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Connectivity of Planar Graphs

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Abstract

We give here three simple linear time algorithms on planar graphs: a 4-connexity test for maximal planar graphs, an algorithm enumerating the triangles and a 3-connexity test. Although all these problems got already linear-time solutions, the presented algorithms are both simple and efficient. They are based on some new theoretical results.

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1 Introduction

The study of graphs by means of special orientations is relatively recent. For instance, bipolar orientations became a basic tool in many graph drawing problems. We give here an example of relations between orientation and topological properties. Constrained orientations (i.e. orientations with bounded indegrees) lead to new characterizations on connexity of planar undirected graphs. Although usual 3-connexity testing of planar graphs are heavily related to planarity testing algorithms (see [10][17] and PQ-tree algorithms), the algorithm we present here assume that a graph is already embedded in the plane and a the problem drastically reduces to the acyclicity testing of a particular orientation. Concerning the 4-connexity testing of a maximal planar graph, the use of an indegree bounded orientation was already used in [2] to enumerate triangles. Here, the use of a specific orientation allows a further simplification of the algorithm. The 4-connexity test itself also reduces to an acyclicity test. It should be noticed that no special data structure is used for these algorithms as, in the planar case, the acyclicity of an orientation may be efficiently tested using a dual topological sort.

2 Preliminaries

In the following we consider plane graphs, that is planar graphs embedded in the plane. Each connected component of the complement in the plane of the vertex and edge sets is a *face region* of the graph. The *external face region* of G is the unbounded one. A *face* is the clockwise walk of the boundary of a face region. When considering an orientation of a graph, such walks also define a *dual orientation* of the dual graph: the outgoing edges of a vertex f of the dual are those traversed according to their orientation in a clockwise walk of the face corresponding to f.

If G is a graph, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. We denote G_A the subgraph of G induced by a subset A of vertices. We denote $d^-_G(x)$ the indegree of the vertex x in the graph G.

Let X and $\overline{X}$ be two complementary subsets of the vertices of an oriented graph. The *cocycle* $\omega(X)$ is the pair $(\omega^+(X), \omega^-(X))$ of the set $\omega^+(X)$ of edges oriented from X to $\overline{X}$ and the set $\omega^-(X)$ of edges oriented from $\overline{X}$ to X. A cocycle $\omega(X)$ is *elementary* if G_X and $G_{\overline{X}}$ are connected. Obviously, any cocycle is the disjoint union of elementary cocycles. A cocycle $\omega(X)$ is a *positive cocircuit* if $\omega^-(X)$ is empty, that is if no edge is directed from $\overline{X}$ to X.

Lemma 2.1 *Let X be a subset of $V(G)$. Then $\omega(X)$ is a positive cocircuit if and only if*

$$|E(G_X)| = \sum_{x \in X} d^-_G(x)$$

□

A *cycle* γ is an Eulerian partial subgraph (i.e. with even vertices only). A cycle is *elementary* (or a *polygon*) if it is connected and 2-regular. A cycle γ is a *circuit* if each of its vertices has in γ an indegree equal to its outdegree. An elementary cycle γ defines a bipartition of the remaining vertices and edges of the graph as *internal* and *external* elements.

Two consecutive edges in the clockwise order at a vertex define an *angle* of the graph. The angle is *lateral* if one of the two edges is incoming and the other is outgoing; otherwise, the angle is *extremal.* The *angle graph* $A(G)$ of a 2-connected plane graph G is the incidence graph of the vertex and face sets of G (the *V-vertices* and *F-vertices* of $A(G)$). The edges of $A(G)$ correspond to the angles of G and their number is twice the number of edges of G. The graph $A(G)$ is maximal bipartite planar. Any embedding of G canonically defines an embedding of $A(G)$, where the faces correspond to the edges of G.

A *k-connected* graph is a graph with at least $k+1$ vertices, such that the deletion of any subset of $k-1$ vertices does not disconnect the graph. A *separating cycle* is an elementary cycle whose vertex set removal disconnects the graph.

Lemma 2.2 *Let X be a vertex subset of plane graph G. If $G_{\overline{X}}$ is connected, then $\overline{X}$ belongs to a same face region of G_X.*

Proof: Assume that two vertices u, v of $\overline{X}$ do not belong to a same face region of G_X. Then a path from u to v in $G_{\overline{X}}$ intersects the boundary of the face region and hence intersects X, which is a contradiction. □

3 A 4-connexity test for maximal planar graphs

The algorithm is based on the following properties:

- A maximal planar graph is 4-connected if and only if it has no separating triangles, i.e. if each of its triangles is a face[19],
- Any maximal planar graph has an orientation where all the vertices (except the 3 external ones) have indegree 3 [3][14],
- In such an orientation, separating triangles corresponds to positive cocircuits (see Lemma 3.4).

An early linear-time algorithm may be found in [11], a more recent one, based on subgraph isomorphism detection, may also be found in [5].

Lemma 3.1 *Let G be a 3-connected planar graph and $\{x, y, z\}$ a cutset of G. Then, $G - \{x, y, z\}$ has 2 connected components.*

Proof: The graph $G-\{x, y, z\}$ has at least 2 connected components, as $\{x, y, z\}$ is a cutset. Assume $G - \{x, y, z\}$ has 3 connected components H_1, H_2, H_3 and let a_1, a_2, a_3 be vertices of H_1, H_2, H_3, respectively. As G is 3-connected, for any $i \neq j$ in $\{1, 2, 3\}$, there exist three internally disjoint paths linking a_i and a_j [18] and these paths respectively include x, y and z. Hence, there exists in G 3 internally paths linking a_1 (resp. a_2, a_3) to x, y, z and whose internal vertices belong to H_1 (resp. H_2, H_3). Thus,a_1, a_2, a_3, x, y, z and these nine paths form a subdivision of $K_{3,3}$, which contradicts the planarity of G. □

Lemma 3.2 *A triangle of a maximal planar graph is a separating triangle if and only if it is not a face.*

Proof: If a triangle is not a face, it separates its interior and exterior vertices. Conversely, assume a face $\{x, y, z\}$ is a separating triangle. A vertex may be added in this face, adjacent to x, y, z, while preserving the planarity. Then, $G - \{x, y, z\}$ has at least 3 components, what contradicts Lemma 3.1. □

Lemma 3.3 (see [19]) *A maximal planar graph G is 4-connected if and only if its has no separating triangle, i.e. a cutset which is the vertex set of a triangle.* □

Lemma 3.4 *Let G be a maximal planar graph (with at least 5 vertices), which is oriented in such a way that all its vertices have indegree 3, except the 3 vertices of the external face which have indegree 1.*

Then, G is 4-connected if and only if it has only one positive cocircuit, namely the one defined by the vertex-set of its external face.

Proof: Let V_0 be the vertex set of the external face. Let us prove that the graph G has a cocircuit different from $\omega(V_0)$ if and only if G has a triangle which is not a face (this is equivalent to the G not being 4-connected, according to Lemma 3.3 and Lemma 3.2):

Algorithm 1 A 4-connexity test for a maximal planar graph G

Require: G is a maximal planar graph
Ensure: *IsFourConnected*=true if and only if G is 4-connected
if G has less than 6 vertices **then**
  $IsFourConnected \leftarrow$ false
else
  $G' \leftarrow G$
  $r_1, r_2, r_3 \leftarrow$ the vertices of some face of G'
  Orient G' in such a way that every vertex has indegree 3 (except r_1, r_2, r_3 which have indegree 1)
  Remove the vertices r_1, r_2, r_3
  Compute the oriented dual H of G'
  if the orientation of H is acyclic **then**
    IsFourConnected $\leftarrow$ true
  else
    IsFourConnected $\leftarrow$ false
  end if
end if

- Let $\omega(X)$ be an elementary positive cocircuit. The sum of the indegrees of the vertices of X is at least $3|X|-6$, since only 3 vertices have indegree 1. Hence, according to Lemma 2.1, G_X has at least $3|X|-6$ edges and then has exactly $3|X|-6$ edges, is maximal planar and contains the vertices of the external face. Thus, according to Lemma 2.2, $\overline{X}$ belongs to a bounded face region of G_X and then is internal to some triangle of G. Thus, either X is the vertex set of the external face of G (i.e. V_0) or G has a triangle which is not a face.
- Let T be a triangle of G which is not a bounded face and let $\overline{X}$ be the set of the vertices internal to T. As G_X is maximal planar and contains r_1, r_2 and r_3, according to Lemma 2.1, the cocycle $\omega(X)$ is a cocircuit. Hence, $\omega(V_0)$ is a cocircuit and any triangle which is not a face defines a cocircuit (different from $\omega(V_0)$).

□

Theorem 3.5 *Algorithm 1 tests in linear time whether a maximal planar graph is* 4*-connected or not.*

Proof: First notice that no 4-connected maximal planar graph has less than 6 vertices. Hence, the preliminary test at line 1: is valid and we may restrict ourselves to the case where G has at least 6 vertices.

The copy of the graph G into a graph G' may be performed in linear time. The orientation of G' performed at line 6: may be computed in linear time [3, 14].

Then, G is 4-connected if and only if G' has only one positive cocircuit, namely the one defined by $\{r_1, r_2, r_3\}$. After the deletion of r_1, r_2, r_3 at line 7:, we get that the graph G is 4-connected if and only if G' has no cocircuit, that is, if and only if its oriented dual H (which is computed in linear time at line 8:) has no circuit. This test (line 9:) can be done in linear time using a topological sort. □

4 Enumerations of the triangles of a planar graph

Linear time algorithms enumerating the triangles of planar graphs may be found in [1] (using tree decompositions) or in [2] (using indegree bounded orientations).

The algorithm we present here has been optimized using Schnyder's decompositions, the definition of which we shall recall here:

Definition 4.1 (Schnyder, [14]) *Let G be a maximal planar graph and $\{r_1, r_2, r_3\}$ one of its faces. A* Schnyder decomposition *relative to $\{r_1, r_2, r_3\}$ is a tricoloration of the edges of G, each color $1 \leq i \leq 3$ forming a directed tree Y_i rooted at r_i such that there exists three total orders $<_1, <_2, <_3$ on the vertex set of G satisfying:*

- *If the arc (u, v) belongs to Y_i then $(u <_j v) \iff (j \neq i)$,*

- *If* $\{x, y\}$ *is an edge of* G*, then*

$$\forall u \notin \{x, y\}, \exists 1 \leq i \leq 3, \quad u >_i x \text{ and } u >_i y$$

Definition 4.2 *Let* G *be a planar graph on* $n \leq 3$ *vertices and let* r_1, r_2, r_3 *be* 3 *vertices of* G*.*

A parent triplet (π_1, π_2, π_3) *relative to* $\{r_1, r_2, r_3\}$ *is a triplet of functions from* $V(G)$ *to* $V(G) \cup \{0\}$*, such that there exists a triangulation* H *of* G *and a Schnyder decomposition of* H *relative to* $\{r_1, r_2, r_3\}$ *which satisfies:* $\pi_i(v)$ *is either the parent of the vertex* v *if these vertices are adjacent in* G*, or* 0 *(otherwise).*

Algorithm 2 Computation of a parent triplet

Require: G is a planar graph with at least 4 vertices
Ensure: π_1, π_2, π_3 are Schnyder parent functions for G relative to r_1, r_2, r_3

$H \leftarrow G$
Triangulate H and mark the added edges
$r_1, r_2, r_3 \leftarrow$ the vertices of some face of H
Compute a Schnyder orientation of H as parent functions π_1, π_2, π_3 (extended with $\pi_i(0) = 0$)
$\pi_1(r_2) \leftarrow r_1$
$\pi_2(r_3) \leftarrow r_2$
$\pi_3(r_1) \leftarrow r_3$
for all marked edge $e = \{u, v\}$ **do**
    for all $i \in \{1, 2, 3\}$ **do**
        if $u = \pi_i(v)$ **then**
            $\pi_i(v) \leftarrow 0$
        else if $v = \pi_i(u)$ **then**
            $\pi_i(v) \leftarrow 0$
        end if
    end for
end for

Theorem 4.1 *A parent triplet of a planar graph* G *may be computed in linear time using algorithm 2.*

Proof: A triangulation is easily performed in linear time. A Schnyder decomposition may also be computed in linear time [15], using the packing algorithm described in [3]. The modification we perform on the functions π is obviously linear. □

Lemma 4.2 *When reversing the orientation of the edges of color* i*, the graph becomes acyclic.*

Proof: According to the definition, if there exists a directed path from x to y in the obtained orientation, then $x <_i y$. Thus, the orientation is acyclic. □

Lemma 4.3 *A triangle of G is a circuit if and only if it is 3-colored; otherwise, it is 2-colored*

Proof: The proof of the lemma will be a consequence of Lemma 4.2:

If a triangle is 2-colored, it does not become a circuit when reversing the orientation of the edges of the third color. Hence, it is not a circuit.

If a triangle is 3-colored, it does not become a circuit when reversing the orientation of any of its edges. Hence, it is a circuit. □

Theorem 4.4 *Algorithm 3 enumerates the triangles of a planar graph in linear time.*

Proof: Associate to each triangle of G either the sink of the triangle if it is acyclic, or the head of the edge colored 1 if it is a circuit. This way, to each triangle is associated exactly one vertex. The algorithm is then an obvious application of Lemma 4.3. □

Lemma 4.5 *Let (a, b, c, d) be a C_4. Then, it is not possible that (a, b) and (c, d) shall be both arcs of the same tree Y_i.*

Proof: Assume such a C_4 exists.

Considering the edge $\{b, c\}$ and according to the definition of a Schnyder decomposition, there exists j such that $a >_j b$ and $a >_j c$. As (a, b) belongs to Y_i, j shall only be equal to i. Hence, $a >_i c$.

Algorithm 3 Enumeration of the triangles of a planar graph

Require: G is a planar graph with at least 4 vertices
Ensure: *NumberOfTriangles* is the number of triangles of G

Compute the Schnyder parent functions π_1, π_2, π_3 of G
$\mathit{NumberOfTriangles} \leftarrow 0$
for all vertex v **do**
    for all $(i, j) \in \{1, 2, 3\}^2, i \neq j$ **do**
        if $(\pi_i(v) \neq 0)$ and $(\pi_j(v) \neq 0)$ and $(\pi_i(\pi_j(v)) = \pi_i(v))$ **then**
            $\mathit{NumberOfTriangles} \leftarrow \mathit{NumberOfTriangles} + 1$
        end if
    end for
    if $(\pi_1(v) \neq 0)$ and $(\pi_2(\pi_1(v)) \neq 0)$ and $(\pi_3(\pi_2(\pi_1(v))) = v)$ **then**
        $\mathit{NumberOfTriangles} \leftarrow \mathit{NumberOfTriangles} + 1$
    end if
    if $(\pi_1(v) \neq 0)$ and $(\pi_3(\pi_1(v)) \neq 0)$ and $(\pi_2(\pi_3(\pi_1(v))) = v)$ **then**
        $\mathit{NumberOfTriangles} \leftarrow \mathit{NumberOfTriangles} + 1$
    end if
end for

Similarly, considering the edge $\{a, d\}$ and the vertex c, we get $c >_i a$ and are led to a contradiction. □

Theorem 4.6 *Algorithm 4 enumerates in linear time the triangles of a planar graph.*

Proof: Algorithm 4 is a reorganized version of Algorithm 3 taking into account some exclusiveness in the cases. The only non-trivial exclusiveness used is that we cannot have simultaneously: $\pi_i(\pi_j(v)) = \pi_i(v)$ and $\pi_k(\pi_j(\pi_i(v))) = v$ (where none of the values taken by the π functions are 0). Otherwise, we would have a C_4: $(\pi_j(v), v, \pi_j(\pi_i(v)), \pi_i(v))$ with arcs $(\pi_j(v), v)$ and $(\pi_j(\pi_i(v)), \pi_i(v))$ colored j, which contradicts Lemma 4.5. □

Remark 4.7 *Algorithm 4 obviously gives the upper bound of* $3n - 8$ *(*1 *in the bloc starting at line 12:, and* $n - 3$ *times* 3 *in the loop at line 17:) for the number of triangles of a planar graph having at least* 4 *vertices.*

Remark 4.8 *This algorithm may be modified to enumerate the separating triangles of* 3*-connected planar graphs, by enumerating the triangles which are not faces.*

Algorithm 4 Optimized enumeration of the triangles of a planar graph

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Require: G is a planar graph with at least 4 vertices
Ensure: *NumberOfTriangles* is the number of triangles of G

1: Compute the Schnyder parent functions π_1, π_2, π_3 of G and the roots r_1, r_2, r_3
2: **if** $\pi_1(r_2) \neq 0)$ and $\pi_2(r_3) \neq 0$ and $\pi_3(r_1) \neq 0$ **then**
3:     $\mathit{NumberOfTriangles} \leftarrow 1$
4: **else**
5:     $\mathit{NumberOfTriangles} \leftarrow 0$
6: **end if**
7: **for all** vertex v different from r_1, r_2, r_3 **do**
8:     $p_1 \leftarrow \pi_1(v), \quad p_2 \leftarrow \pi_2(v), \quad p_3 \leftarrow \pi_3(v)$
9:     **if** $p_1 \neq 0$ **then**
10:         **if** $(p_2 \neq 0)$ and $(\pi_2(p_1) = p_2)$ or $(\pi_1(p_2) = p_1)$ or $(\pi_3(\pi_2(p_1)) = v)$ **then**
11:             $\mathit{NumberOfTriangles} \leftarrow \mathit{NumberOfTriangles} + 1$
12:         **end if**
13:         **if** $(p_3 \neq 0)$ and $(\pi_3(p_1) = p_3)$ or $(\pi_1(p_3) = p_1)$ or $(\pi_2(\pi_3(p_1)) = v)$ **then**
14:             $\mathit{NumberOfTriangles} \leftarrow \mathit{NumberOfTriangles} + 1$
15:         **end if**
16:     **end if**
17:     **if** $(p_3 \neq 0)$ and $(\pi_3(p_2) = p_3)$ or $(p_2 \neq 0)$ and $(\pi_2(p_3) = p_2)$ **then**
18:         $\mathit{NumberOfTriangles} \leftarrow \mathit{NumberOfTriangles} + 1$
19:     **end if**
20: **end for**

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5 A 3-connexity Test for Planar Graphs

The algorithm is based on the following properties we shall prove later:

- A 2-connected planar graph is 3-connected if and only if each of the C_4 of its angle-graph is a face,
- Any planar quadrangulation has an orientation where all the vertices have indegree 2, except the 4 external ones, which have indegree 1.
- In such an orientation, the C_4 which are not faces correspond to positive cocircuits.

Algorithm 5 3-connexity test for a 2-connected planar graph G

Require: G is a 2-connected planar graph
Ensure: x=true if and only if G is 3-connected

if G has less than 4 vertices **then**
  $x \leftarrow$ false
else
  $H \leftarrow \mathcal{A}(G)$
  $b_1, w_1, b_2, w_2 \leftarrow$ the vertices of some face of H
  H is oriented in such a way that every vertex (except b_1, b_2) has 2 incoming edges
  Remove the vertices b_1, w_1, b_2, w_2
  $D \leftarrow$ oriented dual of H
  if D is connected and its orientation is acyclic **then**
    $x \leftarrow$ true
  else {D has a directed circuit}
    $x \leftarrow$ false
  end if
end if

Definition 5.1 *A* 2-articulated subgraph *of a 2-connected graph G is a connected proper induced subgraph H with at least 3 vertices, which may be disconnected from the remaining of the graph by the deletion of two vertices, the* articulation pair *of H.*

Lemma 5.1 *Let G be a 2-connected planar graph. Then G is 3-connected if and only if each C_4 of $A(G)$ is a face.*

Proof: Let γ be a C_4 of $A(G)$ which is not a face and let u, v be its V-vertices. As γ is not a face, there exists at least one vertex of $A(G)$ inside and outside γ. If the only vertices of $A(G)$ inside (resp. outside) γ where F-vertices, the faces inside (resp. outside) γ would correspond to multiple edges of G. Hence, $A(G)$ has at least one V-vertex internal to γ and one V-vertex external to γ. The subgraph H of G induced by the vertices corresponding to u, v and the

V-vertices of $A(G)$ inside γ meets then the requirement of the definition of a 2-articulated subgraph. Thus, G is not 3-connected.

Conversely, if G is not 3-connected, it has a 2-articulated subgraph H with articulation pair u, v. Let f_1 and f_2 be two faces of G adjacent to u and v, such that f_1 does not include the edge $\{u, v\}$ (if this edge exists). Then, f_1, u, f_2, v is not a face of $A(G)$ as it does not correspond to an edge of G. □

Remark 5.2 *There will be no linear-time algorithm to enumerate the C_4 of 3-connected planar graphs, as this number may be quadratic (any double-wheel will do), although it is possible to "implicitly" enumerate them in linear time [1][4].*

Lemma 5.3 *Let G be a 2-connected planar graph with at least 4 vertices and let $A(G)$ its angle graph, oriented in such a way that each of its vertices have indegree 2, except the vertices of the external faces which have indegree 1.*

Then, the graph G is 3-connected if and only if $A(G)$ has only one positive cocircuit, namely the one defined by the vertex-set of its external face.

Proof: Let V_0 be the vertex set of the external face. Let us prove that the graph G has a cocircuit different from $\omega(V_0)$ if and only if $A(G)$ has a C_4 which is not a face (this is equivalent to the 3-connexity of G, according to Lemma 5.1):

- Let $\omega(X)$ be an elementary positive cocircuit. The sum of the indegrees of the vertices of X is at least $2|X|-4$, since only 4 vertices have indegree 1. Hence, according to Lemma 2.1, G_X has at least $2|X|-4$ edges and then has exactly $2|X|-4$ edges, is a planar quadrangulation and contains the vertices of the external face. Thus, according to Lemma 2.2, $\overline{X}$ belongs to a bounded face region of G_X and then is internal to some C_4 of G. Thus, X is the vertex set of the external face (i.e. V_0) or G has a C_4 which is not a face.

- Let C be a C_4 of G which is not a bounded face and let $\overline{X}$ be the set of the vertices internal to C. As G_X is a planar quadrangulation and contains the vertices of the external face, according to Lemma 2.1, the cocycle $\omega(X)$ is a cocircuit. Hence, $\omega(V_0)$ is a cocircuit and any C_4 which is not a face defines a cocircuit (different from $\omega(V_0)$).

□

Definition 5.2 *An* e-bipolar orientation *is an acyclic orientation with exactly one source s and one sink t linked by the edge e. Such an orientation may be computed in linear time [16, 8, 9].*

Lemma 5.4 *Let G be a 2-connected plane graph and let e_0 be an edge of G. Let $\{r_1, r_2, r_3, r_4\}$ be the face of $A(G)$ corresponding to e_0, where r_1 and r_3 are V-vertices.*

Algorithm 6 Optimized 3-connexity test for a 2-connected planar graph G

Require: G is a 2-connected planar graph
Ensure: x=true if and only if G is 3-connected

```
if G has less than 4 vertices then
  x ← false
else
  e0 ← some edge of G
  S ← ∅ (empty stack)
  Compute a minimal e0-bipolar orientation of G [8]
  for all edge e of G do
    d[e] ← number of invertible angles at e
    if d[e] = 0 then
      Push e in the stack S
      Mark all the angles incident to e
    end if
  end for
  while S is not empty do
    Pop e from the stack S
    for all the neighbor edges e' of e do
      Decrement d[e']
      if d[e'] = 0 then
        Push e' in the stack S
        Mark all the angles incident to e'
      end if
    end for
  end while
  Mark all the angles incident to an edge adjacent to e0
  Mark all the angles incident to an edge is a same face than e0
  if all the angles are marked then
    x ← true
  else
    x ← false
  end if
end if
```

Any orientation of G defines an orientation of $A(G)$: an edge of $A(G)$ is directed from its incident V-vertex to its incident F-vertex if the corresponding angle of G is extremal.

If G is e_0-bipolarly oriented, then the induced orientation of $A(G)$ is such that every vertex has indegree 2, except r_1 and r_3 which are sources.

Proof: The poles have no lateral angles, any other vertex has at least two lateral angles and each face has at least two extremal angles. As $A(G)$ has $2|E(G)| = 2|F(G)| + 2(|V(G)| - 2)$ edges, the V-vertices different from the poles and the F-vertices have two incoming edges. □

Theorem 5.5 *Algorithm 5 tests in linear time whether a 2-connected planar graph is 3-connected or not.*

Proof: A bipolar orientation of G will induce, according to Lemma 5.4, an orientation of $A(G)$ such that all the vertices of $A(G)$ (except the V-vertices incident to e_0) have indegree 2. Then, the validity of Algorithm 5 follows from Lemma 5.3. □

Remark 5.6 *Using a particular e_0-bipolar orientation [8], we can ensure that all the circuits of the angle-graph are clockwise (the external face corresponding to e_0). Then, as the vertices and edges of the dual of the angle-graph are nothing but the edges and the angles of the original graph, Algorithm 5 may be translated on the original graph itself. Using the property of the particular e_0-bipolar orientation, we obtain (optimized) Algorithm 6.*

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