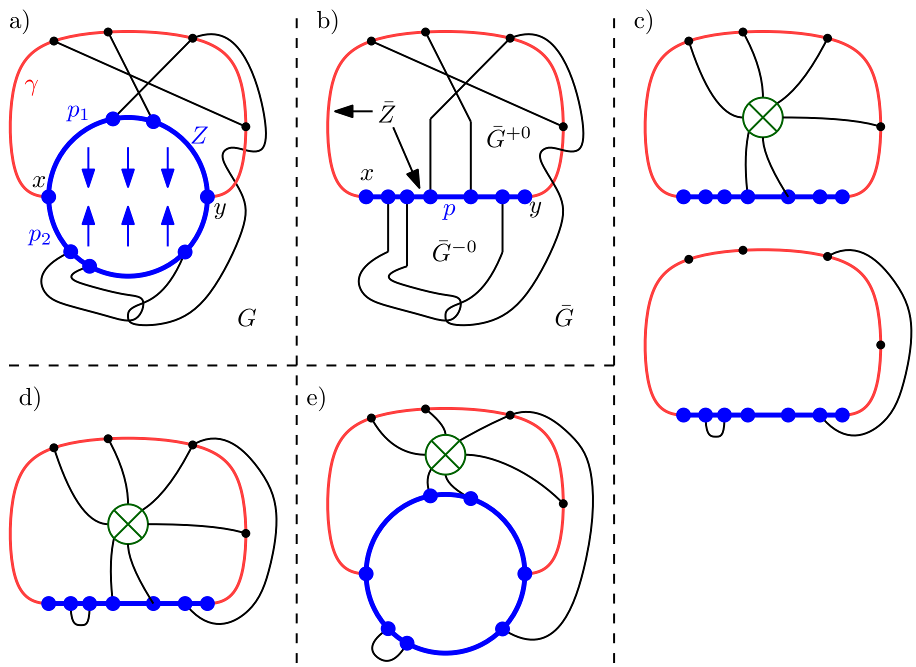

Figure 13: The deformation of the plane that changes G into $\bar{G}$, the redrawing of $\bar{G}$ and the resulting embeddings of $\bar{G}$ and G.

This way, we get a projective HT-drawing $(\bar{D}, \bar{\lambda})$ of a new graph $\bar{G}$: all the vertices of G remain present in $\bar{G}$, that is, $V(G) = V(\bar{G})$. Also the edges of G which are not on Z are present in $\bar{G}$. Only some of the edges of Z may disappear and they are replaced with edges forming a path p between x and y. Note that we did not introduce any multiple edges, because there is no edge in G connecting an inner vertex of p_1 with an inner vertex of p_2 (such an edge would have to cross γ oddly). It also turns out that $\bar{G}$ has one edge less than G. Regarding $\bar{\lambda}$, we have $\lambda(e) = \bar{\lambda}(e)$ if e is an edge of $E(G) \setminus E(Z)$ and we have $\bar{\lambda}(e) = 0$ if e belongs to p.

Now consider the cycle $\bar{Z}$ in $\bar{G}$ formed by γ and p. It is trivial and simple. In particular, we distinguish the inside and the outside according to Definition 9. For example, $\bar{G}^{+0}$ corresponds to the part of G in between γ and p_1 before the flattening; see Fig. 13 a) and b).

Now, we apply Theorem 10 and we get a drawing D' of $\bar{G}$. When we look at the two sides of $\bar{G}$ separately, we get that the drawing of one of the sides, say the drawing of $\bar{G}^{+0}$, is a projective HT-drawing, while there is an ordinary HT-drawing on S^2 on the other side. If, in addition, D were already an ordinary HT-drawing, we get an ordinary HT-drawing on both sides by Theorem 18.

Note also that since G was 2-connected, both parts of $\bar{G}$ are 2-connected as well. Subsequently, we examine each of these two parts separately and use the inductive hypothesis; we obtain an embedding of $\bar{G}^{+0}$ into $\mathbb{R}P^2$ such that $\bar{Z}$ bounds a face homeomorphic to a disk as well as an embedding of $\bar{G}^{-0}$ into S^2 such that $\bar{Z}$ bounds a face homeomorphic to a disk. If, in addition, D were already an ordinary HT-drawing, we get also the required embedding of $\bar{G}^{+0}$ into S^2. We merge these two embeddings along $\bar{Z}$ obtaining an embedding of $\bar{G}$ into $\mathbb{R}P^2$ (or S^2 if D were an ordinary HT-drawing). See Fig. 13 c) and d).

Finally, we need to undo the identification of p_1 and p_2 into p. Whenever we consider a vertex v on p different from x and y, it is uniquely determined whether it comes from p_1 or p_2. In addition, if v comes from p_1, then any edge $e \in E(G) \setminus E(Z)$ incident with v must belong to $\bar{G}^{+0}$. Similarly, if v comes from p_1, then any edge $e \in E(G) \setminus E(Z)$ incident with v must belong to $\bar{G}^{-0}$. Therefore, it is possible to undo the identification and we get the required embedding of G. See Fig. 13 e).

Case 2: All choices of γ are nontrivial. Now we deal with the situation when all possible choices of γ are nontrivial. We will first analyse which situations allow such configuration. Later we will show how to draw each of these situations.

Let us consider the inside arrow graph A^+. Since all choices of γ are nontrivial, Lemma 23 shows that every inside bridge induces a single inside arrow. This allows us to redraw inside bridges separately as is provided by the following claim.

Claim 24.1 *For any inside bridge B there exists a planar drawing of $Z \cup B$ in which Z is the outer face.*

Proof. Since we know that B induces only a single arrow, we get that $Z \cup B$ forms an inside fan, according to Definition 15. It follows from Proposition 17 that $Z \cup B$ admits an ordinary HT-drawing such that Z is an outer cycle. However, the setting of ordinary HT-drawings is already fully resolved in Case 1. That is, we may already use Proposition 24 for this drawing and we get the required conclusion. ⊗

We consider the graph A^{+0} obtained from A^+ by adding the edges of Z to it, where A^+ is the inside arrow graph. (Note that $V(A^+) = V(Z)$ according to our definition of the arrow graph.)

Our main aim will be to find an embedding of A^{+0} to $\mathbb{R}P^2$ such that Z bounds a face. As soon as we reach this task, then we can replace an embedding of each arrow by the embedding of inside bridges inducing this arrow via Claim 24.1 in a close neighbourhood of the arrow. If there are, possibly, more inside bridges inducing the arrow, then they are embedded in parallel.

Finally, we show that it is possible to embed A^{+0} in the required way. By Lemma 13, any two disjoint arrows interleave.

Let us consider two concentric closed disks E_1 and E_2 such that E_1 belongs to the interior of E_2. Let us draw Z on the boundary of E_1. Let a be the number of arrows of A^+ and let us consider $2a$ *points* on the boundary of E_1 making the vertices of regular $2a$-gon. These points will be marked by ordered pairs (x, y) where $\overline{xy}$ is an inside arrow. We mark the points so that the cyclic order of the points respect the cyclic order as on Z in the first coordinate (in particular pairs with the same first coordinate are consecutive). However, for a fixed x, the pairs $(x, y_1), \dots (x, y_k)$ corresponding to all arrows emanating from x are ordered in the reverted order when compared with the order of $y_1, \dots, y_k$ on Z. See Fig. 14.

We show that it follows that the points marked (x, y) and (y, x) are directly opposite on E_1 for every inside arrow $\overline{xy}$. For contradiction, let us assume that (x, y) and (y, x) are not directly opposite for some $\overline{xy}$. Then there is another arrow $\overline{uv}$ such that (x, y) and (y, x) do not interleave with (u, v) and (v, u). Indeed, such an arrow must exist because the arrows induce a matching on the points, and (x, y) and (y, x) do not split the points equally. However, if $\overline{xy}$ and $\overline{uv}$ do not share an endpoint, we get a contradiction with the fact that disjoint arrows interleave. If $\overline{xy}$ and $\overline{uv}$ share an endpoint, we get a contradiction that we have reverted the order on the second coordinate.

Now, we get the required drawing in the following way. For any arrow $\overline{xy}$ we connect x with the point (x, y) and y with (y, x). We can do all the connections simultaneously for all arrows without introducing any crossing since we have respected the cyclic order on the first coordinate. We remove the interior of E_1 and we identify the pairs of opposite points on the boundary. This way we introduce a crosscap. Finally, we glue another disk along its boundary to Z and we get the required drawing on $\mathbb{R}P^2$. $\otimes$

Finally, we prove Theorem 1.

Proof of Theorem 1. We prove the result by induction in the number of vertices of G. We can trivially assume that G has at least three vertices.

If G has at least two blocks of 2-connectivity, G can be written as $G_1 \cup G_2$, where $G_1 \cap G_2$ is a minimal cut of G and, therefore, has at most one vertex. By Lemma 19 we may assume that G_1 is planar and G_2 non-planar. By induction, there exists an embedding D_2 of G_2 into $\mathbb{R}P^2$. So G_1 is planar, G_2 is embeddable into $\mathbb{R}P^2$ and $G_1 \cap G_2$ has at most one vertex. From these two embeddings, we easily derive an embedding of $G = G_1 \cup G_2$ in $\mathbb{R}P^2$.

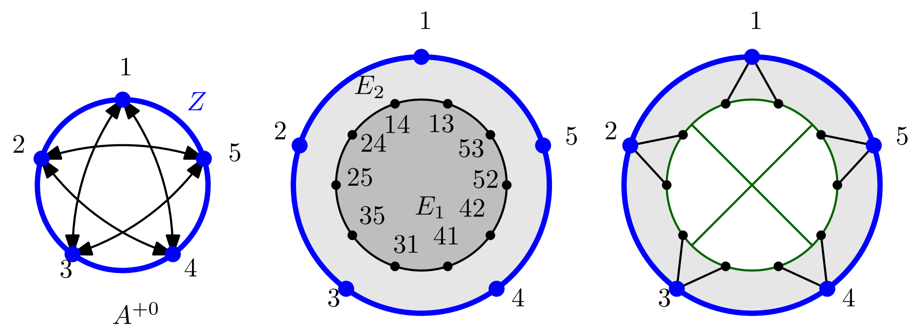

Figure 14: Redrawing the case where every inside bridge induces a single arrow.

We are left with the case when G is 2-connected. By Observation 20, we may assume that there is at least one trivial cycle Z in (D, λ). We can also make each of its edges trivial by Lemma 8 and even by Lemma 21. Then we make Z, in addition, simple using Lemma 22. Hence G, Z and the current projective HT-drawing satisfy the separation assumptions.

Then we use Z to redraw G as follows. At first, we apply Theorem 10 to get a projective HT-drawing (D', λ') that separates G^{+0} and G^{-0}. We define $D^+ := D'(G^{+0})$ and $D^- := D'(G^{-0})$—without loss of generality, D^- is an ordinary HT-drawing on S^2, while D^+ is a projective HT-drawing on S^2.

Finally, we apply Proposition 24 above to D^+ and D^- separately. Thus, we get embeddings of G^{+0} and G^{-0}—one of them in S^2, the other one in $\mathbb{R}P^2$. In addition, Z bounds a face in both of them; hence, we can easily glue them to get an embedding of the whole graph G into $\mathbb{R}P^2$. $\otimes$

5 Labellings of Inside/Outside Bridges and the Proof of Proposition 16

In this section, given an inside (or outside) bridge B, we first describe what are possible combinations of arrows induced by B. Then we use the obtained findings for a proof of Proposition 16, assuming validity of Lemmas 12, 13 and 14 which will be proved in Sect. 6.

Labelling the vertices of the inside/outside bridges. We start with the first step. As usual, we only describe the 'inside' case; the 'outside' case will be analogous. We introduce certain labellings of $V(B) \cap V(Z)$ which will help us to determine arrows.

Definition 25 (Labelling of $V(B) \cap V(Z)$) *A* valid *labelling* $L = L_B$ *for* B *is a mapping* $L\colon V(B) \cap V(Z) \to \{\{0\}, \{1\}, \{0, 1\}\}$ *obtained in the following way.*

If $V(B)\setminus V(Z) \neq \emptyset$ *we pick a* reference vertex $v_B \in V(B)\setminus V(Z)$ *for* L. *Then we fix a* labelling parameter $\alpha_B \in \mathbb{Z}_2$ *for* L. *Finally, for any* $u \in V(B) \cap V(Z)$ *and for any proper walk* ω *with endpoints* u *and* v_B*, the vertex* u *receives the label* $\alpha_B + \lambda(\omega) \in \mathbb{Z}_2$. *Note that* u *may receive two labels after considering all such walks. On the other hand, each vertex of* $V(B) \cap V(Z)$ *obtains at least one label, which follows from the definition of bridges (Definition 11).*

If $V(B) \subseteq V(Z)$*, then* B *comprises only of one edge* $e = uv$ *connecting two vertices of* $V(Z)$. *In such case, there are two valid labellings for* B. *We set* $L(u) = \{\alpha_B\}$ *and* $L(v) = \{\lambda(e)+\alpha_B\}$ *for a chosen labelling parameter* $\alpha_B \in \mathbb{Z}_2$.

If the bridge B is understood from the context we may write just v instead of v_B for the reference vertex and α instead of α_B for the labelling parameter. By alternating the choice of α in the definition we may swap all labels. This means that there are always at least two valid labellings for a given inside bridge. On the other hand, a different choice of the reference vertex either does not influence the resulting labelling, or has the same effect as swapping the value of the labelling parameter α. In other words, there are always exactly two valid labellings of the given inside/outside bridge B corresponding to two possible choices of the labelling parameter α, as is explained below.

To see this, consider a vertex $u \in V(B) \setminus V(Z)$ different from $v = v_B$. By Definition 11, there is a proper uv-walk γ in B not using any vertex of Z. Now, for any $x \in V(B) \cap V(Z)$ and for any proper xv-walk ω_{xv} in B, the concatenation of the walks ω_{xv} and γ is a proper xu-walk in B of type $\lambda(\omega_{xv}) + \lambda(\gamma)$. Also, for any proper xu-walk ω_{xu} in B, the concatenation of the walks ω_{xu} and γ is a proper xv-walk in B of type $\lambda(\omega_{xu}) + \lambda(\gamma)$. As a result, choosing u as the reference vertex with $\alpha + \lambda(\gamma)$ as the labelling parameter leads to the same labelling as the choice of v as the reference vertex with the labelling parameter α.

The idea presented above can be used to establish the following simple observation, which we later use several times in the proofs.

Observation 26 *Let* B *be an inside or an outside bridge containing at least one inside/outside vertex. Moreover, let* L *be a valid labelling for* B *and* v *the reference vertex for* L. *Let* $x, y \in V(B)$ *and let* ω *be a proper* xy*-walk in* B. *Then there is a proper* xy*-walk* ω' *in* B *containing the reference vertex* v *such that* $\lambda(\omega) = \lambda(\omega')$.

Proof. If ω contains inside/outside vertices, we choose one of them and denote it by u. In a degenerate case, when ω does not contain any such vertex, then $x \in V(Z)$ and $x = y$, since B cannot consist of just one edge. That is, ω is a walk with the single vertex x and no edge. In this case we choose $u = x$.

Now we find a proper uv-walk γ in B and use it as a detour. More precisely, ω' starts at x and follows ω to the first occurrence of u in ω. Then it goes to v and back along γ. Finally, it continues to y along ω. It is clear that $\lambda(\omega) = \lambda(\omega')$. By the choice of u, the walk ω' is also proper. ⊗

Now, whenever u and w are two vertices from $V(B) \cap V(Z)$, there is an arrow $\overline{uw}$ arising from B if and only if the vertices u and w were assigned different labels by L_B—this is proved in Proposition 27 below.

Proposition 27 *Let B be an inside bridge and L be a valid labelling for B. Let $x, y \in V(B) \cap V(Z)$ (possibly $x = y$). Then the inside arrow graph A^+ contains an arrow $\overline{xy}$ arising from B if and only if $L(x) \cup L(y) = \{0, 1\}$.*

Proof. It is straightforward to check the claim if B is just an edge e. Indeed, if $x \neq y$, then $e = xy$, and it defines the arrow $\overline{xy}$ arising from B if and only if $\lambda(e) = 1$, which in turn happens if and only if $L(x) \cup L(y) = \{0, 1\}$ according to Definition 25. If $x = y$, then $\overline{xx}$ is not induced by B and $|L(x) \cup L(x)| = 1$.

If $V(B) \setminus V(Z) \neq \emptyset$, let $v = v_B$ be the reference vertex for L. First, let us assume that $L(x) \cup L(y) = \{0, 1\}$. Let us consider a proper xv-walk ω_{xv} and a proper vy-walk ω_{vy} in B such that $\lambda(\omega_{xv}) \neq \lambda(\omega_{vy})$. Such walks exist by Definition 25, since $L(x) \cup L(y) = \{0, 1\}$. Then the concatenation of these two walks is a nontrivial walk which belongs to $W^+_{xy,B}$; therefore, $\overline{xy}$ is induced by B.

On the other hand, let us assume that there is a nontrivial walk ω in $W^+_{xy,B}$ defining the arrow $\overline{xy}$. We can assume that ω is not just an edge, because it would mean that B consists only of that edge. By Observation 26, we may assume that ω contains the reference vertex v. This vertex splits ω into two proper walks ω_1 and ω_2 so that each of them has at least one edge. Since $\lambda(\omega) = 1$, we have $\lambda(\omega_1) \neq \lambda(\omega_2)$. Consequently, $L(x) \cup L(y) = \{0, 1\}$. ⊗

The argument from the last two paragraphs of the proof above can also be used to establish the following lemma.

Lemma 28 *Let B be an inside or an outside bridge, let L be a valid labelling for B, and let $x, y \in V(B) \cap V(Z)$ be two distinct vertices. Moreover, we assume that $|L(x)| = |L(y)| = 1$. Then for any proper xy-walks ω_1, ω_2 in B we have $\lambda(\omega_1) = \lambda(\omega_2)$.*

Proof. If B contains just the edge xy, the observation is trivially true. Therefore, we assume that there is the inside/outside reference vertex $v \in V(B)$ for L. By the assumption, every two proper xv-walks in B have the same λ-value. The same holds also for proper vy-walks in B. By Observation 26, we can assume that both ω_1 and ω_2 contain v. Then the lemma follows. ⊗

We will also need the following description of inside arrows induced by an inside bridge which does not induce any loop.

Lemma 29 *Let B be an inside bridge which does not induce any loop. Then the inside arrows induced by B form a complete bipartite graph. (One of the parts is empty if B does not induce any arrow.)*

Proof. Let us consider a valid labelling L for B. By Proposition 27, $|L(x)| = 1$ for any $x \in V(B) \cap V(Z)$, since B does not induce any loop. By Proposition 27 again, the inside arrows induced by B form a complete bipartite graph, in which one part corresponds to the vertices labelled 0 and the second part corresponds to the vertices labelled 1. ⊗

We conclude this section a by a proof of Proposition 16.

Proof of Proposition 16. We need to distinguish few cases.

First, we consider the case when we have two disjoint inside arrows, but at least one of them is a loop. In this case, it is easy to see that Lemma 12 implies that G forms the outside fan and we are done.

Second, let us consider the case that we have two disjoint inside arrows $\overline{ab}$ and $\overline{cd}$ which are not loops. Lemma 12 implies that the only possible outside arrows are $\overline{ac}$, $\overline{ad}$, $\overline{bc}$, $\overline{bd}$. (In particular, there are no loops outside.) If there are not two disjoint arrows outside, then G forms an outside fan and we are done. Therefore, we may assume that there are two disjoint arrows outside, without loss of generality, $\overline{ac}$ and $\overline{bd}$ (otherwise we swap a and b). By swapping outside and inside in the previous argument, we get that only further possible arrows inside are $\overline{ad}$ and $\overline{bc}$.

Now we distinguish a subcase when there is an inside bridge inducing the inside arrows $\overline{ab}$ and $\overline{cd}$. In this case, $\overline{ad}$ and $\overline{bc}$ must be inside arrows as well by Lemma 29. By Lemma 12, we know that $\overline{ac}$ and $\overline{bd}$ are the only outside arrows (in particular they are induced by different outside bridges by a variant of Lemma 29 for outside) and we get that they must alternate by Lemma 13. That is, up to relabelling of the vertices, we get the right cyclic order for an inside square. In order to check that G indeed forms an inside square, it remains to verify that G has only one nontrivial inside bridge. The inside arrows are $\overline{ab}$, $\overline{bc}$, $\overline{cd}$ and $\overline{ad}$. If any of these arrows, for example $\overline{ab}$, is induced by two bridges, then we get a contradiction with Lemma 13, in this case on arrows $\overline{ab}$ and $\overline{cd}$.

By swapping inside and outside we solve the subcase when there is an outside bridge inducing the outside arrows $\overline{ac}$ and $\overline{bd}$; we get that G forms an outside square.

It remains to consider the subcase when $\overline{ab}$ and $\overline{cd}$ arise from different inside bridges and $\overline{ac}$ and $\overline{bd}$ arise from different outside bridges. However, Lemma 13 applied to the inside and then to the outside reveals that these two events cannot happen simultaneously.

Consequently, we have proved Proposition 16 in case there are two disjoint inside arrows. Analogously, we resolve the case when we have two disjoint arrows outside.

Finally, we consider the case when every pair of inside arrows shares a vertex and every pair of outside arrows shares a vertex. If there is a vertex v common to all the inside arrows, then we get an inside fan and we are done.

It remains to consider the last subcase when there is no vertex common to all inside arrows while every pair of inside arrows shares a vertex. This leaves the only option that there are three distinct vertices a, b and c on Z and all three inside arrows $\overline{ab}, \overline{ac}$ and $\overline{bc}$ are present. Then, the only possible outside arrows are $\overline{ab}, \overline{ac}$ and $\overline{bc}$ as well due to Lemma 12. In addition, all three outside arrows $\overline{ab}$, $\overline{ac}$ and $\overline{bc}$ must be present, otherwise we have an outside fan and we are done.

In the present case, an inside bridge can induce at most two arrows by Lemma 29. Let us consider the three pairs of arrows $\{\overline{ab}, \overline{ac}\}$, $\{\overline{ab}, \overline{bc}\}$, and $\{\overline{ac}, \overline{bc}\}$. If at most one of these pairs is induced by an inside bridge, then G forms an inside split triangle and we are done. Analogously, we are done, if at most one of these pairs is induced by an outside bridge. Therefore, it remains to consider the case that at least two such pairs are induced by inside bridges and at least two such pairs are induced by outside bridges. However, this yields a contradiction to Lemma 14. $\otimes$

6 Forbidden Configurations of Arrows

In this section we show that certain combinations of arrows are not possible. That is, we prove Lemmas 12, 13 and 14. As before, we have a fixed graph G, its drawing (D, λ) on S^2 and a cycle Z in G. Again, we assume that $G, (D, \lambda)$ and Z satisfy the separation assumptions.

Homology and intersection forms. We start with a brief explanation of intersection forms that will help us to prove the required lemmas.

We assume that the reader is familiar with basics of homology theory, otherwise we refer to the introductory books by Hatcher [10] or Munkres [17]. We always work with homology over $\mathbb{Z}_2$ and, unless stated otherwise, we work with singular homology. Let S be a surface. We will mainly work with the first homology group and we denote by $B_1(S)$, $Z_1(S)$ and $H_1(S) := Z_1(S)/B_1(S)$ the group of 1-*boundaries*, of 1-*cycles* and the first *homology group*, respectively. Given a 1-cycle $z \in Z_1(S)$, if there is no risk of confusion, we also consider it as an element of $H_1(S)$, although, formally speaking, we should consider its homology class $[z]$. Similarly, if there is no risk of confusion, we do not distinguish a 1-cycle and its support. Namely, by an intersection of two 1-cycles we actually mean an intersection of their images. We use the same convention for crossings, that is, transversal intersections.

Let S be a surface. The *intersection form* on S is a unique bilinear map $\Omega_S \colon H_1(S) \times H_1(S) \to \mathbb{Z}_2$ with the following property. Whenever $z_1, z_2 \in Z_1(S)$ are two 1-cycles intersecting in finite number of points and crossing in every such point (i. e., intersecting transversally), then $\Omega_S(z_1, z_2)$ is the number of crossings of z_1 and z_2 modulo 2; we refer to [6, Sect. 8.4] for the existence of Ω_S. In particular, Ω_{S^2} is the trivial map since $H_1(S^2)$ is trivial. On the other hand, $\Omega_{\mathbb{R}P^2}$ is already nontrivial:

Lemma 30 (Intersection form on $\mathbb{R}P^2$) *Let z_1 and z_2 be two homologically nontrivial 1-cycles in $\mathbb{R}P^2$. Then $\Omega_{\mathbb{R}P^2}(z_1, z_2) = 1$. In particular, if z_1 and z_2*

have a finite number of intersections and they cross at every intersection, then they have to cross an odd number of times.

Proof. Since the intersection form $\Omega_{\mathbb{R}P^2}$ depends only on the homology class, and since $H_1(\mathbb{R}P^2) = \mathbb{Z}_2$, it is sufficient to exhibit any two nontrivial 1-cycles that intersect an odd number of times on $\mathbb{R}P^2$. This is an easy task. $\otimes$

From sphere to the projective plane. Although it is overall simpler to do the proof of Theorem 1 in the setting of projective HT-drawings on S^2, it is easier to prove Lemmas 12, 13 and 14 in the setting of HT-drawings on $\mathbb{R}P^2$. A small drawback is that we need to check that splitting of S^2 to the inside and outside part works analogously on $\mathbb{R}P^2$ as well.

Lemma 31 *Let (D, λ) be a projective HT-drawing of a graph G on S^2 and let Z be a cycle satisfying the separation assumptions. Let $D_\otimes$ be the HT-drawing of G on $\mathbb{R}P^2$ coming from the proof of Lemma 4. Then $D_\otimes(Z)$ is a simple cycle such that each of its edges is even, which separates $\mathbb{R}P^2$ into two parts, $(\mathbb{R}P^2)^+$ and $(\mathbb{R}P^2)^-$. In addition, every inside edge (with respect to D) which is incident to a vertex of Z points locally into $(\mathbb{R}P^2)^+$ in $D_\otimes$ as well as every outside edge (with respect to D) which is incident to a vertex of Z points locally into $(\mathbb{R}P^2)^-$.*

Proof. By statement of Lemma 4 we already know that $D_\otimes(Z)$ is a simple cycle and that each of its edges is even. For the rest, we need to inspect the construction of $D_\otimes$ in the proof of Lemma 4. However, we get all the required conclusions directly from this construction. $\otimes$

Drawings of walks. We also need to set up a convention regarding drawings of walks in a graph G. Let D be a drawing of a graph G on a surface S. Let ω be a walk in G. Then D induces a continuous map $D(\omega)\colon [0,1] \to S$; it is given by the concatenation of drawings of edges of ω. Here we also allow that ω is a walk of length 0 consisting of a single vertex v. Then $D(\omega)$ is a constant map whose image is $D(v)$. If ω is a closed walk, then we may regard it as an element of $H_1(S)$.

Proofs of the lemmas. Now we have introduced enough tools to prove the required lemmas. In all three proofs, $D_\otimes$ stands for the HT-drawing on $\mathbb{R}P^2$ from Lemma 31. First, we prove Lemma 13 which has a very simple proof. In fact, we prove slightly stronger statement which we plan to reuse later on. Lemma 13 follows directly from Lemma 32 below.

Lemma 32 *Let a, b, x and y be four distinct vertices of Z such that x and y are on the same arc of Z when split by a and b. Then any two walks $\omega^+_{ab} \in W^+_{ab}$ and $\omega^+_{xy} \in W^+_{xy}$ must share a vertex.*

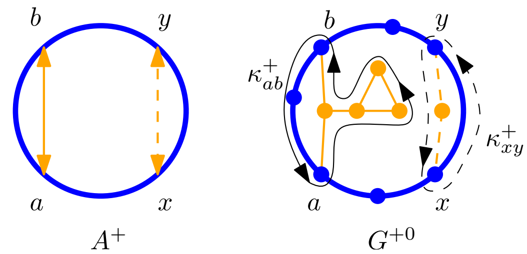

Figure 15: Walks in Lemma 32.

Proof. For contradiction, $\omega^+_{ab} \in W^+_{ab}$ and $\omega^+_{xy} \in W^+_{xy}$ do not share a vertex. We consider a closed walk κ^+_{ab} arising from a concatenation of the walk ω^+_{ab} and the arc of Z connecting a and b not containing x, y. We also consider the closed walk κ^+_{xy} obtained analogously. See Fig. 15. The homological 1-cycles corresponding to $D_\otimes(\kappa^+_{ab})$ and $D_\otimes(\kappa^+_{xy})$ are both non-trivial; therefore, by Lemma 30, $D_\otimes(\kappa^+_{ab})$ and $D_\otimes(\kappa^+_{xy})$ must have an odd number of crossings. (Note that, for example, $D_\otimes(\kappa^+_{ab})$ may have self-intersections or self-touchings, but there is a finite number of intersections between $D_\otimes(\kappa^+_{ab})$ and $D_\otimes(\kappa^+_{xy})$ which are necessarily crossings.) However, as $\omega^+_{ab} \in W^+_{ab}$ and $\omega^+_{xy} \in W^+_{xy}$ do not have a vertex in common, it follows that $D_\otimes(\kappa^+_{ab})$ and $D_\otimes(\kappa^+_{xy})$ have an even number of crossings, because $D_\otimes$ is an HT-drawing by Lemma 4. A contradiction. $\otimes$

We have proved Lemma 13 and we continue with the proofs of the next two lemmas.

Proof of Lemma 12. To the contrary, we assume that we have an inside arrow $\overline{xy}$ and an outside arrow $\overline{uv}$ which do not share any endpoint. However, we allow $x = y$ or $u = v$, that is, we allow loops. As before, we consider a closed walk κ^+_{xy} obtained from the concatenation of a walk from $\omega^+_{xy} \in W^+_{xy}$ and any of the two arcs of Z connecting x and y. If $x = y$, then we do not add the arc from Z. Analogously, we have a closed walk κ^-_{uv} coming from a walk in W^-_{uv} and an arc of Z connecting u and v. Both of these walks are nontrivial and we aim to get a contradiction with Lemma 30.

Unlike the previous proof, this time $D_\otimes(\kappa^+_{xy})$ and $D_\otimes(\kappa^-_{uv})$ may not cross at every intersection. Namely, κ^+_{xy} and κ^-_{uv} may share some subpath of Z, but apart from this subpath the intersections are crossings. We slightly modify these drawings in the following way. Let us recall that $D_\otimes(Z)$ splits $\mathbb{R}P^2$ into two parts $(\mathbb{R}P^2)^+$ and $(\mathbb{R}P^2)^-$ according to Lemma 31. We slightly push into $(\mathbb{R}P^2)^+$ the subpath of κ^+_{xy} shared with Z (possibly consisting of a single vertex). This way, we obtain a drawing $D^+_\otimes$ of κ^+_{xy}. Similarly, we slightly push the subpath of κ^-_{uv} shared with Z into $(\mathbb{R}P^2)^-$, obtaining a drawing $D^-_\otimes$ of κ^-_{uv}. See Fig. 16. Now, $D^+_\otimes(\kappa^+_{xy})$ and $D^-_\otimes(\kappa^-_{uv})$ cross at every intersection and the crossings of $D^+_\otimes(\kappa^+_{xy})$ and $D^-_\otimes(\kappa^-_{uv})$ correspond to the crossings of $D_\otimes(\kappa^+_{xy})$ and $D_\otimes(\kappa^-_{uv})$.

We now consider the crossings of $D_\otimes(\kappa^+_{xy})$ and $D_\otimes(\kappa^-_{uv})$. Whenever e is

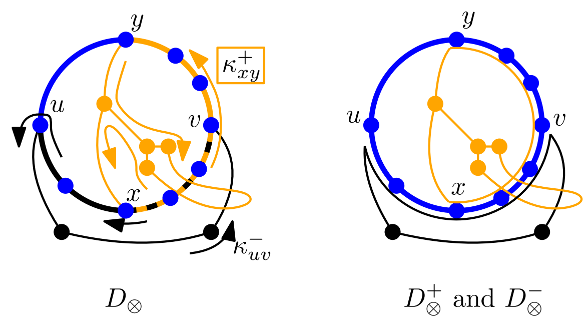

Figure 16: Walks in Lemma 12.

an edge of κ^+_{xy} and f is an edge of κ^-_{uv} such that e and f are independent, then $D_\otimes(e)$ and $D_\otimes(f)$ have an even number of crossings, because $D_\otimes$ is an HT-drawing. However, if e and f are adjacent, then they still cross evenly since one of these edges must belong to Z. Here we crucially use that $\overline{xy}$ and $\overline{uv}$ do not share any endpoint. Therefore, $D_\otimes(\kappa^+_{xy})$ and $D_\otimes(\kappa^-_{uv})$ have an even number of crossings, and consequently, $D^+_\otimes(\kappa^+_{xy})$ and $D^-_\otimes(\kappa^-_{uv})$ as well. This is a contradiction to Lemma 30. $\otimes$

Proof of Lemma 14. For contradiction, there is such a configuration.

Let e^+_a be any edge of $E(B^+)$ incident to a. Analogously, we define edges e^-_a, e^+_b, e^-_b, e^+_c and e^-_c. We observe that there is a walk $\omega^+_{ab} \in W^+_{ab}$ which uses the edges e^+_a and e^+_b. Indeed, it is sufficient to consider an arbitrary proper walk using e^+_a and e^+_b in B^+. This walk is nontrivial by Lemma 28. (The assumptions of the lemma are satisfied by Proposition 27 since B^+ does not induce any inside loops.) We also let κ^+_{ab} be the closed walk obtained from the concatenation of ω^+_{ab} and the arc of Z connecting a and b and avoiding c. Analogously, we define ω^+_{ac}, ω^-_{ab}, ω^-_{bc} and closed walks κ^+_{ac}, κ^-_{ab} and κ^-_{bc}. When defining the closed walks, we always use the arc of Z which avoids the third point among a, b and c. All these eight walks are nontrivial.

Now, we aim to show that e^+_a and e^-_a cross oddly in the drawing $D_\otimes$. We consider the closed walks κ^-_{ab} and κ^+_{ac} and their drawings $D_\otimes(\kappa^-_{ab})$ and $D_\otimes(\kappa^+_{ac})$. The walks κ^-_{ab} and κ^+_{ac} share only the point a; therefore, $D_\otimes(\kappa^-_{ab})$ and $D_\otimes(\kappa^+_{ac})$ cross at every intersection possibly except $D_\otimes(a)$. By Lemma 31 we know that e^+_a and e^-_a point to different sides of Z (in $D_\otimes$); thus, $D_\otimes(\kappa^-_{ab})$ and $D_\otimes(\kappa^+_{ac})$ actually touch in $D_\otimes(a)$. This touching can be removed by a slight perturbation of these cycles, analogously as in the proof of Lemma 12, without affecting other intersections. By Lemma 30 we therefore get that $D_\otimes(\kappa^-_{ab})$ and $D_\otimes(\kappa^+_{ac})$ have an odd number of crossings. However, if we consider any pair of edges (e, f) where e is an edge of κ^-_{ab} and f is an edge of κ^+_{ac} different from (e^-_a, e^+_a), we get that e and f cross an even number of times. Indeed, if we have such $(e, f) \neq (e^-_a, e^+_a)$, then either e or f belongs to Z, or they are independent. Consequently, the odd number of crossings of $D_\otimes(\kappa^-_{ab})$ and $D_\otimes(\kappa^+_{ac})$ has to be realized on e^+_a and e^-_a.

Analogously, we show that e^+_b and e^-_b must cross oddly by considering the walks κ^+_{ab} and κ^-_{bc}.

Now let us consider the closed walk κ^+_{ab} and a closed walk μ^-_{ab} obtained from the concatenation of ω^-_{ab} and the arc of Z connecting a and b which contains c. By analogous ideas as before, we get that $D_\otimes(\kappa^+_{ab})$ and $D_\otimes(\mu^-_{ab})$ touch in $D_\otimes(a)$ and $D_\otimes(b)$; if they intersect anywhere else, they cross there. Using a small perturbation as before, they must have an odd number of crossings by Lemma 30. On the other hand, the pairs of edges (e^+_a, e^-_a) and (e^+_b, e^-_b) cross oddly, as we have already observed. Any other pair (e, f) of edges where e is an edge of κ^+_{ab} and f is an edge of μ^-_{ab} must cross evenly since they are either independent or one of them belongs to Z. This means that $D_\otimes(\kappa^+_{ab})$ and $D_\otimes(\mu^-_{ab})$ intersect evenly, which is a contradiction. ⊗

Intersection of trivial interleaving walks. We conclude this section by a proof of a lemma similar in spirit to Lemma 32. We will need this Lemma in Sect. 7, but we keep the lemma here due to its similarity to previous statements.

Lemma 33 *Let a, b, x and y be four distinct vertices of Z such that x and y are on different arcs of Z when split by a and b. Let ω^+_{ab} and ω^+_{xy} be a proper ab-walk and a proper xy-walk in G^{+0}, respectively, such that $\lambda(\omega^+_{ab}) = \lambda(\omega^+_{xy}) = 0$. Then ω^+_{ab} and ω^+_{xy} must share a vertex.*

Proof. We proceed by contradiction. As usual, we consider closed walks κ^+_{ab} and κ^+_{xy} defined as follows. The walks κ^+_{ab} consists of ω^+_{ab} and an arc of Z connecting a and b, while the walk κ^+_{xy} is formed by ω^+_{xy} and an arc of Z connecting x and y. This time, ω^+_{ab} and ω^+_{xy} are trivial.

We push $D_\otimes(\kappa^+_{ab})$ a bit inside and $D_\otimes(\kappa^+_{xy})$ a bit outside of Z, similarly as in the proof of Lemma 12. This time, however, we introduce one more crossing, because both κ^+_{ab} and κ^+_{xy} are walks in G^{+0}. Since the intersection form of trivial cycles corresponding to the drawings of κ^+_{ab} and κ^+_{xy} is trivial, we get that these drawings have to cross an even number of times. This in turn means that the drawings of ω^+_{ab} and ω^+_{xy} cross an odd number of times. Since $D_\otimes$ is an HT-drawing, this yields a contradiction to the assumption that ω^+_{ab} and ω^+_{xy} do not share a vertex. ⊗

7 Redrawings

We will prove Proposition 17 in this section separately for each case. That is, we show that if G^{+0} forms any of the configurations depicted in Fig. 8, then G^{+0} admits an ordinary HT-drawing on S^2. However, we start with a general redrawing result that we will use in all cases.

Lemma 34 *Let (D, λ) be a projective HT-drawing of G^{+0} on S^2 and Z a cycle satisfying the separation assumptions. Let us also assume that that $D(G^{+0}) \cap S^- = \emptyset$. Let B be one of the inside bridges different from an edge and let L be a valid labelling of B. Let us assume that there is at least one vertex*

$x \in V(B) \cap V(Z)$ such that $|L(x)| = 1$. Then there is a projective HT-drawing (D', λ') of G^{+0} on S^2 such that

(a) D coincides with D' on Z and $D'(G^{+0}) \cap S^- = \emptyset$;

(b) every edge $e \in E(G^{+0}) \setminus E(B)$ satisfies $\lambda(e) = \lambda'(e)$;

(c) every edge $e \in E(B)$ that is not incident to Z satisfies $\lambda'(e) = 0$; and

(d) for every edge $uv = e \in E(B)$ such that $u \in V(Z)$, we have $\lambda'(e) \in L(u)$.

Note that the condition (b) allows that the edges in inside bridges other than B may be redrawn, but only under the condition, that their triviality/nontriviality is not affected.

Proof. Let B^+ be the subgraph of B induced by the vertices of $V(B) \setminus V(Z)$. By the definition of the inside bridge, the graph B^+ is connected; it is also nonempty since we assume that B is not an edge.

Every cycle of the graph B^+ must be trivial. Indeed, if B^+ contained a nontrivial cycle, then this cycle could be used to obtain a nontrivial proper walk from x to x. This would contradict the fact that $|L(x)| = 1$ via Proposition 27. That is, B^+ satisfies the assumptions of Lemma 8. Let $U \subseteq V(B^+)$ be the set of vertices obtained from Lemma 8. That is, if we perform the vertex-crosscap switches on U, we obtain a projective HT-drawing (D_U, λ_U) such that $\lambda_U(e) = 0$ for any edge $e \in E(B^+)$.

Let us recall that every vertex-crosscap switch over a vertex y is obtained from vertex-edge switches of nontrivial edges over y and then from swapping the value of λ on all edges incident to y. The vertex-edge switches do not affect the value of λ. Overall, we get that D_U coincides with D on Z. We also require that all vertex-edge switches are performed in S^+; therefore, D_U does not reach S^-. Altogether, D_U and λ_U satisfy (a), (b) and (c), but we do not know yet whether (d) is satisfied.

In fact, (d) may not be satisfied and we still may need to modify D_U and λ_U. Let e_0 be any edge incident with x. If $L(x) = \{\lambda_U(e_0)\}$, we set $D' := D_U$ and $\lambda' := \lambda_U$. If $L(x) \neq \{\lambda_U(e_0)\}$, we further perform vertex-crosscap switches over all vertices in $V(B^+)$, obtaining D' and λ'. We want to check (a) to (d) for D' and λ'.

It is sufficient to check (a), (b) and (c) only in the latter case. Regarding (a), we again change the drawing only by vertex-edge switches over edges e with $\lambda_U(e) = 1$ inside S^+. Validity of (b) is obvious from the fact that λ_U may be changed only on edges incident with $V(B^+)$. Regarding (c), for any edge $e \in E(B^+)$ we perform the vertex-crosscap switch for both endpoints of e. Therefore, $\lambda'(e) = \lambda_U(e) = 0$. It remains to check (d).

First, we realize that we have set up D' and λ' in such a way that $L(x) = \{\lambda'(e_0)\}$. Indeed, if $L(x) \neq \{\lambda_U(e_0)\}$, then we have made a vertex-crosscap switch over exactly one endpoint of e_0. In particular, we have just checked (d) if $e = e_0$.

Now, let $e = uv \neq e_0$ be an edge from (d). We need to check that $\lambda'(e) \subseteq L(u)$. If $L(u) = \{0, 1\}$, then we are done; therefore, we may assume that $|L(u)| = 1$. Let ω be any proper xu-walk in B containing e_0 and e. Such a walk exists from the definition of an inside bridge (see Definition 11). We have $\lambda(\omega) = \lambda'(\omega)$ because the vertex-crosscap switches over the inner vertices of ω do not affect the triviality of ω. But we also have $\lambda'(\omega) = \lambda'(e_0) + \lambda'(e)$ because $\lambda'(f) = 0$ for any edge $f \in E(B^+)$. Since $L(x) = \{\lambda'(e_0)\}$ and $|L(u)| = 1$, it follows that $L(u) = \{\lambda'(e)\}$ by Proposition 27 and Lemma 28 applied to x and u. ⊗

Inside fan. Now we may prove Proposition 17 for inside fans, which is the simplest case.

Proof of Proposition 17 for inside fans. We assume that G^{+0} forms an inside fan; see Fig. 8. Let $x \in V(Z)$ be the endpoint common to all inside arrows. Let us consider any inside bridge B, possibly trivial. Let $L = L_B$ be a valid labelling of B. It follows from Proposition 27 that $|L(u)| = 1$ for any $u \in V(B) \cap V(Z)$ different from x. (Actually, there is at least one such u, because we assume that G is 2-connected; this is contained in the separation assumptions.) In addition, all $u \in V(B) \cap V(Z)$ different from x have to have the same labels, because there are no arrows among them. Since we may switch all labels in a valid labelling by changing the value of the labelling parameter, we may assume that $L(u) = \{0\}$ for any such u.

Now, we consider all inside bridges $B_1, \dots, B_\ell$ (possibly trivial) and the corresponding labellings $L_{B_1}, \dots L_{B_\ell}$ as above. We apply Lemma 34 to each of these bridges which is not an edge one by one. This way we get a projective HT-drawing (D_1, λ_1) which satisfies:

- (i) D coincides with D_1 on Z and $D_1(G^{+0}) \cap S^- = \emptyset$;
- (ii) every edge $e \in E(G^{+0})$ which is not incident with Z satisfies $\lambda_1(e) = 0$;
- (iii) every edge $e \in E(G^{+0})$ such that $\lambda_1(e) = 1$ is incident with x.

Indeed, property (i) follows from the iterative application of property (a) of Lemma 34. Property (ii) follows from the iterative application of properties (b) and (c) of Lemma 34. Finally, property (iii) follows from (ii), from the iterative application of properties (b) and (d) of Lemma 34 and from the fact that any nontrivial inside bridge which is a single edge must contain x.

Finally, we set $D' := D_1$ and let $\lambda' \colon E(G^{+0}) \to \{0, 1\}$ be the constantly zero function. We observe that from (ii) and (iii), it follows that $\lambda'(e)\lambda'(f) = \lambda_1(e)\lambda_1(f)$ for any pair of independent edges of G^{+0}. Therefore (D', λ') is a projective HT-drawing as well. But, since λ' is identically zero function, D' is also just an ordinary HT-drawing on S^2. ⊗

Inside square. Now we prove Proposition 17 for an inside square. Let B be the inside bridge inducing the inside square and let a, b, c and d be the vertices of $V(B) \cap V(Z)$ labelled according to Definition 15. The main ingredient for our proof of Proposition 17 is the following lemma, which shows that B must have a suitable cut vertex.

Lemma 35 *The inside bridge B, inducing the inside square, contains a vertex v such that the graph $B - v$ is disconnected and the vertices a, b, c and d belong to four different components of $B - v$.*

We first show how Proposition 17 for inside squares follows from Lemma 35. The proof is analogous to the previous proof.

Proof of Proposition 17 for inside squares. We assume that B is the unique inside bridge inducing the inside square and a, b, c and d are vertices of $V(B) \cap V(Z)$ as above. In addition, let v be the vertex from Lemma 35.

First we consider valid labellings of trivial inside bridges. After possibly switching the value of the labelling parameter, we may achieve that all labels of a trivial inside bridge are 0. We apply Lemma 34 to all trivial inside bridges (which are not an edge) and we get a projective HT-drawing (D_1, λ_1) such that $\lambda_1(e) = 0$ for any edge of G^{+0} which does not belong to the nontrivial bridge B. Also, we did not affect λ on edges of B, D_1 coincides with D on Z and we still have $D_1(G^{+0}) \cap S^- = \emptyset$.

Now, we consider a valid labelling L of B. By Proposition 27, every vertex in $V(B) \cap V(Z)$ has just one label. It is easy to check that, up to switching all labels, we have $L(a) = L(c) = \{1\}$ and $L(b) = L(d) = \{0\}$. We apply Lemma 34 to B according to this labelling and we get a projective HT-drawing (D_2, λ_2) such that the only edges e of G^{+0} with $\lambda_2(e) = 1$ are edges of B incident to a or c.

Next, let C_a and C_c be the components of $B - v$ which contains a and c, respectively. We perform vertex-crosscap switches over all vertices of C_a and C_c except a, c and v. We perform the switches inside S^+ as usual. This way we get a projective HT-drawing (D_3, λ_3) such that only edges e of G^{+0} such that $\lambda_3(e) = 1$ are the edges of B incident to v.

Finally, we let $D' = D_3$ and we set $\lambda'(e) = 0$ for any edge e of G^{+0}. Analogously as in the previous proof, $\lambda_3(e)\lambda_3(f) = \lambda'(e)\lambda'(f)$ for any pair of independent edges of G^{+0}. Therefore, (D', λ') is a projective HT-drawing on S^2 and D' is also an ordinary HT-drawing on S^2, as required. ⊗

It remains to prove Lemma 35 to conclude the case of inside squares.

We start with a certain separation lemma in a general graph and then we conclude the proof by verification that the assumptions of this lemma are satisfied.

Lemma 36 *Let G' be an arbitrary connected graph and $A = \{a_1, \ldots, a_4\} \subseteq V(G')$ be a set of four distinct vertices. Let us assume that any $a_i a_j$-path has a common point in $V(G') \setminus A$ with any $a_k a_\ell$-path whenever $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$. Then there is a cut vertex v of G' such that $a_1, \ldots, a_4$ are in four distinct components of $G' - v$.*

Proof. Let us consider an auxiliary graph G'' which is obtained from G' by adding two new vertices x, y and attaching x to a_1, a_2 and y to a_3, a_4. By the assumptions, G'' is connected and moreover, there are no two vertex-disjoint paths connecting x and y. By Menger's theorem (see, e.g., [5, Corollary 3.3.5]), there is a cut-vertex $v \in V(G'') \setminus \{x, y\} = V(G')$ disconnecting x and y. Let C_1 be the connected component of $G'' - v$ containing x and C_2 be the component containing y. Let C'_i, for $i = 1, 2$, be the subgraph of G' induced by v and the vertices of $C_i \cap G'$. Note that, since G' is connected, both C'_1 and C'_2 are connected. We show that v is the desired cut vertex.

Let p_1 be an a_1a_2-path in C'_1 and p_2 an a_3a_4-path in C'_2. Since C'_1 and C'_2 are connected, such paths p_1 and p_2 exist. Moreover, p_1 and p_2 may intersect only in v; however, according to the assumptions, they have to intersect in a vertex outside A. Therefore, they must intersect in v and $v \notin A$. Overall, we have verified that any a_ia_j-path passes through v, for $1 \leq i < j \leq 4$, which shows that v is the desired cut vertex. ⊗

Proof of Lemma 35. We apply Lemma 36 to B and to $A = \{a, b, c, d\}$. Let us consider a valid labelling L of B. Up to swapping the labels, we may assume that $L(a) = L(c) = \{1\}$ and $L(b) = L(d) = \{0\}$. Then Proposition 27 together with Lemma 28 imply that any proper *ab*, *bc*, *cd*, or *ad*-walk is nontrivial, whereas any proper *ac* or *bd*-walk is trivial. Then, the assumptions of Lemma 36 are satisfied due to Lemmas 32 and 33. ⊗

Inside split triangle. Finally, we prove Proposition 17 for an inside split triangle.

Proof of Proposition 17 for an inside split triangle. Let a, b, c be the three vertices of Z from the definition of the inside split triangle; see Definition 15 or Fig. 8.

First, similarly as in the proof for inside squares, we take care of trivial inside bridges via suitable labellings and Lemma 34. We reach a projective HT-drawing (D_1, λ_1) still satisfying the assumptions of Proposition 17, which in addition satisfies $\lambda_1(e) = 0$ for any edge e of G^{+0} that does not belong to a nontrivial bridge.

Now, let us consider nontrivial inside bridges. By the assumptions, each such bridge is either an *a-bridge*, that is, a nontrivial inside bridge which contains a (and b or c or both), or a *bc-bridge* which contains b and c, but not a. We consider valid labellings of these bridges. By Proposition 27, as before, a valid labelling assigns only one label to each vertex of these bridges lying on Z. As usual, we may swap all labels in a valid labelling when needed. This way, it is easy to check that every a-bridge B admits a valid labelling L_B such that $L_B(a) = \{1\}$, whereas all other labels are 0. Similarly, each *bc*-bridge B admits a valid labelling L_B such that $L_B(b) = \{1\}$ and $L_B(c) = \{0\}$. We apply Lemma 34 and we reach a projective HT-drawing (D_2, λ_2) still satisfying the assumptions of Proposition 17, which in addition satisfies the following property. The edges e of G^{+0} with $\lambda_2(e) = 1$ are exactly the edges of an a-bridge which are incident to a or edges of a *bc*-bridge incident to b.

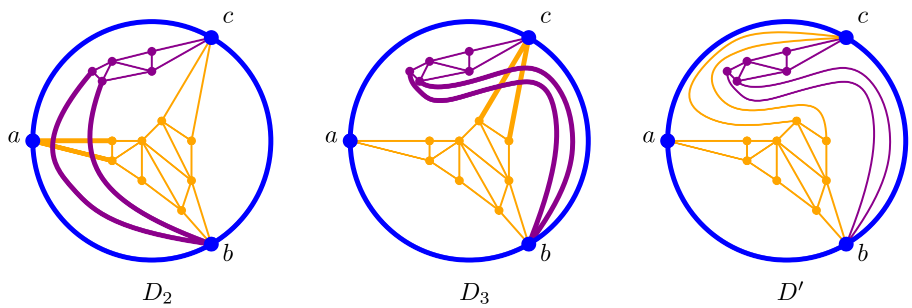

Figure 17: An example of redrawing an inside split triangle with one a-bridge and one bc-bridge. The edges participating in independent pairs crossing oddly are thick. For simplicity of the picture, the drawings D_3 and D' are actually simplified. For example, the vertex-edge switches used to obtain D_3 from D_2 introduce many pairs of independent edges crossing evenly and some pairs of adjacent edges crossing oddly. These intersections are removed in the picture as they do not play any role in the argument. (In particular, the drawing D' is, in fact, typically not a plane drawing.)

If we do not have any bc-bridge, then all nontrivial edges are incident to a and we finish the proof by setting $D' = D_2$ and letting λ' be identically 0, similarly as in the cases of an inside fan or an inside square. However, if we have bc-bridge(s), we need to be more careful.

Let E_a^x and E_{bc}^x be the sets of edges incident to a vertex x in an a-bridge and the set of edges incident to x in a bc-bridge, respectively. Because D_2 is a projective HT-drawing, we have $\lambda_2(e)\lambda_2(f) = \mathrm{cr}_{D_2}(e, f)$ for any pair of independent edges e and f. In particular, $\mathrm{cr}_{D_2}(e, f) = 1$ for a pair of independent edges if and only if one of the edges belongs to E_a^a and the second one to E_{bc}^b.

Now, for every edge $e \in E_{bc}^b$, we perform the vertex-edge switch over each vertex different from a, b, c of each a-bridge obtaining a drawing D_3. We perform the switches inside S^+. This way, we change the crossing number of such e with edges from E_a^a, E_a^b and E_a^c. In particular, after this redrawing, we get $\mathrm{cr}_{D_3}(e, f) = 1$ for a pair of independent edges if and only if one of the edges belongs to E_a^c and the second one to E_{bc}^b. See Fig. 17.

Finally, for every edge $e \in E_a^c$, we perform the vertex-edge switch over each vertex different from b and c of each bc-bridge obtaining the final drawing D'. Again, we perform the switches inside S^+. This way, we change the crossing number of such e with edges from E_{bc}^b and E_{bc}^c. However, it means that $\mathrm{cr}_{D'}(e, f) = 0$ for any pair of independent edges. That is, D' is the required ordinary HT-drawing on S^2. See Fig. 17. ⊗

8 Redrawing by Pelsmajer, Schaefer and Štefankovič

It remains to prove Theorem 18. As mentioned above, our proof is almost identical to the proof of Theorem 2.1 in [19]. The only notable difference is that we avoid contractions.[10] As noted before, the proof of Lemma 3 in [8] can also be extended to yield the desired result.

Proof. First, we want to get a drawing such that there is only one edge of Z which may be intersected by other edges. Here, part of the argument is almost the same as the analogous argument in the proof of Lemma 22.

Let us consider an edge $e = uv \in E(Z)$ intersected by some other edges and let $f = vw \in E(Z)$ be a neighbouring edge of e. We again almost-contract e so that we move the vertex v towards u until we remove all intersection of e with other edges. This way, e is now free of crossings and these crossings appear on f. Since both e and f were even edges in the initial drawing, f remains even after the redrawing as well. Also we do not affect parity of the other intersections, and we remove possible self-intersections of the edges incident with v similarly as in the proof of Lemma 22. Finally, since we want to keep the position of Z, we consider a self-homeomorphism of S^2 which sends v back to its original position. See Fig. 11.

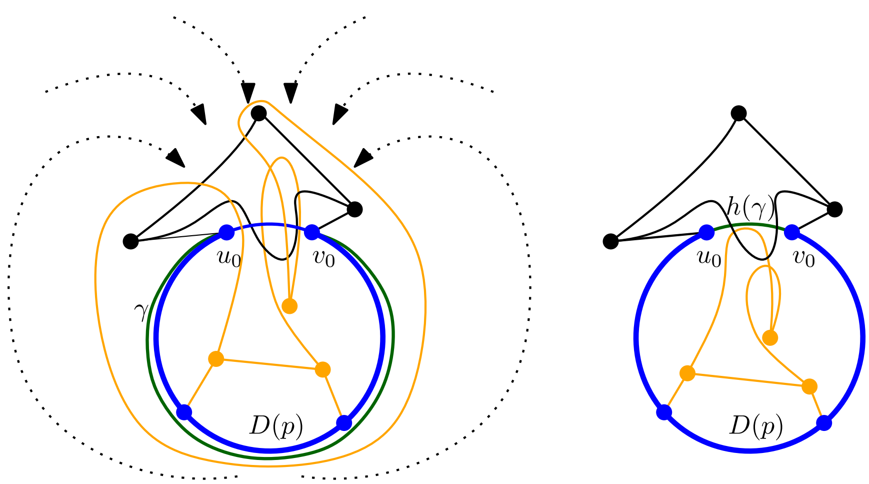

Figure 18: An illustration of the self-homeomorphism h, which maps B to S^+, applied to the drawing of $G^{+0} - e_0$ (where $e_0 = u_0v_0$).

By such redrawings, it can be achieved that only one edge $e_0 = u_0v_0$ of Z may be intersected by other edges while keeping Z fixed and e_0 even. Without

[10]Our reason why we avoid contractions is mainly for readability issues. Contractions yield multigraphs and, formally speaking, we would have to redo several notions for multigraphs. Introducing multigraphs in the previous sections would be disturbing and it is not convenient to repeat all the definitions in such setting now.

loss of generality, we may assume that the original drawing D satisfies these assumptions.

Let p be the path in Z connecting u_0 and v_0 avoiding e_0. Let us also consider an arc γ connecting u_0 and v_0 outside (that is in S^-) close to $D(p)$ such that it does not cross any inside edge. The closed arc obtained from γ and $D(p)$ bounds two disks (2-balls). Let B be the open disk which contains S^+. Finally, we consider a self-homeomorphism h of S^2 that keeps $D(p)$ fixed and maps B to S^+. Considering the drawing $h \circ D$ on $G^{+0} - e_0$, it turns out that $G^{+0} - e_0$ is now drawn in S^+, up to p, which stays fixed. For the edge e_0, we also keep its original position, that is, we do not apply h to this edge. See Fig. 18.

Since the redrawing is done by a self-homeomorphism, we do not change the number of crossings among pairs of edges in G^{+0}. Analogously, we map G^{-0} to S^- and we get the required drawing. $\otimes$

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