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Embeddings Between Hypercubes and Hypertrees

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Abstract

Graph embedding problems have gained importance in the field of interconnection networks for parallel computer architectures. Interconnection networks provide an effective mechanism for exchanging data between processors in a parallel computing system. In this paper, we embed the rooted hypertree $RHT(r)$ into r-dimensional hypercube Q^r with dilation 2, $r \geq 2$. Also, we compute the exact wirelength of the embedding of the r-dimensional hypercube Q^r into the rooted hypertree $RHT(r)$, $r \geq 2$.

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1 Introduction

A suitable interconnection network is an important part for the design of a multicomputer or multiprocessor system. This network is usually modeled by a symmetric graph, where the nodes represent the processing elements and the edges represent the communication channels. Desirable properties of an interconnection network include symmetry, embedding capabilities, relatively small degree, small diameter, scalability, robustness, and efficient routing [21]. One of the most efficient interconnection networks is the hypercube due to its structural regularity, potential for parallel computation of various algorithms, and the high degree of fault tolerance [22]. The hypercube has many excellent features and thus becomes the first choice of topological structure of parallel processing and computing systems. The machines based on hypercubes such as the Cosmic Cube from Caltech, the iPSC/2 from Intel and Connection Machines have been implemented commercially [8]. Hypercubes are very popular models for paralled computation because of their symmetry and relatively small number of interprocessor connections. The hypercube embedding problem is the problem of mapping a communication graph into a hypercube multiprocessor. Hypercubes are known to simulate other structures such as grids and binary trees [7, 16].

Graph embedding is an important technique that maps a logical graph into a host graph, usually an interconnection network. Many applications can be modeled as graph embedding. In architecture simulation, graph embedding has been known as a powerful tool for implementation of parallel algorithms or simulation of different interconnection networks. A parallel algorithm can be modeled by a task interaction graph, where nodes and edges represent tasks and direct communications between tasks, respectively. Thus, the problem of efficiently executing a parallel algorithm A on a parallel computer M can be often reduced to the problem of mapping the logical graph G, representing A, on the host graph H, representing M, so that the communication overhead is minimized [15]. In parallel computing, a large process is often decomposed into a set of small sub-processes that can execute in parallel with communications among these sub-processes. The problem of allocating these sub-processes into a parallel computing system can be again modeled by graph embedding [6].

The quality of an embedding can be measured by certain cost criteria. One of these criteria which is considered very often is the *dilation*. The dilation of an embedding is defined as the maximum distance between a pair of vertices of host graph that are images of adjacent vertices of logical graph. It is a measure for the communication time needed when simulation one network on another [15]. Another important cost criteria is the *wirelength*. The wirelength of an embedding is the sum of the dilations in host graph of edges in guest graph. The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [14, 24]. Graph embeddings have been well studied for a number of networks [2, 3, 7, 16, 17, 18, 19, 20].

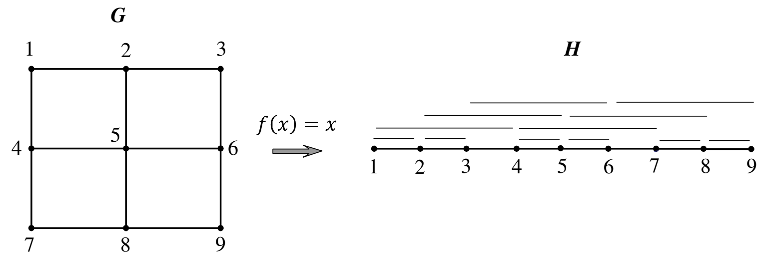

Figure 1: Wiring diagram of a grid G into path H with $EC_f(G, H) = 4$

Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds [2]. The embedding discussed in this paper produce exact wirelength.

2 Preliminaries

In this section we give the basic definitions and preliminaries related to embedding problems.

Definition 2.1 [2] *Let G and H be finite graphs. An embedding of G into H is a pair (f, P_f) defined as follows:*

1. *f is a one-to-one map from $V(G) \to V(H)$*
2. *P_f is a one-to-one map from $E(G)$ to $\{P_f(u,v) : P_f(u,v)$ is a path in H between $f(u)$ and $f(v)$ for $(u,v) \in E(G)\}$.*

For brevity, we denote the pair (f, P_f) as f.

Definition 2.2 [2] *If $e = (u,v) \in E(G)$, then the length of $P_f(u,v)$ in H is called the dilation of the edge e. The maximum dilation over all edges of G is called the dilation of the embedding f. The dilation of embedding G into H is the minimum dilation taken over all embeddings f of G into H and denote it by $dil(G,H)$.*

The *expansion* [2] of an embedding f is the ratio of the number of vertices of H to the number of vertices of G. In this paper, we consider embeddings with expansion one.

Definition 2.3 [2] *Let $f : G \to H$ be an embedding. For $e \in E(H)$, let $EC_f(e)$ denote the number of edges (u,v) of G such that e is in the path $P_f(u,v)$ between $f(u)$ and $f(v)$ in H. In other words,*

$$EC_f(e) = |\{(u,v) \in E(G) : e \in P_f(u,v)\}| .$$

Then the edge congestion of $f : G \to H$ *is* $EC_f(G,H) = \max EC_f(e)$*, where the maximum is taken over all edge* e *of* H*.*

The edge congestion of G *into* H *is defined as* $EC(G,H) = \min EC_f(G,H)$*, where the minimum is taken over all embeddings* $f : G \to H$*. On the other hand, if* S *is any subset of* $E(H)$*, then* $EC_f(S) = \sum_{e \in S} EC_f(e)$*.*

If we think of G as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion $EC(G,H)$ is the minimum, over all embeddings $f : V(G) \to V(H)$, of the maximum number of wires that cross any edge of H [3]. See Figure 1.

Definition 2.4 [16] *The wirelength of an embedding* f *of* G *into* H *is given by*

$$WL_f(G,H) = \sum_{(u,v) \in E(G)} d_H(f(u), f(v)) = \sum_{e \in E(H)} EC_f(e)$$

where $d_H(f(u), f(v))$ *denotes the length of the path* $P_f(u,v)$ *in* H*.*

The wirelength of G *into* H *is defined as*

$$WL(G,H) = \min WL_f(G,H)$$

where the minimum is taken over all embeddings f *of* G *into* H*.*

The *wirelength problem* [2, 3, 16, 18] of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G,H)$. The following two versions of the edge isoperimetric problem of a graph $G(V,E)$ have been considered in the literature [4], and are NP-complete [10].

Problem 1 : Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given m, if $\theta_G(m) = \min_{A \subseteq V,\ |A|=m} |\theta_G(A)|$ where $\theta_G(A) = \{(u,v) \in E : u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $\theta_G(m) = |\theta_G(A)|$.

Problem 2 : Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = \max_{A \subseteq V,\ |A|=m} |I_G(A)|$ where $I_G(A) = \{(u,v) \in E : u, v \in A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $I_G(m) = |I_G(A)|$.

For a given m, where $m = 1, 2, \ldots, n$, we consider the problem of finding a subset A of vertices of G such that $|A| = m$ and $|\theta_G(A)| = \theta_G(m)$. Such subsets are called optimal. We say that optimal subsets are nested if there exists a total order $\mathcal{O}$ on the set V such that for any $m = 1, 2, \ldots, n$, the first m vertices in this order is an optimal subset. In this case we call the order $\mathcal{O}$ an optimal order [4, 12]. This implies that $WL(G, P_n) = \sum_{m=0}^{n} \theta_G(m)$ [10].

Further, if a subset of vertices is optimal with respect to Problem 1, then its complement is also an optimal set. But, it is not true for Problem 2 in general. However for regular graphs a subset of vertices S is optimal with respect to Problem 1 if and only if S is optimal for Problem 2 [4]. In the literature, Problem 2 is defined as the maximum subgraph problem [10].

Lemma 2.5 (Congestion Lemma) [16] *Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:*

(i) *For every edge $(a, b) \in G_i$, $i = 1, 2, P_f(a, b)$ has no edges in S.*

(ii) *For every edge (a, b) in G with $a \in G_1$ and $b \in G_2$, $P_f(a, b)$ has exactly one edge in S.*

(iii) *G_1 is an optimal set.*

Then $EC_f(S)$ is minimum and $EC_f(S) = \sum_{e \in S} EC_f(e) = r\,|V(G_1)| - 2\,|E(G_1)|$.

Lemma 2.6 (Partition Lemma) [16] *Let $f : G \to H$ be an embedding. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H satisfying the conditions of Congestion Lemma. Then*

$$WL_f(G, H) = \sum_{i=1}^{p} EC_f(S_i).$$

Lemma 2.7 (2-Partition Lemma) [1] *Let $f : G \to H$ be an embedding. Let $[2E(H)]$ denote a collection of edges of H repeated exactly 2 times. In other words, $[2E(H)]$ comprises of 2 copies of the edge set of H. Let $\{S_1, S_2, \ldots, S_m\}$ be a partition of $[2E(H)]$ such that each S_i is an edge cut of H. Then*

$$WL_f(G, H) = \frac{1}{2} \sum_{i=1}^{m} EC_f(S_i).$$

Definition 2.8 [24] *For $r \geq 1$, let Q^r denote the r-dimensional hypercube. The vertex set of Q^r is formed by the collection of all r-dimensional binary strings. Two vertices $x, y \in V(Q^r)$ are adjacent if and only if the corresponding binary strings differ exactly in one bit.*

Equivalently if $n = 2^r$ then the vertices of Q^r can also be identified with integers $0, 1, \ldots, n-1$ so that if a pair of vertices i and j are adjacent then $i - j = \pm 2^p$ for some $p \geq 0$.

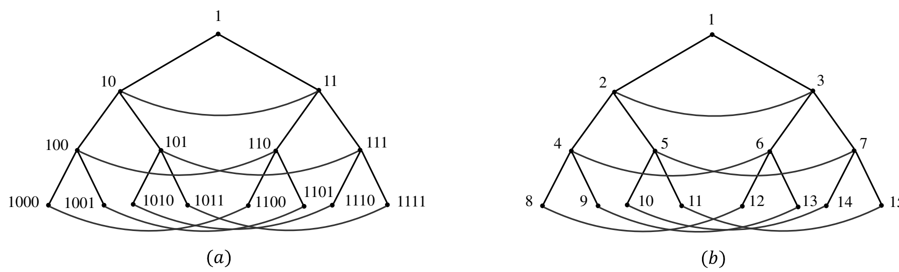

Figure 2: (a) $HT(4)$ with binary labels (b) $HT(4)$ with decimal labels

Definition 2.9 [13] *An incomplete hypercube on i vertices of Q^r is the subcube induced by $\{0, 1, \ldots, i-1\}$ and is denoted by L_i, $1 \leq i \leq 2^r$.*

Definition 2.10 [11] *The basic skeleton of a hypertree is a complete binary tree T_r. Here the nodes of the tree are numbered as follows: The root node has label 1. The root is said to be at level 1. Labels of left and right children are formed by appending a 0 and 1, respectively, to the label of the parent node. See Figure 2(a). The decimal labels of the hypertree in Figure 2(a) is depicted in Figure 2(b). Here the children of the node x are labeled as $2x$ and $2x+1$. Additional links in a hypertree are horizontal and two nodes are joined in the same level i of the tree if their label difference is 2^{i-2}. We denote an r-level hypertree as $HT(r)$. It has $2^r - 1$ vertices and $3\ (2^{r-1} - 1)$ edges. The rooted hypertree $RHT(r)$ is obtained from the hypertree $HT(r)$ by attaching to its root a pendant edge. The new vertex is called the root of $RHT(r)$, $r \geq 2$.*

Theorem 2.11 [12] *Let Q^r be an r-dimensional hypercube. For $1 \leq i \leq 2^r$, L_i is an optimal set on i vertices.*

Lemma 2.12 [16] *Let Q^r be an r-dimensional hypercube. Let $m = 2^{t_1} + 2^{t_2} + \cdots + 2^{t_l}$ such that $r \geq t_1 > t_2 > \cdots > t_l \geq 0$. Then $|E(Q^r[L_m])| = [t_1 \cdot 2^{t_1-1} + t_2 \cdot 2^{t_2-1} + \cdots + t_l \cdot 2^{t_l-1}] + [2^{t_2} + 2 \cdot 2^{t_3} + \cdots + (l-1)2^{t_l}]$.*

3 Main Results

In this section, we embed the rooted hypertree $RHT(r)$ into r-dimensional hypercube Q^r with dilation 2. Further we compute the minimum wirelength of embedding Q^r into $RHT(r)$.

The concept of embedding is widely studied in the area of fixed interconnection parallel architectures. A parallel architecture is embedded into another architecture to simulate one on another. An important feature of an interconnection network is its ability to efficiently simulate programs written for other architectures [15].

A tree is a connected graph that contains no cycles. The most common type of tree is the binary tree. It is so named because each node can have at most two descendents. A binary tree is said to be a complete binary tree if each internal node has exactly two descendents. These descendents are described as left and right children of the parent node. Binary trees are widely used in data structures because they are easily stored, easily manipulated, and easily retrieved. Also, many operations such as searching and storing can be easily performed on tree data structures. Furthermore, binary trees appear in communication pattern of divide-and-conquer type algorithms, functional and logic programming, and graph algorithms [24].

There are several useful ways in which we can systematically order all nodes of a tree. The three most important ordering are called *preorder*, *inorder* and *postorder*. To achieve these orderings the tree is traversed in a particular fashion. Starting from the root, the tree is traversed counter clockwise staying as close to the tree as possible. For preorder, we list a node the first time we pass it. For inorder, we list a node the second time we pass it. For postorder, we list a node the last time we pass it [9].

For any non-negative integer r, the complete binary tree of height $r-1$, denoted by T_r, is the binary tree where each internal vertex has exactly two children and all the leaves are at the same level. Clearly, a complete binary tree T_r has r levels and level i, $1 \leq i \leq r$, contains 2^{i-1} vertices. Thus, T_r has exactly $2^r - 1$ vertices. The rooted complete binary tree RT_r is obtained from a complete binary tree T_r by attaching to its root a pendant edge. The new vertex is called the root of RT_r and is considered to be at level 0 [24].

A hypertree is a hypergraph H if there is a tree T such that the hyperedges of H induce subtrees in T [5]. In the literature, hypertree is also called a subtree hypergraph or arboreal hypergraph [5, 23].

A hypertree is an interconnection topology for incrementally expansible multicomputer systems, which combines the easy expansibility of tree structures with the compactness of the hypercube; that is, it combines the best features of the binary tree and the hypercube. These two properties make this topology particulary attractive for implementation of multiprocessor networks of the future, where a complete computer with a substantial amount of memory can fit on a single VLSI chip [11].

Algorithm Dilation (Hypertree, Hypercube)

Input : The rooted hypertree $RHT(r)$ and the r-dimensional hypercube Q^r, $r \geq 2$.

Algorithm : Removal of the horizontal edges in rooted hypertree $RHT(r)$ leaves a rooted complete binary tree RT_r. Label the vertices of RT_r using binary codes corresponding to the inorder labeling [9]. Label the vertices of Q^r by using binary code corresponding to the lexicographic order [2] from 0 to $2^r - 1$. See Figure 3.

Output : An embedding f of $RHT(r)$ into Q^r given by $f(x) = x$ with dilation 2. See Figure 3.

Theorem 3.1 *The rooted hypertree $RHT(r)$ can be embedded into the r-dimensional hypercube Q^r with dilation 2, $r \geq 2$.*

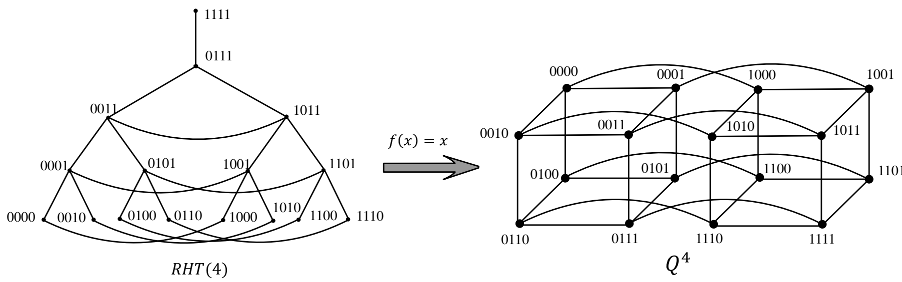

Figure 3: Embedding of $RHT(4)$ into Q^4 with dilation 2

Proof: Label the vertices of $RHT(r)$ and Q^r using Dilation Algorithm. $RHT(r)$ and Q^r are not isomorphic, since $RHT(r)$ contains a cycle of length 3 and Q^r is a bipartite graph. Hence the dilation of $RHT(r)$ into Q^r is ≥ 2.

Consider any edge $e = (u, v)$ in $RHT(r)$. We have the following two cases.

Case 1 ($e \in RT_r$): Since, the children of the node u in level i, $1 \leq i \leq r-1$ are labeled as $u-2^{r-(i+1)}$ and $u+2^{r-(i+1)}$, the binary codes of u and $u-2^{r-(i+1)}$ will differ in exactly one position and the binary codes of u and $u+2^{r-(i+1)}$ will differ in exactly two positions. Suppose u is the root of RT_r, then the binary code of u and v will differ in exactly one position.

Case 2 ($e \notin RT_r$): By the labeling of $RHT(r)$, the binary codes of u and v will differ in exactly one position.

Hence the distance between $f(u)$ and $f(v)$ in Q^r is not larger than 2 in both the cases. $\square$

Next, we compute the exact wirelength of embedding r-dimensional hypercube Q^r into rooted hypertree $RHT(r)$. For proving the main result, we need the following Lemmas.

Lemma 3.2 [19] *For* $i = 1, 2, ..., r-1$, $NcutS_i^{2^i} = \{2^i, 2^i+1, ..., 2^{i+1}-1\}$ *is an optimal set in* Q^r.

Lemma 3.3 *For* $i = 1, 2, ..., r-1$, $NcutS_i^{2^i} = \{2^i, 2^i+1, ..., 2^{i+1}-2\}$ *is an optimal set in* Q^r.

Proof: Let L_{2^i} denote the incomplete hypercube on 2^i vertices. Define $\varphi : NcutS_i^{2^i} \to L_{2^i}$ by $\varphi(2^i + k) = k$. If the binary representation of $2^i + k$ is $\alpha_1\alpha_2 \cdots \alpha_r$ then the binary representation of k is $\underbrace{00\cdots00}_{r-i \text{ times}}\alpha_{r-i+1}\alpha_{r-i+2}\cdots\alpha_r$. Thus the binary representation of two numbers x and y differ in exactly one bit $\Leftrightarrow$ the binary representation of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore (x, y) is an edge in $NcutS_i^{2^i} \Leftrightarrow (\varphi(x), \varphi(y))$ is an edge in L_{2^i}. Hence $NcutS_i^{2^i}$ and L_{2^i} are isomorphic. By Theorem 2.11, $NcutS_i^{2^i}$ is an optimal set in Q^r. $\square$

Lemma 3.4 *For* $i = 1, 2, \ldots, r-2$, $NcutS_1^i = \{0, 1, 2, \ldots, 2^i-2, 2^{r-1}, 2^{r-1}+1, \ldots, 2^{r-1}+2^i-2\}$ *is an optimal set in* Q^r.

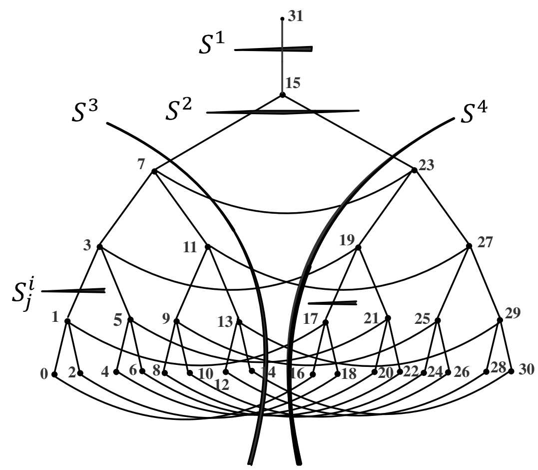

Figure 4: Edge cut of $RHT(5)$

Proof: By Theorem 2.11, the set $\{0, 1, 2, \ldots, 2^i - 2\}$ is optimal and by Lemma 3.2, the set $\{2^{r-1}, 2^{r-1}+1, \ldots, 2^{r-1}+2^i-2\}$ is optimal in Q^r. Also the binary representation of k and $2^{r-1}+k$, $0 \leq k \leq 2^i - 2$, differ exactly in one bit. Therefore $\left|E(Q^r[NcutS_1^i])\right| = 2\left|E(Q^r[L_{2^i-1}])\right| + 2^i - 1 = 2i(2^{i-1} - i) + 2^i - 1 = (i+1)2^i - 2i - 1$. But by Lemma 2.12, $\left|E(Q^r[L_{2(2^i-1)}])\right| = (i+1)2^i - 2i - 1$ and hence by Theorem 2.11, $NcutS_1^i$ is an optimal set in Q^r. □

Algorithm Wirelength (Hypercube, Hypertree)

Input : The r-dimensional hypercube Q^r and the rooted hypertree $RHT(r)$, $r \geq 2$.
Algorithm : Label the vertices of Q^r by lexicographic order [2] from 0 to $2^r - 1$. Removal of the horizontal edges in rooted hypertree $RHT(r)$ leaves a rooted complete binary tree RT_r. Label the vertices of RT_r using inorder labeling [9]. See Figure 4.
Output : An embedding f of Q^r into $RHT(r)$ given by $f(x) = x$ with optimal wirelength.

Theorem 3.5 *The exact wirelength of Q^r into $RHT(r)$, $r \geq 2$ is given by*

$$WL(Q^r, RHT(r)) = 2^{r-1}(r^2 - 5r + 11) - (r + 3).$$

Proof. Label the vertices of Q^r and $RHT(r)$ using Wirelength Algorithm. We assume that the labels represent the vertices to which they are assigned.

For $1 \leq i \leq r-2$, $1 \leq j \leq 2^{r-(i+1)}$ and j is odd, let S_j^i and R_j^i be edge cuts in $RHT(r)$ given by $S_j^i = R_j^i = \{(2^{i-1}(2j-1)-1, j2^i - 1), (2^{r-1} + 2^{i-1}(2j-1) - 1, 2^{r-1} + j2^i - 1)\}$.

For $1 \leq i \leq r-2$, $1 \leq j \leq 2^{r-(i+1)}$ and j is even, let S_j^i and R_j^i be edge cuts in $RHT(r)$ given by $S_j^i = R_j^i = \{(2^{i-1}(2j-1)-1, 2^{i-1}(2j-2)-1), (2^{r-1} + 2^{i-1}(2j-1)-1, 2^{r-1}+2^{i-1}(2j-2)-1)\}$.

For $i = 1$, let S^i and R^i be edge cuts in $RHT(r)$ given by $S^i = R^i = \{((2i-1)2^{r-2}-1, 2^{r+i-2}-1)\}$. For $i = 2$, let $S^i = \{(2^{r-2}-1, 2^{r-1}-1), (2^{r-1}-1, 2^{r-2}+2^{r-1}-1)\}$. For $i = 3$, let $S^i = \{(k-1, k+2^{r-1}-1), (2^{r-2}-1, 2^{r-1}-1) : 1 \leq k \leq 2^{r-1}-1\}$. For $i = 4$, let $S^i = \{(k-1, k+2^{r-1}-1), (2^{r-1}-1, 2^{r-2}+2^{r-1}-1) : 1 \leq k \leq 2^{r-1}-1\}$.

Then $\{S_j^i, R_j^i : 1 \leq i \leq r-2, 1 \leq j \leq 2^{r-(i+1)}\} \cup \{S^1, R^1\} \cup \{S^i : 2 \leq i \leq 4\}$ is a partition of $[2E(RHT(r))]$. See Figure 4.

For each i, j, $1 \leq i \leq r-2$, $1 \leq j \leq 2^{r-(i+1)}$, $E(RHT(r)) \backslash S_j^i$ has two components H_{j1}^i and H_{j2}^i, where $V(H_{j1}^i) = \{(j-1)2^i, (j-1)2^i+1, \ldots, j2^i-2, 2^{r-1}+(j-1)2^i, 2^{r-1}+(j-1)2^i+1, \ldots, 2^{r-1}+j2^i-2\}$. Let $G_{j1}^i = f^{-1}(H_{j1}^i)$ and $G_{j2}^i = f^{-1}(H_{j2}^i)$. By Lemma 3.4, G_{j1}^i is an optimal set and each S_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_j^i)$ is minimum. Similarly, $EC_f(R_j^i)$ is minimum.

For $i = 1$, $E(RHT(r)) \backslash S^i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{2^{r-2+i}-1\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Theorem 2.11, G_{i1} is an optimal set and S^i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S^i)$ is minimum. Similarly, $EC_f(R^i)$ is minimum.

For $i = 2$, $E(RHT(r)) \backslash S^i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{2^{r-1}-1, 2^r-1\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. Since, G_{i1} is an optimal set and S^i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S^i)$ is minimum.

For $i = 3$, $E(RHT(r)) \backslash S^i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{0, 1, \ldots, 2^{r-1}-2\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Theorem 2.11, G_{i1} is an optimal set and S^i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S^i)$ is minimum.

For $i = 4$, $E(RHT(r)) \backslash S^i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{2^{r-1}, 2^{r-1}+1, \ldots, 2^r-2\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Lemma 3.3, G_{i1} is an optimal set and S^i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S^i)$ is minimum. The 2-Partition Lemma implies that the wirelength is minimum.

By Congestion Lemma, $EC_f(S^1) = EC_f(R^1) = r$, $EC_f(S^2) = 2r-2$, $EC_f(S^3) = EC_f(S^4) = 2^{r-1}+r-2$. For each i, j, $1 \leq i \leq r-2$, $1 \leq j \leq 2^{r-(i+1)}$, $EC_f(S_j^i) = EC_f(R_j^i) = 2^{i+1}(r-i-1)-2r+4i$. Then, by 2-Partition Lemma,

$$\begin{aligned} WL(Q^r, RHT(r)) &= 3r-3+2^{r-1}+\sum_{i=1}^{r-2} 2^{r-(i+1)}[2^{i+1}(r-i-1)-2r+4i] \\ &= 2^{r-1}(r^2-5r+11)-(r+3). \quad \square \end{aligned}$$

4 Concluding Remarks

We provide an embedding of the rooted hypertree $RHT(r)$ into r-dimensional hypercube Q^r with dilation 2. Further, we compute the exact wirelength of embedding r-dimensional hypercube Q^r into rooted hypertree $RHT(r)$, $r \geq 2$. Finding the dilation of embedding hypercube into rooted hypertree is under investigation.

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