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Bounded, minimal, and short representations of unit interval and unit circular-arc graphs. Chapter I: theory

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Abstract

This is the first of two chapters of a work in which we consider the unrestricted, minimal, and bounded representation problems for unit interval (UIG) and unit circular-arc (UCA) graphs. In the unrestricted version, a proper circular-arc (PCA) model $\mathcal{M}$ is given and the goal is to obtain an equivalent UCA model $\mathcal{U}$. In the bounded version, $\mathcal{M}$ is given together with some lower and upper bounds that the beginning points of $\mathcal{U}$ must satisfy. In the minimal version, we have to find a *minimal model* equivalent to $\mathcal{M}$, in which the circumference of the circle and length of the arcs must be simultaneously as small as possible. In this chapter we motivate these problems from an historical perspective, and we develop the theoretical framework required for the algorithms in Chapter II. We present new characterizations of those PCA models that have equivalent UCA models, and of those UCA models with a circle of circumference c and the arcs of length ℓ. We also prove that every UCA model is equivalent to a minimal one. We remark that all our results are of an algorithmic nature and can be readily employed to solve the problems at hand, even though these algorithms are not as efficient as those in Chapter II.

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1 Introduction

In this work we are concerned with some recognition and representation problems for unit interval and unit circular-arc graphs. A *proper circular-arc (PCA) model* is a pair $\mathcal{M} = (C, \mathcal{A})$ where C is a circle and $\mathcal{A}$ is a family of inclusion-free arcs of C in which no pair of arcs in $\mathcal{A}$ cover C. If some point of C is crossed by no arcs, then $\mathcal{M}$ is a *proper interval (PIG)* model. *Unit circular-arc (UCA)* and *unit interval (UIG)* models correspond to the PCA and PIG models in which all the arcs have the same length, respectively. Every PCA model $\mathcal{M}$ is associated with a graph $G(\mathcal{M})$ that contains a vertex for each of its arcs, where two vertices are adjacent if and only if their corresponding arcs have a nonempty intersection. A graph G is a *proper circular-arc (PCA)* graph when it is isomorphic to $G(\mathcal{M})$ for some PCA model $\mathcal{M}$. In such a case, G is said to *admit* the model $\mathcal{M}$, while $\mathcal{M}$ is said to *represent* G. *Proper interval (PIG)*, *unit circular-arc (UCA)*, and *unit interval (UIG)* graphs are defined analogously.

The recognition problem is well solved for UIG graphs. Indeed, Roberts' PIG=UIG Theorem states that every PIG graph admits a UIG model [41]. Hence, it suffices to determine if G is a PIG graph, a task that can be accomplished in linear time (e.g. [21]) or logspace [30]. Moreover, there are *certifying* algorithms that exhibit either a PIG model or a forbidden induced subgraph according to whether the input graph is PIG or not.

Knowing that G is a UIG graph and has a PIG model tells us nothing about its UIG models. There are numerous applications in which a UIG model is to be found (e.g. [8, 9, 15, 16, 18, 20, 45] and [40, Chapter 2]), and others that benefit from having a UIG model as input [40, Chapter 2]. Usually, the problem is to transform an input PIG model $\mathcal{M}$ into an equivalent UIG model $\mathcal{U}$, where *equivalent* means that the extremes of $\mathcal{U}$ must appear in the same order as in $\mathcal{M}$. We refer to this problem as the *(unrestricted) representation* (Rep) problem.

Rep is a classical problem whose research is even older than the notion of PIG graphs. Indeed, Rep is one of the motivations in the pioneering philosophical work by Goodman [20], which dates back to the 1940's. Moreover, Fine and Harrop [14] developed, in 1957, an effective method to transform a weak mapping of an array (i.e., a PIG model) into a uniform mapping of the same array (i.e., a UIG model of a power of a path); this algorithm is actually the first proof of Robert's PIG=UIG theorem, as far as our knowledge extends. Linear-time algorithms for Rep are known since more than two decades [7, 33, 37] and, recently, a logspace implementation has been devised [30].

Rep can be generalized to the *partial representation extension* (RepExt) problem in which some arcs of $\mathcal{M}$ are *pre-drawn*, and $\mathcal{U}$ must contain these arcs. RepExt is in turn a special case of the more general *bounded representation* (BoundRep) problem in which a length $\ell \in \mathbb{Q}$ is given together with lower and upper bounds $d_\ell(A), d_r(A) \in \mathbb{Q}$ for each arc A of $\mathcal{M}$, and the goal is to produce an equivalent UIG model $\mathcal{U}$ in which all the arcs have length ℓ in such a way that $d_\ell(A) \leq s(A) \leq d_r(A)$ for every arc A. Here $s(A) \in \mathbb{Q}$ represents the beginning point of A. In this work we consider a variant of BoundRep in which ℓ, $d_\ell(A)$, $d_r(A)$ are integers, and each beginning point $s(A)$ of $\mathcal{U}$ is required to be an integer as well. We refer to this problem as the IntBoundRep; as far as we know, IntBoundRep has not been considered before.

The research on RepExt and BoundRep did not begin until recently and, consequently, they are not as studied as Rep. Even though RepExt and BoundRep are natural generalizations of Rep, these problems are not motivated by real-world applications. The reason for studying them resides in the increasing interest for partial and bounded representation problems, that were studied for several graph classes (e.g. [1, 3, 4, 5, 26, 28, 29]). Concerning PIG graphs, Balko et al. [1] show that the bounded representation problem is solvable in $O(n^2)$ time. Regarding UIG graphs, Klavík et al. [28] designed an $O(n^2+nD)$ time algorithm for BoundRep, where D is the cost of multiplying large numbers (requiring r bits, where r is the total space consumed by the bounds). As the main open problem, the authors inquire if there exists an algorithm running in less than $O(n^2 + nD)$ time. In [28], a generalization of BoundRep in which the output UIG model $\mathcal{U}$ need not be equivalent to the input PIG model $\mathcal{M}$ is also considered; what the authors ask is for $G(\mathcal{U})$ to be isomorphic to $G(\mathcal{M})$. Whereas BoundRep is polynomial, this generalization is NP-complete [28].

While introducing their research on RepExt, in the preprint version of the article [27, p. 2], Klavík et al. state that "specific properties of unit interval representations were never investigated since it is easier to work with combinatorially equivalent proper interval representations".[1] This strong assertion reflects a state of affairs within the graph theory community about the research on UIG graphs, but it is not true in a literal sense. In 1990, Pirlot proved that every PIG graph admits a *minimal* UIG model [38]. Tough Pirlot's work is not of an algorithmic nature, the main tool he uses is a space efficient representation of PIG models called the *synthetic graph*. With the aid of an appropriate weighing, this graph reflects the separation constraints that all the equivalent UIG models must satisfy. As part of his work, Pirlot solves the ℓ-Rep problem of determining if a PIG graph admits a UIG model in which all the arcs have integer endpoints and a given length ℓ. Clearly, this is a specific property of UIG models. Moreover, Pirlot introduces synthetic graphs to solve the linear program in [28, Proposition 4.4] (except for the bound constraints) and, vice versa, the graph used in [28, Proposition 4.4] is a synthetic graph (plus two vertices for modeling the bounds). Isaak [23, 24] considers the representation problem for intervals of bounded length, obtaining similar results for unit intervals, and Balof et al. [2] study the polyhedron defined by Pirlot's linear program.

UIG models with integer endpoints where considered at least three times after Pirlot's publication. In 2004, Czyzowicz et al. [9, Problem 3.2] ask whether ℓ-Rep is NP-complete, while, in a recent poster[2], Durán et al. present a characterization of those UIG graphs for which ℓ-Rep is solvable. Interestingly, both

[1] N.B.: this assertion was replaced with "specific properties of unit interval representations were not much investigated" in [28]

[2] Durán, G., Grippo, L., Oliveira, F., Slezak, F., Szwarcfiter, J.: Characterizations of (k)-interval graphs (2014). Poster session presented at Foundations of Computational Mathematics 2014 (`http://focm2014.dm.uba.ar/session/sessionAllSchedule.pdf`, accessed December 2014).

problems where settled by Pirlot in 1990 [38], as he shows that ℓ-Rep can be solved in $O(n^2)$ time, providing a forbidden induced subgraph in case of failure. In a conference paper of the aforementioned poster by Durán et al. [10], the authors write that "those semiorders representable with intervals of length k were characterized [by Pirlot] by forbidden suborders for every integer k". Instead, they "generate the family" of forbidden subgraphs "without dealing with suborders as the other articles did", referring to the work by Pirlot and a preprint version of the present manuscript. Actually, Pirlot [38], and the treatment of his work that we present, describes the forbidden configurations in terms of cycles in the synthetic graph (see Theorem 1). Thus, taking into account that the vertices of a PIG graph are in a one-to-one correspondence with the vertices of (any of) its synthetic graphs, the characterization by forbidden induced subgraphs follows easily. (We remark that the forbidden subgraphs as described in [10] correspond to cycles of the synthetic graph as well, where the **Left-Choice** options correspond to noses and the **Right-Choice** options correspond to hollows; see Section 3.)

Similarly as above, Gardi [19, p. 2908] claims that, up to 2007, the algorithm by Corneil et al. [7] was the **only** one able to solve Rep in linear time. Again, by Pirlot's theorem, it makes sense to consider the *minimal UIG representation* (MinUIG) problem, in which an input PIG model has to be transformed into an equivalent minimal UIG model. By taking a deeper look to synthetic graphs, Mitas [37] devised a linear-time algorithm to solve MinUIG and, thus, Rep. In the present work we show that Mitas' algorithm sometimes fails to find the minimal model. Yet, her algorithm correctly solves Rep in linear time. We remark that Mitas' (1994) algorithm is contemporary to the one by Corneil et al. (1995).

The problem of finding a minimal model is as old as Rep [20]. However, there has been some controversy about what a minimal model is, and which properties should it hold, as many researchers expect the minimal model to be unique [14, 15, 16, 18]. In this work we follow the definition given by Pirlot [38]. In short, a PIG model is *minimal* when both the length and the beginning point of every interval are as small as possible (see Section 6). The advantage of Pirlot's definition is that the minimal model is unique; the disadvantage is that proving the existence of minimal models is not a trivial task. Thus, the first algorithm that solves MinUIG with this definition of minimality is the one given by Pirlot [38]. As mentioned before, Mitas' algorithm fails to solve MinUIG, and in this work we fix it at the expense of increasing its complexity to $O(n^2)$ time and linear space. In a recent conference, Durán et al. [10] presented an $O(n^3)$ time algorithm that, given a UIG model $\mathcal{M}$, finds a solution $\mathcal{U}$ to ℓ-Rep such that ℓ and the rightmost interval of $\mathcal{U}$ are as small as possible. Of course, any solution to MinUIG serves as a solution to the problem studied by Durán et al. (the converse is not true). Although Durán et al. acknowledge that our patch to Mitas' algorithm finds a solution to ℓ-Rep for the minimum value of ℓ, they omit to mention that it also solves their problem as well (see Section 6).

MinUIG is implicitly solved in a recent article by Costa et al. [8], where the authors devise an $O(n^2)$ time and space algorithm to solve the MinP_q^k problem.

In the $\textsc{MinP}_q^k$ problem we are given a PIG model $\mathcal{M}$ and the goal is to find a UIG model $\mathcal{U}$ representing a power of a path P_q^k in such a way that $\mathcal{M}$ is equivalent to some induced submodel $\mathcal{U}'$ of $\mathcal{U}$ and q, k are as small as possible. As proven in [14, 32], $\textsc{MinP}_q^k$ is always solvable. Moreover, $\mathcal{U}$ need not be explicitly constructed, as it is implied by $\mathcal{U}'$. In fact, it suffices to take $\mathcal{U}'$ to be the solution to MinUIG, as it follows from [32] (see Section II.6). Soulignac [42, Chapter 9] mentions that Mitas' algorithm can be used to find $\mathcal{U}'$ in linear time. Yet, Costa et al. omit this fact although they reference [42] to explain the strong relation between Rep and $\textsc{MinP}_q^k$.

In this work we consider the unrestricted, bounded, and minimal representation problems for the broader class of unit circular-arc graphs. As far as our knowledge extends, only the unrestricted version has been considered, while Lin and Szwarcfiter [34] leave some open problems related to the minimal representation problem as we conceive it.

As for PIG graphs, the recognition problem for PCA graphs is solvable in linear time [25, 43] or logspace [31]. Again, a PCA model or a forbidden induced subgraph is obtained according to whether the input graph is PCA or not. As before, knowing that a UCA graph has a PCA model tells us nothing about its UCA models. Thus, there is a motivation for finding UCA models for applications that require them as input (e.g. [12, 17]). Goodman [20] already considers the representation problem for UCA graphs, when he deals with the adjustment of polygonal weak arrays (i.e., PCA models). Goodman [20, p. 215] gives a good reason for dealing with more complex structures: "We can by no means take it for granted that all the categories of qualia are linear arrays; indeed, it is clear that the order of colors, for example, is much more complex." (Luce [35] contains more examples of circular quantities.) Consequently, the same motivations for studying MinUIG on UIG graphs hold for UCA graphs. The problem is that, as far as our knowledge extends, there is no satisfactory definition of what a minimal UCA model is.

About the representation problems, Goodman [20, p. 239] declares: "for linear and polygonal arrays do we have anything even approaching adequate rules for adjustment". These rules, however, will fail when applied to certain PCA models because not every PCA graph is UCA. In 1974, Tucker showed a characterization by forbidden subgraphs of those PCA graphs that are UCA [46]. His proof yields an effective method to transform a PCA model $\mathcal{M}$ into an equivalent UCA model $\mathcal{U}$. Unfortunately, the extremes of $\mathcal{U}$ are not guaranteed to be of a polynomial size and, thus, the corresponding representation algorithm cannot be regarded as polynomial. More than three decades later, in 2006, Durán et al. [11] described how to obtain a forbidden subgraph in $O(n^2)$ time, thus solving the recognition problem. The representation problem remained unsolved until Lin and Szwarcfiter showed how to transform any PCA model into an equivalent UCA model in linear time [34]. Their algorithm, however, does not output a *negative* witness when the input graph is not UCA. The problem of finding a forbidden subgraph in linear time was solved by Kaplan and Nussbaum [25]. Yet, up to this date, there is no *unified* algorithm for solving the transformation problem while providing a negative witness when the input model has no equivalent UCA models. Regarding the space complexity, Köbler et al. [31] mention that the representation problem in logspace is still open.

1.1 Contributions and outline

Synthetic graphs appeared more than two decades ago, and they are covered in detail in a book by Pirlot and Vincke [40, Chapter 4]. Pirlot and Mitas' articles are written in terms of semiorders; their emphasis is on preference modeling and order theory. This could be, perhaps, the reason why synthetic graphs have gone unnoticed by many researchers in the field of algorithmic graph theory. Similar considerations hold for many results on this subject; for instance, some recent ideas in [32, 33] were already present in [13, 14]. In this work we generalize synthetic graphs to PCA models and we apply them to show improved algorithms for Rep, (Int)BoundRep, MinUIG (and its restricted generalization IntMinUCA), and MinP_q^k (and its generalization MinC_q^k) for UCA graphs. Our main meta-contribution is to show that synthetic graphs provide a simpler theoretical ground for understanding PCA models with separation constraints. For this reason, we re-prove some known theorems and rewrite some known algorithms in terms of synthetic graphs.

Because of its length, this work has been divided in two chapters during the reviewing process. In Chapter I, i.e. the present manuscript, we provide a thorough motivation to study the problems at hand, and we prepare the theoretical ground that is required for Chapter II. Even though the work as a whole is of an algorithm nature, Chapter I provides different characterizations of UCA graphs that can be of interest in non-algorithmic applications. In Chapter II [44], instead, we provide efficient algorithms for the different problems. Both chapters are self-contained and each could be read independently of the other. Yet, we feel that our meta-contribution applies only when the work is seen as a unit. The next paragraphs describe our main contributions; each reference here provided is prefixed with the number of the corresponding chapter. Also, the number of each theorem here presented corresponds to the number of a theorem in one of the chapters.

Characterization of u-CA models. A u-CA model is a constrained UCA model that satisfies several restrictions imposed by a *descriptor* u. These restrictions have to do with the circumference of the circle, the length of the arcs, and the position and distance between the endpoints. In Section I.3, we characterize those PCA models that are equivalent to some u-CA model. The characterization is in terms of forbidden cycles of the synthetic graph. Specifically, we prove a theorem of the following form.

Theorem I.1 *A PCA model is equivalent to a u-CA model if and only if every cycle of its synthetic graph has a nonnegative weight.*

The bounded representation problem. In Section II.3 we develop algorithms to solve BoundRep and IntBoundRep for UCA graphs. Both algorithms require $O(n^2)$ time and $O(n)$ space, improving over the algorithm in [28] even when restricted to UIG graphs.

Theorem II.1 BoundRep *and* IntBoundRep *are solvable in* $O(n^2)$ *time and* $O(n)$ *space.*

Certified recognition of UCA graphs. The advantage of certifying algorithms [36] over their non-certifying counterparts is that they provide a *witness* guaranteeing the validity of the YES-NO answer. The end user can *authenticate* the witness to be confident that the answer is correct, even in the presence of an incorrect implementation. Regarding the representation problem of UCA graphs, the algorithm LS by Lin and Szwarcfiter [34] outputs a UCA model for any YES instances, while the algorithm KN by Kaplan and Nussbaum [25] outputs a forbidden subgraph for NO instances. The pair (LS, KN) can be regarded as a certifying algorithm. However, an erroneous implementation of LS could claim that a UCA graph G admits no UCA models, while a correct implementation of KN claims that G is UCA. In such a case, no witness is obtained at all, defeating the purpose of a certifying algorithm. Is for this reason that Kaplan and Nussbaum [25] leave open the problem of finding a *unified* certification algorithm. In Section II.4.3, we provide this unified algorithm.

Theorem II.2 *There is a unified certifying algorithm that solves* Rep *in linear time.*

New proof of Tucker's theorem, and a revision of the algorithm by Kaplan and Nussbaum. The correctness of our recognition algorithm depends on a new proof of Tucker's theorem. In short, Tucker [46] proved that $\mathcal{M}$ is equivalent to some UCA model if and only if $a/b < x/y$ for every (a, b)-independent and every (x, y)-circuit of $\mathcal{M}$. The negative witness provided by the algorithm by Kaplan and Nussbaum [25] is an (a, b)-independent plus an (x, y)-circuit with $a/b \geq x/y$. In Section I.4 we provide a new proof of Tucker's theorem that contains more equivalent statements, including one to construct a UCA model of polynomial size (see Theorem I.2). In Section II.4.2 we provide a new implementation of the algorithm by Kaplan and Nussbaum, by taking advantage of synthetic graphs. Even though we claim that our new implementation is equivalent to the one provided by Kaplan and Nussbaum, this fact is not obvious. The forbidden structure that we employ to characterize UCA graphs is a cycle of the synthetic graph which, a priori, is unrelated to the (a, b)-independents and (x, y)-circuits employed by Tucker. In Section I.5 we show that, in fact, these structures are strongly related as follows.

Theorem I.4 *A PCA model* $\mathcal{M}$ *contains an* (a, b)*-independent (resp.* (x, y)*-circuit) if and only if the synthetic graph of* $\mathcal{M}$ *contains a nose (resp. hollow) circuit with ratio* $a/b - h$ *(resp.* $x/y - h$*), where* h *is the height of* $\mathcal{M}$*.*

Logspace recognition of UCA graphs. Köbler et al. [31] solve the recognition of PCA graphs in logspace. They leave open the problem of recognizing UCA graphs in logspace. We solve this problem in Section II.4.4; the correctness of our algorithm depends on our new proof of Tucker's theorem.

Theorem II.5 *There is an algorithm that solves* Rep *in logspace.*

Minimal UCA models. In Section I.6 we introduce the concept of minimal models for UCA graphs, following the one proposed by Pirlot for UIG graphs. We prove that every UCA model is equivalent to some minimal model, and we provide a polynomial algorithm to find it. In Section I.7 we discuss some open problems regarding minimal UCA models.

Theorem I.5 *Every UCA graph admits a minimal UCA model.*

Minimal UIG models. In Section II.5, we consider the MinUIG problem. We show that, even though Mitas' algorithm correctly solves Rep, it sometimes fails to provide a minimal model. We propose a patch but, unfortunately, the new algorithm runs in $O(n^2)$ time. The minimal representation problem in linear time remains, thus, an open problem.

Theorem II.7 MinUIG *can be solved in $O(n^2)$ time and linear space.*

Powers of paths and cycles. In Section II.6 we show how MinUIG and its generalization IntMinUCA can be used so as to solve MinP_q^k and MinC_q^k, respectively. The obtained algorithm for MinP_q^k runs in $O(n^2)$ and linear space, thus improving the results obtained by Costa et al. [8].

2 Preliminaries

In this article we consider (simple) graphs, (simple) digraphs, and q-digraphs. A *q-digraph*, for $q \in \mathbb{N}$, is a $(q+1)$-tuple $G = (V, E_1, \ldots, E_q)$ such that (V, E_i) is a digraph, for $1 \leq i \leq q$. Clearly, every digraph is a 1-digraph. For the sake of simplicity, we refer to the directed edges in E_i as being *edges* of G, unless otherwise stated. For a (q-di)graph G, we write $V(G)$ and $E(G)$ to denote the sets of vertices and (bag of) edges of G, respectively, while we use n and m to denote $|V(G)|$ and $|E(G)|$, respectively. For any pair $u, v \in V(G)$, we write uv to denote the pair (u, v); note that uv is an unordered pair when G is a graph, while it is an ordered pair when G is a q-digraph. To avoid confusion, we write $u \to v$ as an equivalent of uv when G is a q-digraph. Sometimes we may refer to the pair uv as being the (directed) edge between u and v (*from or starting at u to or ending at v*), regardless of whether $uv \in E(G)$.

A walk W of a (q-di)graph G is a sequence of edges $v_1v_2, v_2v_3 \ldots, v_{k-1}v_k$ of G. Walk W goes *from (or starts at) v_1 to (or ends at) v_k*. We say that W is a circuit when $v_k = v_1$, that W is a *path* when $v_i \neq v_j$ for every $1 \leq i < j \leq k$, and that W is a *cycle* when it is a circuit and $v_1v_2, \dots, v_{k-2}v_{k-1}$ is a path. If G contains no cycles, then G is an *acyclic* (q-di)graph. For the sake of notation, we could say that W is a *circuit* when $v_1 \neq v_k$; this means that W, v_kv_1 is a circuit. Moreover, we may write that a sequence of vertices $v_1, \dots, v_k$ is a *walk* of G to express that some sequence of edges $v_1v_2, \dots, v_{k-1}v_k$ is a walk of G. Both conventions are ambiguous when G is a q-digraph for $q > 1$, as there could be q edges from v_i to v_{i+1} (or from v_k to v_1 in the former case). In general, the edge represented by v_iv_{i+1} is clear by context.

An *edge weighing*, or simply a *weighing*, of a (q-di)graph G is a function $w\colon E(G) \to \mathbb{R}$. The value $w(uv)$ is referred to as the *weight* of uv (with respect to w). For any bag of edges E, the *weight* of E (with respect to an edge weighing w) is $w(E) = \sum_{uv \in E} w(uv)$. We use two distance measures on a (q-di)graph G with a weighing w. For $u, v \in V(G)$, we denote by $\mathbf{d}^*w(G, u, v)$ the maximum $w(W)$ among the walks W from u to v, while $\mathbf{d}w(G, u, v)$ denotes the maximum $w(W)$ among the paths W starting at u and ending at v. Note that $\mathbf{d}w(G, u, v) < \infty$ for every u, v, while $\mathbf{d}^*w(G, u, v) = \mathbf{d}w(G, u, v)$ when G contains no cycle of positive weight [6]. For a weighing w', we write $(\mathbf{d}w \circ \mathbf{d}w')(G, u, v) = \max\{w(W) \mid W \text{ is a path from } u \text{ to } v \text{ with } w'(W) = \mathbf{d}w'(G, u, v)\}$. In other words, $\mathbf{d}w \circ \mathbf{d}w'$ measures the "w-distance" from u to v when only those walks that impose the maximum "w'-distance" from u to v are considered. For the sake of notation, we omit the parameter G when there are no ambiguities.

A *proper circular-arc* (PCA) model $\mathcal{M}$ is a pair $(C, \mathcal{A})$, where C is a circle and $\mathcal{A}$ is a collection of open arcs of C such that no arc contains another arc and no pair of arcs in $\mathcal{A}$ cover C. When traversing the circle C, we always choose the clockwise direction. If s, t are points of C, we write (s, t) to mean the arc of C defined by traversing the circle from s to t, and $|s, t|$ to mean the length of (s, t). Sometimes we refer to $|s, t|$ as being the *separation* from s to t. Points s and t are the *extremes* of (s, t), while s is its *beginning point* and t its *ending point*. For $A \in \mathcal{A}$, we write $A = (s(A), t(A))$. The *extremes* of $\mathcal{A}$ are those of all arcs in $\mathcal{A}$. In this article we assume that no pair of extremes of $\mathcal{A}$ coincide. An ordered pair of extremes s_1s_2 of $\mathcal{M}$ is *consecutive* when there is no extreme $s \in (s_1, s_2)$ (note that s_2s_1 is not consecutive in this case, unless $|\mathcal{A}| = 1$). We assume C has a special point 0 with the property that $s(A_i) = |0, s(A_i)|$ and $t(A_i) = |0, t(A_i)|$, for every $1 \leq i \leq n$. For every pair of points p_1, p_2, we write $p_1 < p_2$ to indicate that p_1 appears before p_2 in a traversal of C from 0. Similarly, we write $A_1 < A_2$ to mean that $s(A_1) < s(A_2)$ for any pair of arcs A_1, A_2 on C.

A *unit circular-arc* (UCA) model is a PCA model $\mathcal{M}$ in which all the arcs have the same length. Let $A_1 < \ldots < A_n$ be the arcs of $\mathcal{M} = (C, \mathcal{A})$, $c, \ell \in \mathbb{Q}_{>0}$, $d, d_s \in \mathbb{Q}_{\geq 0}$, and $d_\ell, d_r\colon \mathcal{A} \to \mathbb{Q}_{\geq 0}$. We say that $\mathcal{M}$ is a $(c, \ell, d, d_s, d_\ell, d_r)$-*CA model* when:

(unit$_1$) C has circumference c,

(unit$_2$) all the arcs of $\mathcal{A}$ have length ℓ, (unit$_3$) $|p_1, p_2| \geq d$ for every pair of consecutive extremes $p_1 p_2$,

(unit$_4$) $|s_1, s_2| \geq d + d_s$ for any pair of beginning points s_1, s_2, and

(unit$_5$) $d_\ell(A_i) \leq s(A_i) \leq c - d_r(A_i)$ for every $1 \leq i \leq n$.

Intuitively, $\mathcal{M}$ is a UCA model in which consecutive extremes are separated by at least d space, the beginning points are separated by $d + d_s$ space, and $d_\ell(A_i)$ and $d_r(A_i)$ are lower bounds of the separation from 0 to $s(A_i)$ and from $s(A_i)$ to 0, respectively. We simply write that $\mathcal{M}$ is a (c, ℓ, d, d_s)-*CA model* to indicate that $d_\ell = d_r = 0$, and that $\mathcal{M}$ is a (c, ℓ)-CA model to mean that $\mathcal{M}$ is a $(c, \ell, 1, 0)$-CA model. To further simplify the notation, we refer to the tuple $u = (c, \ell, d, d_s, d_\ell, d_r)$ as a *UCA descriptor*, and we say that u is *integer* when c, ℓ, d, d_s, d_ℓ, and d_r are integers. Similarly, a u-CA model $\mathcal{M}$ is *integer* when c, ℓ and all the extremes of $\mathcal{M}$ are integers.

A *proper interval* (PIG) model is a PCA model $\mathcal{M}$ in which no arc crosses 0; if $\mathcal{M}$ is also UCA, then $\mathcal{M}$ is a *unit* interval (UIG) model. Any UIG model $\mathcal{M}$ is a u-CA model for some large enough c; for simplicity, we just write $c = \infty$ in this case. Moreover, we write that $\mathcal{M}$ is an (ℓ, d, d_s)-*IG model* to mean that $\mathcal{M}$ is a (∞, ℓ, d, d_s)-CA model.

Each PCA model $\mathcal{M}$ defines a graph $G(\mathcal{M})$ that contains a vertex for each arc of $\mathcal{M}$ where two vertices are adjacent if and only if their corresponding arcs have a nonempty intersection. We say that $\mathcal{M}$ *represents* a graph G, and that G *admits* $\mathcal{M}$, when G is isomorphic to $G(\mathcal{M})$. A graph is a *proper circular-arc (PCA)*, *unit circular-arc (UCA)*, *proper interval (PIG)*, or *unit interval (UIG)* graph when it admits a PCA, UCA, PIG, or UIG model, respectively.

Clearly, two PCA models $\mathcal{M}_1 = (C_1, \mathcal{A}_1)$ and $\mathcal{M}_2 = (C_2, \mathcal{A}_2)$ are equal when $C_1 = C_2$ and $\mathcal{A}_1 = \mathcal{A}_2$. We say that $\mathcal{M}_1$ is *equivalent* to $\mathcal{M}_2$ when the extremes of $\mathcal{M}_1$ appear in the same order as in $\mathcal{M}_2$ in the traversals of C_1 and C_2 from their respective 0 points. Formally, $\mathcal{M}_1$ and $\mathcal{M}_2$ are equivalent if there exists $f\colon \mathcal{A}_1 \to \mathcal{A}_2$ such that $e(f(A)) < e'(f(B))$ if and only if $e(A) < e'(B)$, for $e, e' \in \{s, t\}$. By definition, $\mathcal{M}_1$ and $\mathcal{M}_2$ are equivalent whenever they are equal.

In this work we consider several recognition problems. In the *representation* (Rep) problem a UCA model equivalent to an input PCA model $\mathcal{M}$ must be generated. Of course, Rep is unsolvable when $\mathcal{M}$ is equivalent to no UCA model, a *negative witness* is desired in such a case. In the u-Rep problem, a (an integer) UCA descriptor u is given together with $\mathcal{M}$, and the goal is to build a (an integer) u-CA model $\mathcal{U}$. We remark that an integer $\mathcal{U}$ equivalent to $\mathcal{M}$ exists whenever u is integer and u-Rep is solvable. The *bounded representation* (BoundRep) problem is a slight variation of u-Rep in which a feasible $d > 0$ must be found by the algorithm, as it is not given as input. That is, we are given a PCA model $\mathcal{M} = (C, \mathcal{A})$ together with $c, \ell \in \mathbb{Q}_{>0}$, $d_s \in \mathbb{Q}_{\geq 0}$ and $d_\ell, d_r\colon \mathcal{A} \to \mathbb{Q}_{\geq 0}$, and we ought to find a u-CA model equivalent to $\mathcal{M}$ for some UCA descriptor $u = (c, \ell, d, d_s, d_\ell, d_r)$ with $d \in \mathbb{Q}_{>0}$. The *integer bounded representation* (IntBoundRep) problem is a variant of BoundRep in which all the input values are integers and the output model must be integer as well.

We also study the MinUIG and IntMinUCA problems that are related to *minimal* models. We postpone their definitions to Section 6.

By its generic nature, the same specification of u-Rep can describe different problems; all we have to do is change the (meaning of the) parameters in the input UCA descriptor. Similarly, the algorithm for u-Rep is a building block that we applied to solve the remaining problems. The six parameters included in UCA descriptors have to do with the problems that we consider in this work. Yet, not all the parameters are equally important for all the problems: some are required for the problem to make sense, while the remaining are accessory and their inclusion makes no harm. For instance, in Rep the goal is to find a (c, ℓ, d, d_s)-CA model, regardless of the values of d and d_s. Yet, we know that Rep is solvable if and only if it is solvable for $d = 1$ and $d_s = 0$, thus defining Rep as the problem of finding a $(c, \ell, 1, 0)$-CA model is not a serious restriction. Similarly, d_s is not required for defining BoundRep, yet it makes no harm to solve the generalization in which d_s is given as input. As it turns out, d_s only plays a fundamental role in the solution of the MinP$_q^k$ and MinC$_q^k$ problems that are defined in Chapter II. Yet, we include them in this chapter so that we can focus in a unique solution for u-Rep.

2.1 Restrictions on the input models

As it is customary in the literature, in this work we assume that all the arcs of a PCA model $\mathcal{M}$ are open and no two extremes of $\mathcal{M}$ coincide. The reason behind these assumptions is that $\mathcal{M}$ can always be transformed into an equivalent model $\mathcal{M}'$ that satisfies these properties. A word of caution is required, though, as in this work we deal with the lengths of the arcs. If we allow coincidences in the extremes of $\mathcal{M}$, for instance, it is possible to shrink the length of the arcs or the circle of some UCA models. We emphasize, nevertheless, that all the arguments in this work, with the obvious adjustments, are equally true when these assumptions are drop. In particular, note that the articles by Klavík et al., Mitas, and Pirlot allow coincident extremes [28, 37, 38, 39].

By our definition, PCA models cannot have two arcs covering the circle. This is a somehow artificial restriction that we impose for the sake of simplicity. In general, this class of models is said to be *normal*. However, it is well known that every non-normal PCA model can be transformed into a normal PCA model in linear time or logspace (see e.g. [25]). Moreover, note that if two arcs in a UIG model cover the circle, then such a model represents a complete graph. The complete graph on n vertices admits the integer $(1, 0)$-*minimal* UIG model $\{(i, i+n+1) \mid 1 \leq i \leq n\}$, thus we do not lose much by excluding these non-normal models when dealing with Rep, IntMinUCA, and MinUIG. In turn, the fact that $\mathcal{M}$ is normal is not used in Theorem 1, thus (Int)BoundRep is also solvable for non-normal models.

Finally, we require two additional restrictions on the input PCA models for technical reasons. We say that a PCA model $\mathcal{M}$ with arcs $A_1 < \ldots < A_n$ is *trivial* when either

1. $s(A_n) < t(A_1)$, or
2. $s(A_i)t(A_i)$ are consecutive for some $1 \leq i \leq n$.

If 1. holds, then we cannot claim that $h(\mathcal{M}) \geq 1$ in Section 3.3. However, in this case $\mathcal{M}$ represents a complete graph and $\{(i, i+n+1) \mid 1 \leq i \leq n\}$ is the unique integer $(1,0)$-minimal model equivalent to $\mathcal{M}$. Thus all the considered problems are trivial in this case. If 2. is true, then $A_i \to A_i$ is a loop of the digraph $\mathcal{B}(\mathcal{M})$ defined in Section 3. We can certainly allow the existence of such a loop in $\mathcal{B}$. However, this edge plays no role in the considered problems as $\operatorname{sep}(A_i \to A_i) < 0$ by (sep_3). Hence, we assume from now on that no PCA model is trivial.

3 The synthetic graph of a model

Pirlot [38, 39] introduced the synthetic graph of a PIG model to represent the separation constraints of its beginning points in any equivalent $(\ell, d, 0)$-IG representation. Recently, Klavík et al. [28] rediscovered and extended synthetic graphs to represent the separations constraints of the bounded representation problem. In this section we further extend synthetic graphs to PCA models, and we show that they correctly reflect the separation constraints in an equivalent u-CA model, for any UCA descriptor u. Before doing so, however, we informally present a simple algorithm to solve u-Rep to motivate the definition of synthetic graphs. Readers familiar with one of [28, 38, 40] can safely skip Section 3.1.

3.1 A simple algorithm for u-Rep

Recall that the input of u-Rep is a PCA model $\mathcal{M}$ with arcs $A_1 < \ldots < A_n$ and a UCA descriptor u, and the output is a u-CA model $\mathcal{U}$ equivalent to $\mathcal{M}$ with arcs $U_1 < \ldots < U_n$. All we have to do to solve u-Rep is to find an appropriate position for $s(U_i)$, for every $1 \leq i \leq n$; such a position can be described using a family of "separation constraints". A *separation constraint* is an inequality that dictates how far or close must $s(U_i)$ be from either 0 or another beginning point of $\mathcal{U}$. For instance, it could specify that $s(U_i) \geq 3$ or $s(U_i) \geq s(U_j) - 7$. If we regard 0 as the beginning point of a fictitious arc U_0, then each separation constraint is an inequality of the form $s(U_j) \geq s(U_i) + \delta$.

To provide the family of separation constraints for u-Rep, let $A_i < A_j$ be arcs of $\mathcal{M}$ and suppose, for the time being, that $\mathcal{M}$ is equivalent to some u-CA model $\mathcal{U}$; as usual $u = (c, \ell, d, d_s, d_\ell, d_r)$. Since $\mathcal{U}$ is equivalent to $\mathcal{M}$, it follows that $U_i < U_j$, thus $s(U_j)$ satisfies the separation constraint $s(U_j) \geq s(U_i) + d + d_s$. Similarly, as $\mathcal{M}$ and $\mathcal{U}$ are circular structures, $s(U_i)$ satisfies the corresponding separation constraint $s(U_i) \geq s(U_j) + d + d_s - c$. Analogously, A_i crosses A_j if and only if U_i crosses U_j. Again, we have two possibilities according to whether A_i crosses 0 or not. In the latter case, $s(U_j) + d \leq t(U_i) = s(U_i) + \ell$, thus $s(U_i)$ satisfies the constraint $s(U_i) \geq s(U_j) + d - \ell$. In the former case, and by the circular nature of $\mathcal{M}$ and $\mathcal{U}$, it follows that $s(U_j) + d \leq t(U_i) + c = s(U_i) + \ell$, thus $s(U_i)$ satisfies $s(U_i) \geq s(U_j)+d+c-\ell$. The complete family of separations constraints for u-REP is given below; here A_i and A_j represent any pair of arcs, whereas $q \in \{0,1\}$ equals to 0 when $A_i < A_j$.

step constraints: $s(U_j) \geq s(U_i) + d + d_s - cq$.

nose constraints: $s(U_j) \geq s(U_i) + \ell + d - cq$ when $t(A_i) < s(A_j)$.

hollow constraints: $s(U_i) \geq s(U_j) + d - \ell + cq$ when $t(A_i) > s(A_j)$.

bound constraints: $s(U_i) \geq s(U_0) + d_\ell(A_i)$ and $s(U_0) \geq s(U_i) + d_r(A_i) - c$.

This set S of separation constraints forms a system of inequalities such that $\mathcal{U}$ is a solution to u-REP if and only if its beginning points are a solution to S. Let the *separation graph* of $\mathcal{M}$ be the 4-digraph G that has one vertex v_i for each arc U_i and one edge $v_i \to v_j$ for each constraint $s(U_j) \geq s(U_i) + \delta$ of S, and let sep be a weighing that assigns δ to the corresponding edge $v_i \to v_j$. The solutions to S are commonly referred to as *potential function* of G; it is well known that S has a solution if and only if G has no cycles of positive weight (c.f. [40]). Moreover, $\{s(U_i) = \mathbf{d}\text{sep}(G, v_0, v_i) \mid 0 \leq i \leq n\}$ is a solution to S.

The above discussion yields a simple algorithm to solve u-REP: just invoke Bellman-Ford's shortest path algorithm on G and v_0. We can improve this algorithm by observing that most of the separation constraints are *implied* (in the sense that they can be derived from other separation constraints), and thus their corresponding edges can be removed from G. This is obvious if we observe that we are interested only in a spanning tree of G defined by the longest paths from v_0. However, the idea is to remove some of the implied edges before solving the longest path problem. For instance, the step constraint $s(U_i) \geq s(U_j) + d$ is implied by the constraints $s(U_i) \geq s(U_{i+1}) + d$ and $s(U_{i+1}) \geq s(U_j) + d$, for every $j > i+1$. Thus, only $O(n)$ of the step constraints should be kept before invoking Bellman-Ford. Summing up, the idea is to keep a minimal set of non-implied constraints plus some other "small" set of implied constraints.

The set S' of kept constraints is what we call a synthesis of S. Formally, $S' \subseteq S$ is *a u-synthesis of* S when all the separation constraints in S, defined using the values in u, are implied by those in S'. On the other hand, $S' \subseteq S$ is a *synthesis of* S when S' is a u-synthesis of S for every UCA descriptor u. Clearly, S' yields a subgraph G' of G that is a *(u-)synthesis* of G. A key observation by Pirlot [38] for PIG models is that there exists a synthesis $\mathcal{S}$ of G with $O(n)$ edges. Pirlot's synthesis is what we call the *synthetic graph* of the PIG model $\mathcal{M}$. Besides being a synthesis, $\mathcal{S}$ has the particularity that it represents $\mathcal{M}$, i.e., there is a one-to-one correspondence between PIG models and synthetic graphs. In the next section we extend Pirlot's synthetic graphs to represent general PCA models.

3.2 The synthetic graph of a PCA model

Let $\mathcal{M} = (C, \mathcal{A})$ be a PCA model with arcs $A_1 < \ldots < A_n$. The *bounded synthetic graph* of $\mathcal{M}$ is the 4-digraph $\mathcal{B}(\mathcal{M})$ (see Figure 1) that has a vertex

$v(A_i)$ for each $A_i \in \mathcal{A}$ and a vertex A_0, and whose edge set is $E_\sigma \cup E_\nu \cup E_\eta \cup E_\beta$, where:

- $E_\sigma = \{v(A_i) \to v(A_{i+1}) \mid 1 \leq i \leq n,\ A_{n+1} = A_1\}$,
- $E_\nu = \{v(A_i) \to v(A_j) \mid t(A_i)s(A_j)$ are consecutive in $\mathcal{M}\}$,
- $E_\eta = \{v(A_i) \to v(A_j) \mid s(A_i)t(A_j)$ are consecutive in $\mathcal{M}\}$, and
- $E_\beta = \{A_0 \to v(A_i), v(A_i) \to A_0 \mid 1 \leq i \leq n\}$.

The edges in E_σ, E_ν, E_η, and E_β are said to be the *steps*, *noses*, *hollows*, and *bounds* of $\mathcal{B}(\mathcal{M})$, respectively.[3] (Note that E_σ, E_ν and E_η could have a nonempty intersection, even if this is not the common case. However, $\mathcal{B}(\mathcal{M})$ has no loops as $\mathcal{M}$ is not trivial.) For the sake of simplicity, we usually drop the parameter $\mathcal{M}$ from $\mathcal{B}(\mathcal{M})$ when no ambiguities are possible. Moreover, we regard the arcs of $\mathcal{M}$ as being the vertices of $\mathcal{B}$, thus we may say that $A_i \to A_j$ is a nose instead of writing that $v(A_i) \to v(A_j)$ is a nose.

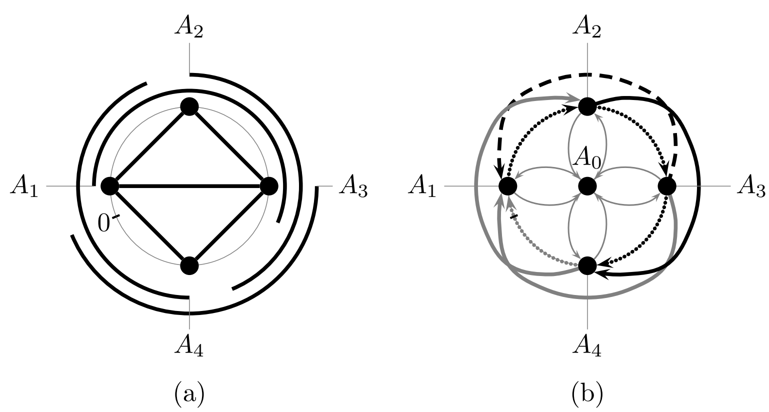


FIGURE 1. *(a) A PCA model $\mathcal{M}$ and the graph $G(\mathcal{M})$ it represents. (b) The synthetic graph of $\mathcal{M}$ where thick solid, dashed, dotted, and thin solid lines represent noses, hollows, steps, and bounds, respectively. Internal edges are black, whereas external edges are gray.*

We classify the edges of $\mathcal{B}$ in two classes according to the positions of their arcs. We say a step (resp. nose) $A_i \to A_j$ is *internal* when $i < j$, while a hollow is *internal* when $i > j$. Non-internal edges are referred to as *external*; in particular, all the bounds are external. Observe that every step is internal except $A_n \to A_1$. Similarly, a nose $A_i \to A_j$ is internal if and only if the arc

[3] Pirlot [38] explains that the terms "*nose*" and "*hollow*" are employed because the matrix representation of a semiorder looks like a "staircase". We follow Pirlot's terminology because it has been applied in many subsequent articles. We emphasize that the intersection matrix of PIG and PCA models (i.e., the augmented adjacency matrix of the corresponding PIG and PCA graphs) have the so-called consecutive-ones and circular-ones properties, respectively, which can also be interpreted as "staircases" as well.

$(s(A_i), s(A_j))$ does not cross 0, while a hollow $A_i \to A_j$ is internal if and only if $(s(A_j), s(A_i))$ does not cross 0. Since the purpose of A_0 is to represent the point 0 of $\mathcal{M}$, we can say, in short, that $A_i \to A_j$ is internal when 0 is not crossed in the traversal of the extremes involved in the definition of $A_i \to A_j$.

We now define the special edge weighing sep_u of $\mathcal{B}$ that models the separation constraints that any u-CA model equivalent to $\mathcal{M}$ must satisfy. For a UCA descriptor u, the edge weighing sep_u is such that, for every $1 \leq i, j \leq n$

(sep_1) $\text{sep}_u(A_i \to A_j) = d + d_s - cq$ if $A_i \to A_j$ is a step,

(sep_2) $\text{sep}_u(A_i \to A_j) = d + \ell - cq$ if $A_i \to A_j$ is a nose,

(sep_3) $\text{sep}_u(A_i \to A_j) = d + cq - \ell$ if $A_i \to A_j$ is a hollow, and

(sep_4) $\text{sep}_u(A_0 \to A_i) = d_\ell(A_i)$ and $\text{sep}(A_i \to A_0) = d_r(A_i) - c$,

where $q \in \{0, 1\}$ equals 0 if and only if $A_i \to A_j$ is internal. For the sake of notation, we omit the subscript u from sep when no ambiguities are possible. Note that (sep_1), (sep_2), (sep_3), and (sep_4) model step, nose, hollow, and bound constraints as defined in Section 3.1, respectively, assuming that A_0 represents 0 in $\mathcal{M}$.

As we shall see in Theorem 1, a u-CA model equivalent to $\mathcal{M}$ exists when the longest path problem with weight sep has a feasible solution on $\mathcal{B}$. In such case, a u-CA model can be generated by observing the distances from A_0. With this in mind, we define $\mathcal{U}(\mathcal{M}, u)$ to be the u-CA model with arcs $U_1, \ldots, U_n$ such that $s(U_i) = \mathbf{d}\text{sep}(A_0, A_i)$, for every $1 \leq i \leq n$ (we assume arithmetic modulo c). For simplicity, we omit $\mathcal{M}$ and u from $\mathcal{U}$ as usual.

Theorem 1 *The following statements are equivalent for a PCA model $\mathcal{M}$ with arcs $A_1 < \ldots < A_n$ and a (an integer) UCA descriptor u:*

(i) $\mathcal{M}$ is equivalent to a u-CA model.

(ii) $\text{sep}(\mathcal{W}) \leq 0$ for every cycle $\mathcal{W}$ of $\mathcal{B}$.

(iii) $\mathcal{U}$ is a (an integer) u-CA model equivalent to $\mathcal{M}$.

Proof: (i) $\Rightarrow$ (ii). Suppose $\mathcal{M}$ is equivalent to a u-CA model $\mathcal{M}'$ with arcs $A'_1 < \ldots < A'_n$. Write $s(A'_0)$ to mean the point 0 of $\mathcal{M}'$. It is not hard to see (see Section 3.1) that $s(A'_j) \geq s(A'_i) + \text{sep}(A_i \to A_j)$ for every edge $A_i \to A_j$ of $\mathcal{B}$. Hence, by induction, $s(A'_i) \geq s(A'_i) + \text{sep}(\mathcal{W})$ for every cycle $\mathcal{W}$ of $\mathcal{B}$ that contains A_i.

(ii) $\Rightarrow$ (iii). Let $U_1 < \ldots < U_n$ be the arcs of $\mathcal{U}$, $U_{n+1} = U_1$, $A_{n+1} = A_1$, and note that $\mathbf{d}^*\text{sep}(A_i, A_j) = \mathbf{d}\text{sep}(A_i, A_j)$ for every $0 \leq i, j \leq n$ as $\mathcal{B}$ has no cycles of positive length. Thus, by (sep_4), $\mathcal{U}$ satisfies (unit_5) as $s(U_i) = \mathbf{d}\text{sep}(A_0, A_i) \geq d_\ell(A_i)$ and $s(U_i) + d_r(A_i) - c = \mathbf{d}\text{sep}(A_0, A_i) + d_r(A_i) - c \leq \mathbf{d}\text{sep}(A_0, A_0) = 0$. Since $\mathcal{U}$ satisfies (unit_1)–(unit_2) by definition, it follows that $\mathcal{U}$ is a $(c, \ell, d', d'_s, d_\ell, d_r)$-CA model for some d', d'_s. To prove that $\mathcal{U}$ is a u-CA model equivalent to $\mathcal{M}$, it suffices to see that (a) $s(U_i) + d + d_s \leq s(U_{i+1})$ for every $1 \leq i \leq n$, (b) $s(U_i) + d \leq t(U_j)$ when $s(A_i)t(A_j)$ are consecutive in $\mathcal{M}$, and (c) $t(U_i) + d \leq s(U_j)$ when $t(A_i)s(A_j)$ are consecutive in $\mathcal{M}$.

(a) $A_i \to A_{i+1}$ is a step, thus $s(U_i) + d + d_s = \mathbf{d}\text{sep}(A_0, A_i) + d + d_s \leq \mathbf{d}\text{sep}(A_0, A_{i+1}) = s(U_{i+1})$.

(b) $A_i \to A_j$ is a hollow of $\mathcal{B}$; let $q \in \{0, 1\}$ be 1 if and only if $(s(A_j), s(A_i))$ crosses 0. Note that, equivalently, $q = 1$ if and only if $A_i \to A_j$ is external. Thus, $t(U_j) = s(U_j) + \ell - cq = \mathbf{d}^*\text{sep}(A_0, A_j) + \ell - cq \geq \mathbf{d}^*\text{sep}(A_0, A_i) + \text{sep}(A_i \to A_j) + \ell - cq = s(U_i) + d$.

(c) $A_i \to A_j$ is a nose of $\mathcal{B}$; if $q \in \{0, 1\}$ equals 1 when $(s(A_i), s(A_j))$ crosses 0, then $s(U_j) \geq \mathbf{d}^*\text{sep}(A_0, A_i) + 1 + \ell - cq \geq t(U_i) + d$.

(iii) $\Rightarrow$ (i). Trivial. □

When restricted to PIG models, Theorem 1 is a somehow alternative formulation of Proposition 2.5 in [38]; see also Proposition 4.4 in [28].

Though simple enough, Theorem 1 allows us to solve u-Rep as follows. First, we build the digraph $\mathcal{B}$ in which every edge $A_i \to A_j$ is weighed with $s_{ij} = \text{sep}_u(A_i, A_j)$. Then, we invoke the Bellman-Ford shortest path algorithm [6] on $\mathcal{B}$ to obtain $s_i = \mathbf{d}^*\text{sep}(A_0, A_i)$ for every $0 \leq i \leq n$. If Bellman-Ford ends in success, then we output $\mathcal{U}(\mathcal{M}, u)$; otherwise, we output the cycle of positive weight found as the negative witness. This algorithm costs $O(n^2)$ time and $O(n)$ space when u is integer. In Chapter II we generalize this algorithm for the case in which u is not integer, and we show how Theorem 1 can be used to solve BoundRep and IntBoundRep as well.

Theorem II.1 (Int)BoundRep *and* u-Rep *can be solved in* $O(n^2)$ *time and* $O(n)$ *space.*

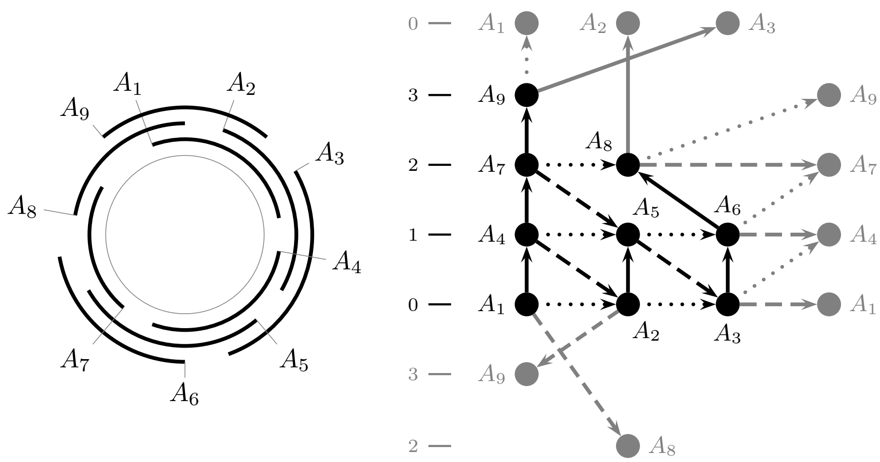
3.3 The separation of a boundless walk

The cycles of $\mathcal{B}$ with maximum sep-values play a fundamental role when deciding if $\mathcal{M}$ admits an equivalent u-CA model, as shown in Theorem 1. The purpose of this section is to analyze these separations in the boundless synthetic graph. The *(boundless) synthetic graph* of $\mathcal{M}$ is just $\mathcal{S}(\mathcal{M}) = \mathcal{B}(\mathcal{M}) \setminus A_0$; for the sake of simplicity, we drop the parameter $\mathcal{M}$ as usual. The main tool that we apply is a pictorial description of $\mathcal{S}$, that generalizes the work of Mitas [37] on PIG models (see Figure 2 and Section II.5). Roughly speaking, Mitas arranges the vertices of $\mathcal{S}$ into a matrix, where the row and column of A_i correspond to its height (cf. below) and the number of internal hollows of some paths from A_1, respectively.

Let $\mathcal{M} = (C, \mathcal{A})$ be a PCA model with arcs $A_1 < \ldots < A_n$. The *height* $h(A_i)$ of A_i ($1 \leq i \leq n$) is recursively defined as follows:

$$h(A_i) = \begin{cases} 0 & \text{if } s(A_i) < t(A_1) \\ 1 + h(A_j) & \text{otherwise, where } A_j = \max\{A_j \mid t(A_j) < s(A_i)\}. \end{cases}$$

The *height* of $\mathcal{M}$ is defined as $h(\mathcal{M}) = h(A_n)$; note that $h(\mathcal{M}) \geq 1$ (because $\mathcal{M}$ is not trivial). For the sake of notation, we drop the parameter $\mathcal{M}$ as usual. In Figure 2, the vertices are drawn in levels according to their height.



(a) A PCA model $\mathcal{M}$. **(b)** The synthetic graph $\mathcal{S}(\mathcal{M})$.

FIGURE 2. *(Boundless) synthetic graph $\mathcal{S}$ of a PCA model $\mathcal{M}$. Each gray vertex corresponds to a black vertex (we separate them for the sake of exposition) and each edge is drawn only once. The edges are solid, dashed, and dotted, according to whether they are noses, hollows, and steps, respectively. The height of $\mathcal{M}$ is $h = 3$ and each vertex is drawn in a row that corresponds to its height; the height is indicated to the left. Note that there are* 1-, $(-h)$-, *and* $(1-h)$*-noses,* 0-, (-1)-, h-, *and* $(h-1)$*-hollows, and* 0-, 1-, *and* $(-h)$*-steps.*

It is important to note that internal noses and steps "jump" to higher or equal levels, while external noses and steps "jump" to lower levels. Similarly, internal hollows "jump" to equal or lower levels while external hollows "jump" to higher levels. We need a more explicit description of how does the height change when an edge is traversed. In general, we say that $A_i \to A_j$ *has* δ *jump* for $\delta = h(A_j) - h(A_i)$. For the sake of notation, we refer to noses (resp. steps, hollows) with δ jump simply as δ-noses (resp. δ-steps, δ-hollows).

It is not hard to see (check Figure 2) that $\mathcal{S}$ has three kinds of noses, namely 1-, $(-h)$-, and $(1-h)$-noses. Moreover, if $A_i \to A_j$ is either a $(-h)$- or $(1-h)$-nose, then $h(A_j) = 0$. Similarly, there are three kinds of steps, namely 0-, 1-, and $(-h)$-steps, and four kinds of hollows, namely 0-, (-1)-, h-, and external $(h-1)$-hollows. Note that we need to differentiate between internal 0-hollows and external $(h-1)$-hollows when $h = 1$. For the sake of simplicity, we will refer to $A_i \to A_j$ as an $(h-1)$*-hollow* to mean that $A_i \to A_j$ is an **external** $(h-1)$-hollow. We emphasize that no confusions are possible because $\mathcal{M}$ has no external 0-hollows; otherwise A_i and A_j would cover the circle of $\mathcal{M}$. Observe that, as it happens with noses, $h(A_i) = 0$ for every h- or $(h-1)$-hollow $A_i \to A_j$, while $h(A_1) = 0$ for the unique $(-h)$-step $A_n \to A_1$. Obviously, the jump of a walk $\mathcal{W}$ depends exclusively on the number of different kinds of noses, hollows and steps that it contains. We write $\nu_\delta(\mathcal{W})$, $\eta_\delta(\mathcal{W})$, and $\sigma_\delta(\mathcal{W})$ to indicate the number of δ-noses, δ-hollows, and δ-steps of $\mathcal{W}$, respectively. As usual, we do not write the parameter $\mathcal{W}$ when it is clear from context. The following observation describes the jump of $\mathcal{W}$.

Lemma 1 *If $\mathcal{W}$ is a walk from A_i to A_j in $\mathcal{S}$, then*

$$h(A_j) - h(A_i) = \nu_1 + \sigma_1 - \eta_{-1} + h(\eta_h - \nu_{-h} - \sigma_{-h}) + (h-1)(\eta_{h-1} - \nu_{1-h}) \tag{1}$$

We now define two kinds of walks that are of particular interest for us. These walks correspond to what Tucker calls by the names of (a,b)-*independent* and (x,y)-*circuits* of a PCA model (see [46] and Section 5). We say that a walk of $\mathcal{S}$ is a *nose walk* when it contains no hollows, while it is a *hollow walk* when it contains no noses and $\eta_h + \eta_{h-1} \geq \sigma_{-h}$. Note that a walk is both a nose and a hollow walk only if all its edges are steps; in general, walks that contain only steps are referred to as *step walks*. Nose and hollow walks are important because they impose lower and upper bounds for the circumference of the circle in a UCA model.

By Theorem 1, if $\mathcal{M}$ is equivalent to a u-CA model, then $\operatorname{sep}(\mathcal{W}_N) \leq 0$ for every nose cycle $\mathcal{W}_N$. By definition,

$$\operatorname{sep}(\mathcal{W}_N) = \nu_1(\ell+d) + (\nu_{-h}+\nu_{1-h})(\ell+d-c) + (d+d_s)(\sigma_0+\sigma_1) + (d+d_s-c)\sigma_{-h}$$

while by (1)

$$\nu_1 = -\sigma_1 + h(\nu_{-h} + \sigma_{-h}) + (h-1)\nu_{1-h}$$

thus

$$c \geq (\ell+d)\left(h + \frac{\nu_{-h} - \sigma_1}{\nu_{1-h} + \nu_{-h} + \sigma_{-h}}\right). \tag{2}$$

For any nose walk $\mathcal{W}_N$, the value $r(\mathcal{W}_N) = \frac{\nu_{-h}-\sigma_1}{\nu_{1-h}+\nu_{-h}+\sigma_{-h}}$ is referred to as the *ratio* of $\mathcal{W}_N$, while the *nose ratio* of $\mathcal{M}$ is $r(\mathcal{M}) = \max\{r(\mathcal{W}_N) \mid \mathcal{W}_N \text{ is a nose cycle of } \mathcal{M}\}$.

A similar analysis is enough to conclude (assuming $x/0 = \infty$) that

$$c \leq (\ell-d)\left(h + \frac{\eta_0 + \eta_h + \sigma_1}{\eta_h + \eta_{h-1} - \sigma_{-h}}\right) \tag{3}$$

for any hollow cycle $\mathcal{W}_H$.[4] This time, the value $R(\mathcal{W}_H) = \frac{\eta_0+\eta_h+\sigma_1}{\eta_h+\eta_{h-1}-\sigma_{-h}}$ is said to be the *ratio* of $\mathcal{W}_H$, while $R(\mathcal{M}) = \min\{R(\mathcal{W}_H) \mid \mathcal{W}_H \text{ is a hollow cycle of } \mathcal{M}\}$ is the *hollow ratio* of $\mathcal{M}$. The following observation sums up (2) and (3); note that, as usual, we omit the parameter $\mathcal{M}$ from r and R.

Lemma 2 *For every u-CA model,*

$$(\ell+d)(h+r) \leq c \leq (\ell-d)(h+R) \tag{4}$$

[4] This is the reason why hollow walks are restricted to $\eta_h + \eta_{h-1} \geq \sigma_{-h}$ by definition. Of course, there is no impediment to define hollow walks in which $\eta_h + \eta_{h-1} < \sigma_{-h}$. Note, however, that only one of such walks is a cycle: the step path that joins A_0 with itself. This cycle imposes a lower bound on c as described in (2). Thus, we prefer to restrict the global definition of hollow cycles instead of excluding this case from all the other definitions.

By (4), $\mathcal{M}$ is equivalent to a u-CA model only if $c = (\ell + d)(h + r) + e$ for some $e \geq 0$. The factor $(\ell + d)(h + r)$ is required for each nose cycle to fit in the model when considered in isolation, while the *extra space* e serves to accommodate the interactions between all the arcs. Note that, in general, $\mathcal{M}$ need not be equivalent to a u-CA model. This is not important, though, as we can always write c as $(\ell + d)(h + r) + e$; just observe that e could be negative in some cases.

Theorem 1 gives us a method to determine if $\mathcal{M}$ is equivalent to a given u-CA model $\mathcal{U}$. When u is not given, we have two variables that we can adjust to find a suitable model $\mathcal{U}$: ℓ and e (or c). Once these variables are fixed, the existence of $\mathcal{U}$ is fully determined by $\mathcal{S}$ and the weighing sep_u so obtained. So, in a nutshell, we would like to find the appropriate values of ℓ and e to guarantee that $\text{sep}(\mathcal{W}) \leq 0$ for every cycle $\mathcal{W}$ of $\mathcal{S}$. The advantage of using ℓ and e, instead of ℓ and c, is that we can express $\text{sep}(\mathcal{W})$ as a linear polynomial. That is, we can write $\text{sep}(\mathcal{W}) = \text{len}\,\ell + \text{ext}\,e + \text{const}$, where len, ext, and const are values that do not depend on ℓ and e. Doing so provides a convenient way to manipulate the values of ℓ and e, as changing one of them has no effects on the other. (Note however that any change in either ℓ or e impacts on c because c depends on both ℓ and e.) With this idea in mind, let $\mathcal{W}$ be any walk of $\mathcal{S}$ and observe that, by definition

$$\begin{aligned}\text{sep}(\mathcal{W}) = \nu_1(\ell + d) + (\nu_{-h} + \nu_{1-h})(\ell + d - c) +\\ (\eta_0 + \eta_{-1})(d - \ell) + (\eta_h + \eta_{h-1})(c + d - \ell) +\\ (d + d_s)(\sigma_0 + \sigma_1) + (d + d_s - c)\sigma_{-h}.\end{aligned}$$

Applying Equation (1) and some algebraic manipulation, we conclude that

$$\text{sep}(\mathcal{W}) = (\ell + d)(h(A_j) - h(A_i) + \text{len}(\mathcal{W})) + e\,\text{ext}(\mathcal{W}) + \text{const}(\mathcal{W}) \quad (5)$$

where:

$$\text{len}(\mathcal{W}) = \nu_{-h} - \eta_h - \sigma_1 - \eta_0 + r\,\text{ext}(\mathcal{W}) \quad (6)$$

$$\text{ext}(\mathcal{W}) = \eta_h + \eta_{h-1} - \nu_{-h} - \nu_{1-h} - \sigma_{-h} \quad (7)$$

$$\text{const}(\mathcal{W}, d, d_s) = 2d(\eta_{-1} + \eta_0 + \eta_h + \eta_{h-1}) + (d + d_s)(\sigma_1 + \sigma_0 + \sigma_{-h}) \quad (8)$$

The values $\text{len}(\mathcal{W})$, $\text{ext}(\mathcal{W})$, and $\text{const}(\mathcal{W}, d, d_s)$ are the *length*, *extra*, and *constant factors* of $\mathcal{W}$, and $\mathcal{W}$, d, and d_s are omitted as usual. We remark that len, ext and const are technical values with no special meaning; there sole purpose is to express sep as a linear polynomial on ℓ and e. Nevertheless, we can think of the triplet $(\text{len}, \text{ext}, \text{const})$ as being some sort of generalization of what Mitas takes as the column in his pictorial representation (see [37] and Section II.5). The main difference is that the external edges can be disregarded from $\mathcal{S}$ when $\mathcal{M}$ is a PIG model. Mitas also discards the 0-hollows and the steps to define the column, thus $(\text{len}, \text{ext}, \text{const})$ gets reduced to $(0, 0, 2\eta_{-1})$.

We can already envision two advantages of expressing sep as a linear polynomial with indeterminate ℓ and e and coefficients len, ext, and const. First, by Theorem 1, we can see at first sight that $\mathcal{M}$ is equivalent to a (c, ℓ)-CA model whenever either $\text{len}(\mathcal{W}) < 0$ or $\text{len}(\mathcal{W}) = 0$ and $\text{ext}(\mathcal{W}) < 0$ for every cycle $\mathcal{W}$. Just take large enough values for ℓ and e. In particular, observe that $\text{len}(\mathcal{W}) < 0$ for every cycle when $\mathcal{M}$ is a PIG model; thus, this is just one more proof of the fact that every PIG model is equivalent to an UIG model. The second advantage is that the factors depend only on the structure of $\mathcal{S}$ and not on the weighing function sep. Thus, we can compute $\text{ext}(\mathcal{W})$ by means of the edge weighing ext (the overloaded notation is intentional) of $\mathcal{S}$ such that

$$\text{ext}(A_i \to A_j) = \begin{cases} 1 & \text{if } A_i \to A_j \text{ is an external hollow} \\ -1 & \text{if } A_i \to A_j \text{ is an external nose or step} \\ 0 & \text{otherwise} \end{cases}$$

We can compute $\text{len}(\mathcal{W})$ and $\text{const}(\mathcal{W}, d, d_s)$ in a similar fashion with the corresponding edge weighings len and const_{d,d_s}.

4 Efficient Tucker's characterization

In this section we give an alternative proof of Tucker's characterization, taking advantage of the framework of synthetic graphs. In short, Tucker's theorem states that $\mathcal{M}$ is equivalent to some UCA model if and only if $a/b < x/y$ for every (a, b)-independent and every (x, y)-circuit of $\mathcal{M}$ [46]. As already mentioned (and proven in Section 5) the nose and hollow cycles of $\mathcal{S}$ are the equivalents of the (a, b)-independents and (x, y)-circuits of $\mathcal{M}$. Moreover, the maximal and minimal values of a/b and x/y are somehow related to r and R, respectively. Thus, intuition tells us that we should be able to prove that $\mathcal{M}$ is equivalent to a UCA model if and only if $r < R$. This is equivalence (i) $\Leftrightarrow$ (ii) of Theorem 2 below.

Though equivalence (i) $\Leftrightarrow$ (ii) is not completely new, our proof of this fact is new and somehow simple. One of the main features about Theorem 2 is that it exhibits other characterizations that can be used for positive and negative certification. In particular, it shows how to obtain an integer (c, ℓ)-CA model equivalent to $\mathcal{M}$ with c and ℓ polynomial in n. The existence of such models was questioned by Durán et al. [11] and proved by Lin and Szwarcfiter [34] by means of feasible circulations.

Before stating Theorem 2, we study the relation between sep and the ratios of $\mathcal{M}$. Recall that the sep-values of nose and hollow cycles impose the lower and upper bounds described by (4), respectively. The reason to consider only nose and hollow cycles is that they have the largest sep-values when c and ℓ are large, as it follows from (5) and the next lemma.

Lemma 3 *For any walk $\mathcal{W}$ of $\mathcal{S}$ there exists either a nose or hollow walk $\mathcal{W}'$ of $\mathcal{S}$ starting and ending at the same vertices as $\mathcal{W}$ such that* $\text{len}(\mathcal{W}) \leq \text{len}(\mathcal{W}')$ *and* $\text{ext}(\mathcal{W}) \leq \text{ext}(\mathcal{W}')$.

Proof: The proof is by induction on $\nu(\mathcal{W}) \cdot \eta(\mathcal{W})$ and $\sigma(\mathcal{W}) + 1$, the base case of which is trivial. Suppose, then, that $\mathcal{W}$ has at least one nose and one hollow. So, $\mathcal{W}$ must have a subwalk $\mathcal{W}_{1i} = B_1, \ldots, B_i$ such that $B_1 \to B_2$ is a nose, $B_2, \ldots, B_{i-1}$ is a step walk, and $B_{i-1} \to B_i$ is a hollow. Observe that $i \geq 3$, because $t(B_1)s(B_2)$ are consecutive and thus $B_2 \to B_1$ is not a hollow.

Consider first the case in which $\mathcal{W}_{1i}$ is not a path, thus it contains a cycle $\mathcal{W}_{jk} = B_j, \ldots, B_k = B_j$ ($j < k$) that must have at least one 1-step or 0-hollow. Note that $\mathcal{W}_{jk} \neq \mathcal{W}_{1i}$ because otherwise $B_2, \ldots, B_{i-1}$ would pass through $B_1 = B_i$ contradicting the fact that $\mathcal{W}_{jk}$ is a cycle. Hence, $\mathcal{W}_{jk}$ does not contain both a nose and a hollow and $\text{len}(\mathcal{W}_{jk}) \leq 0$; recall (6). Moreover, if $\mathcal{W}_{jk}$ has an external hollow (which must be $B_{i-1} \to B_i$), then it must contain the unique external step of $\mathcal{S}$. Therefore, $\text{ext}(\mathcal{W}_{ji}) \leq 0$ by (7), and the proof follows by induction on $\mathcal{W} \setminus \mathcal{W}_{ji}$.

Consider now the case in which $\mathcal{W}_{1i}$ is a path and let $\mathcal{W}'_{1i}$ be the step path from B_1 to B_i. We claim that $\text{len}(\mathcal{W}_{1i}) = \text{len}(\mathcal{W}'_{1i})$ and $\text{ext}(\mathcal{W}_{1i}) = \text{ext}(\mathcal{W}'_{1i})$, in which case the proof follows by induction on $(\mathcal{W} \setminus \mathcal{W}_{1i}) \cup \mathcal{W}'_{1i}$. Since $\mathcal{W}_{1i}$ is a path, it follows that either $B_2 > B_i$ or $B_i > B_1$, which leaves us with only five possible combinations for the heights of B_1, B_2, B_{i-1}, and B_i, all of which are analyzed in the table below. The claim is therefore true.

h				len		ext	
B_1	B_2	B_{i-1}	B_i	$\mathcal{W}_{1i}$	$\mathcal{W}'_{1i}$	$\mathcal{W}_{1i}$	$\mathcal{W}'_{1i}$
x	y	y	x	0	0	0	0
h	0	0 or 1	0	$-r$	$-r$	-1	-1
$h-1$	0	0 or 1	0	$-1-r$	$-1-r$	-1	-1
$h-1$	h or 0	0	h	-1	-1	0	0
$h-1$	h	0	$h-1$	0	0	0	0

□

The above lemma brings us closer to Tucker's characterization, as it shows that any cycle with sep > 0 can be transformed into a hollow or nose cycle; thus, the existence of a UCA model equivalent to $\mathcal{M}$ is reduced to how its ratios look. One of the salient features of our proof is that it builds an efficient UCA model $\mathcal{U}$ equivalent to $\mathcal{M}$. The idea is to take $\mathcal{U} = \mathcal{U}(\mathcal{M}, c, \ell)$ as in Theorem 1 for some appropriate values of c and ℓ. Observe that $d_\ell = d_r = 0$ in this case, hence we can replace $\mathcal{B}$ and A_0 with $\mathcal{S}$ and A_1 in the definition of $\mathcal{U}$.

If we rely on Bellman-Ford, then we will inevitably pay $O(n^2)$ time to compute $\mathbf{d}\text{sep}(A_1, A_i)$. As discussed in Section 3.1, we can speed up the computation of $\mathbf{d}\text{sep}$ if we removing some, but perhaps not all, of the implied edges. Say that $A_i \to A_j$ is *implied* when $\mathbf{d}\text{sep}(A_1, A_i) + \text{sep}(A_i \to A_j) < \mathbf{d}\text{sep}(A_1, A_j)$, i.e., $A_i \to A_j$ is implied when it can be safely removed from $\mathcal{S}$ before solving the longest path problem. As discussed in Section 3.1, some edges could be implied for whichever values of c and ℓ are taken, while others edges are implied only for certain values of c and ℓ. Since this time we get to chose the appropriate values for c and ℓ, we can build a particular subgraph $\mathcal{R}$ of $\mathcal{S}$ (i.e., a (c, ℓ)-synthesis in the terms of Section 3.1). The idea, then, is to remove as many implied edges as needed to make $\mathcal{R}$ acyclic. If we succeed, then we will only pay $O(n)$ time for computing $\mathbf{d}$sep. With this in mind, we say that an edge $A_i \to A_j$ of $\mathcal{S}$ is *redundant* when either

(red$_1$) $\mathbf{d}\mathrm{len}(A_1, A_j) > \mathbf{d}\mathrm{len}(A_1, A_i) + \mathrm{len}(A_i \to A_j)$, or

(red$_2$) $\mathbf{d}\mathrm{len}(A_1, A_j) = \mathbf{d}\mathrm{len}(A_1, A_i) + \mathrm{len}(A_i \to A_j)$ and $(\mathbf{d}\mathrm{ext} \circ \mathbf{d}\mathrm{len})(A_1, A_j) > (\mathbf{d}\mathrm{ext} \circ \mathbf{d}\mathrm{len})(A_1, A_i) + \mathrm{len}(A_i \to A_j)$.

Roughly speaking, $A_i \to A_j$ is redundant only if it is implied for large values of ℓ and not-so-large values of e; the converse need not be true. (Recall that $(\mathbf{d}\mathrm{ext} \circ \mathbf{d}\mathrm{len})$ is the ext-distance restricted only to those paths with maximum len-distance.) The reduction of $\mathcal{S}(\mathcal{M})$ is the digraph $\mathcal{R}(\mathcal{M})$ obtained after removing all the redundant edges of $\mathcal{S}(\mathcal{M})$; as usual, we omit the parameter $\mathcal{M}$. Theorem 2 includes Tucker's characterization as equivalence (i) $\Leftrightarrow$ (ii).

Theorem 2 *Let $\mathcal{M}$ be a PCA model with arcs $A_1 < \ldots < A_n$, and $r_1, r_2 \in \mathbb{N}$ be such that $r(\mathcal{M}) = r_1/r_2$. Then, the following statements are equivalent:*

(i) *$\mathcal{M}$ is equivalent to a UCA model.*

(ii) $r < R$.

(iii) $\mathrm{len}(\mathcal{W}_H) < 0$ *for every hollow cycle* $\mathcal{W}_H$ *of* $\mathcal{S}$.

(iv) *either* $\mathrm{len}(\mathcal{W}) < 0$ *or* $\mathrm{len}(\mathcal{W}) = 0$ *and* $\mathrm{ext}(\mathcal{W}) < 0$, *for each cycle* $\mathcal{W}$ *of* $\mathcal{S}$.

(v) *$\mathcal{R}$ is acyclic.*

(vi) $\mathbf{d}^*\mathrm{sep}_{(c,\ell)}(\mathcal{S}, A_1, A_i) = \mathbf{d}\mathrm{sep}_{(c,\ell)}(\mathcal{R}, A_1, A_i)$ *for every* $1 \leq i \leq n$, *where* $c = (\ell+1)(h+r)+e$, $(\ell+1) = r_2 e^2$, *and* $e = 4n$.

(vii) *$\mathcal{U}(\mathcal{M}, c, \ell)$ is an integer (c,ℓ)-CA model equivalent to $\mathcal{M}$ for c and ℓ as in (vi).*

Proof: (i) $\Rightarrow$ (ii). This is direct consequence of (4).

(ii) $\Rightarrow$ (iii). If $\mathrm{len}(\mathcal{W}_H) \geq 0$, then

$$0 \leq -\sigma_1 - \eta_0 - \eta_h + r(\eta_h + \eta_{h-1} - \sigma_{-h})$$

implying (recall (3) observing that $\eta_h + \eta_{h-1} \geq \sigma_{-h}$)

$$r \geq \frac{\eta_h + \eta_0 + \sigma_1}{\eta_h + \eta_{h-1} - \sigma_{-h}} = R(\mathcal{W}_H) \geq R.$$

(iii) $\Rightarrow$ (iv). Suppose either $\mathrm{len}(\mathcal{W}) > 0$ or $\mathrm{len}(\mathcal{W}) = 0$ and $\mathrm{ext}(\mathcal{W}) \geq 0$ for some cycle $\mathcal{W}$ of $\mathcal{S}$. By Lemma 3, $\mathcal{S}$ has a nose or hollow circuit $\mathcal{W}'$ with $f(\mathcal{W}') \geq f(\mathcal{W})$ for $f \in \{\mathrm{len}, \mathrm{ext}\}$. Then, since $f(\mathcal{W}') = \sum_{i=1}^{k} f(\mathcal{W}_i)$ for the family of cycles $\{\mathcal{W}_1, \ldots, \mathcal{W}_k\}$ that partitions the edges of $\mathcal{W}'$, we obtain that either $\mathrm{len}(\mathcal{W}_i) > 0$ or $\mathrm{len}(\mathcal{W}_i) = 0$ and $\mathrm{ext}(\mathcal{W}_i) \geq 0$ for some $1 \leq i \leq k$.

By construction, $\mathcal{W}_i$ is either a nose or a hollow cycle. In the latter case the statement is true, while the former case is impossible as it implies

$$0 < \nu_{-h} - \sigma_1 - r(\nu_{-h} + \nu_{1-h} + \sigma_r)$$

from which we obtain that (recall (2))

$$r < \frac{\nu_{-h} - \sigma_1}{\nu_{-h} + \nu_{1-h} + \sigma_{-h}} = r(\mathcal{W}) \leq r.$$

(iv) $\Rightarrow$ (v). Suppose $\mathcal{R}$ has some cycle $\mathcal{W} = B_1, \ldots, B_k$ with $B_1 = B_k$. By (red$_1$),

$$\mathbf{d}\text{len}(A_1, B_{i+1}) \leq \mathbf{d}\text{len}(A_1, B_i) + \text{len}(B_i \to B_{i+1}). \tag{a}$$

Then, by induction,

$$\mathbf{d}\text{len}(A_1, B_1) = \mathbf{d}\text{len}(A_1, B_k) \leq \mathbf{d}\text{len}(A_1, B_1) + \text{len}(\mathcal{W}),$$

which implies that $\text{len}(\mathcal{W}) \geq 0$. Moreover, $\text{len}(\mathcal{W}) = 0$ only if (a) holds by equality for every $1 \leq i \leq k$, thus

$$(\mathbf{d}\text{ext} \circ \mathbf{d}\text{len})(A_1, B_{i+1}) \leq (\mathbf{d}\text{ext} \circ \mathbf{d}\text{len})(A_1, B_i) + \text{ext}(B_i \to B_{i+1})$$

by (red$_2$), implying $\text{ext}(\mathcal{W}) \geq 0$ by induction.

(v) $\Rightarrow$ (vi). Taking into account that $\mathcal{R}$ is acyclic and every walk of $\mathcal{R}$ is also a walk of $\mathcal{S}$, it follows that $\mathbf{d}^*\text{sep}(\mathcal{S}, A_1, A_i) \geq \mathbf{d}\text{sep}(\mathcal{R}, A_1, A_i)$ for every $1 \leq i \leq n$.

For the remaining inequality note that, by induction, it suffices to prove that $\text{sep}(\mathcal{W}) \leq \mathbf{d}\text{sep}(\mathcal{R}, A_1, A_i)$ for every walk $\mathcal{W}$ whose length is at most n. We prove this fact by induction on the length k of $\mathcal{W}$. The base case $k = 0$ is trivial. For the inductive step, consider any walk $\mathcal{W} = B_1, \ldots, B_{k+1}$ of length $k \leq n$ that goes from $B_1 = A_1$ to $B_{k+1} = A_i$, and let, for $1 \leq j \leq k+1$,

- $\mathcal{W}_j^{\mathcal{R}}$ be a walk of $\mathcal{R}$ from B_1 to B_j with $\text{sep}(\mathcal{W}_j^{\mathcal{R}}) = \mathbf{d}\text{sep}(\mathcal{R}, B_1, B_j)$, and
- $\mathcal{W}_{k+1}^{\mathcal{S}}$ be the walk obtained by traversing $B_k \to B_{k+1}$ after $\mathcal{W}_k^{\mathcal{R}}$.

By inductive hypothesis, $\text{sep}(\mathcal{W}) \leq \text{sep}(\mathcal{W}_k^{\mathcal{R}}) + \text{sep}(B_k \to B_{k+1}) = \text{sep}(\mathcal{W}_{k+1}^{\mathcal{S}})$, thus $\text{sep}(\mathcal{W}) \leq \text{sep}(\mathcal{W}_{k+1}^{\mathcal{R}})$ when $B_k \to B_{k+1}$ is an edge of $\mathcal{R}$. Suppose, then, that $B_k \to B_{k+1}$ is redundant in $\mathcal{S}$, and consider the two possibilities according to (red$_1$) and (red$_2$).

Case 1: (red$_1$) is true. Note that since no edge of $\mathcal{W}_{k+1}^{\mathcal{R}}$ is redundant, it follows by induction that $\text{len}(\mathcal{W}_j^{\mathcal{R}}) = \mathbf{d}\text{len}(A_1, B_j)$ for every $1 \leq j \leq k+1$. Hence,

$$\begin{aligned}\text{len}(\mathcal{W}_{k+1}^{\mathcal{R}}) &= \mathbf{d}\text{len}(A_1, B_{k+1}) \\ &> \mathbf{d}\text{len}(A_1, B_k) + \text{len}(B_k \to B_{k+1}) = \text{len}(\mathcal{W}_{k+1}^{\mathcal{S}}).\end{aligned}$$

Now, taking into account that every term of the length factor is a multiple of either 1 or $r = r_1/r_2$ in (6), we obtain that

$$(\ell+1)(\mathrm{len}(\mathcal{W}^{\mathcal{R}}_{k+1}) - \mathrm{len}(\mathcal{W}^{\mathcal{S}}_{k+1})) = r_2 e^2(\mathrm{len}(\mathcal{W}^{\mathcal{R}}_{k+1}) - \mathrm{len}(\mathcal{W}^{\mathcal{S}}_{k+1})) \geq e^2.$$

By (5), we obtain that

$$\begin{aligned}\mathrm{sep}(\mathcal{W}^{\mathcal{R}}_{k+1}) &\geq (\ell+1)(h(B_{k+1}) + \mathrm{len}(\mathcal{W}^{\mathcal{R}}_{k+1})) - en\\ \mathrm{sep}(\mathcal{W}^{\mathcal{S}}_{k+1}) &\leq (\ell+1)(h(B_{k+1}) + \mathrm{len}(\mathcal{W}^{\mathcal{S}}_{k+1})) + ek + 2k,\end{aligned}$$

thus,

$$\mathrm{sep}(\mathcal{W}^{\mathcal{R}}_{k+1}) - \mathrm{sep}(\mathcal{W}^{\mathcal{S}}_{k+1}) \geq e^2 - e(k+n) - 2k \geq 8n - 2k \geq 2.$$

Case 2: (red_1) is false, thus (red_2) holds. As before, we observe by induction that $\mathrm{len}(\mathcal{W}^{\mathcal{R}}_j) = \mathbf{d}\mathrm{len}(A_1, B_j)$ and, thus, $\mathrm{ext}(\mathcal{W}^{\mathcal{R}}_j) = (\mathbf{d}\mathrm{ext} \circ \mathbf{d}\mathrm{len})(A_1, B_j)$. Consequently, by ($\mathrm{red}_2$),

$$\mathrm{ext}(\mathcal{W}^{\mathcal{R}}_{k+1}) > (\mathbf{d}\mathrm{ext} \circ \mathbf{d}\mathrm{len})(A_1, B_k) + \mathrm{ext}(B_k \to B_{k+1}) = \mathrm{ext}(\mathcal{W}^{\mathcal{S}}_{k+1}).$$

Since (red_1) is true, it follows that $\mathrm{len}(\mathcal{W}^{\mathcal{S}}_{k+1}) = \mathbf{d}\mathrm{len}(A_1, B_{k+1})$. Then, by (5),

$$\begin{aligned}\mathrm{sep}(\mathcal{W}^{\mathcal{R}}_{k+1}) - \mathrm{sep}(\mathcal{W}^{\mathcal{S}}_{k+1}) &\geq e(\mathrm{ext}(\mathcal{W}^{\mathcal{R}}_{k+1}) - \mathrm{ext}(\mathcal{W}^{\mathcal{S}}_{k+1})) - \mathrm{const}(\mathcal{W}^{\mathcal{S}}_{k+1})\\ &\geq 4n - 2k \geq 2.\end{aligned}$$

Whichever the case, $\mathrm{sep}(\mathcal{W}) \leq \mathrm{sep}(\mathcal{W}^{\mathcal{R}}_{k+1}) = \mathbf{d}\mathrm{sep}(\mathcal{R}, A_1, A_{k+1})$ as desired.

(vi) $\Rightarrow$ (vii) Since $\mathbf{d}^*\mathrm{sep}(\mathcal{S}, A_1, A_i) = \mathbf{d}\mathrm{sep}(\mathcal{R}, A_1, A_i)$ for every $1 \leq i \leq n$ and $\mathcal{S}$ is strongly connected, it follows that $\mathbf{d}^*\mathrm{sep}(\mathcal{S}, A_i, A_j) = \mathbf{d}\mathrm{sep}(\mathcal{S}, A_i, A_j)$ for every $1 \leq i, j \leq n$. Hence $\mathrm{sep}(\mathcal{W}) \leq 0$ for every cycle $\mathcal{W}$ of $\mathcal{S}$ and the implication follows by Theorem 1 (note that c and ℓ are integer values).

(vii) $\Rightarrow$ (i). Trivial. □

Theorem 2 has some nice algorithmic consequences on Rep when combined with Theorem 1. For any input PCA model $\mathcal{M}$ we solve u-Rep for the UCA descriptor u implied by statement (vi). As a byproduct, we either obtain a UCA model $\mathcal{U}$ equivalent to $\mathcal{M}$ or a cycle of $\mathcal{S}$ that can be used for negative certification. The algorithm costs $O(n^2)$ time, plus the time and space required so as to compute $r(\mathcal{M})$. It turns out that r can be computed in linear time with the algorithm by Kaplan and Nussbaum [25], or directly from the synthetic graph (see Section II.4.2), while $\mathcal{U}$ can be obtained in $O(n)$ time by taking advantage of the reduction of $\mathcal{S}$ as discussed above (see Section II.4.3). Furthermore, a slight variation of this algorithm can be used to solve the problem in logspace as well (see Section II.4.4)

Theorem II.2 *There is a unified certifying algorithm that solves* Rep *in $O(n)$ time.*

Theorem II.5 *There is an algorithm that solves* Rep *in logspace.*

5 (a, b)-independents and (x, y)-circuits

The unified certifying algorithm that we describe in Section II.4.3 outputs two cycles when $\mathcal{M}$ is not equivalent to a UCA model: a nose cycle $\mathcal{W}_N$ of $\mathcal{S}$ with ratio $r(\mathcal{M})$ and a cycle $\mathcal{W}_H$ of $\mathcal{R}$. As in the proofs of implications (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v), $\mathcal{W}_H$ is a hollow cycle with a nonnegative length factor. Moreover, as in implication (ii) $\Leftrightarrow$ (iii), $R(\mathcal{W}_H) \leq r(\mathcal{W}_N)$.

Viewing each vertex of $\mathcal{S}$ as the arc of $\mathcal{M}$ it represents, the preceding argument is enough to conclude that $\mathcal{W}_N \cup \mathcal{W}_H$ induces a forbidden submodel of $\mathcal{M}$. By Tucker's characterization, this means that $\mathcal{W}_N \cup \mathcal{W}_H$ contains an (a, b)-independent $\mathcal{A}_I$ and an (x, y)-circuit $\mathcal{A}_C$ with $a/b \geq x/y$. We stated before that $\mathcal{W}_N$ and $\mathcal{W}_H$ are equivalent to $\mathcal{A}_I$ and $\mathcal{A}_C$, respectively. We remark that this equivalence does not imply equality; $\mathcal{W}_N \cup \mathcal{W}_H$ could contain more arcs than $\mathcal{A}_I \cup \mathcal{A}_C$. These added arcs are, nevertheless, redundant and can be eliminated from $\mathcal{W}_N \cup \mathcal{W}_H$ to obtain a minimal forbidden induced submodel as the negative witness. The purpose of this section is to describe the equivalence between nose (resp. hollow) cycles and (a, b)-independents (resp. (x, y)-circuits) and how to transform one into the other and vice versa. We begin describing what are (a, b)-independents and (x, y)-circuits.

For two arcs A_i, A_j of a PCA model $\mathcal{M}$, we define the *ss arc of* A_i, A_j to be the arc $(s(A_i), s(A_j))$. For a sequence of arcs $\mathcal{A} = B_1, \ldots, B_k$, the *ss traversal of* $\mathcal{A}$ is the family of arcs $\mathcal{T}$ that contains the *ss* arc of B_i, B_{i+1} for every $1 \leq i \leq k$ (where $B_1 = B_{k+1}$). The number of *turns* of $\mathcal{T}$ is the number of its arcs that contain the point 0 of $C(\mathcal{M})$. In simple terms, the *ss* traversal of $\mathcal{A}$ is obtained by traversing $C(\mathcal{M})$ from $s(B_1)$ to $s(B_2)$ to ... to $s(B_k)$ to $s(B_1)$, while its number of turns is the number of complete loops to the circle in such a traversal.

An (a, b)-independent of a PCA model $\mathcal{M}$ is a sequence of arcs $\mathcal{A} = B_1$, $\ldots$, B_a such that $s(B_{i+1}) \notin B_i$ for every $1 \leq i \leq a$ and whose *ss* traversal takes b turns. Similarly, an (x, y)-circuit is a sequence of arcs $B_1, \ldots, B_x$ such that $s(B_{i+1}) \in B_i$ for every $1 \leq i \leq x$ and whose *ss* traversal takes y turns. Note that $x > 2y$ as no pair of arcs of $\mathcal{M}$ cover the circle, while $a/b < c < x/y$ when $\mathcal{M}$ is a (c, ℓ)-CA model. An (a, b)-independent is *maximal* when a/b is maximum and a, b are relative primes, while an (x, y)-circuit is *minimal* when x/y is minimum and x, y are relative primes. As we shall shortly see, statement (i) $\Leftrightarrow$ (ii) of Theorem 2 is equivalent to the following theorem by Tucker.

Theorem 3 ([46]) *A PCA model $\mathcal{M}$ is equivalent to an UCA model if and only if $a/b < x/y$ for every maximal (a, b)-independent and every minimal (x, y)-circuit.*

Say that an (a, b)-independent $\mathcal{A} = B_1, \ldots, B_a$ is *standard* when $s(B_i)$ is immediately preceded by an ending point in $\mathcal{M}$, for every $1 \leq i \leq a$. Note that if $s(B_i)$ is preceded by the beginning point of an arc A, then B_1, $\ldots$, B_{i-1}, A, B_{i+1}, $\ldots$, B_a is also an (a, b)-independent of $\mathcal{M}$. Consequently, $\mathcal{M}$ has an (a, b)-independent if and only if it has a standard (a, b)-independent.

There is a one-to-one correspondence between the standard (a,b)-independents of $\mathcal{M}$ and the nose circuits of $\mathcal{S}$, as follows. Let $\mathcal{A} = B_1, \ldots, B_a$ be a standard (a,b)-independent and $\mathcal{W}_i$ be the step path of $\mathcal{S}$ that goes from B_i to B'_i, where B'_i is the arc whose ending point immediately precedes $s(B_{i+1})$. Clearly, $B'_i \to B_{i+1}$ is a nose of $\mathcal{S}$, thus $\mathcal{W}(\mathcal{A}) = \mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_a$ is a nose circuit of $\mathcal{S}$. Conversely, if $\mathcal{W}$ is a nose circuit, and $B'_1 \to B_1, \ldots, B'_a \to B_a$ are its noses, then $\mathcal{A}(\mathcal{W}) = B_1, \ldots, B_a$ is a standard (a,b)-independent for some b. It is not hard to see that $\mathcal{A}(\mathcal{W}(\mathcal{A})) = \mathcal{A}$ and $\mathcal{W}(\mathcal{A}(\mathcal{W})) = \mathcal{W}$, thus the correspondence is one-to-one.

Observe that the number of turns b in the ss traversal of $\mathcal{A}$ is precisely the number of external noses and steps of $\mathcal{W} = \mathcal{W}(\mathcal{A})$. In other words,

$$b = \nu_{-h} + \nu_{1-h} + \sigma_{-h}.$$

Similarly, the number a of arcs of $\mathcal{A}$ equals the number of noses of $\mathcal{W}$; by (1),

$$a = h(\nu_{-h} + \nu_{1-h} + \sigma_{-h}) + \nu_{-h} - \sigma_1 = hb + \nu_{-h} - \sigma_1.$$

Hence, $a/b = h + r(\mathcal{W})$.

A similar analysis holds for (x,y)-circuits. Say that an (x,y)-circuit $\mathcal{A} = B_1, \ldots, A_x$ is *standard* when $s(B_i)$ immediately precedes an ending point $t(B'_i)$ in $\mathcal{M}$. Note that $\mathcal{M}$ contains an (x,y)-circuit if and only if it contains an standard (x,y)-circuit; in such circuit, $B_i \to B'_i$ is a hollow of $\mathcal{S}$. Then, $\mathcal{A}$ is in a one-to-one correspondence with the hollow circuit $\mathcal{W} = \mathcal{W}(\mathcal{A})$ that goes through $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_x$ where each $\mathcal{W}_i$ is the step path going from B'_i to B_{i+1}. As before, the number of turns in the ss traversal of $\mathcal{A}$ is the number of external hollows minus the number of external steps, i.e.,

$$y = \eta_h + \eta_{h-1} - \sigma_{-h},$$

while the number x of arcs in $\mathcal{A}$ is its number of hollows; by (1)

$$x = h(\eta_h + \eta_{h-1} - \sigma_{-h}) + \eta_h + \eta_0 + \sigma_1.$$

Hence, $x/y = h + R(\mathcal{W})$.

Clearly, we can obtain $\mathcal{W}(\mathcal{A})$ in $O(n)$ time for any standard (a,b)-independent (resp. (x,y)-circuit) $\mathcal{A}$, and vice versa. Moreover, note that, as stated, $a/b \geq x/y$ if and only if $r \geq R$. We summarize this section in the next theorem.

Theorem 4 *A PCA model $\mathcal{M}$ has an (a,b)-independent (resp. (x,y)-circuit) $\mathcal{A}$ if and only if $\mathcal{S}$ contains a nose (resp. hollow) circuit $\mathcal{W}$ with ratio $a/b - h$ (resp. $x/y - h$). Moreover, such a circuit $\mathcal{W}$ of $\mathcal{S}$ can be obtained in $O(n)$ time when $\mathcal{A}$ is given as input. Conversely, $\mathcal{A}$ can be obtained in $O(n)$ time when $\mathcal{W}$ is given as input.*

To end this section we recall that the witness provided by our unified certifying algorithm is composed by a nose cycle $\mathcal{W}_N$ with ratio $r(\mathcal{M})$ and a cycle $\mathcal{W}_H$ of $\mathcal{R}$. By Theorem 4, $\mathcal{A}(\mathcal{W}_N)$ is a maximal (a,b)-independent. Unfortunately,

$\mathcal{A}(\mathcal{W}_H)$ need not be a minimal (x, y)-circuit, because we have no guarantees that $R(\mathcal{W}_H) = R(\mathcal{M})$. This is not a problem, though, as in Section II.4.2 we show how to obtain a hollow cycle with minimum ratio. In fact, the algorithm in Section II.4.2 is just a translation of the algorithm by Kaplan and Nussbaum [25].

6 Minimal UCA and UIG models

Theorem 1 gives us a procedure to check if $\mathcal{M}$ is equivalent to a u-CA model, when u is given as input. However, not much is known about the sets of feasible values c and ℓ. In this aspect, unit circular-arc models are much less studied than unit interval models. In this section we prove that every UCA model admits an equivalent *minimal* UCA model. Minimal UCA models are a generalization of minimal UIG models, as defined by Pirlot [38]. An (ℓ, d, d_s)-IG model with arcs $A_1 < \ldots < A_n$ is (∞, d, d_s)-*minimal* when

(min-uig$_1$) $\ell \leq \ell'$, and

(min-uig$_2$) $s(A_i) \leq s(A'_i)$ for every $1 \leq i \leq n$,

for every equivalent (ℓ', d, d_s)-IG model.

Condition (min-uig$_2$) as expressed above does not make much sense for general UCA models, as there is not a natural left-to-right order of the arcs; when $c < \infty$, the 0 point of the circle is just a denotational tool. However, we can translate condition $s(A_n) < s(A'_n)$ by asking the circumference of the circle to be minimized. With this in mind, we say that a (c, ℓ, d, d_s)-CA model $\mathcal{M}$ is (d, d_s)-*minimal* when

(min-uca$_1$) $\ell \leq \ell'$, and

(min-uca$_2$) $c \leq c'$,

for every equivalent (c', ℓ', d, d_s)-CA model. The fact that every UCA model is equivalent to a minimal UCA model follows from the next lemma.

Lemma 4 *If $\mathcal{M}$ is equivalent to a $(c, \ell + y, d, d_s)$- and a $(c + x, \ell, d, d_s)$-CA models for $x, y \geq 0$, then $\mathcal{M}$ is also equivalent to a $(c+a, \ell+b, d, d_s)$-CA model, for every $0 \leq a \leq x$ and $0 \leq b \leq y$.*

Proof: For the sake of notation, write $\langle a', b'\rangle$ to denote the UCA descriptor $(c + a', \ell + b', d, d_s)$, for every $0 \leq a' \leq x$ and $0 \leq b' \leq y$. Suppose, to obtain a contradiction, that $\mathcal{M}$ is equivalent to no $\langle a, b\rangle$-CA model for some $0 \leq a \leq x$ and $0 \leq b \leq y$. Then, by Theorem 1, $\text{sep}_{\langle a,b\rangle}(\mathcal{W}) > 0$, $\text{sep}_{\langle x,0\rangle}(\mathcal{W}) \leq 0$, and $\text{sep}_{\langle 0,y\rangle}(\mathcal{W}) \leq 0$ for some cycle $\mathcal{W}$ of $\mathcal{S}$. By (4), there exists $e \geq 0$ such that

$$\begin{aligned} c + a' &= (\ell + y + d)(h + r) + e + a' \\ &= (\ell + b' + d)(h + r) + (y - b')(h + r) + e + a' \end{aligned} \tag{i}$$

for every $0 \leq a' \leq x$ and $0 \leq b' \leq y$. Thus, by (5),

$$\text{sep}_{\langle a,b\rangle} = (\ell + b + d)\,\text{len} + ((y-b)(h+r) + e + a)\,\text{ext} + \text{const} > 0 \quad \text{(ii)}$$
$$\text{sep}_{\langle 0,y\rangle} = (\ell + y + d)\,\text{len} + e\,\text{ext} + \text{const} \leq 0 \quad \text{(iii)}$$
$$\text{sep}_{\langle x,0\rangle} = (\ell + d)\,\text{len} + (y(h+r) + e + x)\,\text{ext} + \text{const} \leq 0. \quad \text{(iv)}$$

Recall that len ≤ 0 by Theorem 2. Then, as $0 <$ (ii) $-$ (iv), we obtain that (v) ext ≤ -1 (recall ext $\in \mathbb{Z}$), while (vi) $(h+r)\,\text{ext} > \text{len}$ follows by $0 <$ (ii) $-$ (iii) and (v). Then,

$$\begin{aligned} 0 <& (\ell + b + d)\,\text{len} + ((y-b)(h+r) + e + a)\,\text{ext} + \text{const} && \text{(by (ii))} \\ <& (\ell + b + d)(h+r)\,\text{ext} + ((y-b)(h+r) + e + a)\,\text{ext} + \text{const} && \text{(by (vi))} \\ =& (c+a)\,\text{ext} + \text{const} && \text{(by (i))} \\ \leq& -c - a + \text{const} && \text{(by (v))} \end{aligned}$$

This is impossible, because $c \geq \max\{2d, d+d_s\}n$ as all the extremes of the $\langle 0, y\rangle$-CA model equivalent to $\mathcal{M}$ are separated by d and each of its n beginning points is separated from the next by $d + d_s$, while $a \geq 0$ and const $\leq \max\{2d, d+d_s\}n$ by definition (8). □

Theorem 5 *Every UCA graph admits a (d, d_s)-minimal UCA model for every $d, d_s \in \mathbb{Q}$.*

Proof: Let (c, ℓ) be the minimum pair (in a lexicographic ordering) such that a UCA model $\mathcal{M}$ is equivalent to a (c, ℓ, d, d_s)-CA model. By construction and Lemma 4, $c \leq c'$ and $\ell \leq \ell'$ for every (c', ℓ', d, d_s)-CA model equivalent to $\mathcal{M}$. □

For the rest of this section, we restrict ourselves to the case in which d and d_s are integers. By (2), $r = 0$ for every UIG model $\mathcal{M}$, thus, by (5), $\text{sep}(\mathcal{W}) \in \mathbb{N}$ if and only if ℓ and c are integers. We obtain, therefore, the following corollary that was first proved by Pirlot [38].

Corollary 1 *Every (∞, d, d_s)-minimal UIG model is integer for all $d, d_s \in \mathbb{N}$.*

We were not able to prove or disprove the above corollary for the general case. For this reason, we say that an integer (c, ℓ, d, d_s)-CA model $\mathcal{M}$ is $(\mathbb{N}, d, d_s)$-*minimal* when if satisfies (min-uca$_1$) and (min-uca$_2$) for every integer (c', ℓ', d, d_s)-CA model.

A natural algorithmic problem is IntMinUCA in which we ought to find a $(\mathbb{N}, d, d_s)$-minimal UCA model equivalent to an input (c, ℓ, d, d_s)-CA model $\mathcal{M}$. A simple solution is to apply Theorem 1 for every $1 \leq \ell^* \leq \ell$ and every $1 \leq c^* \leq c$ with a total cost of $O(\ell^* c^* n^2)$ time. We can easily improve this algorithm by replacing the linear search of c^* with a binary search.

Corollary 2 *Let $\mathcal{M}$ be a PCA model. If $\text{sep}_{(c,\ell,d,d_s)}(\mathcal{W}) > 0$ for some cycle $\mathcal{W}$ of $\mathcal{S}$, then $\mathcal{M}$ is not equivalent to a $(c + x\,\text{sg}, \ell - y, d, d_s)$-CA model, for every $x, y \geq 0$, where sg is the sign of $\text{ext}(\mathcal{W})$.*

Proof: Let e be such that $c = (\ell + d)(h + r) + e$; note that e need not be positive. By (5),

$$\text{sep}_{(c+x\,\text{sg},\ell,d,d_s)} = (\ell+d)\,\text{len} + (e + x\,\text{sg})\,\text{ext} + \text{const} = x\,\text{sg}\,\text{ext} + \text{sep}_{(c,\ell,d,d_s)} > 0.$$

Then, by Theorem 1, $\mathcal{M}$ cannot be equivalent to a $(c + x\,\text{sg}, \ell, d, d_s)$-CA model. □

Corollary 2 provides us with a somehow efficient algorithm to binary search the minimum $c^* \in \mathbb{N}$ such that $\mathcal{M}$ is equivalent to a (c^*, ℓ, d, d_s)-CA model, when $\ell, d, d_s \in \mathbb{N}$ are given as input. By definition $d < \ell$, while we assume $d_s + d < \ell$ as otherwise $G(\mathcal{M})$ has no edges and the problem is trivial. The idea of the algorithm is simply to assume that such a value c^* exists and belongs to some range $[a, c]$; initially $a = 0$ and $c = n(\ell + 1)$. Then, we query if $\mathcal{M}$ is equivalent to some (b, ℓ, d, d_s)-CA model, where $b \in \mathbb{N}$ is the middle of $[a, c]$. If affirmative, then $c^* \in [a, b]$ by definition. Otherwise, we search some cycle $\mathcal{W}$ with $\text{sep}_{(b,\ell,d,d_s)} > 0$. By Corollary 2, $c^* \in [0, b)$ if $\text{ext}(\mathcal{W}) \geq 0$, while $c^* \in (b, c]$ otherwise. Regardless of whether c^* exists or not, this algorithm requires $O(\log(n\ell))$ queries.

Every time we need to query if $\mathcal{M}$ is equivalent to a (b, ℓ, d, d_s)-CA model, we solve u-Rep as in Section 3 (note that u is integer). Since $\ell^* = O((d+d_s)n^2)$ [34], we conclude the following.

Theorem 6 IntMinUCA *is can be solved in* $O((d+d_s)n^4 \log(n(d+d_s)))$ *time, for every* $d, d_s \in \mathbb{N}$.

As mentioned above, Pirlot [38] proved that every UIG model $\mathcal{M}$ is equivalent to a (∞, d, d_s)-minimal UIG model. However, it was Mitas [37] who showed that such a model can be found in linear time by transforming $\mathcal{M}$ into an equivalent UIG model. Thus, Mitas' algorithm solves the *minimal UIG representation* (MinUIG) problem in which $\mathcal{M}$ and $d, d_s \in \mathbb{Q}_{\geq 0}$ are given and a (∞, d, d_s)-minimal UIG model $\mathcal{M}^*$ equivalent to $\mathcal{M}$ must be generated. Unfortunately, her proof has a flaw that invalidates the minimality arguments (see Section II.5). Though $\mathcal{M}^*$ is equivalent to $\mathcal{M}$, it need not be (∞, d, d_s)-minimal. On the other hand, the algorithm for IntMinUCA above can be implemented to solve MinUIG in $O(n^2 \log n)$ time (the idea is to apply a binary search on ℓ). Luckily, we can patch Mitas' algorithm to solve MinUIG in $O(n^2)$ time; the algorithm is postponed to Section II.5.

Theorem II.7 MinUIG *can be solved in* $O(n^2)$ *time and linear space, for any* $d, d_s \in \mathbb{Q}$.

7 Further remarks

Synthetic graphs proved to be an important tool for studying what the UIG representations of PIG graphs look like. The generalization to PCA models is direct; the fact that some arcs wrap around the circle is not important for defining the synthetic graph. To represent the separation constraints that an equivalent UCA model must satisfy, all we had to include to Pirlot's original formulation was the variable c representing the circumference of the circle. Generalizations of simple ideas from PIG to PCA graphs are not always as easy to obtain. Unfortunately, Pirlot's ideas were introduced in the context of semiorders and were not exploited in the context of PCA graphs; the recognition problem of UCA graphs in polynomial time could have been solved more than a decade earlier. To close the chapter we provide some further remarks and open problems.

Our definition of UCA descriptors states that every pair of beginning points should be separated by $d+d_s$ distance. An obvious generalization to u-REP and (INT)BOUNDREP is to replace d_s with a function $d_s\colon \mathcal{A} \to \mathbb{Q}_{\geq 0}$ that indicates, for each arc A_i, the separation between $s(A_i)$ and the next beginning point $s(A_{i+1})$. The reader can check that Theorem 1 holds for this generalization as well. All we need to do is to replace the value d_s with $d_s(A_i)$ for each step $A_i \to A_{i+1}$. Moreover, we can use similar functions to further separate $t(A_i)$ from $s(A_j)$ for every nose $A_i \to A_j$, and $s(A_i)$ from $t(A_j)$ for any hollow $A_i \to A_j$. We did not consider these generalizations for the sake of simplicity and notation.

In Section 6 we gave a simple pseudo-polynomial algorithm to transform a UCA model $\mathcal{M}$ into a $(\mathbb{N}, d, d_s)$-minimal (c^*, ℓ^*)-CA model. The algorithm works by performing a linear search on ℓ^* and a binary search on c^*. An obvious idea to improve its running time is to replace the linear search on ℓ^* with a binary search. Unfortunately, this idea is not feasible at first sight because we cannot claim

$$L = \{\ell \in \mathbb{N} \mid \mathcal{M} \text{ is equivalent to a } (c, \ell, d, d_s)\text{-CA model for some } c \in \mathbb{N}\}$$

to be a range. For instance, C_{11}^4 admits a $(22, 9)$-CA model, but it admits no $(c, 10)$-CA model, whatever the value of c is. This is just one more example of a property that is lost when the linear structure of PIG models is replaced by the circular structure of PCA graphs as $L = [\ell^*, \infty)$ when $\mathcal{M}$ is PIG.

As calculated in Section 6, the running time of the minimization algorithm is $O((d + d_s)n^4 \log(n(d + d_s)))$. This bound is not tight, as the actual running time is $O(\ell^* n^2 \log(n\ell^*))$, and ℓ^* could be much lower than $(d + d_s)n^2$. As a matter of fact, we developed a simple program for testing if a UCA model is equivalent to some $(c, 2n)$-CA model. We tested it on many input UCA models and, in all cases, the program was successful.

Finally, it should be noted that a UCA graph may admit many *non-equivalent* (d, d_s)-minimal UCA models. Indeed, a UCA graph may admit an exponential number of non-equivalent models, each of which is equivalent to a (d, d_s)-minimal UCA model. It makes sense, then, to say that a model $\mathcal{M}$ is (d, d_s)-*minimum* when it satisfies (min-uca$_1$) and (min-uca$_2$) for every model $\mathcal{M}'$ such that $G(\mathcal{M})$ is isomorphic to $G(\mathcal{M}')$. As it was noted by Huang [22], every connected and co-connected PCA graph admits a unique PCA model, up to equivalence and full reversal. Thus, any (d, d_s)-minimal model of a connected and co-connected PCA graph is (d, d_s)-minimum. Similarly, every disconnected PCA graph is PIG, and all its models can be obtained from a model

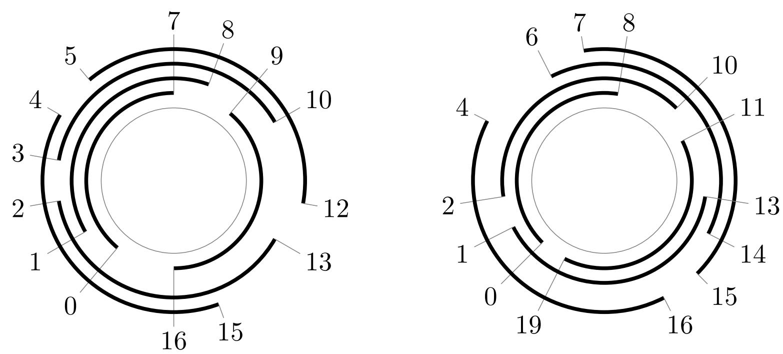


FIGURE 3. *Two* $(1,0)$*-minimal UCA models representing the same graph.*

$\mathcal{M}$ by exchanging the order in which their components appear from 0, and reversing some of the components. Thus, again, any (d, d_s)-minimal model of $G(\mathcal{M})$ is (d, d_s)-minimum (note that we are not taking (min-uig$_2$) into account, as (d, d_s)-minimal models are not a strict generalization of (∞, d, d_s)-minimal models for non-connected graphs). Co-disconnected PCA graphs share a similar property: all their PCA models can be obtained from a PCA model $\mathcal{M}$ by exchanging the order in which its co-components appear, plus reversing some co-components [22]. Thus, one is tempted to think that all the (d, d_s)-minimal PCA models are (d, d_s)-minimum, yet this is not the case. Figure 3 shows two $(1, 0)$-minimal $(18, 7)$-CA and $(20, 8)$-CA models that represent the graph whose co-components are P_3 and P_4. We leave as open the problem of computing the (d, d_s)-minimum UCA model.

Note. Soulignac and Terlisky[5] recently claimed to have proved that c and ℓ are integer when $\mathcal{M}$ is a (d, d_s)-minimal (c, ℓ)-CA model and $d, d_s \in \mathbb{N}$. They also show that MinUCA can be solved in $O(n^3)$ time and $O(n^2)$ space.

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