

Algorithms and Characterizations for 2-Layer Fan-planarity: From Caterpillar to Stegosaurus

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Abstract

In a *fan-planar drawing* of a graph there is no edge that crosses two other independent edges. We study *2-layer fan-planar drawings*, i.e., fan-planar drawings such that the vertices are restricted to two distinct horizontal layers and edges are straight-line segments that connect vertices of different layers. We characterize 2-layer fan-planar drawable graphs and describe a linear-time testing and embedding algorithm for biconnected graphs. We also study the relationship between 2-layer fan-planar graphs and 2-layer right-angle crossing graphs.

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1 Introduction

In a *2-layer drawing* of a graph, each vertex is drawn as a point on one of two distinct horizontal layers and each edge is drawn as a straight-line segment that connects vertices of different layers. Clearly, a graph admits such a drawing if and only if it is bipartite. The study of 2-layer drawings has a long tradition in Graph Drawing for two main reasons: (i) 2-layer drawings are a natural way to visually convey bipartite graphs; (ii) algorithms that compute such drawings represent a building block for the popular *Sugiyama's framework* [32, 33], used to draw graphs on multiple horizontal layers.

Since it is commonly accepted that edge crossings negatively affect the readability of a diagram (see, e.g., [29, 30, 35]), the study of 2-layer drawings has focused for a long time on the minimization of edge crossings. Eades *et al.* proved that a connected bipartite graph admits a crossing-free 2-layer drawing if and only if it is a *caterpillar* [16], i.e., a tree for which the removal of all vertices of degree one produces a path. Eades and Whitesides proved that the problem of minimizing edge crossings in a 2-layer drawing is NP-hard [17] and, as a consequence, many papers focused on efficient heuristics or exact exponential techniques for computing 2-layer drawings with minimum number of edge crossings; a limited list of these papers includes [15, 18, 23, 27, 34].

More recently, a growing attention has been devoted to the study of graph drawings where edge crossings are allowed under some specific restrictions, which still guarantee a good readability of the layout. In particular, motivated by cognitive experiments of Huang *et al.* [22], several papers investigated *right angle crossing drawings* (*RAC drawings* for short) [13], in which the edges can cross only at right angles (see [14] for a survey on the subject). Di Giacomo *et al.* characterized the class of bipartite graphs that admit a RAC drawing on two layers, and described a linear-time testing and embedding algorithm for 2-layer RAC drawable graphs [10]. Heuristics for computing the maximum 2-layer RAC subgraph of a given graph are also described in the literature [11].

In this paper we concentrate only on *2-layer fan-planar drawings*, i.e., 2-layer drawings that are also *fan-planar*. In a fan-planar drawing an edge can only cross edges having a common end-vertex, thus an edge cannot cross two independent edges (see Figure 1). Fan-planar drawings were introduced by Kaufmann and Ueckerdt [24], who showed that fan-planar graphs with n vertices have at most $5n - 10$ edges, which is a tight bound. Subsequent papers proved that recognizing fan-planar graphs is NP-hard and studied restricted classes of fan-planar graphs in terms of density and recognition algorithms [4, 6]. In particular, it is shown that 2-layer fan-planar drawings have at most $2n - 4$ edges, which is also a tight bound [6]. From an application perspective, it has been observed that fan-planar drawings may be used to create confluent drawings with few edge crossings per edge [6]. Our contribution is as follows:

- (i) We first study biconnected graphs (Section 3). We prove that a biconnected graph is 2-layer fan-planar if and only if it is a spanning subgraph of a *snake* graph (Section 3.1), which is a chain of complete bipartite

graphs $K_{2,h}$ (see Definition 1). We also describe a linear-time algorithm that tests whether a biconnected graph admits a 2-layer fan-planar drawing, and that computes such a drawing if it exists (Section 3.2).

- (ii) We then give a characterization of the class of graphs that admit a 2-layer fan-planar drawing (Section 4). We prove that a connected graph is 2-layer fan-planar if and only if it is a subgraph of a *stegosaurus* graph, a further generalization of a snake (see Definition 2). Since every 2-layer crossing-free drawing is also fan-planar, but not vice versa, caterpillars are a proper subclass of stegosaurus.
- (iii) We explore the relationship between 2-layer fan-planar and 2-layer RAC drawable graphs (Section 5). We prove that, for biconnected graphs the first class is properly included in the second one, while there is no inclusion relationship for general graphs.

We conclude this introduction by observing that our results fall in a research line often referred to as “beyond planarity”. The general framework of this line is to relax the planarity constraint by allowing edge crossings, but still forbidding those configurations that would affect the readability of the drawing too much. Different types of forbidden edge-crossing configurations give rise to different families of beyond planar graphs. RAC graphs and fan-planar graphs, discussed above, are examples of these families. Other remarkable examples are *k-quasi planar graphs* and *k-planar graphs*. For any integer $k \geq 3$, the family of *k-quasi planar graphs* is the set of graphs that have a drawing with no k mutually crossing edges (see, e.g., [1, 2, 19]). For any positive integer k , the family of *k-planar graphs* is the set of graphs that admit a drawing with at most k crossings per edge [28]; in particular, *1-planar graphs* have been widely studied in the literature (see, e.g., [3, 5, 8, 12, 20, 21, 25, 26, 31]). Several classical problems concerned with computing crossing-free drawings or drawings that minimize the number of crossings, where vertices are constrained to be on specific lines, point sets, or surfaces, can be reinterpreted in terms of beyond planarity.

2 Preliminaries

We assume familiarity with basic concepts of graph drawing and planarity [9]. Throughout the paper, a graph with a fixed planar (outerplanar) embedding is also called a *plane (outerplane) graph*. Let G be a graph. For each vertex v of G , the set of edges incident to v is called the *fan of v* . Each edge (u, v) of G belongs to the fan of u and to the fan of v at the same time. Two edges that do not share a vertex are called *independent edges*, and always belong to distinct fans. A *fan-planar drawing* Γ of G is a drawing such that: (a) no edge is crossed by two independent edges (the forbidden configuration of Figure 1(a)); (b) there are not two adjacent edges (u, v) , (u, w) that cross an edge e from different “sides” while moving from u to v and from u to w (the forbidden configuration of Figure 1(b)).

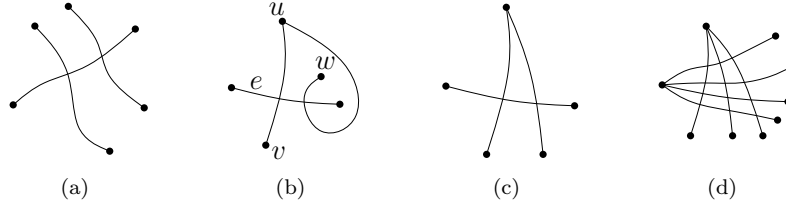


Figure 1: (a)-(b) Forbidden and (c)-(d) allowed configurations of fan-planar drawings.

Two allowed configurations of a fan-planar drawing are in Figures 1(c) and 1(d). A *fan-planar graph* is a graph that admits a fan-planar drawing. Observe that in a straight-line drawing, the forbidden case (b) cannot happen. By definition, a fan-planar drawing does not contain 3 mutually crossing edges.

In a *2-layer drawing* of a graph, each vertex is drawn as a point on one of two distinct horizontal lines, called *layers*, and each edge is drawn as a straight-line segment that connects vertices of different layers. A *2-layer fan-planar drawing* is a 2-layer drawing that is also fan-planar. A *2-layer fan-planar graph* is a graph that admits a 2-layer fan planar drawing. Clearly, every graph that has a 2-layer drawing is bipartite. For a given 2-layer drawing of a bipartite graph $G = (V_1, V_2, E)$, denote by ℓ_i the horizontal line on which the vertices of V_i are drawn ($i = 1, 2$). We always assume that ℓ_1 is above ℓ_2 . Two 2-layer drawings of G are equivalent if they have the same left-to-right order π_i of the vertices of V_i along ℓ_i ($i = 1, 2$). A *2-layer embedding* is an equivalence class of 2-layer drawings and it is described by a pair of linear orderings (i.e., permutations) $\gamma = (\pi_1, \pi_2)$ of the vertices in V_1 and V_2 , respectively. Given any two vertices u and v of V_i , we write $u \prec v$ if $\pi_i(u) < \pi_i(v)$ ($i = 1, 2$). Also, the first (last) vertex of π_1 and the first (last) vertex of π_2 are the *leftmost vertices* (*rightmost vertices*) of γ . The edge connecting the leftmost (rightmost) vertices of γ (if it exists) is called the *leftmost edge* (the *rightmost edge*) of γ . If Γ is a drawing within class γ , we say that γ is the *embedding* of Γ . If Γ is a 2-layer fan-planar drawing, we also say that γ is a *2-layer fan-planar embedding*. Since any geometric position of the vertices that respects the two linear orderings defined by γ yields a 2-layer fan-planar drawing in linear time, we will concentrate on embeddings in the following. We say that γ is *maximal* if for any two vertices u and v that are not adjacent in G , the embedding obtained from γ by adding the edge (u, v) is no longer 2-layer fan-planar. Similarly, a 2-layer fan-planar graph is *maximal*, if it cannot be augmented by an edge without losing the property of being 2-layer fan-planar.

3 Biconnected 2-Layer Fan-planar Graphs

Let G_1 and G_2 be two graphs. The operation of merging G_1 and G_2 by identifying an edge e_1 of G_1 with an edge e_2 of G_2 (in one of the two possible ways) is called an *edge merging*; the resulting graph G is called a *merger* of G_1 and

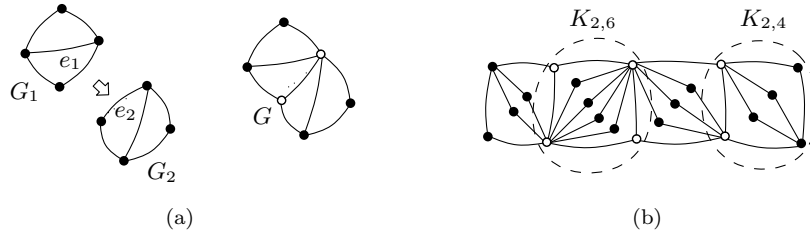


Figure 2: (a) Edge merging of two graphs. (b) A snake.

G_2 with respect to e_1, e_2 . The end-vertices of the edge obtained by identifying e_1 with e_2 are *merged vertices* of G . In Figure 2(a), the white vertices in the merger graph are the merged vertices.

Definition 1 A snake is a graph recursively defined as follows:

- (i) A complete bipartite graph $K_{2,h}$ ($h \geq 2$) is a snake.
- (ii) A merger of two snakes G_1 and G_2 with respect to edges e_1 of G_1 and e_2 of G_2 , with the property that none of the end-vertices of e_i is a merged vertex of G_i ($i = 1, 2$), is a snake.

Intuitively, a snake is a bipartite planar graph consisting of a chain of complete bipartite graphs $K_{2,h}$ (see Figure 2(b)). Note that we exclude the case $h = 1$ in our definition of a snake as we require biconnectivity. An alternative definition of a snake can be derived from the definition of *ladder*, i.e., a bipartite outerplanar graph consisting of two paths of the same length $\langle u_1, u_2, \dots, u_{\frac{n}{2}} \rangle$ and $\langle v_1, v_2, \dots, v_{\frac{n}{2}} \rangle$ plus the edges (u_i, v_i) ($i = 1, 2, \dots, \frac{n}{2}$) (see also [10]); the edges (u_1, v_1) and $(u_{\frac{n}{2}}, v_{\frac{n}{2}})$ are called the *extremal edges* of the ladder. A snake is a planar graph obtained from an outerplane ladder, by adding, inside each internal face, an arbitrary number (possibly none) of paths of length two connecting a pair of non-adjacent vertices of the face.

3.1 Characterization

The characterization of the biconnected graphs that admit a 2-layer fan-planar embedding is given by Theorem 3. The proof is based on the next two lemmas.

Lemma 1 Let G be a biconnected graph. If G admits a maximal 2-layer fan-planar embedding γ then G is a snake.

Proof: Due to maximality, the leftmost and the rightmost edges of γ always exist, and do not cross any other edge. Therefore, γ contains at least two uncrossed edges. We prove the statement by induction on the number $l \geq 2$ of uncrossed edges in γ . Recall that, since G is biconnected, it has vertex-degree at least two.

Base case: $l = 2$. In this case, we prove that G is a $K_{2,r}$ for some $r \geq 2$, which implies that G is a snake. Note that G cannot be a $K_{1,r}$, since it has vertex-degree at least two. If G contains only four vertices, then G is a $K_{2,2}$, as desired, since there are exactly two uncrossed edges. Suppose now that G has more than four vertices.

Claim 1.1 *Let (u, v) and (w, x) be a pair of crossing edges in γ , such that $u \prec w$ on ℓ_1 and $x \prec v$ on ℓ_2 . Then the edges (u, x) and (w, v) exist.*

Claim's proof: We first prove that (w, v) exists. Denote by \overline{wv} the segment connecting w and v . We distinguish between two cases: (i) There is no edge traversing \overline{wv} ; in this case, since w and v can be connected by an edge without crossings, then (w, v) is the rightmost edge of γ , and thus it exists due to maximality, as already observed. (ii) There is an edge e traversing \overline{wv} ; in this case e must be either incident to u or to x , otherwise one between (u, v) and (w, x) would cross two independent edges. Without loss of generality, assume that e is incident to u and to another vertex z of G , where $v \prec z$. Since v has degree at least two, there exists another edge (y, v) such that $y \neq u$. If $y \prec u$, or $w \prec y$, or $u \prec y \prec w$, then there would be always an edge crossed by two independent edges in γ . It follows that $y = w$, i.e., (w, v) exists. With a symmetric argument we can prove that (u, x) also exists. ■

Claim 1.2 *If G' is a subgraph of G such that G' is a $K_{2,r'}$ (for some $r' > 2$) and G' contains the leftmost and the rightmost edges of γ , then G is a $K_{2,r}$ (for some $r > r'$).*

Claim's proof: Without loss of generality, assume that the $r' > 2$ vertices of G' belong to layer ℓ_2 ; denote by V' this subset of vertices; also, let u and w be the two vertices of G' that belong to ℓ_1 , and assume that $u \prec w$. We show that γ contains only u and w on layer ℓ_1 . Suppose by contradiction that another vertex z exists on ℓ_1 . Since by hypothesis u and w are the leftmost and the rightmost vertices on ℓ_1 , respectively, we have that $u \prec z \prec w$, and z must be adjacent to a vertex v of ℓ_2 occurring between the leftmost and rightmost vertices of V' (v can also coincide with one of these two vertices). It is immediate to verify that (z, v) would cause two independent edges crossed by a third one, a contradiction. Now, any other vertex on ℓ_2 must be connected to both u and w , as it has degree at least two and u and w are the only two vertices on ℓ_1 . ■

We now continue the proof of the base case by using Claims 1.1 and 1.2 to prove that G is a $K_{2,r}$, for some $r > 2$. Consider the rightmost vertex w on ℓ_1 and the rightmost vertex v on ℓ_2 in γ . Due to maximality, edge (w, v) exists and is uncrossed. Also, since w and v have degree at least two, they both have one more incident edge, which we denote by (w, x) and (u, v) . Since w and v are the rightmost vertices, (w, x) and (u, v) cross each other, and thus, by Claim 1.1, edge (u, x) exists. Let H be the $K_{2,2}$ subgraph of G induced by u, v, x , and w . Since we are assuming that G has more than four vertices, there exists a vertex z other than the vertices of H . Without loss of generality, assume that z is on layer ℓ_2 .

If (u, x) is the leftmost edge of γ , then $x \prec z \prec v$, and this implies that z can be adjacent to u and w only, as otherwise (w, x) , (u, v) , and an edge incident to z would form three mutually crossing edges. Also, since z has degree at least two, z is adjacent to both u and w . Thus subgraph G' of G induced by $\{u, v, w, x, z\}$ is a $K_{2,3}$ containing the left- and rightmost edges of γ . By Claim 1.2, G is a $K_{2,r}$, with $r > 2$.

If (u, x) is not the leftmost edge of γ , then since $l = 2$, (u, x) is crossed in γ , and, as observed in the proof of Claim 1.1, it is crossed by an edge having either w or v as an end-vertex. Without loss of generality, suppose that (u, x) crosses an edge (w, z) . By applying Claim 1.1 to (u, x) and (w, z) , edge (u, z) exists. Hence, again, the subgraph G' induced by the vertices of H plus z is a $K_{2,3}$ graph. If (u, z) is the leftmost edge of γ , then by Claim 1.2, G is a $K_{2,r}$, with $r > 2$. If (u, z) is not the leftmost edge, then again it is crossed by an edge having either w or v as an end-vertex. However, since (u, x) is already crossed by (w, z) , (u, z) can only be crossed by edges having w as an end-vertex. Denoted by (w, y) one of the edges that cross (u, z) , we have that edge (u, y) exists by Claim 1.1, and therefore the subgraph induced by the vertices of H plus vertices z and y is a $K_{2,4}$ that contains the rightmost edge of γ . By iterating this argument, we eventually obtain a subgraph $K_{2,r'}$ ($r' > 2$) of G that contains the rightmost and also the leftmost edge of γ , which by Claim 1.2 implies that G is a $K_{2,r}$, with $r > 2$.

Inductive case: $l > 2$. Assume by induction that the statement holds for $l - 1$, with $l > 2$. Consider an uncrossed edge (u, v) different from the leftmost and the rightmost edge of γ . Let γ_1 (resp., γ_2) be the embedding induced by the vertices to the left (resp., right) of (u, v) plus u and v . Clearly, γ_1 and γ_2 are 2-layer fan-planar. Let G_i be the subgraph of G consisting of the vertices and edges of γ_i ($i = 1, 2$). Since (u, v) is uncrossed in γ , G_1 and G_2 are biconnected. Also, each of the two γ_i contains a number $l_i < l$ of uncrossed edges, and thus G_i is a snake by induction hypothesis. Since G is a merger of G_1 and G_2 with respect to (u, v) , G is a snake. \square

Lemma 2 *Every n -vertex snake admits a 2-layer fan-planar embedding, which can be computed in $O(n)$ time.*

Proof: Let G be a snake. We compute a 2-layer fan-planar embedding $\gamma = (\pi_1, \pi_2)$ of G . By definition, G is a chain of graphs G_1, \dots, G_k , such that each G_i is a complete bipartite graph K_{2,h_i} , sharing a pair of merged vertices with G_{i+1} ($i = 1, \dots, k - 1$). We say that the vertices of G to be placed on layer ℓ_1 (ℓ_2) are *white* (*black*, respectively). Choose any ordering π_1 of the white vertices such that, for each $i = 1, \dots, k - 1$: (i) the white vertices of each G_i precede all the white vertices of G_{i+1} ; (ii) the last white vertex of G_i is one of the two merged vertices shared with G_{i+1} (which will be the first white vertex of G_{i+1}). Analogously, do the same for the black vertices, obtaining π_2 . See Figure 3 for an illustration.

It is immediate to see that γ is a 2-layer fan-planar embedding of G , where the only uncrossed edges are those connecting pairs of merged vertices shared

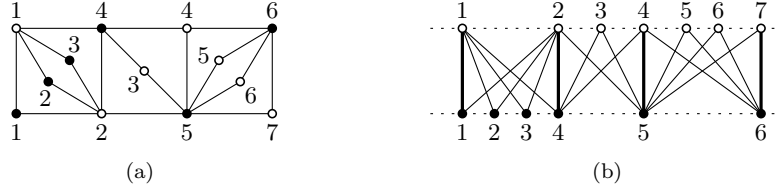


Figure 3: Illustration for the proof of Lemma 2. (a) A snake G ; the vertices of each partite set are ordered (i.e., numbered) according to the rules given in the proof. (b) A 2-layer fan-planar drawing of G whose embedding reflects the vertex ordering; the uncrossed edges are in bold.

by G_i and G_{i+1} ($i = 1, \dots, k-1$), and those two connecting the first and last white and black vertices. Furthermore, γ can be computed in $O(n)$ time. \square

Theorem 3 *A biconnected graph G is 2-layer fan-planar if and only if G is a spanning subgraph of a snake.*

Proof: Suppose first that G has a 2-layer fan-planar embedding γ . If γ is maximal, then G is a snake by Lemma 1. Else, there is a maximal 2-layer fan-planar embedding γ' of a graph G' such that: (i) $G \subset G'$, (ii) G' has the same vertex set of G , and (iii) the restriction of γ' to G coincides with γ . Hence, by Lemma 1, G is a spanning subgraph of a snake. Conversely, let G be a spanning subgraph of a snake. Since any spanning subgraph of a 2-layer fan-planar graph is also 2-layer fan-planar, G is 2-layer fan-planar by Lemma 2. \square

3.2 Testing and Embedding Algorithm

We now describe an algorithm to test whether a given biconnected bipartite graph G is 2-layer fan-planar. Since every biconnected 2-layer fan-planar graph is a spanning subgraph of a snake (Theorem 3), the algorithm must check whether G can be augmented to a snake by only adding a suitable set of edges. In what follows we assume that the input graph G is not a simple (even) cycle, as otherwise it is clearly 2-layer fan-planar.

We require some further definitions that will be used to describe the algorithm and to prove its correctness. A *chain* $P = \langle u, v_1, v_2, \dots, v_k, v \rangle$ of G is a maximal path of G such that all its internal vertices v_i have degree 2 in G ($i = 1, \dots, k$). *Contracting P* is to transform G into a new graph G' obtained from G by replacing P with a single edge $e_P = (u, v)$ of weight $w(e_P) = k$. Reversely, we can say that G is obtained from G' by *expanding e_P* (P is the *expansion* of e_P). Note that G' may have multiple edges that connect u and v . If G is a plane graph, we assume that the contraction of P preserves the embedding of G . See also Figure 4 for an illustration. The *weighted contraction* of G is the edge-weighted multi-graph $\mathcal{C}(G)$ obtained from G by contracting every maximal chain of G ; all edges of $\mathcal{C}(G)$ that are also in G are assigned weight 0.

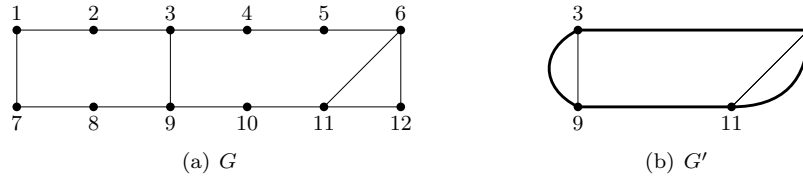


Figure 4: The graph G in (a) is transformed into the new graph G' in (b) by contracting the paths $P_1 = \{3, 2, 1, 7, 8, 9\}$, $P_2 = \{3, 4, 5, 6\}$, $P_3 = \{9, 10, 11\}$, and $P_4 = \{6, 11, 12\}$. Graph G can be obtained from G' by expanding the edges $e_1 = (3, 9)$, $e_2 = (3, 6)$, $e_3 = (9, 11)$, and $e_4 = (6, 12)$.

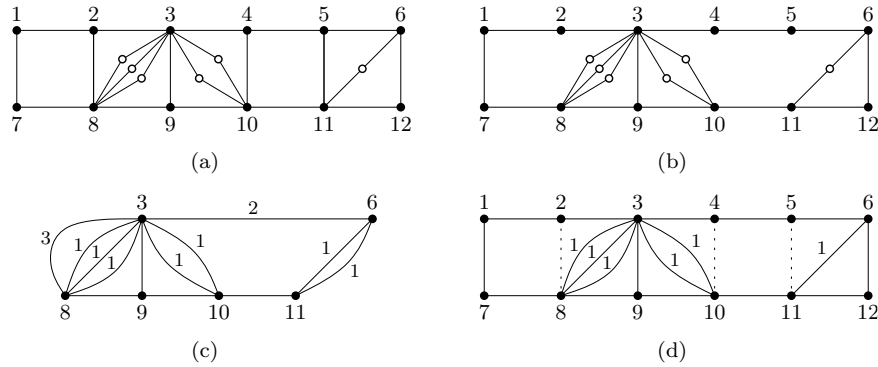


Figure 5: Illustration for Lemma 4. (a) A plane snake \overline{G} consisting of an outerplane ladder (black vertices) with arbitrary paths of length two inside each internal face. (b) A (plane) biconnected spanning subgraph G of \overline{G} . (c) The (plane) weighted contraction $\mathcal{C}(G)$; only edge weights greater than 0 are shown. (d) The plane multi-graph G^* of property (c) in the statement of Lemma 4.

Figure 5(c) shows the weighted contraction of the graph in Figure 5(b). Based on weighted contractions, we can reinterpret the characterization of 2-layer fan-planar graphs as follows (see Figure 5):

Lemma 4 *Let G be a bipartite biconnected graph that is not a simple cycle. G is a spanning subgraph of a snake if and only if its weighted contraction $\mathcal{C}(G)$ has a planar embedding such that:*

- (a) *All vertices of $\mathcal{C}(G)$ are on the external face.*
- (b) *All edges e_P of $\mathcal{C}(G)$ with $w(e_P) \geq 2$ are on the external face.*
- (c) *Let G^* be the plane multi-graph obtained from $\mathcal{C}(G)$ by expanding all edges e_P of the external face. It is possible to add to G^* internal edges of weight 0, such that the resulting graph H^* is outerplane and the removal of the internal edges of weight 1 from H^* produces a ladder.*

Proof: Suppose first that $\mathcal{C}(G)$ has a planar embedding that verifies properties (a), (b), and (c). We prove that G is a spanning subgraph of a snake. Consider the multi-graph H^* of property (c). By property (a) and (b), H^* is an outerplane multi-graph that does not have internal edges of weight greater than 1. The expansion of each internal edge of weight 1 is a path of length 2. Therefore, by property (c) and since G is bipartite, the graph \overline{G} obtained from H^* by expanding each internal edge of weight 1 is an outerplane ladder plus a set of paths of length two connecting non-adjacent vertices of internal faces of the ladder. Therefore, by definition, \overline{G} is a snake. Since \overline{G} is super-graph of G with the same vertex-set of G , then G is a spanning subgraph of a snake.

Suppose vice versa that G is a spanning subgraph of a snake \overline{G} . By definition, \overline{G} can be planarly embedded so that it is an outerplane ladder such that inside each internal face there is an arbitrary number (possibly none) of paths of length two connecting a pair of non-adjacent vertices of the face. Since G is spanning and biconnected, the only edges of \overline{G} that are not in G are chords of the outerplane ladder (the removal of any other type of edge, without the removal of vertices, will produce either a cut-vertex or an isolated vertex in the subgraph). Let $\mathcal{C}(G)$ be the (plane) weighted contraction of G . Each vertex that is not on the external face in the planar embedding of \overline{G} is a degree-two vertex of a path P of length two; since P will be replaced by an edge e_P in $\mathcal{C}(G)$ such that $w(e_P) = 1$, it follows that properties (a) and (b) hold. Property (c) immediately follows by observing that the outerplane graph H^* can be obtained by adding to G^* the chords of \overline{G} that do not belong to G . \square

We now give a linear-time algorithm, called **Bic2LFPTest**, that tests whether a bipartite biconnected graph G has a 2-layer fan-planar embedding, and that constructs such an embedding in the positive case. The algorithm checks whether $\mathcal{C}(G)$ admits a planar embedding with the properties (a), (b), and (c) of Lemma 4. If such an embedding exists, a snake for which G is a spanning subgraph is obtained by expanding the edges of weight 1 in the multi-graph H^* of property (c); a 2-layer fan-planar embedding of this snake (and hence of G) is obtained using the construction in the proof of Lemma 2. Figure 6 shows an example of a 2-layer fan-planar embedding computed by algorithm **Bic2LFPTest**.

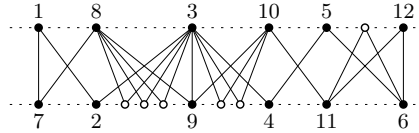


Figure 6: A 2-layer fan-planar drawing of the spanning subgraph of Figure 5(b).

Algorithm **Bic2LFPTest** (G)

Step 1. Compute the weighted contraction $\mathcal{C}(G)$ of G , and compute, if any, an outerplanar embedding of $\mathcal{C}(G)$ (i.e., property (a) of Lemma 4). This can

be done in linear time: temporarily add to $\mathcal{C}(G)$ a dummy vertex u and a dummy edge (u, v) for every vertex v of $\mathcal{C}(G)$; then run a linear-time planarity testing and embedding algorithm (e.g. [7]) on it. Note that, since $\mathcal{C}(G)$ is still biconnected, the outerplanar embedding of $\mathcal{C}(G)$ is unique (if it exists), except for the permutation of multi-edges. If $\mathcal{C}(G)$ is not outerplanar, the whole test is negative and the algorithm stops, otherwise an outerplanar embedding is found and the algorithm goes to the next step.

Step 2. Check whether the outerplanar embedding can be modified (if needed) so that all edges with weight greater than 1 can be put on the external face (property (b) of Lemma 4), keeping all vertices on the external face. This is possible if and only if: (i) for every pair of consecutive vertices $\{u, v\}$ on the boundary of the external face there is at most one copy of edge $e = (u, v)$ with $w(e) \geq 2$ (which can be then put on the external face), and (ii) there is no chord with weight greater than 1. Both conditions (i) and (ii) can be checked in linear time. If this check fails, then the whole test is negative, otherwise the new outerplanar embedding with the heaviest edges on the external face is computed and the algorithm goes to the next step.

Step 3. Expand the external edges with weight greater than 0 to get the multi-graph G^* in property (c) of Lemma 4; this can be done in linear time if we suitably store the chain P associated with each edge e_P when $\mathcal{C}(G)$ is computed in Step 1. Then, check whether it is possible to add to G^* a suitable set of internal edges (chords) connecting vertices of the external face such that the resulting multi-graph H^* is still outerplane and becomes a ladder if we subsequently remove the internal edges of weight 1 (property (c) of Lemma 4). This can be done with the following procedure. If H^* already contains a chord of weight 0, then: (i) temporarily remove the edges with weight 1; (ii) verify whether the resulting graph can be augmented with extra chords to an outerplane ladder, using the linear-time algorithm described by Di Giacomo *et al.* [10]. We remark that, if such an augmentation exists it is unique under the assumption that H^* already contains a chord of weight 0; (iii) check whether the removed edges with weight 1 can be reinserted inside the outerplane ladder without violating planarity (which can be done in linear time by verifying that, for each removed edge, its two end-vertices are in the same face of the outerplane ladder). If H^* does not contain a chord with weight 0, then H^* contains at least one chord $e = (u, v)$ with weight 1 (we assumed that G is not a simple cycle, hence H^* contains at least one chord). In this case, consider the two vertices u_1, u_2 that are adjacent to u on the boundary of the external face, and the two vertices v_1, v_2 that are adjacent to v on the boundary of the external face (some of these vertices may coincide). It can be seen that any edge augmentation of H^* that leads to an outerplane ladder with the edges of weight 1 inside its internal faces, must include at least one chord $e' \in C = \{(u, v_1), (u, v_2), (v, u_1), (v, u_2)\}$ (in particular, in the outerplane ladder either two edges of C are chords or one is a chord and one is an extremal edge of the ladder). Hence, for each of these (at most four) chords e' , try to add e' to H^* and then repeat the substeps (i)–(iii) described above. If the augmentation fails for all possible choices of e' , the

whole test is negative, otherwise it is positive and a snake that contains G as a spanning subgraph is obtained. A 2-layer fan-planar embedding of this snake coincides with that of G , and is computed using the construction of Lemma 2.

Theorem 5 *Let G be a bipartite biconnected graph with n vertices. There exists an $O(n)$ -time algorithm that tests whether G is 2-layer fan-planar, and that computes a 2-layer fan-planar embedding of G in the positive case.*

4 Simply Connected 2-Layer Fan-Planar Graphs

We have proved that a biconnected graph is 2-layer fan-planar if and only if it is a spanning subgraph of a snake. We now show that a (simply) connected graph is 2-layer fan-planar if and only if it is a subgraph of a *stegosaurus* (see Definition 2 hereunder). Clearly, a non-connected graph is 2-layer fan-planar if and only if every connected component is a 2-layer fan-planar graph.

Recall that snakes are obtained by merging edges of a sequence of several $K_{2,h}$ ($h \geq 2$). We may denote the partite set with more than 2 vertices (if any) the *large side* of a $K_{2,h}$. Given a snake G , a vertex in G is *mergeable* if it is an end-vertex of a mergeable edge and belongs to the large side of an original $K_{2,h}$. Note that a snake always has at most two mergeable vertices; by definition, a $K_{2,2}$ on either end of the snake prohibits a mergeable vertex. The graph resulting from merging two graphs G_1 and G_2 by identifying a mergeable vertex of G_1 with a mergeable vertex of G_2 is a *vertex merger*.

Definition 2 *A stegosaurus is either a fan (a trivial stegosaurus) or a graph recursively defined as follows (Figure 7(a)):*

- (i) *A snake is a stegosaurus, whose mergeable vertices are the mergeable vertices of the snake.*
- (ii) *The vertex merger of two stegosaurus G_1, G_2 is a stegosaurus. Its mergeable vertices are those (at most one per G_1, G_2) not used in this merging.*
- (iii) *Let v be a mergeable vertex of a stegosaurus G_1 . Adding a new vertex v' and an edge (v, v') gives a stegosaurus with the same mergeable vertices as G_1 .*

Observation 1 *Consider merging two snakes G_1, G_2 at vertices v_1, v_2 . Assume that v_1 is an end-vertex of a mergeable edge but not from a large side; v_2 may be chosen as v_1 or be a mergeable vertex. Then, the merged graph would be a subgraph of a snake (Figure 7(b)). Thus, only vertices from the large side have to be considered in Definition 2.*

In the following, a block of a graph (i.e., a biconnected component) is called *trivial* if it consists of a single edge. Let an edge e be a trivial block. If e has an end-vertex of degree 1, e is a *stump*, otherwise, it is a *bridge*. Observe that, in

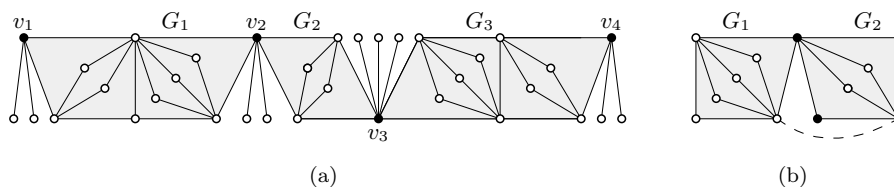


Figure 7: (a) A stegosaurus composed of three snakes G_1, G_2, G_3 that have been merged at v_2, v_3 and several edges have been attached to v_1, \dots, v_4 . (b) The result of merging snakes G_1, G_2 using a non-mergeable vertex can be augmented into one snake by adding the dashed edge.

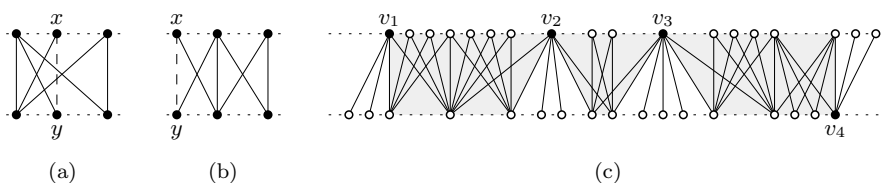


Figure 8: (a) A maximal 2-layer fan-planar drawing and (b) a different embedding to which one may add the edge (x, y) . (c) A 2-layer fan-planar drawing of the stegosaurus from Figure 7(a).

contrast to the biconnected case, we have the situation that an embedding (or drawing) of G is maximal 2-layer fan-planar (i.e., we cannot add an edge within this embedding), but the graph is not maximal 2-layer fan-planar; it “simply” requires a different 2-layer fan-planar embedding into which we can add another edge. Figures 8(a) and 8(b) show an example. By definition and Theorem 3, a biconnected graph is 2-layer fan-planar if and only if it is the subgraph of a snake, and thus, of a stegosaurus. Also, a simply connected graph that is a subgraph of a snake is 2-layer fan-planar. We will first show that stegosaurus are 2-layer fan-planar. Then, we will show that every 2-layer fan-planar graph is a subgraph of a stegosaurus.

Lemma 6 *Every stegosaurus has a 2-layer fan-planar embedding.*

Proof: Figure 8(c) outlines the idea. We already know that snakes are 2-layer fan-planar and how to draw them, and, by definition, that the non-trivial blocks of a stegosaurus are snakes. Drawing a stegosaurus hence means drawing the individual snakes simply next to each other. Moreover, we can draw additional trivial blocks (arising from *iii*) in the definition) at the left and right “ends” of the stegosaurus, as well as at its cut vertices. \square

Clearly a trivial stegosaurus is a maximal 2-layer fan-planar graph. Hence, in the following, we only consider non-trivial stegosaurus. We start with proving a property that holds for all 2-layer fan-planar drawings, not only for maximal ones:

Lemma 7 *Let B be a non-trivial block of a 2-layer fan-planar graph G , and e an independent edge, i.e., none of its end-vertices belongs to B . No edge of B can be crossed by e in any 2-layer fan-planar embedding of G .*

Proof: Assume there is an embedding where some edge $b \in E[B]$ is crossed by e . Since B is a non-trivial block, b is part of a cycle $C \subseteq E[B]$ with $|C| \geq 4$. Hence, by the properties of 2-layer embeddings, e needs to cross another edge $c \in C$ as well. The edges b, c need to be adjacent, as otherwise we would get pairwise crossings between three independent edges. Embedding a cycle, in our case C , on two layers, requires a crossing of every edge except for two non-adjacent edges. Hence either b or c will have a crossing with another (non-adjacent and hence independent) edge of C and edge e , which is independent w.r.t. B ; a contradiction. \square

From the above lemma, we obtain a simple but useful observation:

Corollary 8 *In a 2-layer fan-planar embedding, two non-trivial blocks cannot cross.*

Hence we know that in a 2-layer fan-planar drawing, non-trivial blocks are “nicely” placed next to each other from left to right without crossings between them. We now show several properties of maximal 2-layer fan-planar graphs. Clearly, a maximal 2-layer fan-planar graph will be connected.

Lemma 9 *Let G be a maximal 2-layer fan-planar graph. There exists an embedding γ of G in which no stump is crossed.*

Proof: We need not to discuss trivial stegosaurus, as there the lemma is obvious. Assume there are multiple stumps incident to the same vertex v , and let e be one of them. If e is feasibly drawn, we can also draw all other these stumps directly next to e (i.e., the stumps’ end-vertices form a contiguous vertex subsequence on the respective layer). Hence, w.l.o.g., we can in the following assume that every vertex is incident to at most one stump. Let γ be a 2-layer fan-planar embedding with the least number of crossings between stumps.

Claim 9.1 *Two stumps do not cross.*

Claim’s proof: Assume to the contrary that there exist two stumps (v, w) and (a, b) that cross, where b and w are the degree-one vertices. We distinguish two cases based on the location of the cut vertices.

In the first case, the two cut vertices of the stumps are on the same layer and w.l.o.g. $v \prec a$ and $b \prec w$ (cf. Figure 9(a)). Every vertex between b and w must be adjacent to v and a , because otherwise there would exist a second stump at v or a (recall that we assumed that every vertex is incident to at most one stump) or a second independent crossing on (v, w) or (a, b) . Hence, we may just swap b and w , thereby, resolving one crossing between two stumps without introducing another one. This contradicts the minimality of the embedding γ w.r.t. to stump crossings.

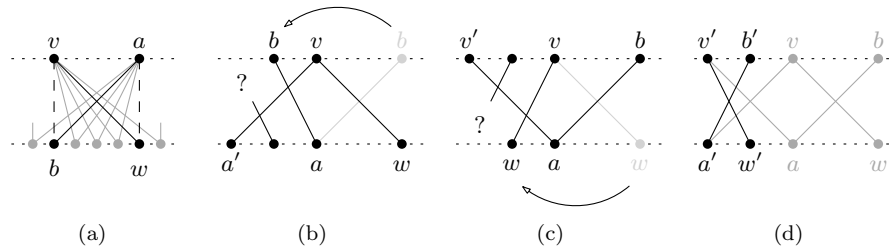


Figure 9: (a) Two crossing stumps that have their cut vertices on the same layer. (b) An edge (v, a') prevents the repositioning of b such that it precedes v . (c) A symmetric situation when moving w directly to the left of a . (d) The situation when we cannot move w nor b . There exist two stumps (v', w') , (a', b') that create the two crossings on (v', a) , (v, a') .

Now, assume the two cut vertices are on different layers. W.l.o.g. assume that b and w are to the right of the cut vertices v and a , respectively, i.e., $v \prec b$ and $a \prec w$. Furthermore, we choose (v, w) and (b, a) such that they are the two leftmost crossing stumps having their cut vertices on different layers and to the left of the involved degree-one vertices. Notice that unlike in the first case, there are no vertices between v and b , and a and w , because these can only be adjacent to a and v , respectively. However, such a vertex implies then a second stump at one of the two vertices, a contradiction.

We show that we can resolve the crossing between the two stumps by either moving b directly to the left of v or w directly to the left of a . Assume that we cannot move b such that it immediately precedes v (cf. Figure 9(b)), because (b, a) would cross an edge (which has to be incident to v due to the absence of vertices between v and the original position of b). Let this edge be (v, a') , and (v, a') is already crossed by some other edge independent of (b, a) . Clearly, such a scenario is possible and prevents us from moving b , but now we may try to move w instead. By a symmetric argument, we may assume that in case we cannot move w , there exists an edge (v', a) that would cross (v, w) and is already crossed by some other edge that is independent of (v, w) (cf. Figure 9(c)). Hence, if we can neither move b nor w , both, (v, a') and (v', a) , exist and they cross. Furthermore, each of them is crossed a second time to prevent the movement of w or b . These second crossings are both not allowed to be independent to the crossing of (v, a') and (v', a) , but the corresponding edges are neither incident to v nor a , respectively. Hence, there exist two stumps (v', w') and (b', a') such that $v' \prec b' \prec v$ and $a' \prec w' \prec a$ holds (cf. Figure 9(d)). However, this contradicts our choice of (v, w) and (a, b) , since (v', w') and (a', b') are two crossing stumps of the same type being to the left of (v, w) and (a, b) . Thus, we can move either w or b and resolve the stump crossing (again without introducing a new one), contradicting the crossing minimality of the embedding. We conclude that for every maximal 2-layer fan-planar graph there exists an embedding in which no two stumps cross. ■

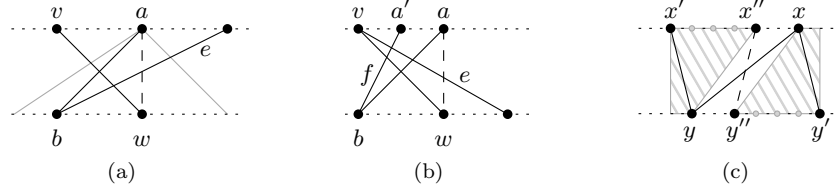


Figure 10: (a) The case in which edge e crosses (a, w) while being incident to b , thus, trapping a such that $\deg(a) = 1$. (b) The other case in which e is incident to v and $f = (a', b)$ is a stump. (c) An uncrossed bridge (x, y) whose adjacent edges do not cross enabling the insertion of (x'', y'') .

Next we use this property to show that a stump does not cross anything at all.

Claim 9.2 *A stump does not cross a non-stump.*

Claim's proof: Again assume to the contrary that a stump (v, w) (with v being the cut vertex) crosses an edge (a, b) that is not a stump. Furthermore, suppose w.l.o.g. that $v \prec a$ and $b \prec w$ holds. We argue now that one can insert the edge (a, w) , contradicting the maximality of G . In order to prevent us from doing so, there must exist two independent edges, say e and f , both independent of (a, w) , possibly crossing each other, and at least one of them crosses (a, w) .

Assume that e crosses (a, w) , then e also crosses either (v, w) or (a, b) . Since (v, w) and (a, b) are independent crossing edges, e must be incident to either v or b . Let us consider the case in which e is incident to b (cf. Figure 10(a)). Since e then crosses also (v, w) , it follows that $\deg(a) = 1$, because any additional edge would create two independent crossings on either (v, w) or e . However, (a, b) is not a stump, because then two stumps would cross. Thus, $\deg(a) \geq 2$ holds, a contradiction.

We may conclude that if e or f crosses (a, w) , then the corresponding edge cannot be incident to b . Since we require e and f to be independent, they cannot be both incident to v . Thus, e, f cannot both cross (a, w) . If f does not cross (a, w) , it must cross e to prevent us from inserting (a, w) . So, w.l.o.g., the only configuration remaining is the one in which the two cross, e is incident to v , crosses (a, w) , and f is incident to b , but does not cross (a, w) . Let $f = (a', b)$ be such that $v \prec a' \prec a$, i.e., the only possible place for a' (cf. Figure 10(b)). With the same argument we used for a , we may now claim that $\deg(a') = 1$ holds. Then, f is a stump, crossing the stump (v, w) , a contradiction to Claim 9.1 that no two stumps cross. Hence there cannot exist e and f , and we thus may insert (a, w) , contradicting that G is maximal 2-layer fan-planar. ■

This establishes the lemma and shows that the embedding γ with the minimum number of stump crossings is one in which no stumps are crossed at all. □

Lemma 10 *A maximal 2-layer fan-planar graph G does not contain bridges.*

Proof: We first show that in case there would exist bridges, we may embed G such that none of them is crossed. This result is then used to show that one may augment G and therefore to contradict its maximality. We choose the embedding γ according to Lemma 9, i.e., no stump is crossed in γ .

Claim 10.1 *No bridge is crossed in γ .*

Claim's proof: Assume that there exists a bridge (x, y) crossed by an edge (a, b) in γ such that w.l.o.g. $x \prec a$ and $b \prec y$ holds. Since both edges are not stumps, all four vertices have degree at least two. Note that, Claim 1.1 in the proof of Lemma 1 only requires G to have vertex-degree at least two (which in the proof of Lemma 1 is implied by biconnectivity). From such a claim it follows that the edges (x, b) , (a, y) exist, inducing a cycle that contains (x, y) which contradicts that (x, y) is a bridge. ■

Claim 10.2 *Two edges adjacent to the same bridge do not cross in γ .*

Claim's proof: Assume that there exist two such crossing edges, say e, f , that are adjacent to a bridge b . Clearly, they have no vertex in common, and none of them is a stump or bridge (due to the choice of γ). Hence, e and f belong to distinct non-trivial blocks, because b is a bridge, thus, contradicting Corollary 8. ■

Assume that there exists a bridge (x, y) in G . By the choice of γ , (x, y) is uncrossed, and by Claim 10.2, w.l.o.g., all neighbors of x are located to the right of y , whereas all neighbors of y are located to the left of x (cf. Figure 10(c)). Therefore, we can add the edge (x'', y'') where x'' (y'') is the direct predecessor of x (direct successor of y). While (x'', y'') crosses (x, y) , there cannot exist any other edge crossing this new edge. Hence, one may insert (x'', y'') , thereby, contradicting maximality of G . □

Corollary 11 *Let G be a maximal 2-layer fan-planar graph. There exists an embedding in which no two blocks cross. Any cut vertex is either contained in two non-trivial blocks, or is a left- or rightmost vertex in this embedding.*

Hence we have that a maximal 2-layer fan-planar graph allows a drawing where non-trivial blocks are neither crossed by other non-trivial nor by trivial blocks. Furthermore, in contrast to the non-biconnected case, if an embedding of a biconnected graph G is maximal 2-layer fan-planar, then G is maximal 2-layer fan-planar. We can deduce:

Corollary 12 *Let G be a maximal 2-layer fan-planar graph. Its non-trivial blocks are maximal 2-layer fan-planar biconnected graphs, i.e., snakes.*

Lemma 6 and Corollary 12 imply the following.

Theorem 13 *A graph is 2-layer fan-planar if and only if it is a subgraph of a stegosaurus.*

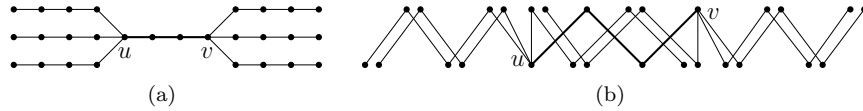


Figure 11: (a) The tree T_3 ; it is 2-layer RAC but not 2-layer fan-planar. (b) A 2-layer RAC embedding (not drawing) of T_3 . In both figures the path connecting u to v has bold edges.

5 Relationship with 2-layer RAC drawings

In a RAC drawing of a graph an edge cannot cross two edges that share a vertex, thus “fan-crossings” are in general not allowed in this drawing model. Conversely, a RAC drawing may contain an edge that crosses other two independent edges, differently from a fan-planar drawing. Hence, a drawing that is RAC and fan-planar at the same time, is necessarily *1-planar*, i.e., each edge crosses at most once. Placing the vertices of a graph on two distinct horizontal layers represents however a strong restriction in both models, which significantly reduces their allowed configurations. Therefore, it is natural to ask for the relationship between 2-layer fan-planarity and 2-layer RAC drawability. In this section we study this problem.

Di Giacomo *et al.* proved that a 2-layer embedding γ is RAC (i.e., there exists a 2-layer RAC drawing w.r.t. γ) if and only if γ has neither 3 mutually crossing edges nor two adjacent edges crossed by a third one [10]. For example, the embedding in Figure 11(b) is 2-layer RAC. They also showed that a biconnected graph has a 2-layer RAC embedding if and only if it is a subgraph of a ladder. Since a ladder is a special snake (but not vice versa), we deduce from Theorem 3:

Corollary 14 *The class of biconnected 2-layer RAC graphs is a proper subclass of the class of biconnected 2-layer fan-planar graphs.*

For general graphs, however, there is no inclusion relationship between those two families. In particular, we exhibit infinitely many trees T_k ($k \geq 3$) that are 2-layer RAC but not 2-layer fan-planar. T_k consists of two vertices u and v connected by a path of length $k \geq 3$, and such that each u and v have further (disjoint) three paths of length $k + 1$ attached to them. Figure 11(a) depicts T_3 . Using the characterization of 2-layer RAC trees [10], one can verify that T_k has a 2-layer RAC embedding, see Figure 11(b).

By Theorem 13, we can show that T_k is not 2-layer fan-planar by observing that it cannot be a subgraph of a stegosaurus. Indeed, suppose that G is some stegosaurus that contains T_k , and suppose that Γ is a planar drawing of G as in Figure 7(a), where all vertices of degree greater than two lie on the external face and are suitably placed on two distinct horizontal lines. Since u and v have degree 4 in T_k , they are external vertices of Γ . Denote by P_{uv} the path from u to v in Γ that corresponds to the path from u to v in T_k . Consider the three

paths of length $k + 1$ attached to u in T_k . Since they only share vertex u , and also share only vertex u with P_{uv} , one of them, call it P_u , is necessarily “routed towards” v in Γ , while the other two can be routed away from v . Analogously, one of the three paths of length $k + 1$ attached to v , call it P_v , must be routed towards u in Γ , while the other two can be routed away from u . Since P_{uv} has length k , it is not difficult to verify that either P_u and P_v share a vertex or at least one of them share a vertex with P_{uv} ; a contradiction. Thus, G cannot exist.

6 Open Problems

The main open problem of our study is to provide, if any, an efficient 2-layer fan-planarity testing algorithm for general (i.e., not necessarily biconnected) graphs, which exploits Theorem 13. A possible approach could be as follows. One could decompose the graph into its blocks, and, for each non-trivial block, check whether it is a spanning subgraph of a snake (Theorem 3). However, in the positive case, this is not sufficient to guarantee that the 2-layer fan-planar drawings of the blocks can be correctly combined into a single 2-layer fan-planar drawing. A further level of difficulty is to manage trivial blocks that are connected together forming a tree.

Another interesting research line is designing algorithms that compute 2-layer drawings that are “as fan-planar as possible”, i.e., whose number of forbidden configurations (two independent edges crossed by a third one) is minimized.

Finally, it would be interesting to study the complexity of the k -layer fan-planarity testing problem, for $k > 2$, both in the setting where the layer of each vertex is given as part of the input and in the setting in which it can be freely decided by the algorithm.

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