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Progress on Partial Edge Drawings

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Abstract

Recently, a new way of avoiding crossings in straight-line drawings of non-planar graphs has been introduced. The idea of *partial edge drawings* (PED) is to drop the middle part of edges and rely on the remaining edge parts called *stubs*. We focus on symmetric partial edge drawings (SPEDs) that require the two stubs of an edge to be of equal length. In this way, the stub at the other endpoint of an edge assures the viewer of the edge's existence. We also consider an additional homogeneity constraint that forces the stub lengths to be a given fraction δ of the edge lengths (δ-SHPED). Given length and direction of a stub, this model helps to infer the position of the opposite stub.

We show that, for a fixed stub–edge length ratio δ, not all graphs have a δ-SHPED. Specifically, we show that K_{165} does not have a 1/4-SHPED, while bandwidth-k graphs always have a $\Theta(1/\sqrt{k})$-SHPED. We also give bounds for complete bipartite graphs. Further, we consider the problem MaxSPED where the task is to compute the SPED of maximum total stub length that a given straight-line drawing contains. We present an efficient solution for 2-planar drawings and a 2-approximation algorithm for the dual problem of minimizing the total amount of erased ink.

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1 Introduction

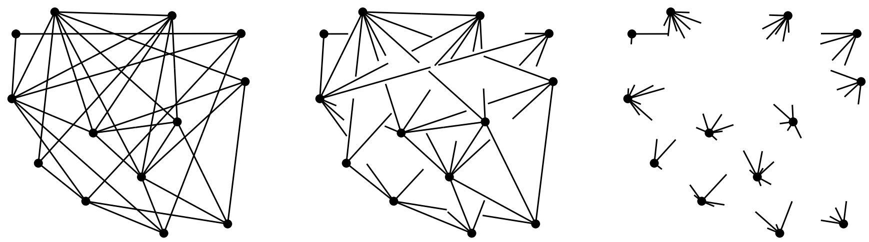
In the layout of graphs, diagrams, or maps, one of the central problems is to avoid *visual clutter*, such as the interference of crossing edges in graph drawings or overlapping labels on maps. Clutter avoidance is the objective of a large body of work in graph drawing, information visualization, and cartography. In this work, we treat a specific aspect of clutter avoidance; we focus on *completely* removing edge crossings in straight-line drawings of non-planar graphs. Clearly, this is not possible in any of the traditional graph drawing styles that insist on connecting the geometric representations of two adjacent vertices (e.g., small disks) by a closed Jordan curve (e.g., segments of straight lines). In such drawings of non-planar graphs, some pairs of edge representations must cross (or overlap), and this becomes even more problematic for dense graphs.

Previous Work. Becker et al. [2] have taken a rather radical approach to escape from this dilemma. They wanted to visualize network overload between the 110 switches of the AT&T long distance telephone network in the U.S. on October 17, 1989, when the San Francisco Bay area was hit by an earthquake. They used straight-line segments to connect pairs of switches struck by overload; the width of the segments indicated the severeness of the overload. Due to the sheer number of edges of a certain width, the underlying map of the U.S. was barely visible. They solved this problem by drawing only a certain fraction (roughly 10%) of each edge; the part(s) incident to the switch(es) experiencing the overload. We call these parts the *stubs* of an edge. The resulting picture is much clearer; it shows a distinct east–west trend among the edges with overload.

Peng et al. [21] used splines to bundle edges, e.g., in the dense graph of all U.S. airline connections. In order to reduce clutter, they increase the transparency of edges towards the middle. They compared their method to other edge bundling techniques [18, 15], concluding that their method, by emphasizing the stubs, is better in revealing directional trends.

Burch et al. [11] investigated the usefulness of partial edge drawings of *directed* graphs. They used a single stub at the source vertex of each edge. They did a user study (with 42 subjects) which showed that, for one of the three tasks they investigated (identifying the vertex with highest out-degree), shorter stubs resulted in shorter completion times *and* smaller error rates. For the two other tasks (deciding whether a highlighted pair of vertices is connected by a path of length one/two) the error rate went up with decreasing stub length; there was just a small dip in the completion time for a stub–edge length ratio of 75%. Recently, Burch et al. [10] presented an interactive graph layout system that combines partial edge drawings with traditional straight-line drawings. Their idea is that partial edges are necessary only in dense and cluttered parts of a graph layout, which are specified by the user.

A similar, but less radical approach, is the use of edge *casing*. Eppstein et al. [13] have investigated how to optimize several criteria that encode the above–below behavior of edges in given graph drawings. They introduce three models (i.e., legal above–below patterns) and several objective functions such as



(a) with crossings (b) as a maxSPED (c) as a maxSHPED

Figure 1: Various drawings of a 13-vertex graph, all using the same vertex positions.

minimizing the total number of above–below switches or the maximum number of switches per edge. For some combinations of models and objectives, they give efficient algorithms, for one they show NP-hardness; others are still open. Edge casings were re-invented by Rusu et al. [23] with reference to Gestalt principles.

Dickerson et al. [12] proposed confluent drawings to avoid edge crossings. In their approach, edges are drawn as locally monotone curves; edges may overlap but not cross.

We build on and extend the work of Bruckdorfer and Kaufmann [6] who formalized the problem of partial edge drawings (PEDs) and suggested several variants. A PED is a straight-line drawing of a graph in which each edge is divided into three segments: a middle part that is not drawn and the two segments incident to the vertices, called stubs, that remain in the drawing. In this article, we require all PEDs to be crossing-free, i.e., that no two stubs intersect. We consider stubs relatively open sets, i.e., to not contain their endpoints. In the symmetric case (SPED), both stubs of an edge must be the same length; in the homogeneous case (HPED), the ratio of stub length over edge length is the same for all edges. A δ-SHPED is a symmetric homogeneous PED with stub-edge length ratio δ. We note that $0 < \delta < 1/2$ since a stub can have at most half the edge length. We remark that a pair of stubs of equal length pointing towards each other at the opposite endpoints of an edge is, for the viewer of a SPED, a valuable witness that the connection actually exists. If the drawing is additionally homogeneous, finding the other endpoint of a stub is made easier since its approximate distance can be estimated from the stub length.

Bruckdorfer and Kaufmann [6] showed that K_n (and thus, any n-vertex graph) admits a $1/\sqrt{4n/\pi}$-SHPED. They also proved that the j-th power of any subgraph of a triangular tiling is a $1/(2j)$-SHPED. They introduced the optimization problem MaxSPED where the aim is to maximize the total stub length (or *ink*) in order to turn a given geometric graph into a SPED. They presented an integer linear program for MaxSPED and conjectured that the problem is NP-hard. Indeed, there is a simple reduction from Planar3SAT [19].

On the practical side, there is a force-directed algorithm [7] that aims to compute drawings that are 1/4-SHPEDs after removing few edges (whose stubs

Graph	Lower bound	Upper bd.	Reference
K_n (for $\delta = \frac{1}{4}$)	$n \geq 16$	$n \leq 164$	[6], Thm. 3
K_n	$n \geq \lfloor \frac{1}{4\pi\delta^2} \rfloor$	$n \leq \frac{4}{\delta^3} \ln \frac{1}{\delta}$	[6, 16]
$K_{n,n}$	$n \geq \lfloor \frac{1}{\delta} \rfloor \cdot \lfloor\lfloor \frac{\log 1/2}{\log(1-\delta)} \rfloor\rfloor \in \Theta(\frac{1}{\delta^2})$	–	Thm. 4
$K_{2k,n}$	$k \geq \lfloor \frac{\log \delta}{\log(1-\delta)} \rfloor \quad \in \Theta(\frac{1}{\delta} \ln \frac{1}{\delta})$	–	Thm. 5
bandwidth k	$k \geq \lfloor \frac{1}{8\delta^2} \rfloor$	–	Thm. 6

Table 1: Summary of known lower and upper bounds on the size of the largest complete, bipartite complete, and bandwith-k graphs that are realizable as δ-SHPED, for a fixed $0 < \delta < 1/2$. Dashes mean that no bound is known, and $\lfloor\lfloor x \rfloor\rfloor$ denotes the largest integer that is strictly less than x.

cross) and preliminary user studies [4, 8] that investigate the usefulness of PEDs. In particular, in [4] it is shown that the SHPED model is more effective than the SPED model, even if the absence of homogeneity allows for more ink and fewer crossings on the computed drawings. The 1/4-SHPED model has also been applied to orthogonal graph drawing [9].

We refer to Fig. 1 for some examples. Figure 1b depicts a maxSPED of the straight-line drawing in Fig. 1a, i.e., a SPED that is a solution to MaxSPED. We have slightly shrunken the stubs in the maxSPED so that they do not touch. For comparison, Fig. 1c depicts a SHPED with maximum ratio $\delta \approx 0.16$.

Contribution. The contribution of this article is as follows, see also Table 1 where some of our results are reported.

- We show that not all graphs admit a 1/4-SHPED; see Sect. 2. Indeed, K_n does not have a 1/4-SHPED for any $n > 164$. If we restrict vertices to be mapped to points in one-sided convex position, the bound drops to $n \geq 16$. On the other hand, the complete graph K_{16} admits a 1/4-SHPED [6]. Our proof technique carries over to other values of the stub–edge length ratio δ, i.e., for every $0 < \delta < 1/2$ there is an integer N_δ such that K_n has no δ-SHPED for any $n \geq N_\delta$.

- Recall that Bruckdorfer and Kaufmann [6] showed that K_n (and thus, any n-vertex graph) has a $1/\sqrt{4n/\pi}$-SHPED. We improve their result for specific graph classes, namely for complete bipartite graphs and for bandwidth-k graphs; see Sect. 3. The latter we show to have $\Theta(1/\sqrt{k})$-SHPEDs independently of their sizes. We note that Granacher showed in her diploma thesis [16] that for $n > 96$ the complete bipartite graph $K_{n,n}$ has no axis-symmetric 1/4-SHPED drawing such that all edges cross the axis of symmetry. She also proved that any bandwidth-k graph admits a

$1/(4.7\sqrt{k})$-SHPED – even if k is not known. For given k, our drawings have a slightly better stub–edge length ratio (of at least $1/(2.83\sqrt{k})$).

- Then we turn to the optimization problem MaxSPED; see Sect. 4. For the class of 2-planar graphs (i.e., the graphs that admit a drawing with at most two crossings per edge, see e.g. [20]), we can solve the problem efficiently; given a 2-planar drawing of a 2-planar graph with n vertices, our algorithm runs in $O(n \log n)$ time. For general graphs, we have a 2-approximation algorithm with respect to the dual problem MinSPED: minimize the amount of ink that has to be erased in order to turn a given drawing into a SPED.

Notation. In this article, we always identify the vertices of the given graph with the points in the plane to which we map the vertices. The graphs we consider are undirected; we use uv as shorthand for the edge connecting u and v. If we refer to the stub uv then we mean the piece of the edge uv incident to u; the stub vu is incident to v.

2 Upper Bounds for Complete Graphs

In this section, we show that not every graph can be drawn as a 1/4-SHPED. Note that 1/4 is an interesting value since it balances the drawn and the erased parts of each edge. Yet, our proof techniques generalize to δ-SHPEDs for arbitrary but fixed $0 < \delta < 1/2$. We start with a simple proof for the scenario where we insist that vertices are mapped to specific point sets, namely point sets in convex or one-sided convex position. We say that a convex point set is *one-sided* if its convex hull contains an edge of a rectangle enclosing the point set.

Theorem 1 *There is no set of 16 points in one-sided convex position on which the graph K_{16} can be embedded as 1/4-SHPED.*

Proof: We assume, to the contrary of the above statement, that there is a set P of 16 points in one-sided convex position that admits an embedding of K_{16} as a 1/4-SHPED. Consider the edge $e = uv$ that witnesses the one-sidedness of P. We can choose our coordinate system such that $u = (0, 0)$, $v = (1, 0)$ and all other points lie above e. We split the area above e into seven interior-disjoint vertical strips of different widths, see Fig. 2. The two outer strips have width 1/4, their neighboring strips have width 1/12, and the three inner strips have width 1/9. The width of each inner strip S is chosen in such a way that for any point p in S the stub pu intersects the left boundary of S and the stub pv intersects the right boundary of S. This holds because the distance of the left (right) boundary of S to u (v) is at least three times the width of S.

We now show that there are is at most one point of P in any inner strip S. Suppose, to the contrary, that there is a strip S that contains two distinct points $a, b \in P$. Let a be the one closer to u. Since S is an inner strip, the

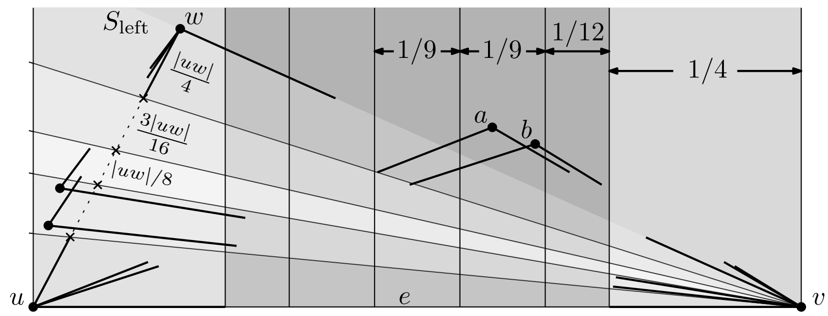


Figure 2: Sketch of the argument why no 16 points in one-sided convex position can be used to embed K_{16} as a 1/4-SHPED.

stub av intersects the right boundary of S (below the stub bv), and the stub bu intersects the left boundary of S (below the stub au). Point a lies above stub bu and point b lies above stub av. (Otherwise the points u, a, b, v would not be in convex position.) Hence, stubs av and bu intersect.

So at least eleven points of P must lie in the union of the two outer strips. We may assume that the left strip S_{left} contains at least six points. Let w be the rightmost point of P in S_{left}. We subdivide the edge uw into five pieces whose lengths are 1/4, 3/16, 1/8, 3/16, and 1/4 of the length of uw. Each piece contains its endpoint that is closer to one of the endpoints of uw. The innermost piece contains both of its endpoints. Now consider the cones with apex v spanned by the five pieces of uw. We claim that no cone contains more than one point.

Our main tool is the following. Let t be a point of $P \setminus \{u, w\}$ in S_{left}. Then the stub tv intersects the right boundary of S_{left} and, hence, also the edge uw that separates $P \cap S_{\text{left}}$ from $P \setminus S_{\text{left}}$. It remains to note that, in each cone, every point has a stub to u or w (whichever is further away from the cone) that intersects the boundary of the cone. Hence no two points of P in S_{left} can be in the same cone without their stubs intersecting as shown in Fig. 2. □

The construction of vertical strips used in the proof of Theorem 1 can be extended to δ-SHPEDs for any $0 < \delta < 1/2$ as follows.

The two outer strips have width δ. The remaining inner part of width $1-2\delta$ is divided into strips of increasing width towards the center such that the stubs of any point inside one such strip to the points u and v must cross its strip boundaries. More precisely, we start on both sides and create two strips of width $\delta^2/(1-\delta)$. In general, the two strips added in the i-th step, $i \geq 1$ have width $\delta^2/(1-\delta)^i$. By a similar argument as in the proof of Theorem 1 no strip may contain more than one point, otherwise two of their stubs would intersect. The total number of strips is bounded by $2k$ where k is the minimum integer satisfying

$$2 \cdot \sum_{i=1}^{k} \frac{\delta^2}{(1-\delta)^i} \geq 1 - 2\delta \quad \Leftrightarrow \quad k \geq \left\lceil \frac{\ln 2\delta}{\ln(1-\delta)} \right\rceil .$$

A similar division pattern is used for defining the cones for the points in the two outer strips. The edge uw is split into pieces of increasing length starting on both endpoints. The length of the ith piece is $\delta(1-\delta)^i$ times the length of uw, until the midpoint of uw is reached. This division pattern again has the property that each of the respective cones may contain at most one point, otherwise two stubs would intersect. Likewise, the number ℓ of half of the cones in one of the two outer strips is obtained from

$$\sum_{i=0}^{\ell-1} \delta(1-\delta)^i \geq \frac{1}{2} \quad \Leftrightarrow \quad \ell \geq \left\lceil \frac{\ln \frac{1}{2}}{\ln(1-\delta)} \right\rceil.$$

From these bounds we obtain the following more general result.

Theorem 2 *For $n > 10 \cdot \frac{1}{\delta} \ln \frac{1}{\delta}$ with $0 < \delta < 1/2$ the graph K_n cannot be drawn as a δ-SHPED on a one-sided convex point set.*

Proof: From the construction of vertical strips and cones described above it follows that there can be at most one point in each strip or cone, otherwise two stubs would cross. We have $2k$ vertical strips and 2ℓ cones emanating from each of u and v. Thus for $n > 2k + 4\ell$ no δ-SHPED exists on a one-sided convex point set.

Next we will show that $f(\delta) = 2\left\lceil \frac{\ln 2\delta}{\ln(1-\delta)} \right\rceil + 4\left\lceil \frac{\ln \frac{1}{2}}{\ln(1-\delta)} \right\rceil = \Theta(\frac{1}{\delta} \ln \frac{1}{\delta})$ for $\delta \to 0$. Let $0 < \delta < 1/4$. We first observe that $\frac{\ln \delta}{\ln(1-\delta)} \leq f(\delta) \leq 5 \frac{\ln \delta}{\ln(1-\delta)}$. We can rewrite $\frac{\ln \delta}{\ln(1-\delta)} = \frac{\ln(1/\delta)}{\ln(1+\delta/(1-\delta))}$. From the Taylor expansion $\ln(1+x) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{x^j}{j}$ for $0 < x \leq 1$ we get the bounds $\frac{x}{2} < \ln(1+x) < x$. This yields the desired bounds $\frac{1}{2} \cdot \frac{1}{\delta} \ln \frac{1}{\delta} < \frac{\ln \delta}{\ln(1-\delta)} < 2 \cdot \frac{1}{\delta} \ln \frac{1}{\delta}$ and shows that for $n > 10 \cdot \frac{1}{\delta} \ln \frac{1}{\delta}$ and $0 < \delta < 1/4$ no δ-SHPED of K_n exists on the point set.

For $1/4 \leq \delta < 1/2$ we first evaluate $f(3/8) = 10$. Since the bound $10 \cdot \frac{1}{\delta} \ln \frac{1}{\delta}$ is increasing as δ decreases, we can set $\delta = 1/2$ and obtain $10 \cdot 2 \ln 2 > 10 = f(3/8)$, which shows the bound of the theorem for $3/8 \leq \delta < 1/2$. Similarly we have $f(1/4) = 18 < 10 \cdot 8/3 \ln 8/3$, which covers the case $1/4 \leq \delta < 3/8$. □

Theorem 1 can be used to derive a first upper bound on general point sets as follows.

Corollary 1 *For any $n > \binom{28}{14} \approx 4.01 \cdot 10^7$, the graph K_n does not admit a 1/4-SHPED.*

Proof: By a result of Erdős and Szekeres [14], any set of more than $\binom{2k-4}{k-2}$ points in general position contains a subset of k points that form a one-sided convex set. Combining this with Theorem 1 and plugging in $k = 16$ yields the claimed bound. □

With the same reasoning, Theorem 2 would yield a huge upper bound for general point sets and $\delta < 1/4$. Yet, for the rest of this section we stick to $\delta = 1/4$ and vastly improve upon the bound of Corollary 1. Let P be the point

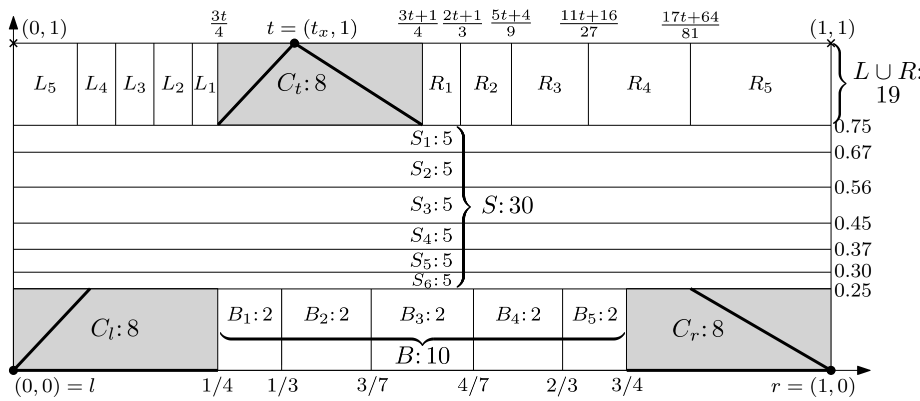


Figure 3: Partition of the enclosing rectangle A_t into cells. We have labeled each cell or group of cells with the maximum number of points that it can contain.

set in the plane, and let l and r be two points on the convex hull that define the diameter of P, which is the largest distance between any two points in P. We rotate P such that the line lr is horizontal and l is on the left-hand side. Now let A be the smallest enclosing axis-aligned rectangle that contains P, and let t and b be the top- and bottommost points in A, respectively. Accordingly, let A_t be the part of A above (and including) lr and let $A_b = A \setminus A_t$. We consider the two rectangles separately and assume that the interior of A_t is not empty. (In our proof we argue, for any interior point, using only its stubs towards the three boundary points l, r, and t.)

We subdivide A_t into 26 cells such that, for any point in a cell, the three stubs to l, r, and t intersect the boundary of that cell; see Fig. 3. For each cell, we prove, in the remainder of this section, an upper bound on the maximum number of points it can contain. Summing up these numbers (see again Fig. 3), we get a bound of 83 points in total. Since we may have a symmetric subproblem below lr, we double this number, subtract 2 because of double-counting l and r, and finally get the following theorem.

Theorem 3 *For any $n > 164$, the graph K_n does not admit a 1/4-SHPED.*

We now prove Theorem 3 by bounding from above the number of points that each cell in Fig. 3 contains.

Note that, by the choice of r and l, there are no other points on the left and right boundary of A_t (otherwise lr would not be the diameter of P). For ease of presentation, we stretch A_t in the y-direction to make it a square. Clearly, this operation does not change the crossing properties. We assume that the side length of A_t is 1. We further assume that the coordinates of l, r, and t, are $(0, 0)$, $(1, 0)$, and $(t_x, 1)$, respectively.

By symmetry, we may further assume that $0 < t_x \leq 1/2$. For a point p, we call stub pt the *upper stub* of p, pr its *right stub*, pl its *left stub*, and both pr and pl its *lower stubs*.

For $p \in \{l, r, t\}$, let $C_p \subset A_t$ be the axis-parallel rectangle spanned by p and the endpoints of the two stubs that go from p to the two other boundary points. Note that C_l, C_r, and C_t (all shaded in Fig. 3) are squares of size $1/4 \times 1/4$.

The middle strip. We first consider the middle strip $S = [0, 1] \times [1/4, 3/4]$. The following proof also includes ideas from Granacher [16]. In order to bound from above the number of points that S contains, we subdivide S into six horizontal strips, $S_1, \ldots, S_6$, from top to bottom. For $i = 1, \ldots, 6$, let a_i and b_i be the y-coordinates of the lower and upper boundaries of S_i. We will fix a_i and b_i such that, for any point p in S_i, each of pl, pr, and pt intersects the boundary of S_i.

Observe that, for any point in S_i, it holds that its lower stubs intersect the line $y = a_i$ if

$$\frac{3b_i}{4} \leq a_i, \tag{1}$$

whereas its upper stubs intersect the line $y = b_i$ if $a_i + (1-a_i)/4 = (3a_i+1)/4 \geq b_i$, that is, if

$$a_i \geq \frac{4b_i - 1}{3}. \tag{2}$$

In addition to the conditions in (1) and (2), we want to determine the height $b_i - a_i$ of strip S_i such that S_i contains at most 5 points. Let

$$c_i = \frac{3}{4} \cdot b_i$$

be the y-coordinate where the lower stubs of points on the line $y = b_i$ end. For any point p in S_i, let I_p be the part of the line $y = c_i$ delimited by the lower stubs of p. Observe that, for $p, q \in S_i$ with $q \neq p$, it holds that I_p and I_q are disjoint. This is due to the fact that the upper stubs of p and q both intersect the line $y = b_i$.

Let δ_p be the length of I_p. We say that p *consumes* δ_p. By the intercept theorem, we obtain that $\delta_p/1 \geq (a_i - c_i)/a_i$ (which is what a point on the lower boundary of S_i would consume).

Assume now that there were six points $q_1, \ldots, q_6$ in S_i from left to right. We may assume that there are at least three indices $j \in \{1, \ldots, 5\}$ such that the y-coordinate of q_j is less than or equal to that of q_{j+1} (otherwise consider the points from right to left).

Consider now such a j. Consider the parallel g of the right stub of q_{j+1} through the endpoint of the left stub of q_{j+1}. Since the right stub of q_{j+1} is steeper than the right stub of q_j it follows that g intersects the line $y = c_i$ to the right of I_{q_j}. Hence, we may assume that q_{j+1} consumes even the segment on the line $y = c_i$ between the intersection point of g and $y = c_i$ and the intersection point of the right stub with $y = c_i$. This segment has the same length as the distance between the end points of the two lower stubs of q_{j+1}, i.e., $1/4$. Hence, the six points together consume at least

$$3 \cdot \frac{1}{4} + 3 \cdot \frac{a_i - c_i}{a_i} = \frac{3}{4} + 3 \cdot \frac{a_i - 3/4 \cdot b_i}{a_i}$$

which has to be less than one. Hence, if we choose a_i such that

$$\frac{3}{4} + 3 \cdot \frac{a_i - 3/4 \cdot b_i}{a_i} \geq 1$$

then there cannot be 6 points in S_i. Resolving this equation, we obtain

$$a_i \geq \frac{9}{11} \cdot b_i \tag{3}$$

Note that Equation 3 automatically implies Equation 1. Combining Equations 2 and 3 yields the following values:

$$\begin{array}{ccccc} & & 3/4 & = & b_1, \\ a_1 & = & 2/3 & = & b_2, \\ a_2 & = & 5/9 & = & b_3, \\ a_3 & = & 5/11 & = & b_4, \\ a_4 & = & 45/121 & = & b_5, \\ a_5 & = & 405/1331 & = & b_6, \text{ and} \\ a_6 & = & 1/4. & & \end{array}$$

Hence, the middle part consists of six strips, each with at most 5 points. Summarizing, we obtain the following lemma.

Lemma 1 *The middle strip S contains at most 30 points.*

The middle part of the bottom strip. We consider the rectangle $B = [1/4, 3/4] \times [0, 1/4]$ of length 1/2 and height 1/4 between the cells C_l and C_r. Similarly as for the middle strip, we construct five cells B_1 to B_5 such that all stubs to the extreme points r, l and t cross the cell boundaries. We denote the x-coordinates of the left and right boundaries of cell B_i by a_i and b_i and set $a_1 = 1/4$, $b_1 = a_2 = 1/3$, $b_2 = a_3 = 4/9$, $b_3 = a_4 = 5/9$, $b_4 = a_5 = 2/3$, $b_5 = 3/4$. Clearly, for every point p in any cell B_i, it holds that the stubs pl and pr intersect the left and right boundaries of B_i, respectively. For the upper stub pt, we only know that it crosses the horizontal line $y = 1/4$, but not necessarily the upper boundary of B_i. We say that point p in cell B_i is *medial* if its stub pt does intersect the upper boundary of B_i. We observe that no two medial points can lie in the same cell B_i without causing stub intersections. Thus, for another point q in B_i, the stub qt must intersect the vertical line $x = a_i$ or $x = b_i$. We call such a point a *lateral* point.

In the following, we show that there can be at most one lateral point in any cell B_i. Without loss of generality, we can assume that the x-coordinate of t is less than a_i. We consider the two rays s_0 and s_1 from t through the two corners $(a_i, 0)$ and $(a_i, 1/4)$ of B_i, see Fig. 4. These two rays define a wedge W. Let p and q be two lateral points in B_i; then p and q must lie in $B_i \cap W$. Let p be the point whose ray from t is left of the ray from t through q. Then the two line segments $\overline{ql}$ and $\overline{pt}$ must intersect in a point z: (otherwise the stubs pr and qt or the stubs $q\ell$ and pr would cross). Let $\delta_q = |\overline{qz}|/|\overline{zl}|$. To avoid a crossing

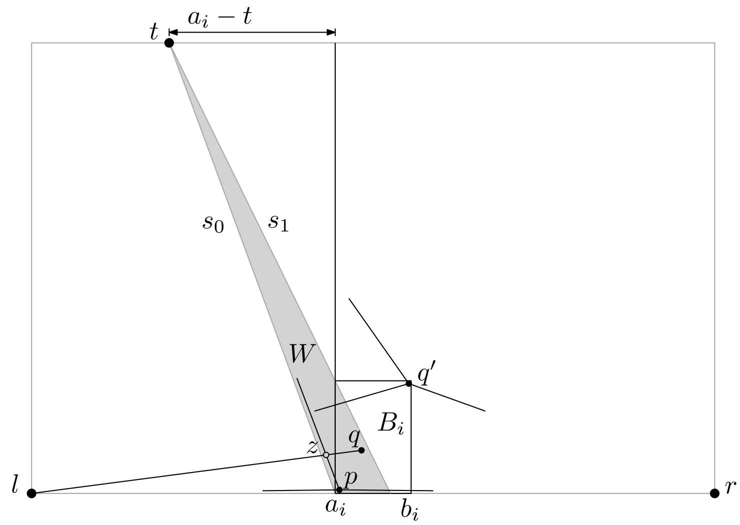


Figure 4: A cell B_i with a medial point q' and a lateral point p whose stubs do not intersect. The two lateral points p and q cannot both exist since the stubs pt and ql must intersect. (The vertical axis in this figure is scaled by 2/3.)

between the stubs pt and ql, we need that $|qz| \geq 1/4(|qz|+|zl|)$, or equivalently, that $\delta_q \geq 1/3$.

Using the intercept theorem, we observe that δ_q is maximized if p lies on s_0, q lies on s_1, and they both lie on the x-axis. We apply the intercept theorem once more for the line $x = a_i$ and the line supported by s_1 to show that in this case $|\overline{qz}| = (a_i - t_x)/3$. Using $|\overline{zl}| = a_i$, we get $\delta_q = 1/3 \cdot (1 - t_x/a_i) < 1/3$. Thus, $\delta_q \geq 1/3$ is not possible. Therefore, each cell B_i contains at most one lateral and at most one medial point.

We summarize this paragraph as follows.

Lemma 2 *The lower rectangle B contains at most 10 points.*

The left and the right part of the upper strip. In the following, we consider the rectangles $L = [0, 3t_x/4] \times [3/4, 1]$ and $R = [(3t_x+1)/4, 1] \times [3/4, 1]$, separated by the upper central square C_t, which has size $1/4 \times 1/4$, is adjacent to $t = (t_x, 1)$, and is defined by the stubs tl and tr.

We subdivide R into five height-1/4 rectangles, which we label $R_1, \ldots, R_5$ from left to right, and analyze how many points each rectangle can contain at most. The analysis for L is symmetric.

Note that the two lower stubs of any point in R intersects the horizontal line $y = 3/4$. To make sure that the upper stub of any point in cell R_i intersects the left boundary with x-coordinate a_i of R_i, the x-coordinate b_i of the right boundary of R_i has to fulfill $(b_i - t_x)/4 \geq b_i - a_i$ and, hence, $b_i \leq (4a_i - t_x)/3$. (Note that we assume that the right boundary of R_i is not part of R_i.) This yields the following boundaries.

$$\begin{array}{rcccl} & & 3/4 \cdot t_x + 1/4 & = & a_1, \\ b_1 & = & 2/3 \cdot t_x + 1/3 & = & a_2, \\ b_2 & = & 5/9 \cdot t_x + 4/9 & = & a_3, \\ b_3 & = & 11/27 \cdot t_x + 16/27 & = & a_4, \\ b_4 & = & 17/81 \cdot t_x + 64/81 & = & a_5, \text{ and} \\ b_5 & = & 1. & & \end{array}$$

We say that the lower stubs of two points p and q are *nested* if p is contained in the triangle qlr or q is contained in the triangle plr.

Claim 1 *The lower stubs of two points in the same cell are nested.*

Proof: Let p be a point in a cell R_i and consider the line $\overline{lp}$ through l and p and the line $\overline{rp}$ through r and p. Assume that there is a point q above p such that the lower stubs of p and q are not nested. Then (1) q has to be to the right of $\overline{lp}$ or (2) to the left of $\overline{rp}$, respectively. Given the way the width of R_i is constructed, however, the left stubs of q and p, respectively, intersect the vertical line $x = a_i$. So, in the first case the left stub of q intersects the right stub of p and in the second case the right stub of q intersects the left stub of p. □

Now we analyze how many points can be stacked on top of each other in each subrectangle, depending on its width, its distance to t, and the *l-shadow* of the stub tr, i.e., the set of all points p such that the stubs tr and pl intersect. We first consider R_1.

Claim 2 R_1 *contains at most one point.*

Proof: Observe that the left stub of any point p in R_1 must leave R_1 through its bottom edge, see Fig. 5. Otherwise p would lie in the l-shadow of tr. It follows that there are no two points with nested lower stubs in R_1: Otherwise either the left or the right stub of the upper point would intersect the upper stub of the lower point. There are also no two points side by side in R_1, because otherwise the right bottom stub of the left point crosses the left bottom stub of the right point. □

We now consider $R_2, \ldots, R_5$; see Fig. 6 for an illustration. Let

$$\sigma_t = t + (r - t)/4$$

be the endpoint of the stub tr. We use the following two observations to bound the number of points in R_i.

(O1) No point of R_i is above the line $\overline{\ell\sigma_t}$; otherwise its left stub would intersect the stub tr.

(O2) The y-coordinate of any point in R_i is between 3/4 and 1.

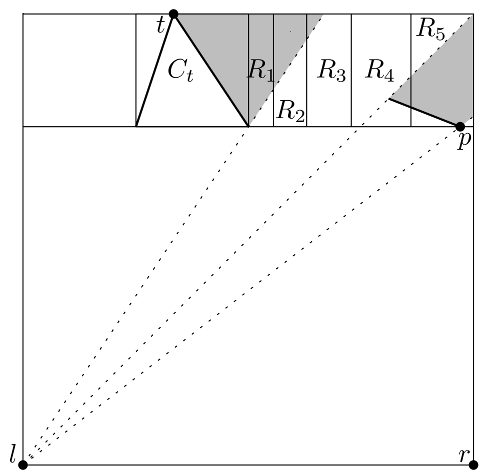


Figure 5: The lines from point l to the endpoints of the stubs tr and pt give rise to (gray) l-shadows where no point can be placed.

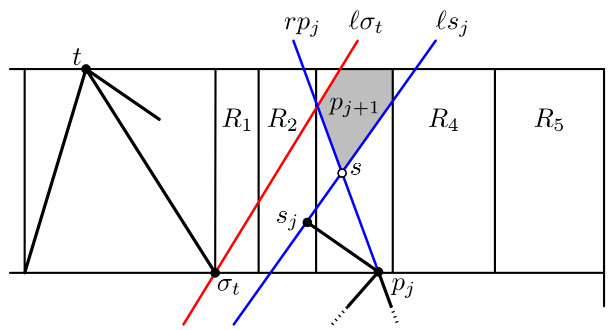


Figure 6: The shaded polygon indicates the possible area for p_{j+1}.

Depending on t, these two observations yield upper bounds on the number of points in $R_2, \ldots, R_5$ and, symmetrically, in $L_2, \ldots, L_5$. In summary, we obtain the following.

Lemma 3 *Together, L and R contain at most 19 points.*

Proof: Let $i \in \{2, 3, 4, 5\}$ and, for $j = 1, \ldots, m$, let $p_j = (x_j, y_j)$ be the points in R_i with ascending y-coordinates. For $j = 1, \ldots, m-1$, let

$$s_j = p_j + (t - p_j)/4$$

be the endpoint of the top stub $p_j t$. To make sure that the left stub of p_{j+1} and the upper stub of p_j do not intersect, p_{j+1} has to be above the line $\overline{ls_j}$. If the x-coordinate of t is large, we exploit observation (O1); otherwise, we exploit (O2). Putting things together, we will obtain, for each region R_i, upper bounds on the number of points that R_i can contain.

Case I: The x-coordinate t_x of t is large.
The condition that no point of R_i is above the line $\overline{\ell\sigma_t}$ means that

$$\text{slope}(\overline{\ell\sigma_t}) \geq \text{slope}(\overline{\ell p_j}) \text{ for } j = 1, \ldots, m. \tag{4}$$

Further, we know that

$$\text{slope}(\overline{\ell p_j}) < \text{slope}(\overline{\ell s_j}) \leq \text{slope}(\overline{\ell p_{j+1}}) \text{ for } j = 1, \ldots, m-1. \tag{5}$$

Observe that the slope of $\overline{\ell s_j}$ increases with the slope of $\overline{\ell p_j}$. Now we claim the following.

Claim 3 *If the slope of $\overline{\ell p_j}$ is fixed, then the slope of $\overline{\ell s_j}$ increases when the distance of p_j from ℓ decreases.*

Proof: We first show this claim. Let q_1 and q_2 be two points in R that are co-linear with $\ell = (0,0)$. Then, there is a $0 < c < 1/t_x$ (actually, we even have $c < 3/(3t_x + 1)$) such that $q_i = (x_i, cx_i), i = 1, 2$. Let the endpoint of stub $q_i t$ be

$$s^i := q_i + (t - q_i)/4 = (3x_i + t_x, 3cx_i + 1)/4.$$

So the slope of $\overline{ls^i}$ is $(3cx_i + 1)/(3x_i + t_x)$. Hence, the slope of $\overline{ls^1}$ exceeds the slope of $\overline{ls^2}$ if and only if

$$\begin{aligned} \frac{3cx_1 + 1}{3x_1 + t_x} > \frac{3cx_2 + 1}{3x_2 + t_x} \quad &\Leftrightarrow \quad 3x_2 + 3cx_1 t_x > 3x_1 + 3cx_2 t_x \\ &\Leftrightarrow \quad ct_x(x_1 - x_2) > (x_1 - x_2) \\ &\Leftrightarrow \quad x_1 < x_2. \end{aligned}$$

The last equivalence holds since $ct_x < 1$ for any t_x. This finishes the proof of our claim. □

Now we continue the proof of Lemma 3. By Claim 3, the number of points $p_1, \ldots, p_m$ in R_i with the properties in Equations 4 and 5 is *maximized* if $p_1 = (b_i, 3/4)$ and p_{j+1} is the intersection point of the line $\overline{\ell s_j}$ with the vertical line at $x = b_i$; see Fig. 7. Then the x-coordinate of s_j is $a'_i := b_i - (b_i - t_x)/4 = (3b_i + t_x)/4$. For $i = 1, \ldots, 4$, we have $a'_i = a_i$. We obtain the following y-coordinates:

$$\begin{aligned} y(s_j) &= y_j + (1 - y_j)/4 = (3y_j + 1)/4 \\ y(p_{j+1}) &= \frac{y_j + (1 - y_j)/4}{a'_i} \cdot b_i = \frac{b_i(3y_j + 1)}{3b_i + t_x} \\ y(s_1) &= 3/4 + \frac{1 - 3/4}{4} = \frac{13}{16}. \end{aligned}$$

The slope of the line $\overline{\ell\sigma_t}$ is less than the slope of the line $\overline{\ell p_{j+1}}$ if the intersection point

$$z = \left(a'_i, \frac{3}{3t_x + 1} a'_i\right)$$

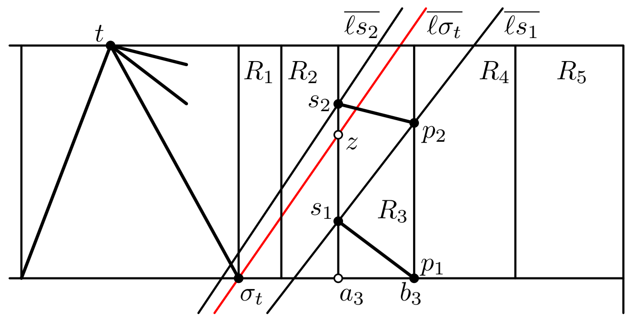


Figure 7: What is the smallest j for which the slope of $\overline{\ell s_j}$ exceeds that of $\overline{\ell \sigma_t}$?

of the line $\overline{\ell \sigma_t}$ with the vertical line at $x = a'_i$ is below s_j. This means that, if z is below s_j, p_{j+1} cannot lie in R_i (due to observation (O1)). Hence, depending on t, we obtain the following upper bounds on the number of points in R_i.

For $i = 2$, we plug in $b_2 = (5t_x + 4)/9$ and $a'_2 = a_2 = (2t_x + 1)/3$ into the equation above and obtain the following y-coordinates:

$$\begin{aligned}
y(p_{j+1}) &= \frac{1/9(5t_x+4)(3y_j+1)}{1/3(5t_x+4)+t_x} = \frac{(5t_x+4)(3y_j+1)}{24t_x+12} \\
y(z) &= \frac{2t_x+1}{3t_x+1} \\
y(p_2) &= \frac{5t_x+4}{24t_x+12} \cdot 13/4 \\
y(s_2) &= \frac{1}{4}\left(\frac{5t_x+4}{24t_x+12} \cdot \frac{39}{4} + 1\right) = \frac{1}{4} \cdot \frac{97t_x+68}{32t_x+16}
\end{aligned}$$

Hence, R_2 contains at most one point if $13/16 = y(s_1) > y(z) = (2t_x+1)/(3t_x+1)$, that is, if $t_x > 3/7$, which holds if $t_x \geq 0.43$. There are at most two points in R_2 if $y(s_2) > y(z)$, that is, if

$$\frac{1}{4} \cdot \frac{97t_x+68}{32t_x+16} > \frac{2t_x+1}{3t_x+1},$$

which turns out to be true for any $t_x > 0$.

For $i = 3$, we get the following y-coordinates:

$$\begin{aligned}
y(p_{j+1}) &= \frac{1/27(11t_x+16)(3y_j+1)}{1/9(11t_x+16)+t_x} = \frac{(11t_x+16)(3y_j+1)}{60t_x+48} \\
y(z) &= \frac{5t_x+4}{9t_x+3} \\
y(p_2) &= \frac{11t_x+16}{60t_x+48} \cdot 13/4 \\
y(s_2) &= \frac{1}{4}\left(\frac{11t_x+16}{60t_x+48} \cdot 39/4 + 1\right) = \frac{1}{4} \cdot \frac{223t_x+272}{80t_x+64}
\end{aligned}$$

Hence, there is at most *one* point in R_3 if $13/16 = y(s_1) > y(z) = (5t_x + 4)/(9t_x + 3)$, that is, if $t_x > 25/37$, which holds if $t_x \geq 0.68$. There are at most *two* points in R_2 if

$$\frac{1}{4} \cdot \frac{223t_x + 272}{80t_x + 64} > \frac{5t_x + 4}{9t_x + 3}, \text{ that is, if } t_x > -\frac{557}{2 \cdot 407} + \sqrt{\left(\frac{557}{2 \cdot 407}\right)^2 + \frac{208}{407}},$$

which holds if $t_x \geq 0.31$.

For $i = 4$, we analogously obtain that R_4 contains at most *one* point if

$$\frac{13}{16} > \frac{11t_x + 16}{27t_x + 9}, \text{ that is, if } t_x > 139/175,$$

which holds, for example, if $t_x \geq 0.8$. The cell R_4 contains at most *two* points if $t_x \geq 0.55$.

For $i = 5$, we have that R_5 contains at most *one* point if $13/16 > (3t_x + 9)/(12t_x + 4)$, that is, if $t_x > 23/27$, which holds, for example, if $t_x \geq 0.86$. There are at most two points in R_5 if $t_x \geq 0.68$.

We summarize our upper bounds in the following table.

	$t_x \geq 0$	$t_x \geq 0.31$	$t_x \geq 0.43$	$t_x \geq 0.55$	$t_x \geq 0.68$	$t_x \geq 0.8$	$t_x \geq 0.86$
R_2	2	2	1	1	1	1	1
R_3		2	2	2	1	1	1
R_4				2	2	1	1
R_5					2	2	1

Case II: t_x is small.

It remains to compute upper bounds on the number of points in R_3, R_4, and R_5 for small values of t_x. For this, we use (O2). To bound the y-coordinates of the points in R_i, observe that since the lower stubs of any points in R_i have to be nested, p_{j+1} has to lie above the line $\overline{rp_j}$. Hence, p_{j+1} has to be above the intersection point s of $\overline{rp_j}$ and $\overline{ls_j}$; see Fig. 6.

Recall that a line through two points (X_1, Y_1) and (X_2, Y_2) can be expressed by

$$y = \frac{Y_1 - Y_2}{X_1 - X_2} \cdot x + \frac{Y_2 X_1 - Y_1 X_2}{X_1 - X_2}$$

So we obtain the following equations for the lines $\overline{rp_j}$ and $\overline{ls_j}$:

$$\overline{rp_j}: \qquad y = -\frac{y_j}{1 - x_j} x + \frac{y_j}{1 - x_j}$$

$$\overline{ls_j}: \qquad y = \frac{3y_j + 1}{3x_j + t_x} x$$

Intersecting these two lines yields the intersection point

$$s = \left(\frac{(3x_j + t_x) y_j}{y_j(t_x + 3) + 1 - x_j}, \frac{(3y_j + 1) y_j}{y_j(t_x + 3) + 1 - x_j} \right).$$

Given that p_{j+1} lies above s and given that $x_j \geq a_i$, we obtain the recursion

$$y_{j+1} > \frac{(3y_j + 1)y_j}{y_j(t_x + 3) + 1 - a_i} \tag{6}$$

for $j \geq 1$. Recall now that $a_i = \lambda_i t_x + 1 - \lambda_i$ with $\lambda_i = 1 - 4^{i-2}/3^{i-1} < 3/4$ for $i = 2, \ldots, 5$ also depends on t. Hence, the lower bound

$$\frac{(3y_j + 1)y_j}{y_j(t_x + 3) + 1 - a_i} = \frac{3y_j + 1}{t_x + 3 + (1 - t_x)\lambda_i/y_j} = \frac{3y_j + 1}{t_x(1 - \lambda_i/y_j) + 3 + \lambda_i/y_j}$$

for y_{j+1} decreases with increasing t_x and decreasing y_j. This means that the number of points that fit into R_i such that Equation 6 is still fulfilled becomes larger if we choose the y-coordinates of the points as small as possible.

Hence, starting with $y_1 = 3/4$, we get lower bounds for $y_2, \ldots, y_m$ by assuming equality in Equation 6. The largest i with $y_i \leq 1$ is an upper bound for the number of points in R_i.

Depending on t_x, observation (O2), that is, $y_j \leq 1$ for $j = 1, \ldots, m$, yields the following upper bounds on the number of points in R_i.

	$t_x \leq 0.31$	$t_x \leq 0.54$	$t_x \leq 0.55$	$t_x \leq 0.68$
R_3	3			
R_4	2	3	3	
R_5	2	2	3	3

This finishes the analysis of case II.

Now we can put things together. The resulting upper bounds for the regions of R and, symmetrically, for L are summarized in Table 2. Adding up the bounds for each resulting subinterval of $[0, 1]$ yields that $L \cup R$ contains at most 19 points, namely if $0.32 < t_x < 0.43$, $0.45 < t_x < 0.46$, $0.54 < t_x < 0.55$, or $0.57 < t_x < 0.68$. □

The $(1/4 \times 1/4)$-squares C_l, C_r, and C_t. Our approach for this part follows a suggestion of Gašper Fijavž. We consider the square C_l; for the two other squares C_r and C_t, we can argue analogously and get the same bound. Let $l, p_1, \ldots, p_k$ be the set of points contained in C_l.

First, we observe that the stubs from $p_1, \ldots, p_k$ to t and r intersect the upper and right boundary of C_l. Hence, the points $p_1, \ldots, p_k$ together with their stubs to r and t form a nested structure. This means that we can order the points such that, for $i = 2, \ldots, k$, the point p_i lies between the stubs of p_{i-1}, the point p_k being innermost. Now we define $\alpha_1, \alpha_2, \ldots, \alpha_k$ to be the angles at point r formed by the lines $\overline{rl}$ and $\overline{rp_i}$. Analogously, we have angles $\beta_1, \ldots, \beta_k$ at point t. We consider only the angles of type α_i. Analogous observations hold for the angles of type β_i, and the resulting bounds are the same.

From the nesting, we see that the sequence α_i, $i = 1, \ldots, k$ is monotonically increasing. Even stronger, we have the following claim.

<table>
<tr><th>$t_x \geq$</th><th>0.00</th><th>0.14</th><th>0.20</th><th>0.31</th><th>0.32</th><th>0.43</th><th>0.45</th><th>0.46</th><th>0.54</th><th>0.55</th><th>0.57</th><th>0.68</th><th>0.69</th><th>0.80</th><th>0.86</th></tr>
<tr><td>R_1</td><td colspan="15">1</td></tr>
<tr><td>L_1</td><td colspan="15">1</td></tr>
<tr><td>R_2</td><td colspan="5">2</td><td colspan="10">1</td></tr>
<tr><td>L_2</td><td colspan="10">1</td><td colspan="5">2</td></tr>
<tr><td>R_3</td><td colspan="3">3</td><td colspan="8">2</td><td colspan="4">1</td></tr>
<tr><td>L_3</td><td colspan="4">1</td><td colspan="8">2</td><td colspan="3">3</td></tr>
<tr><td>R_4</td><td colspan="3">2</td><td colspan="6">3</td><td colspan="4">2</td><td colspan="2">1</td></tr>
<tr><td>L_4</td><td colspan="2">1</td><td colspan="4">2</td><td colspan="6">3</td><td colspan="3">2</td></tr>
<tr><td>R_5</td><td colspan="8">2</td><td colspan="3">3</td><td colspan="3">2</td><td>1</td></tr>
<tr><td>L_5</td><td>1</td><td colspan="3">2</td><td colspan="3">3</td><td colspan="8">2</td></tr>
<tr><td>total</td><td>15</td><td>16</td><td>17</td><td>17</td><td>19</td><td>18</td><td>19</td><td>18</td><td>19</td><td>18</td><td>19</td><td>17</td><td>17</td><td>16</td><td>15</td></tr>
</table>

Table 2: Upper bounds for the number of points in the cells of L and R; $L \cup R$ contains at most 19 points.

Claim 4 *For $1 < i \leq k$ it holds that $\alpha_i \geq 1.3 \cdot \alpha_{i-1}$.*

Proof: Consider the segment from point l to p_i. We subdivide it into four segments of equal length; see Fig. 8. This defines the four angles $\gamma_1, \dots, \gamma_4$ by the connecting lines from point r. We have $\alpha_i = \gamma_1 + \dots + \gamma_4$ and $\alpha_{i-1} \leq \gamma_1 + \gamma_2 + \gamma_3 =: \gamma$. It remains to prove that $\alpha_i \geq 1.3\gamma$.

The ratio of γ_4 and γ and, hence, the ratio of α_i and γ is smallest if the segment lp_i is vertical. Hence, we may assume that p_i lies in the upper left corner of C_l. Hence, $\alpha_i/\gamma \geq \arctan(1/4)/\arctan(3/16) > 1.3$. Note that in general p_i must be to the right of stub lt, so the ratio is even slightly better. □

Next, we restrict the range of the smallest and largest angle. Then we can easily compute the number k of points.

Let s be the endpoint of the stub lp_k. Consider the two lines $\overline{ts}$ and $\overline{rs}$. The point p_1, which defines the angle α_1 and β_1 respectively, has to lie either above $\overline{rs}$ or to the right of $\overline{ts}$ or both. We assume, without loss of generality, the first case, so the angle formed by $\angle l, r, s := \bar{\alpha} \leq \alpha_1$. We call the length of the base line, which is the distance between l and r, to be $d = 1$.

We compute $\tan(\bar{\alpha}) = h_s/(d - l_s) = (h_k/4)/(d - l_s)$ and $\tan(\alpha_k) = h_k/(d - 4l_s)$; see Fig. 9, where h_k and h_s are the minimum distances of p_k and s, respectively, to the base line $\overline{lr}$. This yields the ratio

$$\frac{\tan \bar{\alpha}}{\tan \alpha_k} = \frac{h_k}{4(d - l_s)} \frac{d - 4l_s}{h_k} = 1 - \frac{3d}{4(d - l_s)} = 1 - \frac{3}{4(1 - l_s/d)}.$$

Using $l_s \leq (1/4)^2$, this yields $\tan \bar{\alpha}/\tan \alpha_k \geq 1/5$. From the Taylor series expansion of the tangent function we know that $\tan \alpha > \alpha$, for all $0 < \alpha < \pi/2$, in particular $\tan \alpha = c_\alpha \alpha$ with $c_\alpha > 1$ monotonically increasing with α.

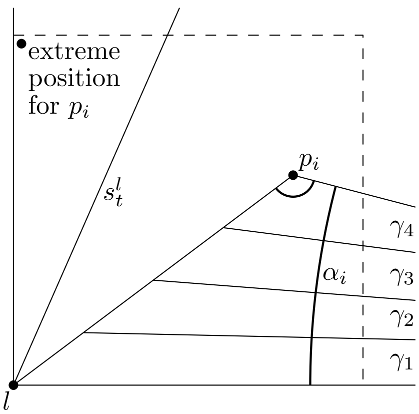


Figure 8: Angles increase by a factor of at least 1.3.

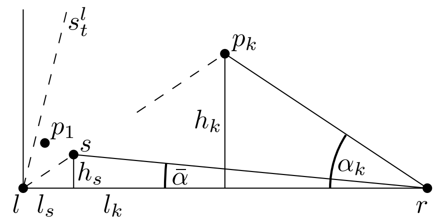


Figure 9: Ratio between the smallest angle, $\bar{\alpha}$, and the largest angle, α_k.

Hence, we conclude that

$$\frac{\alpha_1}{\alpha_k} = \frac{c_{\alpha_k} \tan \alpha_1}{c_{\alpha_1} \tan \alpha_k} > \frac{\tan \bar{\alpha}}{\tan \alpha_k} \geq 1/5.$$

This yields $\alpha_k \geq (1.3)^{k-1} \cdot \alpha_1 > (1.3)^{k-1} \cdot 1/5 \cdot \alpha_k$, which in turn implies $k < \log 5 / \log 1.3 + 1 \leq 6.2$.

If p_1 lies to the right of $\overline{ts}$, we analogously obtain

$$\frac{\beta_1}{\beta_k} > 1 - \frac{3}{4(1 - l_s/d)}$$

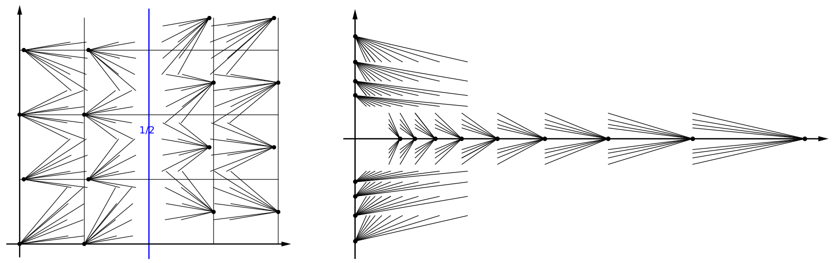
where $d > 1$ is the distance of l and t and $l_s \leq 1/4 \cdot \sqrt{2}/4$ is the length of the projection of the segment $\overline{ls}$ to the line lt, hence $\beta_1 > 1/6 \cdot \beta_k$.

Arguing along the lines of the first case, we get $\beta_k > (1.3)^{k-1} \cdot 1/6 \cdot \beta_k$, and derive $k < 7.9$.

Lemma 4 *The squares C_l, C_t, and C_r each contain at most eight points.*

This finishes the proof of Theorem 3. We would like to mention that Granacher showed in her diploma thesis [16] how the techniques used in the proof of Theorem 2 can be generalized to obtain the following result for arbitrary $k \in \mathbb{Z}_{>2}$: The complete graph with n vertices has no $1/k$-SHPED representation if $n > 4k^3 \ln k$.

Granacher [16] also showed that, for $n > 96$, the complete bipartite graph $K_{n,n}$ has no axis-symmetric 1/4-SHPED drawing such that all edges cross the axis of symmetry. In the next section we will – among others – discuss drawings of the bipartite graph $K_{n,n}$ for smaller n.



(a) 1/4-SHPED of $K_{8,8}$ (b) 1/4-SHPED of $K_{8,9}$

Figure 10: Two methods for drawing complete bipartite graphs as SHPEDs.

3 Improved Bounds for Specific Graph Classes

In this section, we improve, for specific graph classes, the result of Bruckdorfer and Kaufmann [6] which says that K_n (and thus, any n-vertex graph) has a $1/\sqrt{4n/\pi}$-SHPED. In other words, K_n has a δ-SHPED if $n \leq \pi/(4\delta^2)$. We give two constructions for complete bipartite graphs and one for graphs of bounded bandwidth.

Complete Bipartite Graphs. Our first construction is especially suitable if both sides of the bipartition have about the same size. The drawing is illustrated in Fig. 10a. Note that the figure scales x-axis and y-axis differently. In the following two results, there are fractions where both numerator and denominator are logarithmic expressions; therefore, we do not need to specify their bases.

Theorem 4 *The complete bipartite graph $K_{n,n}$ has a δ-SHPED if*

$$n \leq \left\lfloor \frac{1}{\delta} \right\rfloor \cdot \left\lfloor\left\lfloor \frac{\log 1/2}{\log(1-\delta)} \right\rfloor\right\rfloor \in \Theta\left(\frac{1}{\delta^2}\right)$$

where $\lfloor\lfloor x \rfloor\rfloor$ denotes the largest integer that is strictly less than x.

Proof: Let $k = \lfloor \frac{1}{\delta} \rfloor$ and $\ell = \left\lfloor\left\lfloor \frac{\log 1/2}{\log(1-\delta)} \right\rfloor\right\rfloor$. The latter implies that $(1-\delta)^\ell > \frac{1}{2}$.

Divide the plane at the vertical line $x = 1/2$ into two half planes, one for each side of the bipartition, to which we will refer as the right-hand side and the left-hand side. In each half plane, draw the n vertices on a (perturbed) $k \times \ell$ grid. More precisely, for a horizontal line, let $\epsilon \geq 0$ such that $(1-\delta)^\ell > \frac{1}{2} + \epsilon$. For $i = 0, \dots, \ell - 1$, draw the vertices with x-coordinates

$$(1-\delta)^i - \epsilon \text{ and } 1 - (1-\delta)^i + \epsilon.$$

Draw the vertices on the left-hand side with y-coordinates $0, \dots, k-1$ and the vertices on the right-hand side with y-coordinates $0 + \sigma, \dots, k - 1 + \sigma$, where $0 < \sigma < 1$ is chosen such that no two vertices on the right-hand side are collinear with a vertex on the left-hand side and vice versa. All edges are between a vertex on the left-hand side and a vertex on the right-hand side.

Then for any two vertices the bounding boxes of their incident stubs are disjoint up to their boundaries. Intersections of the stubs on the boundaries can be avoided by a suitable choice of ϵ.

1. Let v be a vertex on the right-hand side with x-coordinate $(1-\delta)^i-\epsilon$. Then the projection to the x-axis of the longest edge incident to v has length $(1-\delta)^i-\epsilon$. Hence all stubs incident to v are in the vertical strip bounded by $x=(1-\delta)^i-\epsilon$ and $x=(1-\delta)^i-\epsilon-\delta((1-\delta)^i-\epsilon)\geq(1-\delta)^{i+1}-\epsilon>1/2$. The latter inequality follows since $i+1\leq\ell$.

2. Let v_i be a vertex with y-coordinate $\sigma+i$, $i=0,\ldots,k-1$. Then the projection to the y-axis of the longest edge incident to v_i and above v has length $k-1-i-\sigma$ while the projection to the y-axis of the longest edge incident to v_i and below v has length $i+\sigma$. Hence the projection to the y-axis of the stubs incident to v_i and v_{i+1} do not intersect if $\delta(k-1-i-\sigma)+\delta(i+1+\sigma)<1$ which is fulfilled if $k<1/\delta$. If $k=1/\delta$ then draw the vertices on the horizontal lines $y=i$ and $y=i+\sigma$ for even i with $\epsilon=0$ and the vertices on the other horizontal lines with a slightly positive ϵ such that the endpoints of the stubs do not intersect.

A symmetric argument holds for the vertices on the left-hand side.

For the asymptotic bound, recall from the proof of Theorem 2 that

$$\ln 2\cdot\left(\frac{1}{\delta}-1\right)\leq\frac{\log 1/2}{\log(1-\delta)}\leq 2\ln 2\cdot\left(\frac{1}{\delta}-1\right).$$

It follows that

$$\frac{1}{5}\cdot\frac{1}{\delta^2}\leq\left\lfloor\frac{1}{\delta}\right\rfloor\cdot\left\lfloor\left\lfloor\frac{\log 1/2}{\log(1-\delta)}\right\rfloor\right\rfloor\leq 2\ln 2\frac{1}{\delta^2}$$

for $0<\delta<1/2$ (where the lower bound is obtained by distinguishing the cases $0<\delta\leq 1/5$, $1/5<\delta<1-1/\sqrt{2}$, $1-1/\sqrt{2}\leq\delta\leq 1/3$, and $1/3<\delta<1/2$). □

Our second construction is especially suitable if one side of the bipartition is much larger than the other. The drawing is illustrated in Fig. 10b.

Theorem 5 *For any integers $n>0$ and $k<\log\delta/\log(1-\delta)\in\theta(\frac{1}{\delta}\log\frac{1}{\delta})$, the complete bipartite graph $K_{2k,n}$ has a δ-SHPED.*

Proof: Draw the n vertices on the x-axis with x-coordinate $x_i=1/(1-\delta)^{i-1}, i=1,\ldots,n$ and the $2k$ vertices on the y-axis with y-coordinate $y_i=1/(1-\delta)^{i-1}, i=1,\ldots,k$ and $-y_i, i=1,\ldots,k$. All edges are between a vertex on the y-axis and a vertex on the x-axis. To show that no stubs intersect, we establish the following two properties on the regions that contain the stubs.

1. The stubs incident to $(0, \pm y_i), i = 2, \ldots, k$ are in the horizontal strip bounded by $y = \pm y_i$ and $y = \pm y_{i-1}$:

   The projection to the y-axis of any stub incident to $(0, y_i)$ has length $\delta \cdot y_i$, hence it stops at $y = (1-\delta) \cdot \left(\frac{1}{1-\delta}\right)^{i-1} = \left(\frac{1}{1-\delta}\right)^{i-2} = y_{i-1}$.

2. The stubs incident to $(0, x_i), i = 2, \ldots, n$ are in the rectangle bounded by $y = \pm(1-\delta)$, $x = x_{i-1}$, and $x = x_i$ (where $x_0 = 1-\delta$):

   As above, the projection of any stub incident to $(0, x_i)$ stops at $x = x_{i-1}$. The absolute value of the projection to the y-axis is bounded by $\delta \cdot y_k = \delta \cdot \left(\frac{1}{1-\delta}\right)^{k-1}$ which is less than $1-\delta$ if $k < \log \delta / \log(1-\delta)$.

Since the stubs incident to $(0, \pm y_1)$ lie in the horizontal strip bounded by $y = \pm 1$ and $y = \pm(1-\delta)$, it follows that any two stubs are disjoint.

For the asymptotic bound, we obtain similarly as in the proof of Theorem 2 that

$$\frac{1}{2} \cdot \frac{1}{\delta} \ln \frac{1}{\delta} \leq \left(\frac{1}{\delta} - 1\right) \cdot \ln \frac{1}{\delta} \leq \frac{\log \delta}{\log(1-\delta)} \leq 2 \cdot \left(\frac{1}{\delta} - 1\right) \cdot \ln \frac{1}{\delta} \leq 2 \frac{1}{\delta} \ln \frac{1}{\delta}$$

for $0 < \delta < 1/2$. □

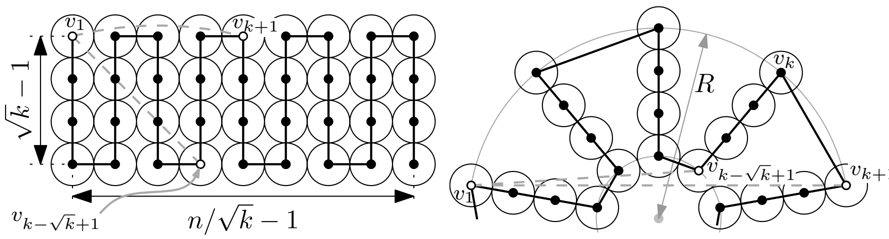
Graphs of Bounded Bandwidth. Recall that the *k-circulant* graph C_n^k with n vertices and $0 \leq k < n$ is the undirected simple graph whose vertex set is $\{v_0, \ldots, v_{n-1}\}$ and whose edge set is $\{v_i v_j : |j-i| \leq k\}$. When we specify the index of a vertex, we implicitly assume calculation modulo n. Note that $C_n^1 = C_n$ and $C_n^{n/2} = K_n$. Recall that a graph has bandwidth k if its vertices can be ordered $v_1, \ldots, v_n$ and for each edge $v_i v_j$ it holds that $|j-i| \leq k$.

Granacher [16] used the Gosper curve in order to show that any bandwidth-k graph admits a $1/(4.7\sqrt{k})$-SHPED even if k is not known. For the case that k is known, we give drawings with a better stub–edge length ratio. In particular, for any bandwidth-k graph, we guarantee that $\delta \geq 1/(2.83\sqrt{k})$. For ease of presentation, we assume that $\sqrt{k}$ and $n/\sqrt{k}$ are integers.

First, let G be a graph of bandwidth k. We draw G as a δ-SHPED as follows. We map the vertices of G to the vertices of an integer grid of $\left(n/\sqrt{k} \times \sqrt{k}\right)$ points such that the sequence of vertices $v_1, \ldots, v_n$ traverses the grid column by column in a snake-like fashion, see Fig. 11a.

The distance from any vertex to its k-th successor is at most $\sqrt{\left(\sqrt{k}-1\right)^2 + k} < \sqrt{2k}$, see the two dashed line segments in Fig. 11a. Setting $\delta = 1/\left(2\sqrt{2k}\right)$ ensures that each stub is contained in the radius-1/2 disk centered at the vertex to which it is incident; see Fig. 11a. Since the disks are pairwise disjoint, so are the stubs.

For the k-circulant graph C_n^k, we modify this approach such that the start and the end of the snake coincide. In other words, we deform our rectangular section of the integer grid into an annulus; see Fig. 11b. We additionally assume that $n/\sqrt{k}$ is even.



(a) path case (b) circle case

Figure 11: SHPEDs for bandwidth-k and k-circulant graphs.

The inner circle circumscribes a regular $\left(n/\sqrt{k}\,\right)$-gon Π of edge length 1. We place the vertices of C_n^k on rays that go from the center of the annulus through the vertices of Π. On each ray, we place $\sqrt{k}$ vertices at distance 1 from one another, starting from the inner circle and ending at the outer circle. The sequence again traverses the stacks of vertices in a snake-like fashion.

For $j < k$, a vertex v can be reached from its j-th successor s, by traversing at most $3\sqrt{k}-2$ segments of length 1: at most $\sqrt{k}-1$ segments from s to the inner circle, at most $\sqrt{k}$ segments on the inner circle, and at most $\sqrt{k}-1$ segments from the inner circle to v. Hence, the maximum distance of two adjacent vertices is less than $3\sqrt{k}$, and we can choose $\delta = 1/(6\sqrt{k})$.

Theorem 6 *Let $2 \le k \le n$ and assume that $\sqrt{k}$ and $n/\sqrt{k}$ are integers. Then any graph of bandwidth k has a $1/\left(2\sqrt{2k}\,\right)$-SHPED. If additionally $n/\sqrt{k}$ is even, the k-circulant graph C_n^k has a $1/\left(6\sqrt{k}\,\right)$-SHPED.*

4 Geometrically Embedded SPEDs

Bruckdorfer and Kaufmann [6] gave an integer-linear program for MaxSPED and conjectured that the problem is NP-hard. Indeed, there is a simple reduction from Planar3SAT [19]. In this section, we first show that the problem can be solved efficiently for the special case of 2-planar geometric graphs. Then we turn to the dual problem MinSPED of minimizing the ink that has to be erased in order to turn a given drawing into a SPED.

4.1 Maximizing Ink in Drawings of 2-Planar Graphs

In this section we prove that, given a 2-planar straight-line drawing Γ of a 2-planar graph G with n vertices, we can compute a maxSPED, i.e., a SPED that maximizes the total stub length, in $O(n \log n)$ time. Recall that a graph G is 2-planar if it admits a simple drawing on the plane where each edge is crossed at most twice.

Given the graph G and its drawing Γ, we define a simple undirected graph C as follows. For each crossed edge e of G, the graph C has a vertex v_e. Two

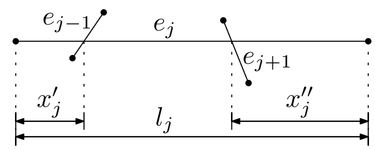


Figure 12: Notation used in the DP.

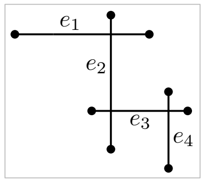
vertices v_e and $v_{e'}$ of C are connected by an edge if and only if e and e' form a crossing in Γ. The maximum degree of C is 2; hence, a connected component of C is either a path (possibly formed by only one edge) or a cycle.

Let C_i be a connected component of C. We define a total ordering on the vertices of C_i. Namely, if C_i is a path such an ordering is directly defined by the order of its vertices along the path (rooted at an arbitrary end vertex). If C_i is a cycle, we simply delete an arbitrary edge of the cycle, obtaining again a path and the related order. That means, if we consider the subdrawing Γ_i of Γ induced by the vertices of C_i (edges of G_i), such a drawing is formed by an ordered sequence of edges (according to the ordering of the vertices of C_i), $e_1, \ldots, e_{n_i}$, such that e_j crosses $e_{(j+1) \bmod n_i}$ for $j = 1, \ldots, n_i - 1$ in case of a path, and $j = 1, \ldots, n_i$ in case of a cycle.

We will use the following notation: l_j is the total length of the edge e_j; x'_j is the length of the shortest stub of e_j defined by the crossing between e_{j-1} and e_j, called the backward stub; x''_j is the length of the shortest stub of e_j defined by the crossing between e_j and e_{j+1}, called the forward stub. See also Fig. 12.

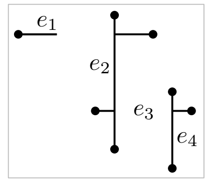
Consider now the subdrawing Γ_i, and assume that $e_1, \ldots, e_{n_i}$ form a path in C_i. If $n_i = 2$, the maximum total length of the stubs is $k_{\text{opt}} = \max\{l_1 + 2x'_2, l_2 + 2x''_1\}$.

In the general case, we can process the path edge by edge, having at most three choices for each edge: (*i*) we can draw it entirely, (*ii*) we can draw only its backward stubs, or (*iii*) we can draw only its forward stubs. The number of choices we have at any step is influenced only by the previous step, while the best choice is determined only by the rest of the path. Following this approach, let γ_i be a maxSPED for Γ_i and consider the choice made for the first edge e_1 of the path. The total length of the stubs in γ_i, minus the length of the stubs assigned to e_1, represents an optimal solution for $\Gamma_i \setminus e_1$, under the initial condition defined by the first step, otherwise, γ_i could be improved, a contradiction. That is, the optimality principle holds for our problem. Thus, we can exploit the following dynamic programming (DP) formulation, where $O_{\text{in}}(e_j)$ describes the maximum total length of the stubs of $e_j, \ldots, e_{n_i}$ under the choice (*i*) for e_j,



	l	x'	x''
e_1	7	3.5	2
e_2	7	1	2
e_3	5	1	1
e_4	4	1	2

(a)



	O_{in}	O'_{out}	O''_{out}
e_4	4	2	4
e_3	7	6	6
e_2	13	9	11
e_1	16	16	17

(b)

Figure 13: (a) A 2-planar drawing Γ and (b) a maxSPED of Γ computed by the DP algorithm.

$O'_{\text{out}}(e_j)$ describes the choice *(ii)* and $O''_{\text{out}}(e_j)$ describes the choice *(iii)*.

$$O_{\text{in}}(e_j) = \begin{cases} l_j + \max\{O'_{\text{out}}(e_{j+1}), O''_{\text{out}}(e_{j+1})\} & \text{if } x'_{j+1} \geq x''_{j+1}, \\ l_j + O'_{\text{out}}(e_{j+1}) & \text{if } x'_{j+1} < x''_{j+1}. \end{cases} \tag{7a}$$

$$O'_{\text{out}}(e_j) = \begin{cases} 2x'_j + \max\{O'_{\text{out}}(e_{j+1}), O''_{\text{out}}(e_{j+1})\} & \text{if } x'_j > x'_j \wedge x'_{j+1} \geq x''_{j+1}, \\ 2x'_j + O'_{\text{out}}(e_{j+1}) & \text{if } x'_j > x''_j \wedge x'_{j+1} < x''_{j+1}, \\ 2x'_j + \max\{O_{\text{in}}(e_{j+1}), O'_{\text{out}}(e_{j+1}), O''_{\text{out}}(e_{j+1})\} & \text{if } x'_j \leq x''_j. \end{cases} \tag{7b}$$

$$O''_{\text{out}}(e_j) = 2x''_j + \max\{O_{\text{in}}(e_{j+1}), O'_{\text{out}}(e_{j+1}), O''_{\text{out}}(e_{j+1})\} \tag{7c}$$

In case of a path, we store in a table the values of $O_{\text{in}}(e_j)$, $O'_{\text{out}}(e_j)$ and $O''_{\text{out}}(e_j)$, for $j = 1, \dots, n_i$, through a bottom-up visit of the path (from e_{n_i} to e_1). Since e_1 and e_{n_i} do not cross, we have $x'_1 = l_1/2$ and $x''_{n_i} = l_{n_i}/2$. Then, the maximal value of ink is given by $k_{\text{opt}} = \max\{O_{\text{in}}(e_1), O'_{\text{out}}(e_1), O''_{\text{out}}(e_1)\}$. See Fig. 13 for an example.

In case of a cycle, we have that e_1 and e_{n_i} cross each other, thus, in order to compute the table of values we must assume an initial condition for e_{n_i}. Namely, we perform the bottom-up visit from e_{n_i} to e_1 three times. The first time we consider as initial condition that e_{n_i} is entirely drawn (choice $O_{\text{in}}(e_{n_i})$), the second time we consider only the backward stubs drawn (choice $O'_{\text{out}}(e_{n_i})$), and the third time we consider only the forward stubs drawn (choice $O''_{\text{out}}(e_{n_i})$). Every initial condition will lead to a table where, in general, we do not have all the three possible choices for e_1 (i.e., some choices are forbidden due to the initial condition). Performing the algorithm for every possible initial condition and choosing the best value yields the optimal solution k_{opt}. The algorithm described above leads to the following result.

Theorem 7 *Let G be a graph with n vertices, and let Γ be a 2-planar straight-line drawing of G. A maxSPED of Γ can be computed in $O(n \log n)$ time.*

Proof: Consider the above described algorithm, based on the DP formulation defined by Equations 7a–7c. We already showed how this algorithm computes a maxSPED of Γ. The construction of the graph C requires time $O(m \log m)$

with a standard sweep-line algorithm for computing the $O(m)$ line-segment intersections [3]. Once C has been constructed, ordering its vertices requires $O(n_C)$ time, where $n_C \in O(m)$ is the number of vertices of C. Performing a bottom-up visit and up to three top-down visits of every path or cycle takes $O(m)$ time. Thus, the overall time complexity is $O(n \log n)$, since for 2-planar graphs $m \in O(n)$ [20]. □

We finally observe that the restricted 0/1-MaxSPED problem for 2-planar drawings, where each edge is either drawn or erased completely, may be solved through a different approach. Indeed, we can exploit a maximum-weight SAT formulation in the CNF+($\leq$2) model, where each variable can appear at most twice and only with positive values [22]. Roughly speaking, we map each edge to a variable, with the weight of the variable equal to the length of its edge, and define a clause for each crossing. Applying an algorithm of Porschen and Speckenmeyer [22] for CNF+($\leq$2) solves 0/1-MaxSPED in $O(n^3)$ time. However, our algorithm solves a more general problem faster.

4.2 Erasing Ink in Arbitrary Graph Drawings

In this section, we consider the problem MinSPED, which is dual to MaxSPED. In MinSPED, we are given a graph with a straight-line drawing, and the task is to erase as little of the edges as possible in order to make it a SPED.

We will exploit a connection between the NP-hard minimum-weight 2-SAT problem (MinW2Sat) and MinSPED. Recall that MinW2Sat, given a 2-SAT formula with real weights assigned to the variables, asks for a satisfying variable assignment that minimizes the total weight of the true variables. There is a 2-approximation algorithm for MinW2Sat that runs in $O(vc)$ time and uses $O(c)$ space, where v is the number of variables and c is the number of clauses of the given 2-SAT formula [1].

Theorem 8 MinSPED *can be 2-approximated in time quadratic in the number of crossings of the given straight-line drawing.*

Proof: Given an instance G of MinSPED, we construct an instance φ of MinW2Sat as follows. Let e be an edge of G with k crossings. Then e is split into $k+1$ pairs of edge segments $e_0, \ldots, e_k$ as shown in Fig. 14. If we order the edges that cross e in increasing order of the distance of their crossing point to the closer endpoint of e, we can assign each segment pair e_i for $i \geq 1$ to the ith edge f^i crossing e, in this order. We also say that edge f^i *induces* segment pair e_i. Any valid maximal (non-extensible) partial edge drawing of e is the union $\bigcup_{i=0}^{j} e_i$ of all pairs of edge segments up to some index $j \leq k$.

We model all pairs of (induced) edge segments as Boolean variables $\hat{e}_1, \ldots, \hat{e}_k$ with the interpretation that the pair e_i is *not* drawn if $\hat{e}_i = true$. The pair e_0 is always drawn. For $i = 1, \ldots, k$, we introduce the clause $(\neg\hat{e}_{i+1} \Rightarrow \neg\hat{e}_i) \equiv (\hat{e}_{i+1} \vee \neg\hat{e}_i)$. This models that e_{i+1} can only be drawn if e_i is drawn. Moreover, for every crossing between two edges e and f, we introduce the clause $(\hat{e}_i \vee \hat{f}_j)$, where e_i is the segment pair of e induced by f and f_j is the segment pair of f

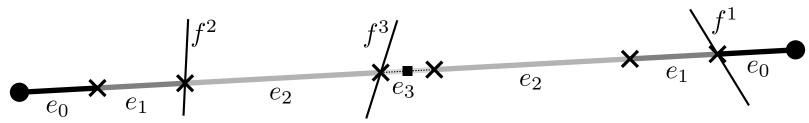


Figure 14: Edge e is split into four pairs of edge segments; pairs are labeled equally.

induced by e. This simply means that at least one of the two induced segment pairs is not drawn and thus the crossing is avoided.

Now we assign a weight $w_{e,i}$ to each variable $\hat{e}_i$, which is either the absolute length $|e_i|$ of e_i if we are interested in ink, or the relative length $|e_i|/(2|e|)$ if we are interested in relative stub lengths (δ). Then minimizing the value $\sum_{\hat{e}_i \in \mathrm{Var}(\varphi)} w_{e,i}\hat{e}_i$ over all valid variable assignments minimizes the weight of the erased parts of the edges in the given geometric graph.

The 2-approximation algorithm for MinW2Sat yields a 2-approximation for the problem to erase the minimum ink from the given straight-line drawing of G. It runs in $O(vc) = O(I^2) \subseteq O(m^4)$ time since our 2-SAT formula has $O(I) \subseteq O(m^2)$ variables and clauses, where m is the number of edges of G and I is the number of intersections in the drawing of G. □

If we encode the primal problem (where we want to maximize ink) using 2SAT, we cannot hope for a similar positive result. The reason is that the tool that we would need, namely an algorithm for the problem MaxW2Sat dual to MinW2Sat would also solve maximum independent set (MIS). For MIS, however, no $(n^{1-\varepsilon})$-approximation exists unless $\mathcal{NP} = \mathcal{ZPP}$ [17].

To see that MaxW2Sat can be used to encode MIS, use a variable $\hat{v}$ for each vertex v of the given (graph) instance G of MIS and, for each edge $\{u, v\}$ of G, the clause $(\hat{u} \vee \hat{v})$. Let φ be the conjunction of all such clauses. Then finding a satisfying truth assignment for φ that maximizes the number of false variables (i.e., all variable weights are 1) is equivalent to finding a maximum independent set in G. Note that this does *not* mean that maximizing ink is as hard to approximate as MIS.

5 Conclusions and Future Work

In this work, we have studied partial edge drawings. We have presented several graph classes that admit SHPEDs with a stub–edge length ratio depending on parameters of the graph class. On the negative side, we have proved that, for any $n > 164$, the complete graph K_n does not admit a 1/4-SHPED. Finally, we have studied symmetric PEDs for geometrically embedded graphs and proved for 2-planar geometric graphs that computing a MaxSPED takes $O(n \log n)$ time using dynamic programming; the dual variant MinSPED can be 2-approximated.

We conclude with a list of open problems that arise from our research.

- Identify other classes of graphs that admit δ-SHPEDs for some $\delta > 0$.
- Find a complete graph smaller than K_{165} such that it does not admit 1/4-SHPEDs.
- Study the complexity of MaxSPED for k-planar drawings with $k > 2$.
- Investigate the complexity of MaxSPED when the input is a topological or an abstract graph (possibly a k-planar graph for $k \geq 2$), rather than a geometric graph. In this scenario the drawing is not given as part of the input, and hence the problem may become NP-hard already for 2-planar graphs.

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