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Drawing Planar Graphs with Reduced Height

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Abstract

A polyline (resp., straight-line) drawing Γ of a planar graph G on a set L_k of k parallel lines is a planar drawing that maps each vertex of G to a distinct point on L_k and each edge of G to a polygonal chain (resp., straight line segment) between its corresponding endpoints, where the bends lie on L_k. The height of Γ is k, i.e., the number of lines used in the drawing. In this paper we establish new upper bounds on the height of polyline drawings of planar graphs using planar separators. Specifically, we show that every n-vertex planar graph with maximum degree Δ, having an edge separator of size λ, admits a polyline drawing with height $4n/9 + O(\lambda)$, where the previously best known bound was $2n/3$. Since $\lambda \in O(\sqrt{n\Delta})$, this implies the existence of a drawing of height at most $4n/9 + o(n)$ for any planar triangulation with $\Delta \in o(n)$. For n-vertex planar 3-trees, we compute straight-line drawings, with height $4n/9 + O(1)$, which improves the previously best known upper bound of $n/2$. All these results can be viewed as an initial step towards compact drawings of planar triangulations via choosing a suitable embedding of the graph.

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1 Introduction

A *polyline drawing* of a planar graph G is a planar drawing of G such that each vertex of G is mapped to a distinct point in the Euclidean plane, and each edge is mapped to a polygonal chain between its endpoints. Let $L_k = \{l_1, l_2, \ldots, l_k\}$ be a set of k horizontal lines such that for each $i \leq k$, line l_i passes through the point $(0, i)$. A polyline drawing of G is called a *polyline drawing on* L_k if the vertices and bends of the drawing lie on the lines of L_k. The *height* of such a drawing is k, i.e., the number of parallel horizontal lines used by the drawing. Such a drawing is also referred to as a k*-layer drawing* in the literature [21, 25]. Let Γ be a polyline drawing of G. We call Γ a t*-bend polyline drawing* if each of its edges has at most t bends. Thus a 0-bend polyline drawing is also known as a *straight-line drawing*. G is called a *planar triangulation* if every face of G is bounded by a cycle of three vertices. Figure 1(a) shows a planar graph G, and Figure 1(b) illustrates a 1-bend polyline drawing of G on L_8.

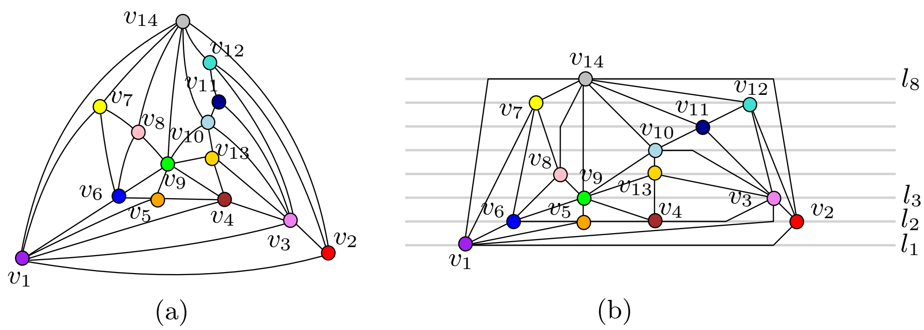

Figure 1: (a) A triangulation G. (b) A polyline drawing of G with height 8.

Drawing planar graphs on a small integer grid is an active research area in graph drawing [3, 8, 17, 24, 15], which is motivated by the need of compact layout of VLSI circuits and visualization of software architecture. In visualization applications, the constraint on area is imposed naturally by the size of the display screen. For VLSI circuit layout, compact drawings reduce the microchip area. Minimizing area often requires the edges to have bends. Since simultaneously optimizing the width and height of the drawing is very challenging, researchers have also focused their attention on optimizing one dimension of the drawing [6, 18, 21, 25], while the other dimension is unbounded.

In this paper we develop new techniques that can produce drawings with small height. We distinguish between the terms 'plane' and 'planar'. A *plane graph* is a planar graph with a fixed combinatorial embedding and a specified outer face. While drawing a planar graph, we allow the output to represent any planar embedding of the graph. On the other hand, while drawing a plane graph, the output is further constrained to respect the input embedding.

Related Work: State-of-the-art algorithms that compute straight-line drawings of n-vertex plane graphs on an $(O(n) \times 2n/3)$-size grid imply an upper bound of $2n/3$ on the height of straight-line drawings [5, 6]. This bound is tight for plane graphs, i.e., there exist n-vertex plane graphs such as plane nested triangles graphs and some plane 3-trees that require a height of $2n/3$ in any of their straight-line drawings [12, 22]. Recall that an n-vertex *nested triangles graph* is a plane graph formed by a sequence of $n/3$ vertex disjoint cycles, $\mathtt{C}_1, \mathtt{C}_2, \ldots, \mathtt{C}_{n/3}$, where for each $i \in \{2, \ldots, n/3\}$, cycle $\mathtt{C}_i$ contains the cycles $\mathtt{C}_1, \ldots, \mathtt{C}_{i-1}$ in its interior, and a set of edges that connect each vertex of $\mathtt{C}_i$ to a distinct vertex in $\mathtt{C}_{i-1}$. Besides, a *plane* 3*-tree* is a triangulated plane graph that can be constructed by starting with a triangle, and then repeatedly adding a vertex to some inner face of the current graph and triangulating that face.

The $2n/3$ upper bound on the height is also the currently best known bound for polyline drawings, even for planar graphs, i.e., when we are allowed to choose a suitable embedding for the output drawing. In the variable embedding setting, Frati and Patrignani [17] showed that every n-vertex nested triangles graph can be drawn with height at most $n/3 + O(1)$, which is significantly smaller than the lower bound of $2n/3$ in the fixed embedding setting. Zhou et al. [28] showed that series-parallel graphs can be drawn with $0.3941n^2$ area, and hence with height $0.628n < 2n/3$. Similarly, Hossain et al. [18] showed that an *universal set* of $n/2$ horizontal lines can support all n-vertex planar 3-trees, i.e., every planar 3-tree admits a drawing with height at most $n/2$. They also showed that $4n/9$ lines suffice for some subclasses of planar 3-trees, and asked whether $4n/9$ is indeed an upper bound for planar 3-trees.

In the context of optimization, Dujmović et al. [13] gave fixed-parameter-tractable (FPT) algorithms, parameterized by pathwidth, to decide whether a planar graph admits a straight-line drawing on k horizontal lines. Drawings with minimum number of parallel lines have been achieved for trees [21]. Recently, Biedl [2] gave an algorithm to approximate the height of straight-line drawings of 2-connected outer planar graphs within a factor of 4. Several researchers have attempted to characterize planar graphs that can be drawn on few parallel lines [7, 16, 26].

Contributions: In this paper we show that every n-vertex planar graph with maximum degree Δ, having an edge separator of size λ, admits a drawing with height $4n/9 + O(\lambda)$, which is better than the previously best known bound of $2n/3$ for any $\lambda \in o(n)$. This result is an outcome of a new application of the planar separator theorem [10]. The resulting drawing is not a grid drawing, i.e., the vertices and bends are not restricted to lie on integer grid points, and it is not obvious whether our technique can be immediately adapted to improve the current best $\frac{8}{9}n^2$-area upper bound [5] on the grid drawings of planar graphs. However, the techniques developed in this paper have the potential to provide powerful tools for computing compact drawings for planar triangulations in the variable embedding setting.

If the input graphs are restricted to planar 3-trees, then we can improve the upper bound to $4n/9 + O(1)$, which settles the question of Hossain et al. [18] affirmatively. Furthermore, the drawing we construct in this case is a straight-line drawing.

2 Preliminary Definitions and Results

Let G be an n-vertex plane graph. G is called *connected* if there exists a path between every pair of vertices in G. We call G a *k-connected* graph, where $k > 1$, if the removal of fewer than k vertices does not disconnect the graph. A plane graph delimits the plane into topologically connected regions called *faces*. The bounded regions are called the *inner faces* and the unbounded region is called the *outer face* of G. The vertices on the boundary of the outer face are called the *outer vertices*, and the remaining vertices are called the *inner vertices* of G. If every face of G (including the outer face) is a cycle of length three, then we call G a *triangulation*, or a *maximal* planar graph. G is called an *internally triangulated* graph if every face except the outer face is a cycle of length three.

Let $G = (V, E)$ be an n-vertex triangulated plane graph. A simple cycle C in G is called a *cycle separator* if the interior and the exterior of C each contains at most $2n/3$ vertices. An *edge separator* of G is a subset of edges M of G such that the graph $G' = (V, E \setminus M)$ consists of two induced subgraphs, each containing at most $2n/3$ vertices. Every planar graph with maximum degree Δ admits an edge separator of size $2\sqrt{2\Delta n}$, where the corresponding edges in the dual graph form a simple cycle [10].

Let v_1, v_n and v_2 be the outer vertices of G in clockwise order on the outer face. Let $\sigma = (v_1, v_2, ..., v_n)$ be an ordering of all vertices of G. By G_k, $2 \leq k \leq n$, we denote the subgraph of G induced by $v_1, v_2, \ldots, v_k$. For each G_k, the notation P_k denotes the path (while walking clockwise) on the outer face of G_k that starts at v_1 and ends at v_2. We call σ a *canonical ordering* of G with respect to the outer edge (v_1, v_2) if for each k, $3 \leq k \leq n$, the following conditions are satisfied [8]:

(a) G_k is 2-connected and internally triangulated.

(b) If $k \leq n$, then v_k is an outer vertex of G_k and the neighbors of v_k in G_{k-1} are consecutive on P_{k-1}.

Let P_k, for some $k \in \{3, 4, \ldots, n\}$, be the path $w_1(= v_1), \ldots, w_l, v_k(= w_{l+1})$, $w_r, \ldots, w_t(= v_2)$. The edges (w_l, v_k) and (v_k, w_r) are the *l-edge* and *r-edge* of v_k, respectively. The other edges incident to v_k in G_k are called the *m-edges*. For example, in Figure 2(c), the edges $(v_6, v_1), (v_6, v_4)$, and (v_5, v_6) are the l-, r- and m-edges of v_6, respectively. Let E_m be the set of all m-edges in G. Then the graph T_{v_n} induced by the edges in E_m is a tree with root v_n. Similarly, the graph T_{v_1} induced by all l-edges except (v_1, v_n) is a tree rooted at v_1 (Figure 2(b)), and the graph T_{v_2} induced by all r-edges except (v_2, v_n) is a tree rooted at v_2. These three trees form the *Schnyder realizer* [24] of G, e.g., see Figure 2(a).

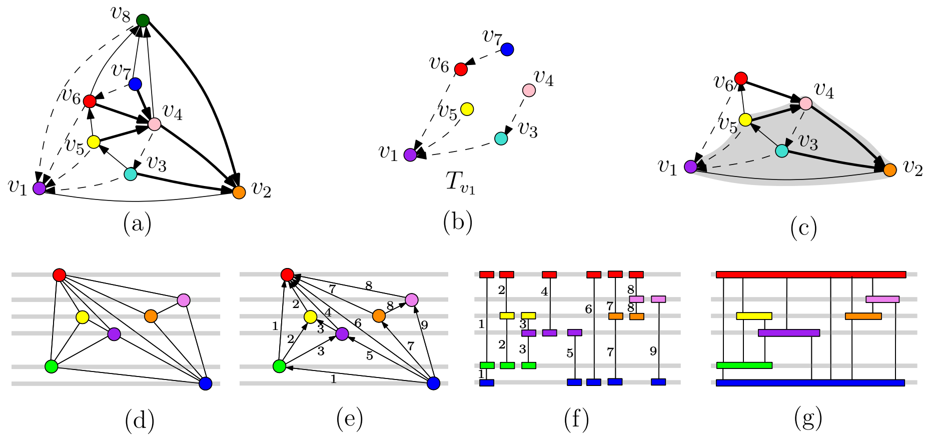

Figure 2: (a) A plane triangulation G with a canonical ordering. The associated realizer, where the l-, r- and m- edges are shown in dashed, bold-solid, and thin-solid lines, respectively. (b) T_{v_1}. (c) Neighbors of v_6 in G_6. (d)–(g) Illustrating Lemma 3.

Lemma 1 (Bonichon et al. [4]) *The total number of leaves in all the trees in any Schnyder realizer of an n-vertex triangulation is at most $2n - 5$.*

Let G be a planar graph and let Γ be a straight-line drawing on k parallel lines. By $l(v)$, where v is a vertex of G, we denote the horizontal line in Γ that passes through v. We now have the following lemma that bounds the height of a straight-line drawing in terms of the number of leaves in a Schnyder tree. Although the lemma can be derived from known straight-line [5] and polyline drawing algorithms [3], we include a proof for completeness.

Lemma 2 *Let G be an n-vertex plane triangulation and let v_1, v_n, v_2 be the outer vertices of G in clockwise order on the outer face. Assume that T_{v_n} has at most p leaves. Then for any placement of v_n on line l_1 or l_{p+2}, there exists a straight-line drawing Γ of G on L_{p+2} such that v_2 and v_1 lie on lines l_{p+2} and l_1, respectively. Symmetrically, there exists a straight-line drawing Γ of G on L_{p+2} such that v_1 and v_2 lie on lines l_{p+2} and l_1, respectively.*

Proof: We construct Γ by a variant of the shift algorithm [8]. The case when G has $n = 3$ vertices is straightforward, and hence we assume that $n > 3$. The construction of Γ is incremental. We start with the drawing of G_3 and then add the other vertices in the canonical order corresponding to T_{v_n}. Let Γ_3 be the drawing of G_3 on L_3, where v_1 and v_2 are placed on l_1 and l_3, respectively, along a vertical line, and v_3 is placed on l_2 to the left of edge (v_1, v_2), e.g., see Figure 3(b). We now add the vertices v_i, where $3 < i < n$, maintaining the following invariants:

(a) P_i is drawn as a strictly y-monotone polygonal chain.

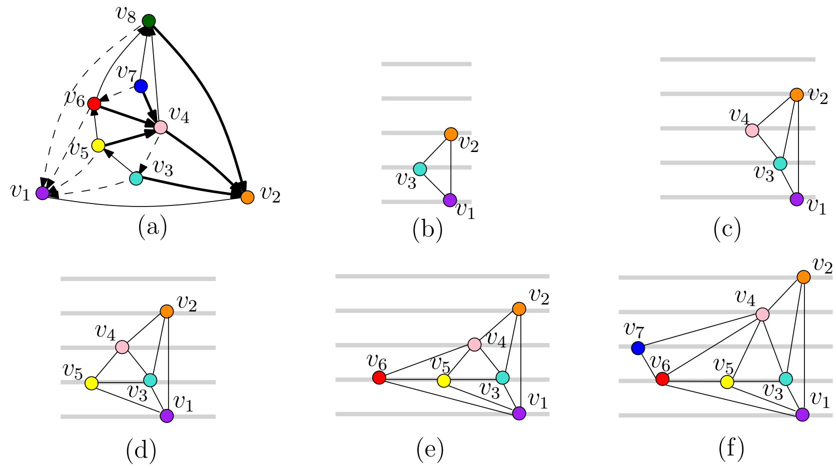

Figure 3: (a) A plane triangulation G with a canonical ordering of its vertices. (b)–(f) Illustration for drawing Γ_i.

(b) Γ_i is a drawing on L_{k+2}, where k is the number of vertices in $v_3, \ldots, v_i$ that are leaves of T_{v_n}.

(c) The vertices v_2 and v_1 lie on the topmost and bottommost lines of L_{k+2}, respectively.

Observe that Γ_3 maintains all the above invariants. We now assume that $i > 3$ and for all $j < i$, Γ_j maintains the above invariants, and consider the insertion of v_i. Let $w_p, \ldots, w_q$ be the neighbors of v_i on P_{i-1}. If $q - p \geq 2$, then v_i is a non-leaf vertex in T_{v_n}. In this case we place v_i on $l(w_{q-1})$ and add the edges (v_i, w), where $w \in \{w_p, \ldots, w_q\}$. Since P_{i-1} is strictly y-monotone, we can place v_i sufficiently far from w_{q-1} to the left such that the edges (v_i, w) do not create any edge crossing, and P_i is strictly y-monotone in Γ_i. Figures 3(d)–(e) illustrate such a scenario. Since the number of leaves in $v_3, \ldots, v_i$ is same as the number of leaves in $v_3, \ldots, v_{i-1}$, Invariants (a)–(c) hold in Γ_i.

In the remaining case, $q - p = 1$, i.e., v_i is a leaf in T_{v_n}. Here we shift the vertices $w_q, \ldots, w_t (= v_2)$ and their descendants in T_{v_n} above by one unit from their current positions. Such a shift does not create edge crossings [8]. Figures 3(b)–(c),(f) illustrate such a scenario. We then place v_i on $l(w_q) - 1$ sufficiently far to the left such that the edges (v_i, w_p) and (v_i, w_q) do not create any edge crossing, and P_i is strictly y-monotone in Γ_i. Since the number of leaves in $v_3, \ldots, v_i$ is one more than the number of leaves in $v_3, \ldots, v_{i-1}$, Invariants (a)–(c) hold in Γ_i.

Since P_{n-1} is strictly y-monotone in Γ_{n-1}, there exists a point c on l_1 (similarly, on l_{p+2}) which is visible to all the vertices on P_{n-1}. We place v_n at c, and draw the edges incident to it, which completes the drawing of G. □

Chrobak and Nakano [6] showed that every planar graph admits a straight-line drawing with height $2n/3$. We now observe some properties of Chrobak and Nakano's algorithm [6]. Let G be a plane triangulation with n vertices and let x, y be two prescribed outer vertices of G in clockwise order on the outer face of G. Let Γ be the drawing of G produced by the Algorithm of Chrobak and Nakano [6]. Then Γ has the following properties:

(CN_1) Γ is a drawing on L_q, where $q \leq 2n/3$.

(CN_2) For the vertices x and y, we have $l(x) = l_1$ and $l(y) = l_q$ in Γ. The remaining outer vertex z lies on either l_1 or l_q.

Note that the placement of z cannot be prescribed to the algorithm, i.e., the algorithm may produce a drawing where $l(x) = l_1, l(y) = l_q$ and $l(z) = l_1$, however, this does not imply that there exists another drawing where $l(x) = l_1, l(y) = l_q$ and $l(z) = l_q$. We end this section with the following lemma.

Lemma 3 *Let G be a plane graph and let Γ be a straight-line drawing of G on a set L_k of k horizontal lines, where the lines are not necessarily equally spaced. Then there exists a straight-line drawing Γ' of G on a set of k horizontal lines that are equally spaced. Furthermore, for every $i \in \{1, 2, \ldots, k\}$, the left to right order of the vertices on the ith line in Γ coincides with that of Γ'.*

Proof: A *flat visibility drawing* of G on L_k maps each vertex of G to a distinct horizontal interval on some horizontal line of L_k, and each edge of G to a horizontal or vertical line segment between the corresponding intervals. Given a straight-line drawing Γ of G on L_k, it is straightforward to transform Γ into a *flat visibility drawing* D on L_k such that for every $i \in \{1, 2, \ldots, k\}$, the left to right order of the vertices on the ith line in Γ coincides with that of D, and for every vertex v in D, the clockwise ordering of the edges around v coincides with the ordering in Γ. One way to construct such a drawing D is to direct the edges of Γ from bottom to top, and then draw the directed paths in a depth-first search order from left to right. Figures 2(d)–(g) illustrate such a construction. In fact, this construction is inspired by the technique for computing visibility representation of planar graphs, as described in [27, 9].

We now adjust the length of the vertical edges so that the layers in D become equally spaced. Biedl [1] showed that such a drawing D can be transformed to the required straight-line drawing Γ', where for every $i \in \{1, 2, \ldots, k\}$, the left to right order of the vertices on the ith line in D coincides with that of Γ'. $\square$

In the following sections we describe our drawing algorithms. For simplicity we often omit the floor and ceiling functions while defining different parameters of the algorithms. One can describe a more careful computation using proper floor and ceiling functions, but that does not affect the asymptotic results discussed in this paper.

3 Drawing Triangulations with Small Height

Every planar triangulation has a simple cycle separator of size $O(\sqrt{n})$ [11]. In the preliminary version of this paper [14], we used this result to prove that every n-vertex planar graph with maximum degree $\Delta \in o(\sqrt{n})$ admits a 4-bend polyline drawing with height at most $4n/9 + o(n)$. In this section we use edge separator, and prove that every planar graph with $\Delta \in o(n)$ can be drawn with 3 bends per edge and height at most $4n/9 + o(n)$.

We first present an overview of our algorithm, and then describe the algorithmic details.

3.1 Algorithm Overview

Let $G = (V, E)$ be an n-vertex planar graph, where $n \geq 9$, and let Γ be a planar drawing of G on the Euclidean plane. Without loss of generality assume that G is a planar triangulation. Let $M \subseteq E$ be an edge separator of G such that the corresponding edges in the dual graph G^* form a simple cycle C^*. Let $V_o \subseteq V$ (respectively, $V_i \subseteq V$) be the vertices that lie outside (respectively, inside) of C^*. Diks et al. [10] proved that there always exists such an edge separator $M \subset E$ such that $|M| \leq 2\sqrt{2\Delta n}$ and $\max\{|V_i|, |V_o|\} \leq 2n/3$. Figures 4(a)–(b) illustrate a planar triangulation G and an edge separator of G. Let $G_i = (V_i, E_i)$ and $G_o = (V_o, E_o)$ be the subgraphs of G induced by the vertices of V_i and V_o, respectively. Since $n \geq 9$, each of G_i and G_o contains at least 3 vertices.

Since G is a planar triangulation, there must be an outer vertex q on G_i or G_o such that q is incident to two or more edges of M. Without loss of generality assume that q lies on G_i, e.g., see vertex v_5 in Figure 4(c). Let a, b, c be three consecutive neighbors of q in G in counter clockwise order such that $a \in V_i$ and $\{b, c\} \subseteq V_o$. We take an embedding G' of G with q, b, c as the outer face, as shown in Figure 4(d) with $q = v_5$, $a = v_3$, $b = v_2$, and $c = v_{11}$. Consequently, G_o and G_i lie on the outer face of each other, as illustrated in Figures 4(d)–(e).

We first draw G_o and G_i separately with small height, and then merge these drawings to compute the final output. The drawings of G_o and G_i are placed side by side. Consequently, the height of the final output can be expressed in terms of the maximum height of the drawings of G_o and G_i, and hence the area of the final drawing becomes small.

3.2 Algorithm Details

Let G' be the embedding obtained from G by choosing q, b, c as the outer face. We first construct a graph G'_o from G_o by adding a vertex w_o on the outer face of G_o, and making w_o adjacent to all the outer vertices of G_o such that the edge (b, c) remains as an outer edge. We remove any resulting multi-edges by subdividing each corresponding inner edge with a dummy vertex, and then by triangulating the resulting graph. Note that we do not need to add dummy vertices on the outer edges. Figure 5(a) illustrates an example of G'_o, where the dummy vertex d removes the multi-edges between v_7 and w_o. Since there are $O(\sqrt{\Delta n})$ edges in M, the number of vertices in G'_o is at most $2n/3 + O(\sqrt{\Delta n})$.

We now use the algorithm of Chrobak and Nakano [6] to compute a straight-line drawing Γ_o of G'_o with height $x = 4n/9 + O(\sqrt{\Delta n})$, where b, c lie on l_1, l_x

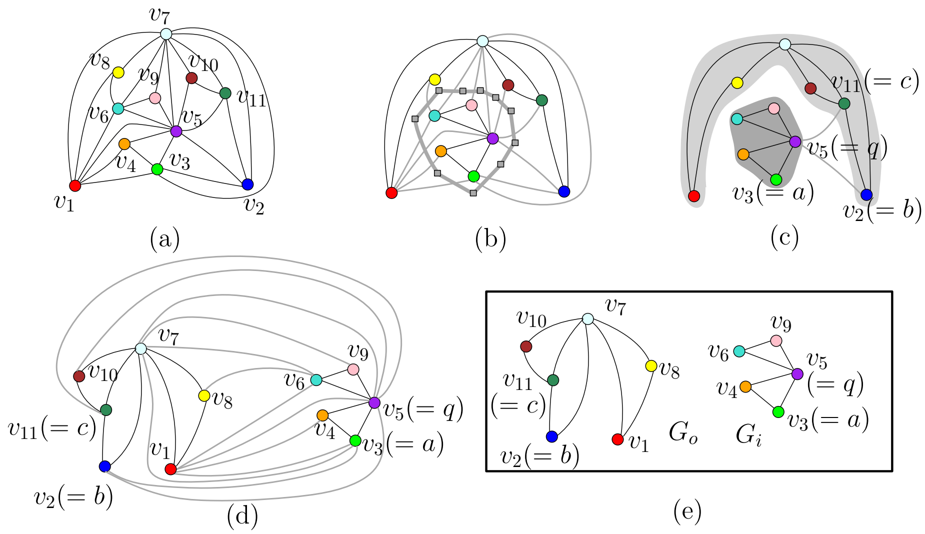

Figure 4: (a) A planar triangulation. (b) An edge separator M of G, and the corresponding simple cycle in the dual graph. The edges of M and C^* are shown in thin and thick gray, respectively. (c) G_o and G_i are shaded in light-gray and dark-gray, respectively. (d)–(e) Choosing a suitable embedding G'.

and w_o lies on either l_1 or l_x. Assume without loss of generality that w_o is in the right half-plane of the line determined by b, c.

We now construct a graph G'_i from G_i, as follows. Observe that the vertex a is an outer vertex of G_i, which appears immediately after q while walking on the outer face of G_i. We add a vertex w_d on the outer face of G_i, and make it adjacent to q and a. We now add another vertex w_i on the outer face, and make it adjacent to w_d and q such that the cycle w_i, q, w_d becomes the boundary of the outer face, e.g., see Figure 5(b).

If w_o lies in l_x in Γ_o, then we make w_i adjacent to all the outer vertices of G_i. Otherwise, we make w_d adjacent to all the outer vertices of G_i. We remove any resulting multi-edges by subdividing each corresponding inner edge with a dummy vertex, and then by triangulating the resulting graph. Figure 5(b) illustrates an example of G'_i, where d' is a dummy vertex. Since there are $O(\sqrt{\Delta n})$ edges in M, the number of vertices in G'_i is at most $2n/3 + O(\sqrt{\Delta n})$.

We now use the algorithm of Chrobak and Nakano [6] to compute a straight-line drawing Γ_i of G'_i with height $y = 4n/9 + O(\sqrt{\Delta n})$ such that w_d, w_i lie on l_1, l_y, respectively, and the segment $w_d w_i$ is vertical. Assume without loss of generality that all the vertices of G'_i are in the right half-plane of the line determined by w_d, w_i.

To construct a drawing of G', we merge the drawings of G'_o and G'_i.

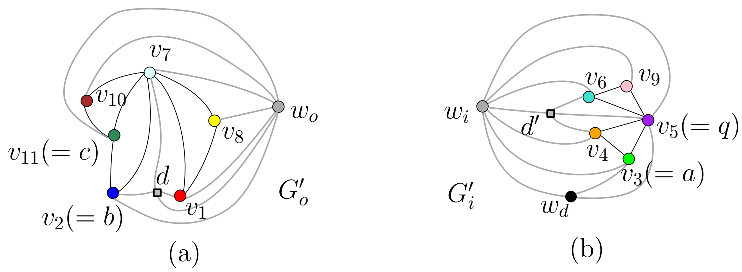

Figure 5: Construction of (a) G'_o and (b) G'_i.

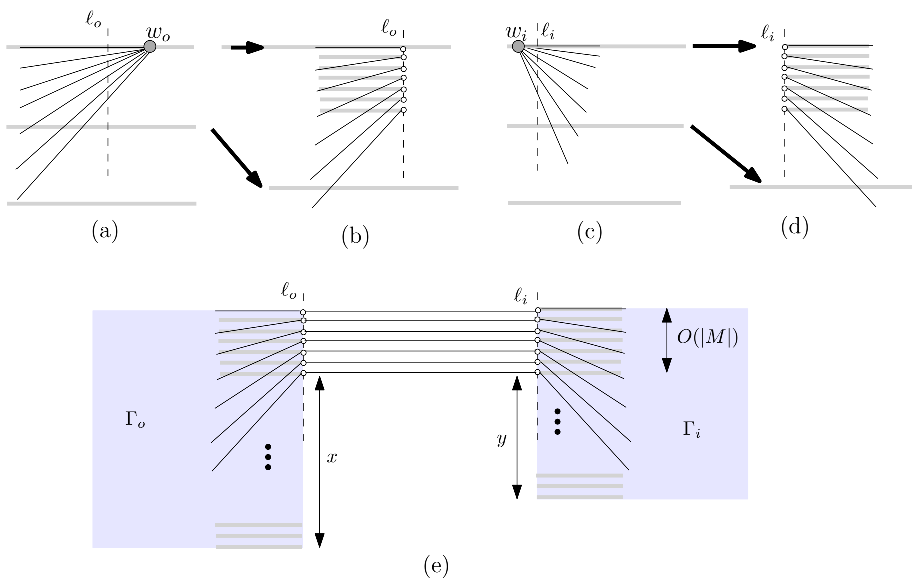

Figure 6: Merging Γ_o and Γ_i.

Merging the drawings of G'_i and G'_o: Without loss of generality assume that $l(w_o) = l_x$ in Γ_o, and recall that in this case w_o and w_i are adjacent to all the outer vertices of G_o and G_i, respectively. Let ℓ_i be a vertical line to the right of segment $w_d w_i$ in Γ_i such that all the other vertices of Γ_i are in the right half-plane of ℓ_i. Furthermore, ℓ_i must be close enough such that all the intersection points with the edges incident to w_i lie in between the horizontal line $l(w_i)$ and the horizontal line immediately below $l(w_i)$. For each intersection point, we insert a division vertex at that point and create a horizontal line through that vertex. We then delete vertex w_i from Γ_i, but not the division vertices. Figures 6(c)–(d) illustrate this scenario. By Lemma 3, we can modify Γ_i such that the horizontal lines are equally spaced. Since $|M| \in O(\sqrt{\Delta n})$, Γ_i is a drawing on at most $y + O(\sqrt{\Delta n})$ horizontal lines. Similarly, we modify Γ_o, as follows.

Let ℓ_o be a vertical line to the left of w_o in Γ_o such that all the other vertices of Γ_o are in the left half-plane of ℓ_o. Furthermore, ℓ_o must be close enough such that all the intersection points with the edges incident to w_o lie in between $l(w_o)$ and $l(w_o) - 1$. For each intersection point, we insert a division vertex at that point and create a horizontal line through that vertex. Delete vertex w_o, but not the division vertices. Finally, by Lemma 3, we can modify Γ_o such that the horizontal lines are equally spaced. Note that Γ_o is a drawing on at most $x + O(\sqrt{\Delta n})$ horizontal lines. Figures 6(a)–(b) illustrate this scenario.

Since the division vertices in Γ_i and Γ_o take a set of consecutive horizontal lines from their respective topmost lines, it is straightforward to merge these two drawings on a set of $\max\{x, y\} + O(\sqrt{\Delta n}) = 4n/9 + O(\sqrt{\Delta n})$ horizontal lines. Let the resulting drawing be $\mathcal{D}$. Figure 6(e) shows a schematic representation of $\mathcal{D}$. Since the division vertices correspond to the bends, each edge may contain at most four bends (one bend inside Γ_o, one bend inside Γ_i, and two bends to merge the drawings Γ_i and Γ_o). Since there are at most $O(\sqrt{\Delta n})$ edges that may have bends, the number of bends is at most $O(\sqrt{\Delta n})$ in total. Note that for every edge containing four bends, two of the bends correspond to w_o and w_i, and they are adjacent one the same horizontal line in the final drawing. Therefore, we can now transform $\mathcal{D}$ into a flat-visibility drawing, where the adjacent pair of bends correspond to a single vertex, and then transform the flat-visibility drawing back into a polyline drawing (similar to the proof of Lemma 3), where the bends that correspond to w_o and w_i are merged to a single bend. Consequently, the number of bends per edge reduces to 3. The following theorem summarizes the result of this section.

Theorem 1 *Let G be an n-vertex planar graph. If G contains a simple cycle separator of size λ, then G admits a 3-bend polyline drawing with height $4n/9 + O(\lambda)$ and at most $O(\lambda)$ bends in total.*

Since every planar triangulation with maximum degree Δ has an edge separator of size $O(\sqrt{\Delta n})$ [10], we obtain the following corollary.

Corollary 1 *Every n-vertex planar triangulation with maximum degree $o(n)$ admits a polyline drawing with height at most $4n/9 + o(n)$.*

Pach and Tóth [23] showed that polyline drawings can be transformed into straight-line drawings while preserving the height if the polyline drawing is monotone, i.e., if every edge in the polyline drawing is drawn as a y-monotone curve. Unfortunately, our algorithm does not necessarily produce monotone drawings.

4 Drawing Planar 3-Trees with Small Height

In this section we examine straight-line drawings of planar 3-trees. We first introduce a few more definitions and recall some known results. Afterwards, we describe the algorithm details.

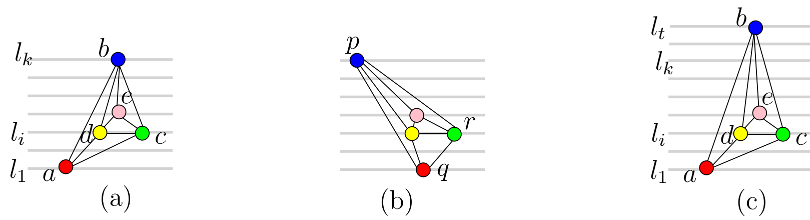

Figure 7: (a)–(b) Illustrating Reshape. (c) Illustrating Stretch.

4.1 Technical Background

Let G be an n-vertex planar 3-tree and let Γ be a straight-line drawing of G. Then Γ can be constructed by starting with a triangle, which corresponds to the outer face of Γ, and then iteratively inserting the other vertices into the inner faces and triangulating the resulting graph. Let a, b, c be the outer vertices of Γ in clockwise order. If $n > 3$, then Γ has a unique vertex p that is incident to all the outer vertices. This vertex p is called the representative vertex of G.

For any cycle i, j, k in G, let G_{ijk} be the subgraph induced by the vertices i, j, k and the vertices lying inside the cycle. Let G^*_{ijk} be the number of vertices in G_{ijk}. The following two lemmas describe some known results.

Lemma 4 (Mondal et al. [22]) *Let G be a plane 3-tree and let i, j, k be a cycle of three vertices in G. Then G_{ijk} is a plane 3-tree.*

Lemma 5 (Hossain et al. [18]) *Let G be an n-vertex plane 3-tree with the outer vertices a, b, c in clockwise order. Let D be a drawing of the outer cycle a, b, c on L_n, where the vertices lie on l_1, l_k and l_i with $k \leq n$ and $i \in \{l_1, l_2, l_n, l_{n-1}\}$. Then G admits a straight-line drawing Γ on L_k, where the outer cycle of Γ coincides with D.*

Let G be a plane 3-tree and let a, b, c be the outer vertices of G. Assume that G has a drawing Γ on L_k, where a, b lie on lines l_1, l_k, respectively, and c lies on line l_i, where $1 \leq i \leq k$. Then the following properties hold for Γ [18].

Reshape. Let p, q and r be three distinct non-collinear points on lines l_1, l_k and l_i, respectively. Then G has a drawing Γ' on L_k such that the outer face of Γ' coincides with triangle pqr (e.g., Figures 7(a)–(b)).

Stretch. For any integer $t \geq k$, G admits a drawing Γ' on L_t such that a, b, c lie on l_1, l_t, l_i, respectively (e.g., Figure 7(c)).

For any triangulation H with the outer vertices a, b, c, let $T_{a,H}, T_{b,H}, T_{c,H}$ be the Schnyder trees rooted at a, b, c, respectively. By $\texttt{leaf}(T)$ we denote the number of leaves in T. The following lemma establishes a sufficient condition for a plane 3-tree G to have a straight-line drawing with height at most $4(n+3)/9+4$.

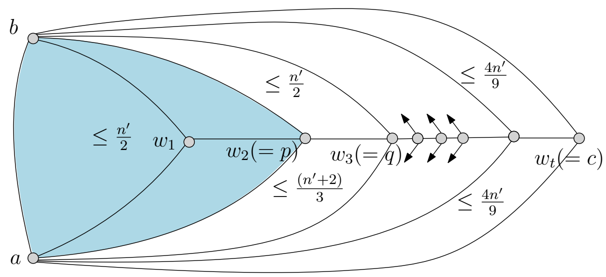

Figure 8: Illustration for Lemma 6, where the graph G_{abp} is in shaded region.

Lemma 6 *Let G be an n-vertex plane 3-tree with outer vertices a, b, c in clockwise order. Let $w_1, \ldots, w_k(=p), w_{k+1}(=q), \ldots, w_t(=c)$ be the maximal path P such that each vertex on P is adjacent to both a and b (e.g., see Figure 8). Assume that $n' = n+3$, and $x = 4n'/9$. If $G^*_{apq} \leq (n'+2)/3$, $G^*_{bpq} \leq G^*_{abp} \leq n'/2$ and $\max_{\forall i>k+1}\{G^*_{aw_iw_{i-1}},\ G^*_{bw_iw_{i-1}}\} \leq 4n'/9$, then G admits a drawing with height at most $4n'/9+4$.*

Proof: To construct the required drawing of G, we distinguish two cases depending on whether $\texttt{leaf}(T_{p,G_{abp}}) \leq x$ or not. Let H be the subgraph of G induced by the vertices $\{a, b\} \cup \{w_k, \ldots, w_t\}$. In each case, we first construct a drawing of H on L_{x+4}, and then extend it to compute the required drawing using Lemmas 2–5.

Case 1 ($\texttt{leaf}(T_{p,G_{abp}}) \leq x$). Since $G^*_{bqp} \leq n'/2$, by Lemma 1, one of the trees in the Schnyder realizer of G_{bqp} has at most $n'/3 \leq x$ leaves. We now draw G_{abq} considering the following scenarios.

Case 1A ($\texttt{leaf}(T_{p,G_{bqp}}) \leq x$). We refer the reader to Figures 9(a)–(b). By Lemma 2 and the Stretch condition, G_{abp} admits a drawing Γ_{abp} on L_{x+2} such that the vertices a, b, p lie on l_1, l_{x+2}, l_{x+2}, respectively. Similarly, since $\texttt{leaf}(T_{p,G_{bqp}}) \leq x$, by Lemma 2 G_{bqp} admits a drawing Γ_{bpq} on L_{x+2} such that the vertices q, b, p lie on l_1, l_{x+2}, l_{x+2}, respectively, as shown in Figure 9(a). By the Stretch property, Γ_{abp} can be extended to a drawing Γ'_{abp} on L_{x+3}, where a, b, p lie on l_1, l_{x+3}, l_{x+2}, respectively. Similarly, Γ_{bqp} can be extended to a drawing Γ'_{bqp} on L_{x+3}, where q, b, p lie on l_1, l_{x+3}, l_{x+2}, respectively. Since $G^*_{apq} \leq (n'+2)/3$, by Lemma 5 and the Stretch condition, G_{apq} admits a drawing Γ_{apq} on $L_{(n'+2)/3}$. Finally, by the Stretch property Γ_{apq} can be extended to a drawing Γ'_{apq} on L_{x+2} such that a, p, q lie on l_1, l_{x+2}, l_1, respectively, and by the Reshape property we can merge these drawings to obtain a drawing of G_{abq} on L_{x+3}. Figure 9(b) depicts an illustration.

Case 1B ($\texttt{leaf}(T_{q,G_{bqp}}) \leq x$). We refer the reader to Figures 9(a)–(b). By Lemma 2 and the Stretch condition, G_{abp} admits a drawing Γ_{abp} on L_{x+2} such that the vertices a, b, p lie on l_1, l_{x+2}, l_1, respectively. Similarly,

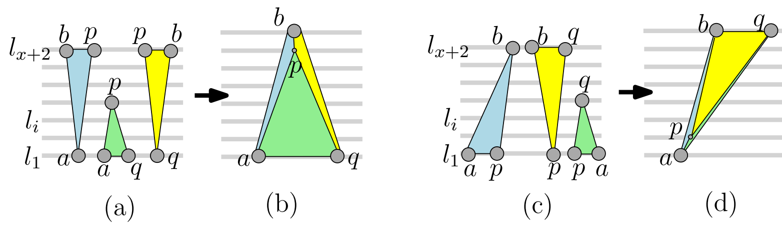

Figure 9: (a)–(b) Illustration for Case 1A. (c)–(d) Illustration for Case 1B.

G_{bqp} admits a drawing Γ_{bpq} on L_{x+2} such that the vertices p, b, q lie on l_1, l_{x+2}, l_{x+2}, respectively. By Lemma 5, G_{apq} admits a drawing Γ_{apq} on $L_{(n'+2)/3}$ such that a, p, q lie on $l_1, l_1, l_{(n'+2)/3}$, respectively. By Stretch, we modify Γ_{apq} such that a, p, q lie on l_1, l_1, l_{x+2}, respectively. Finally, by Stretch and Reshape we can merge these drawings to obtain a drawing of G_{abq} on L_{x+3}. Figures 9(c)–(d) show an illustration.

Case 1C ($\texttt{leaf}(T_{b,G_{bqp}}) \leq x$). The drawing of this case is similar to Case 1B. The only difference is that we use $T_{b,G_{bqp}}$ while drawing G_{bqp}.

Observe that each of the Cases 1A–1C produces a drawing of G_{abq} such that a, b lie on l_1, l_{x+3}, respectively, and q lies on either l_1 or l_{x+3}. We use the Stretch operation to modify the drawing such that a, b lie on l_1, l_{x+4}, respectively, and q lies on either l_2 or l_{x+3}. Specifically, if q is on l_{x+3}, then we push b to l_{l+4}. Otherwise, q is on l_1, and in this case we push a to l_0, and then shift the drawing up by one layer to move a back to l_1.

If q lies on l_{x+3}, then we place the vertices $w_{k+1}, \ldots, w_t (= c)$ on l_2 and l_{x+3} alternatively, as shown in Figure 10(a). Similarly, if q lies on l_2, then we draw the path $w_{k+1}, \ldots, w_t (= c)$ in a zigzag fashion, placing the vertices on l_{x+3} and l_2 alternatively such that each vertex is visible to both a and b. For each $i > k+1$, Lemma 4 ensures that the graphs $G_{aw_iw_{i-1}}$ and $G_{bw_iw_{i-1}}$ are plane 3-trees. Since $\max_{\forall i > k+1}\{G^*_{aw_iw_{i-1}}, G^*_{bw_iw_{i-1}}\} \leq x$, we can draw $G_{aw_iw_{i-1}}$ and $G_{bw_iw_{i-1}}$ using Lemma 5 inside their corresponding triangles.

Case 2 ($\texttt{leaf}(T_{p,G_{abp}}) > x$). Since $G^*_{abp} \leq n'/2$, by Lemma 1, $\texttt{leaf}(T_{a,G_{abp}}) + \texttt{leaf}(T_{b,G_{abp}}) \leq n' - \texttt{leaf}(T_{p,G_{abp}}) \leq 5n'/9$. Hence we draw G_{abq} considering the following scenarios.

Case 2A ($\texttt{leaf}(T_{a,G_{abp}}) \leq x$ and $\texttt{leaf}(T_{b,G_{abp}}) \leq x$). We refer the reader to Figures 10(b)–(c). Since $G^*_{bqp} \leq n'/2$, by Lemma 1, one of the trees in the Schnyder realizer of G_{bqp} has at most $n'/3 \leq x$ leaves.

If $\texttt{leaf}(T_{p,G_{bpq}}) \leq x$, then we draw G_{abq} on L_{x+3}, where a, b, p, q lie on $l_1, l_{x+3}, l_{x+2}, l_1$, respectively, as in Figure 10(b). Specifically, since $\texttt{leaf}(T_{b,G_{abp}}$ and $\texttt{leaf}(T_{p,G_{bpq}})$ both are at most x, we use Lemma 2 to draw G_{abp} and G_{abp}. Since $G^*_{apq} \leq (n'+2)/3$, we can draw G_{apq} using Lemma 5. Finally, we use Stretch and Reshape to merge these drawings.

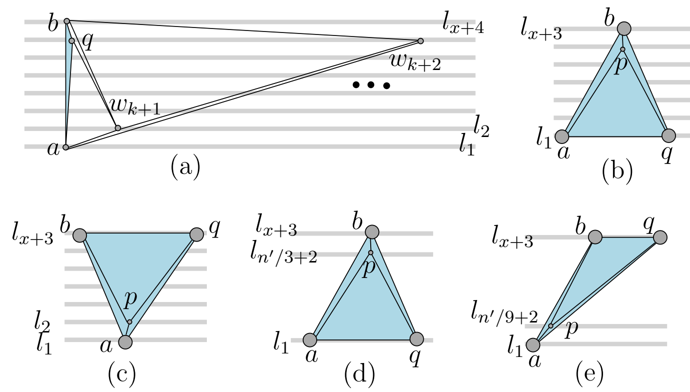

Figure 10: (a) Illustrating Case 1. (b)–(c) Illustrating Case 2A. (d)–(e) Case 2B.

If $\texttt{leaf}(T_{p,G_{bpq}}) > x$, then either $\texttt{leaf}(T_{b,G_{bpq}}) \leq x$ or $\texttt{leaf}(T_{q,G_{bpq}}) \leq x$. In this case we draw G_{abq} on L_{x+3}, where a, b, p, q lie on $l_1, l_{x+3}, l_2, l_{x+3}$, respectively, as in Figure 10(c). Specifically, we use Lemma 2 to draw G_{bpq}. Since $\texttt{leaf}(T_{a,G_{abp}}) \leq x$, we use Lemma 2 to draw G_{abp}, and since $G^*_{apq} \leq (n'+2)/3$, we draw G_{apq} using Lemma 5. Finally, we use Stretch and Reshape to merge these drawings.

Case 2B ($\texttt{leaf}(T_{a,G_{abp}}) > x$ and $\texttt{leaf}(T_{b,G_{abp}}) \leq n'/9$). If $\texttt{leaf}(T_{p,G_{bpq}}) \leq n'/3$, then we first draw G_{bpq} using Lemma 2 such that b, p, q lie on $l_{n'/3+2}$, $l_{n'/3+2}, l_1$, respectively, and then use the Stretch condition to shift b to l_{x+3}. By Lemma 2 and the Stretch condition, there exists a drawing of G_{abp} on L_{x+3} with a, b, p lying on $l_1, l_{x+3}, l_{n'/3+2}$, respectively. Since $G^*_{apq} \leq (n'+2)/3$, we can draw G_{apq} using Lemma 5 inside triangle apq. Figure 10(d) illustrates the scenario after applying Stretch and Reshape.

If $\texttt{leaf}(T_{p,G_{bpq}}) > n'/3$, then by Lemma 1 either $\texttt{leaf}(T_{b,G_{bpq}}) \leq n'/3 - 2$ or $\texttt{leaf}(T_{q,G_{bpq}}) \leq n'/3 - 2$. Hence we can use Lemma 2 and the Stretch condition to draw G_{bpq} such that b, p, q lie on $l_{x+3}, l_{n'/9+2}, l_{x+3}$, respectively. On the other hand, we use Lemma 2 to draw G_{abp} such that a, b, p lie on $l_1, l_{n'/9+2}, l_{n'/9+2}$, respectively, and then use the Stretch condition to move b to l_{x+3}. Since $G^*_{apq} \leq (n'+2)/3$, we can draw G_{apq} using Lemma 5 inside triangle apq. Figure 10(e) illustrates the scenario after applying Stretch and Reshape.

Case 2C ($\texttt{leaf}(T_{a,G_{abp}}) \leq n'/9$ and $\texttt{leaf}(T_{b,G_{abp}}) > x$). The drawing in this case is analogous to Case 2B. The only difference is that we use $T_{a,G_{abp}}$ while drawing G_{abp}.

Each of the Cases 2A–2C produces a drawing of G_{abq} such that a, b lies on l_1, l_{x+3}, respectively, and q lies on either l_1 or l_{x+3}. Hence we can extend these drawings to draw G as in Case 1. □

4.2 Drawing Algorithm

We are now ready to describe our algorithm.

4.2.1 Decomposition.

Let G be an n-vertex plane 3-tree with the outer vertices a, b, c and the representative vertex p. A tree spanning the inner vertices of G is called the *representative tree* T if it satisfies the following conditions [22]:

(a) If $n = 3$, then T is empty.

(b) If $n = 4$, then T consists of a single vertex.

(c) If $n > 4$, then the root p of T is the representative vertex of G and the subtrees rooted at the three clockwise ordered children p_1, p_2 and p_3 of p in T are the representative trees of G_{abp}, G_{bcp} and G_{cap}, respectively.

Recall that every r-vertex tree T' has a vertex v' such that the connected components of $T' \setminus v'$ are all of size at most $r/2$ [19]. Such a vertex v in T corresponds to a decomposition of G into four smaller plane 3-trees G_1, G_2, G_3, and G_4, as follows.

- The plane 3-tree G_i, where $1 \le i \le 3$, is determined by the representative tree rooted at the ith child of v, and thus contains at most $r/2 + 3 = (n-3)/2 + 3 = (n+3)/2$ vertices.
- The plane 3-tree G_4 is obtained by deleting v and the vertices from G that are descendent of v in T, and contains at most $(n + 3)/2$ vertices.

4.2.2 Drawing Technique.

Without loss generality assume that $G_3^* \le G_2^* \le G_1^*$. If G_1 is incident to the outer face of G, then let (a, b) be the corresponding outer edge. Otherwise, G_1 does not have any edge incident to the outer face of G. In this case there exists an inner face f in G that is incident to G_1, but does not belong to G_1. We choose f as the outer face of G, and now we have an edge (a, b) of G_1 that is incident to the outer face of G. Let $P{=}(w_1, \ldots, w_k(= p), w_{k+1}(= q), \ldots, w_t)$ be the maximal path in G such that each vertex on P is adjacent to both a and b, where $\{a, b, p\}, \{a, p, q\}, \{b, q, p\}$ are the outer vertices of G_1, G_2, G_3, respectively, e.g., see Figure 11. Assume that $n' = n + 3$ and $x = 4n'/9$. We draw G on L_{x+4} by distinguishing two cases depending on whether $G_4^* > x$ or not.

Case 1 ($G_4^* > x$). Recall that $G_2^* \le G_1^* \le n'/2$. Since $G_3^* + G_2^* + G_1^* \le G^* - G_4^* + 9 \le n' + 6 - 4n'/9$, we have $G_3^* \le 5n'/27 + 2 \le n'/3$ for sufficiently large values of n. If $\max_{\forall i>k+1}\{G^*_{aw_iw_{i-1}}, G^*_{bw_iw_{i-1}}\} \le x$ holds, then G admits a drawing on L_{x+4} by Lemma 6. We may thus assume that there exists some $j > k + 1$ such that either $G^*_{aw_jw_{j-1}} > x$ or $G^*_{bw_jw_{j-1}} > x$. Hence $\max_{\forall i>k+1, i\ne j}\{G^*_{aw_iw_{i-1}}, G^*_{bw_iw_{i-1}}\} \le n'/9$.

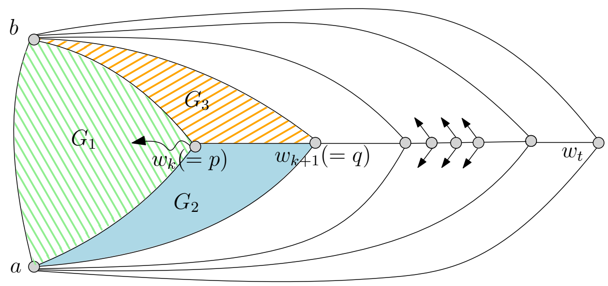

Figure 11: Illustration for G_1, G_2, G_3 and G_4.

We first show that G_{abq} can be drawn on L_{x+3} in two ways: One drawing Γ_1 contains the vertices a, b, q on l_1, l_{x+3}, l_2, respectively, and the other drawing Γ_2 contains a, b, q on l_1, l_{x+3}, l_{x+2}, respectively. We then extend these drawings to obtain the required drawing of G. Consider the following scenarios depending on whether $G_1^* \leq x$ or not.

- If $G_1^* \leq x$, then $G_3^* \leq G_2^* \leq G_1^* \leq x$. Here we draw the subgraph G' induced by the vertices a, b, p, q such that they lie on $l_1, l_{x+3}, l_{x+2}, l_2$, respectively. Since $G_3^* \leq G_2^* \leq G_1^* \leq x$, by Lemma 5, G_1, G_2 and G_3 can be drawn inside their corresponding triangles, which corresponds to Γ_1. Similarly, we can find another drawing Γ_2 of G_{abq}, where the vertices a, b, p, q lie on $l_1, l_{x+3}, l_2, l_{x+2}$, respectively.

- If $G_1^* > x$, then $G_3^* \leq G_2^* \leq n'/9$. Since $G_1^* < n'/2$, we can use Chrobak and Nakano's algorithm [6] and Stretch operation to draw G_1 such that that a, b lie on $l_1, l_{n'/3+1}$, respectively, and p lies either on l_2 or $l_{n'/3}$. First consider the case when p lies on $l_{n'/3}$. We then use the Stretch condition to push b to l_{x+3}. To construct Γ_1, we place q on l_2, and to construct Γ_2, we place q on l_{x+2}. Since $G_3^* \leq G_2^* \leq n'/9$, for each placement of q, we can draw G_2 and G_3 using Lemma 5 inside their corresponding triangles. The case when p lies on l_2 is handled symmetrically, i.e., first by pushing a downward using Stretch operation so that the drawing spans $(x+3)$ horizontal lines, then shifting the drawing upward such that a comes back to l_1, and finally placing the vertex q on l_2 (for Γ_1) and l_{x+2} (for Γ_2) .

We now show how to extend the drawing of G_{abq} to compute the drawing of G. Consider two scenarios depending on whether $G^*_{aw_jw_{j-1}} > x$ or $G^*_{bw_jw_{j-1}} > x$.

- Assume that $G^*_{aw_jw_{j-1}} > x$. Shift b to l_{x+4}, and draw the path $w_{k+1}, \ldots, w_{j-1}$ in a zigzag fashion, placing the vertices on l_2 and l_{x+3} alternatively, such that $l(w_{k+1}) \neq l(w_{k+2})$, and each vertex is visible to both a and b. Choose Γ_1 or Γ_2 such that the edge (a, w_{j-1}) spans at least $x+3$ lines. We now draw $G_{aw_jw_{j-1}}$ using Chrobak and Nakano's algorithm [6]. Since $x < G^*_{aw_jw_{j-1}} \leq n'/2$, we can draw $G_{aw_jw_{j-1}}$ on at most $n'/3$ parallel lines. By the Stretch and Reshape conditions, we merge this drawing with the current drawing such that w_j lies on either l_{x+3} or $l_{n'/9+2}$. Since $G^*_{bw_jw_{j-1}} \leq n'/9$, we can draw $G_{bw_jw_{j-1}}$ inside its corresponding triangle using Lemma 5. Since $\max_{\forall i>j}\{G^*_{aw_iw_{i-1}}, G^*_{bw_iw_{i-1}}\} \leq n'/9$, it is straightforward to extend the current drawing to a drawing of G on $x+4$ parallel lines by continuing the path $w_j, \ldots, w_t$ in the zigzag fashion.

- Assume that $G^*_{bw_jw_{j-1}} > x$. The drawing in this case is similar to the case when $G^*_{aw_jw_{j-1}} > x$. The only difference is that while drawing the path $w_{k+1}, \ldots, w_{j-1}$, we choose Γ_1 or Γ_2 such that the edge (b, w_{j-1}) spans at least $x+3$ lines.

Case 2 ($G^*_4 \leq x$). Observe that $G^*_2 \leq G^*_1 \leq n'/2$. We now show that $G^*_3+G^*_2+G^*_1$ can be at most $n-5$ in the worst case. If $G^*4 = 0$, then G_1, G_2 and G_3 spans the graph G. Let I_1, I_2 and I_3 be the inner vertices of G_1, G_2 and G_3, respectively. Then $G^*_3+G^*_2+G^*_1 = (I_1+I_2+I_3)+9 = (n-4)+9 = n+5 = n'+2$.

Since $G^*_3 \leq G^*_2 \leq G^*_1$, we have $G^*_3 \leq (n'+2)/3$. Hence G admits a drawing on L_{x+4} by Lemma 6.

The following theorem summarizes the result of this section.

Theorem 2 *Every n-vertex planar 3-tree admits a straight-line drawing with height $4(n+3)/9+4 = 4n/9+O(1)$.*

5 Conclusion

In this paper we have shown that every n-vertex planar graph with maximum degree Δ, having an edge separator of size λ, admits a polyline drawing with height $4n/9+O(\lambda)$, which is $4n/9+o(n)$ for any planar graph with $\Delta \in o(n)$. While restricted to n-vertex planar 3-trees, we compute straight-line drawings with height at most $4n/9+O(1)$. In some cases the width of the drawings that we compute for plane 3-trees may be exponentially large over n. Hence it would be interesting to find drawing algorithms that can produce drawings with the same height as ours, but bound the width as a polynomial function of n.

Several natural open question follows.

- Does every n-vertex planar triangulation admit a straight-line drawing with height at most $4n/9+O(1)$?
- What is the minimum constant c such that every n-vertex planar 3-tree admits a straight-line (or polyline) drawing with height at most cn?
- Does a lower bound on the height for straight-line drawings of triangulations determine a lower bound also for their polyline drawings?

Recently, Biedl [1] has examined height-preserving transformations of planar graph drawings, which shed some light on the last open question.

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