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Increasing-Chord Graphs On Point Sets

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Abstract

We tackle the problem of constructing increasing-chord graphs spanning point sets. We prove that, for every point set P with n points, there exists an increasing-chord planar graph with $O(n)$ Steiner points spanning P. The main intuition behind this result is that Gabriel triangulations are increasing-chord graphs, a fact which might be of independent interest. Further, we prove that, for every convex point set P with n points, there exists an increasing-chord graph with $O(n \log n)$ edges (and with no Steiner points) spanning P.

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1 Introduction

A *proximity graph* is a geometric graph that can be constructed from a point set by connecting points that are "close", for some local or global definition of proximity. Proximity graphs constitute a topic of research in which the areas of graph drawing and computational geometry nicely intersect. A typical graph drawing question in this topic asks to characterize the graphs that can be represented as a certain type of proximity graphs. A typical computational geometry question asks to design an algorithm to construct a proximity graph spanning a given point set.

Euclidean minimum spanning trees and Delaunay triangulations are famous examples of proximity graphs. Given a point set P, a *Euclidean minimum spanning tree* of P is a geometric tree with P as vertex set and with minimum total edge length; the *Delaunay triangulation* of P is a triangulation T such that no point in P lies inside the circumcircle of any triangle of T. From a computational geometry perspective, given a point set P with n points, a Euclidean minimum spanning tree of P with maximum degree five exists [16] and can be constructed in $O(n \log n)$ time [5]; also, the Delaunay triangulation of P exists and can be constructed in $O(n \log n)$ time [5]. From a graph drawing perspective, every tree with maximum degree five admits a representation as a Euclidean minimum spanning tree [16] and it is NP-hard to decide whether a tree with maximum degree six admits such a representation [9]; also, characterizing the class of graphs that can be represented as Delaunay triangulations is a deeply studied question, which still eludes a clear answer; see, e.g., [7, 8]. Refer to the excellent survey by Liotta [14] for more on proximity graphs.

While proximity graphs have constituted a frequent topic of research in graph drawing and computational geometry, they gained a sudden peak in popularity even outside these communities in 2004, when Papadimitriou *et al.* [19] devised an elegant routing protocol that works effectively in all the networks that can be represented as a certain type of proximity graphs, called *greedy graphs*. For two points p and q in the plane, denote by $\overline{pq}$ the straight-line segment having p and q as end-points, and by $|\overline{pq}|$ the length of $\overline{pq}$. A geometric path $(v_1, \dots, v_n)$ is *greedy* if $|\overline{v_{i+1}v_n}| < |\overline{v_i v_n}|$, for every $1 \leq i \leq n-1$. A geometric graph G is *greedy* if, for every ordered pair of vertices u and v, there exists a greedy path from u to v in G. A lot is known about the existence of greedy graphs spanning given point sets and about the possibility of representing graphs as greedy graphs; see, e.g., [3, 11, 13, 18, 19]. A result strictly related to our paper is that, for every point set P, the Delaunay triangulation of P is a greedy graph [18].

In this paper we study *self-approaching* and *increasing-chord graphs*, that are types of proximity graphs defined by Alamdari *et al.* [2].

A geometric path $\mathcal{P} = (v_1, \dots, v_n)$ is *self-approaching* from v_1 to v_n if, for every three points a, b, and c in this order on $\mathcal{P}$ from v_1 to v_n (possibly a, b, and c are internal to segments of $\mathcal{P}$), we have $|\overline{bc}| < |\overline{ac}|$. The geometric path in Figure 1 is self-approaching from v_1 to v_n. A geometric graph G is *self-approaching* if, for every ordered pair of vertices u and v, G contains a self-approaching path from u to v.

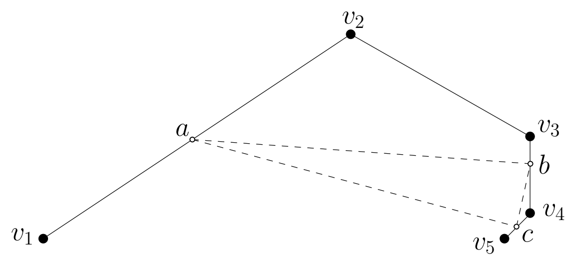

Figure 1: A geometric path $\mathcal{P} = (v_1, \ldots, v_n)$ that is self-approaching from v_1 to v_n, since for every three points a, b, and c in this order on $\mathcal{P}$ from v_1 to v_n, we have $|\overline{bc}| < |\overline{ac}|$. Path $\mathcal{P} = (v_1, \ldots, v_n)$ is not self-approaching from v_n to v_1 (and thus it is not increasing-chord between v_1 and v_n), since for the three points a, b, and c shown in the illustration we have $|\overline{ba}| > |\overline{ca}|$.

A geometric path $\mathcal{P} = (v_1, \ldots, v_n)$ is *increasing-chord* between v_1 and v_n if it is self-approaching both from v_1 to v_n and from v_n to v_1; equivalently, $\mathcal{P}$ is increasing-chord if, for every four points a, b, c, and d in this order on $\mathcal{P}$ from v_1 to v_n, we have $|\overline{bc}| < |\overline{ad}|$ (from which the name *increasing-chord*). A geometric graph G is *increasing-chord* if, for every pair of vertices u and v, G contains an increasing-chord path between u and v. Observe that, by definition, an increasing-chord graph is also self-approaching.

The study of self-approaching and increasing-chord graphs is motivated by their relationship with greedy graphs (by definition, a self-approaching graph is also greedy), and by the fact that such graphs have a small geometric dilation, namely at most 5.3332 for self-approaching graphs [12] and at most 2.094 for increasing-chord graphs [20].

Alamdari *et al.* [2] considered three types of problems about self-approaching and increasing-chord graphs.

1. *Complexity of recognizing self-approaching and increasing-chord graphs:* Alamdari *et al.* [2] showed how to test in $O(n)$ time (in $O(n \log^2 n / \log \log n)$ time) whether an n-vertex path in $\mathbb{R}^2$ (resp. in $\mathbb{R}^3$) is self-approaching. They also exhibit an $\Omega(n \log n)$ lower bound for the same problem in $\mathbb{R}^3$. Further, they proved that it is NP-hard to test the existence of a self-approaching path between two given vertices in a geometric graph in $\mathbb{R}^3$ and left open the intriguing problem of determining the complexity of testing whether a geometric graph is self-approaching or increasing-chord in two or more dimensions.

2. *Realizability of a given abstract graph as a self-approaching or increasing-chord graph:* Alamdari *et al.* [2] characterized the class of trees that can be realized as self-approaching graphs; recently, Nöllenburg *et al.* [17] proved that planar triangulations can be realized as increasing-chord graphs, that planar 3-trees can be realized as increasing-chord planar graphs, and that

triconnected planar graphs can be realized as increasing-chord graphs in the hyperbolic plane.

3. *Existence of a self-approaching and increasing-chord graph spanning a given point set:* Alamdari *et al.* [2] showed how to construct, for every point set P with n points in $\mathbb{R}^2$, an increasing-chord graph that spans P and uses $O(n)$ Steiner points (which are extra points that are added to the input point set). They also proved that the Delaunay triangulation of a point set is not always a self-approaching graph.

In this paper we focus our attention on the third type of problem above, i.e., on the problem of constructing self-approaching and increasing-chord graphs spanning given point sets in $\mathbb{R}^2$. We prove two main results.

- We show how to construct, for every point set P with n points, an increasing-chord planar graph with $O(n)$ Steiner points spanning P. This answers a question of Alamdari *et al.* [2] and improves upon their result mentioned above, since our increasing-chord graphs are planar (while the increasing-chord graphs constructed in [2] are not, although they have thickness at most two) and contain increasing-chord paths between every pair of points, including the Steiner points (which is not the case for the graphs in [2]). It is interesting that our result is achieved by studying Gabriel triangulations, which are proximity graphs strongly related to Delaunay triangulations (the Gabriel graph of a point set P is a subgraph of the Delaunay triangulation of P). On the way to proving our main result, we show that Gabriel triangulations are increasing-chord graphs, which is not the case, in general, for Delaunay triangulations [2].
- We show that, for every convex point set P with n points, there exists an increasing-chord graph that spans P and that has $O(n \log n)$ edges (and no Steiner points).

The rest of the paper is organized as follows. In Section 2 we give some preliminaries. In Section 3 we show how to construct increasing-chord planar graphs with few Steiner points spanning given point sets. In Section 4 we show how to construct increasing-chord graphs with few edges spanning given convex point sets. Finally, in Section 5 we conclude and suggest some open problems.

2 Definitions and Preliminaries

A *geometric graph* (P, S) consists of a point set P in the plane and of a set S of straight-line segments (called *edges*) between points in P. A geometric graph is *planar* if no two of its edges cross. A planar geometric graph partitions the plane into connected regions called *faces*. The bounded faces are *internal* and the unbounded face is the *outer face*. A geometric planar graph is a *triangulation* if every internal face is delimited by a triangle and the outer face is delimited by a convex polygon.

Let p, q, and r be points in the plane. We denote by $\angle pqr$ the angle defined by a clockwise rotation around q bringing $\overline{pq}$ to coincide with $\overline{qr}$.

A *convex combination* of a set of points $P = \{p_1, \dots, p_k\}$ is a point $\sum \alpha_i p_i$ where $\sum \alpha_i = 1$ and $\alpha_i \geq 0$ for each $1 \leq i \leq k$. The *convex hull* $\mathcal{H}_P$ of P is the set of points that can be expressed as a convex combination of the points in P. A *convex point set* P is such that no point is a convex combination of the others. Let P be a convex point set and $\vec{d}$ be a directed straight line not orthogonal to any line through two points of P. Order the points in P as their projections appear on $\vec{d}$; then the *minimum point* and the *maximum point* of P with respect to $\vec{d}$ are the first and the last point in such an ordering. We say that P is *one-sided with respect to* $\vec{d}$ if the minimum and the maximum point of P with respect to $\vec{d}$ are consecutive along the boundary of $\mathcal{H}_P$. See Figure 2. A *one-sided convex point set* is a convex point set that is one-sided with respect to some directed straight line $\vec{d}$.

The proof of our first lemma gives an algorithm to construct an increasing-chord planar graph spanning a one-sided convex point set.

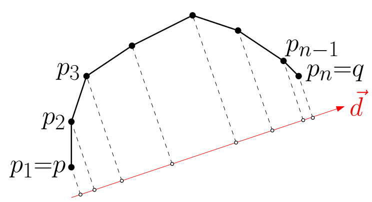

Figure 2: A convex point set that is one-sided with respect to a directed straight line $\vec{d}$.

Lemma 1 *Let P be any one-sided convex point set with n points. There exists an increasing-chord planar graph spanning P with $2n-3$ edges.*

Proof: Assume that P is one-sided with respect to the positive x-axis $\vec{x}$. Such a condition can be met after a suitable rotation of the Cartesian axes. Let $p_1, p_2, \dots, p_n$ be the points in P, ordered as their projections appear on $\vec{x}$. Assume that $p_2, p_3, \dots, p_{n-1}$ are above the straight line through p_1 and p_n, as the case in which they are below such a line is symmetric.

We show by induction on n that an increasing-chord planar graph G spanning P exists, in which all the edges on the boundary of $\mathcal{H}_P$ are in G.

If $n = 2$ then the graph with a single edge $\overline{p_1 p_2}$ is an increasing-chord planar graph spanning P.

Next, assume that $n > 2$ and let p_j be a point with largest y-coordinate in P (possibly $j = 1$ or $j = n$). Point set $Q = P \setminus \{p_j\}$ is convex, one-sided with respect to $\vec{x}$, and has $n-1$ points. By induction, there exists an increasing-chord planar graph G' spanning Q in which all the edges on the boundary of $\mathcal{H}_Q$ are in G'. Let G be the graph obtained by adding vertex p_j and edges $\overline{p_{j-1} p_j}$ and $\overline{p_j p_{j+1}}$ to G' (where $p_{n+1} = p_0$ and $p_{-1} = p_n$). We have that G is planar, given that G' is planar and that edges $\overline{p_{j-1}p_j}$ and $\overline{p_jp_{j+1}}$ are on the boundary of $\mathcal{H}_P$. Further, all the edges on the boundary of $\mathcal{H}_P$ are in G. Moreover, G contains an increasing-chord path between every pair of points in Q, by induction; also, G contains an increasing-chord path between p_j and every point p_i in Q, as one of the two paths on the boundary of $\mathcal{H}_P$ connecting p_j and p_i is both x- and y-monotone, and hence increasing-chord, as proved in [2]. Finally, G is a maximal outerplanar graph, hence it has $2n-3$ edges. □

The *Gabriel graph* of a point set P is the geometric graph that has an edge $\overline{pq}$ between two points p and q if and only if the closed disk whose diameter is $\overline{pq}$ contains no point of $P \setminus \{p,q\}$ in its interior or on its boundary. A *Gabriel triangulation* is a triangulation that is the Gabriel graph of its point set P. We say that a point set P *admits* a Gabriel triangulation if the Gabriel graph of P is a triangulation. A triangulation is a Gabriel triangulation if and only if every angle of a triangle delimiting an internal face is acute [10]. See [10, 14, 15] for more properties about Gabriel graphs.

In Section 3 we will prove that every Gabriel triangulation is increasing-chord. A weaker version of the converse is also true, as proved in the following.

Lemma 2 *Let P be a set of points and let $G(P,S)$ be an increasing-chord graph spanning P. Then all the edges of the Gabriel graph of P are in S.*

Proof: Suppose, for a contradiction, that there exists an increasing-chord graph $G(P,S)$ and an edge $\overline{uv}$ of the Gabriel graph of P such that $\overline{uv} \notin S$. Then consider any increasing-chord path $\mathcal{P} = (u = w_1, w_2, \ldots, w_k = v)$ in G. Since $\overline{uv} \notin S$, it follows that $k > 2$. Assume w.l.o.g. that w_1, w_2, and w_k appear in this clockwise order on the boundary of triangle (w_1, w_2, w_k). Since the closed disk with diameter $\overline{uv}$ does not contain any point in its interior or on its boundary, it follows that $\angle w_k w_2 w_1 < 90°$. If $\angle w_2 w_1 w_k \geq 90°$, then $|w_1 w_k| < |w_2 w_k|$, a contradiction to the assumption that $\mathcal{P}$ is increasing-chord. If $\angle w_2 w_1 w_k < 90°$, then the altitude of triangle (w_1, w_2, w_k) incident to w_k hits $\overline{w_1 w_2}$ in a point h. Hence, $|h w_k| < |w_2 w_k|$, a contradiction to the assumption that $\mathcal{P}$ is increasing-chord which proves the lemma. □

3 Increasing-Chord Planar Graphs with Few Steiner Points Spanning Point Sets

In this section we show that, for any point set P, one can construct an increasing-chord planar graph $G(P', S)$ such that $P \subseteq P'$ and $|P'| \in O(|P|)$.

Our proof consists of two main ingredients. The first one is that Gabriel triangulations are increasing-chord graphs. The second one is a result of Bern *et al.* [4] stating that, for any point set P, there exists a point set P' such that $P \subseteq P'$, $|P'| \in O(|P|)$, and P' admits a Gabriel triangulation. Combining these two facts proves our main result.

The proof that Gabriel triangulations are increasing-chord graphs consists of two parts. In the first one, we prove that geometric graphs having a *θ-path* between every pair of points are increasing-chord. In the second one, we prove that every Gabriel triangulation contains a θ-path between every pair of points.

We introduce some definitions. The *slope* of a straight-line segment $\overline{uv}$ is the angle spanned by a clockwise rotation around u that brings $\overline{uv}$ to coincide with the positive x-axis. Thus, if θ is the slope of $\overline{uv}$, then $\theta + k \cdot 360°$ is also the slope of $\overline{uv}$, $\forall k \in \mathbb{Z}$. A straight-line segment $\overline{uv}$ is a θ-*edge* if its slope is in the interval $[\theta - 45°; \theta + 45°]$. Also, a geometric path $\mathcal{P} = (p_1, \dots, p_k)$ is a θ-*path* from p_1 to p_k if $\overline{p_i p_{i+1}}$ is a θ-edge, for every $1 \leq i \leq k-1$. Consider a point a on a θ-path $\mathcal{P}$ from p_1 to p_k. Then the subpath $\mathcal{P}_a$ of $\mathcal{P}$ from a to p_k is also a θ-path. Moreover, denote by $W_\theta(a)$ the closed wedge with an angle of 90° incident to a and whose delimiting lines have slope $\theta - 45°$ and $\theta + 45°$; then we have that $\mathcal{P}_a$ is contained in $W_\theta(a)$, which easily follows from the fact that $\overline{p_i p_{i+1}}$ is a θ-edge, for every $1 \leq i \leq k-1$ (see Figure 3). We have the following:

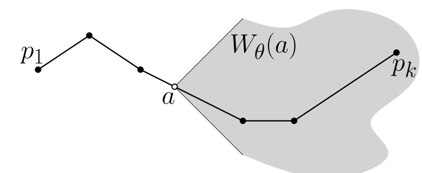

Figure 3: Wedge $W_\theta(a)$ contains path $\mathcal{P}_a$.

Lemma 3 *Let $\mathcal{P}$ be a θ-path from p_1 to p_k, for some angle θ. Then $\mathcal{P}$ is increasing-chord.*

Proof: Lemma 3 in [12] states the following (see also [1]): A curve $\mathcal{C}$ with endpoints p and q is self-approaching from p to q if and only if, for every point a on $\mathcal{C}$, there exists a closed wedge with an angle of 90° incident to a and containing the part of $\mathcal{C}$ between a and q. As observed before the lemma, for every point a on $\mathcal{P}$, the closed wedge $W_\theta(a)$ with an angle of 90° incident to a and whose delimiting lines have slope $\theta - 45°$ and $\theta + 45°$ contains the subpath $\mathcal{P}_a$ of $\mathcal{P}$ from a to p_k. Hence, by Lemma 3 in [12], $\mathcal{P}$ is self-approaching from p_1 to p_k. An analogous proof shows that $\mathcal{P}$ is self-approaching from p_k to p_1, given that $\mathcal{P}$ is a $(\theta + 180°)$-path from p_k to p_1. □

We now prove that Gabriel triangulations contain θ-paths.

Lemma 4 *Let G be a Gabriel triangulation on a point set P. For every two points $s, t \in P$, there exists an angle θ such that G contains a θ-path from s to t.*

Proof: Consider any two points $s, t \in P$. Rotate G clockwise of an angle ϕ so that $y(s) = y(t)$ and $x(s) < x(t)$. Observe that, if there exists a θ-path from s to t after the rotation, then there exists a $(\theta + \phi)$-path from s to t before the rotation.

A θ-path $(p_1, \ldots, p_k)$ in G is *maximal* if there is no $z \in P$ such that $\overline{p_k z}$ is a θ-edge. For every maximal θ-path $\mathcal{P} = (p_1, \ldots, p_k)$ in G, p_k lies on the boundary of $\mathcal{H}_P$. To prove this, assume the converse, for a contradiction. Since G is a Gabriel triangulation, the angle between any two consecutive edges incident to an internal vertex of G is smaller than $90°$, thus there is a θ-edge incident to p_k. This contradicts the maximality of $\mathcal{P}$. A maximal θ-path $(s = p_1, \ldots, p_k)$ is *high* if either (a) $y(p_k) > y(t)$ and $x(p_k) < x(t)$, or (b) $\overline{p_i p_{i+1}}$ intersects the vertical line through t at a point above t, for some $1 \leq i \leq k-1$. Symmetrically, a maximal θ-path $(s = p_1, \ldots, p_k)$ is *low* if either (a) $y(p_k) < y(t)$ and $x(p_k) < x(t)$, or (b) $\overline{p_i p_{i+1}}$ intersects the vertical line through t at a point below t, for some $1 \leq i \leq k-1$. High and low $(\theta + 180°)$-paths starting at t can be defined analogously. The proof of the lemma consists of two main claims.

Claim A. If a maximal θ-path $\mathcal{P}_s$ starting at s and a maximal $(\theta + 180°)$-path $\mathcal{P}_t$ starting at t exist such that $\mathcal{P}_s$ and $\mathcal{P}_t$ are both high or both low, for some $-45° \leq \theta \leq 45°$, then there exists a θ-path in G from s to t.

Claim B. For some $-45° \leq \theta \leq 45°$, there exist a maximal θ-path $\mathcal{P}_s$ starting at s and a maximal $(\theta + 180°)$-path $\mathcal{P}_t$ starting at t that are both high or both low.

Observe that Claims A and B imply the lemma.

We now prove Claim A. Suppose that G contains a maximal high θ-path $\mathcal{P}_s$ starting at s and a maximal high $(\theta + 180°)$-path $\mathcal{P}_t$ starting at t, for some $-45° \leq \theta \leq 45°$. If $\mathcal{P}_s$ and $\mathcal{P}_t$ share a vertex $v \in P$, then the subpath of $\mathcal{P}_s$ from s to v and the subpath of $\mathcal{P}_t$ from v to t form a θ-path in G from s to t. Thus, it suffices to show that $\mathcal{P}_s$ and $\mathcal{P}_t$ share a vertex. For a contradiction assume the converse. Let p_s and p_t be the end-vertices of $\mathcal{P}_s$ and $\mathcal{P}_t$ different from s and t, respectively. Recall that p_s and p_t lie on the boundary of $\mathcal{H}_P$. Denote by $\vec{l_s}$ and $\vec{l_t}$ the vertical half-lines starting at s and t, respectively, and directed toward increasing y-coordinates; also, denote by q_s and q_t the intersection points of $\vec{l_s}$ and $\vec{l_t}$ with the boundary of $\mathcal{H}_P$, respectively. Finally, denote by Q the curve obtained by following the boundary of $\mathcal{H}_P$ clockwise from q_s to q_t.

Assume that $x(p_s) \geq x(t)$, as in Figure 4(a). Path $\mathcal{P}_s$ starts at s and passes through a point r_s on $\vec{l_t}$ (possibly $r_s = q_t$), given that $x(p_s) \geq x(t)$. Path $\mathcal{P}_t$ starts at t and either passes through a point r_t on $\vec{l_s}$, or ends at a point p_t on Q, depending on whether $x(p_t) \leq x(s)$ or $x(p_t) > x(s)$, respectively. Since $\mathcal{P}_s$ is x-monotone and lies in $\mathcal{H}_P$, it follows that r_t and p_t are above or on $\mathcal{P}_s$; also, t is below $\mathcal{P}_s$ given that $\mathcal{P}_s$ is a high path. It follows $\mathcal{P}_s$ and $\mathcal{P}_t$ intersect, hence they share a vertex given that G is planar.

Analogously, if $x(p_t) \leq x(s)$, then $\mathcal{P}_s$ and $\mathcal{P}_t$ share a vertex.

If $x(p_t) = x(p_s)$, then p_s and p_t are the same point, hence $\mathcal{P}_s \cup \mathcal{P}_t$ is a θ-path from s to t.

Next, if $x(s) < x(p_t) < x(p_s) < x(t)$, as in Figure 4(b), then the end-points of $\mathcal{P}_s$ and $\mathcal{P}_t$ alternate along the boundary of the region R that is the intersection of $\mathcal{H}_P$, of the half-plane to the right of $\vec{l_s}$, and of the half-plane to the left of $\vec{l_t}$. Since $\mathcal{P}_s$ and $\mathcal{P}_t$ are x-monotone, they lie in R, thus they intersect, and hence they share a vertex.

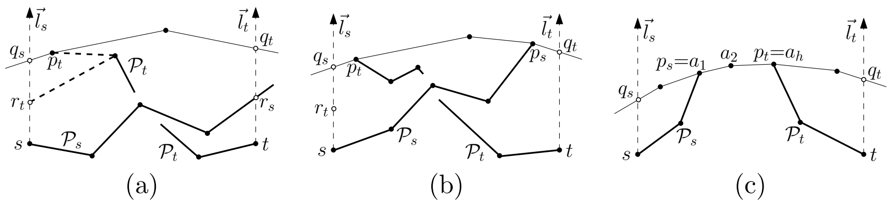

Figure 4: Paths $\mathcal{P}_s$ and $\mathcal{P}_t$ intersect if: (a) $x(p_s) \geq x(t)$, (b) $x(s) < x(p_t) < x(p_s) < x(t)$, and (c) $x(s) < x(p_s) < x(p_t) < x(t)$.

Finally, assume that $x(s) < x(p_s) < x(p_t) < x(t)$, as in Figure 4(c). Let $a_1, \ldots, a_h$ be the clockwise order of the points along Q, starting at $p_s = a_1$ and ending at $a_h = p_t$. By the assumption $x(p_s) < x(p_t)$ we have $h \geq 2$. We prove that $\overline{a_1 a_2}$ is a θ-edge. Suppose, for a contradiction, that $\overline{a_1 a_2}$ is not a θ-edge. Since the slope of $\overline{a_1 a_2}$ is larger than $-90°$ and smaller than $90°$, it is either larger than $\theta + 45°$ and smaller than $90°$, or it is larger than $-90°$ and smaller than $\theta - 45°$. First, assume that the slope of $\overline{a_1 a_2}$ is larger than $\theta + 45°$ and smaller than $90°$, as in Figure 5(a). Since the slope of $\overline{s a_1}$ is between $\theta - 45°$ and $\theta + 45°$, it follows that a_1 is below the line composed of $\overline{s a_2}$ and $\overline{a_2 t}$, which contradicts the assumption that a_1 is on Q. Second, if the slope of $\overline{a_1 a_2}$ is larger than $-90°$ and smaller than $\theta - 45°$, then we distinguish two further cases. In the first case, represented in Figure 5(b), the slope of $\overline{a_1 t}$ is larger than $\theta - 45°$, hence a_2 is below the line composed of $\overline{s a_1}$ and $\overline{a_1 t}$, which contradicts the assumption that a_2 is on Q. In the second case, represented in Figure 5(c), the slope of $\overline{a_1 t}$ is in the interval $[-90°; \theta - 45°]$. It follows that the slope of $\overline{t a_1}$ is in the interval $[90°; \theta + 135°]$; since the slope of $\overline{t a_h}$ is smaller than the one of $\overline{t a_1}$, we have that $\mathcal{P}_t$ is not a $(\theta + 180°)$-path. This contradiction proves that $\overline{a_1 a_2}$ is a θ-edge. However, this contradicts the assumption that $\mathcal{P}_s$ is a maximal θ-path, and hence concludes the proof of Claim A.

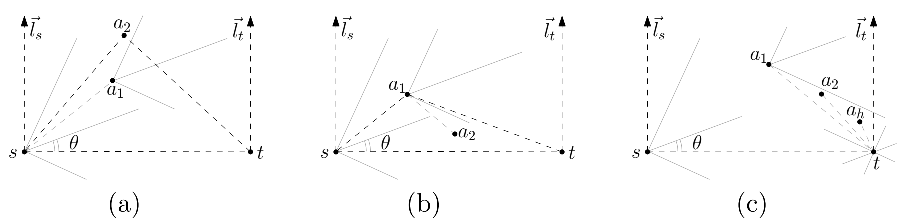

Figure 5: Illustration for the proof that $\overline{a_1 a_2}$ is a θ-edge.

We now prove Claim B. First, we prove that, for *every* θ in the interval $[-45°; 45°]$, there exists a maximal θ-path starting at s that is low or high. Indeed, it suffices to prove that there exists a θ-edge incident to s, as such an edge is also a θ-path starting at s, and the existence of a θ-path starting at s

implies the existence of a maximal θ-path starting at s. Consider a straight-line segment e_θ that is the intersection of a directed half-line incident to s with slope θ and of a disk of arbitrarily small radius centered at s. If e_θ is internal to $\mathcal{H}_P$, then consider the two edges e_1 and e_2 of G that are encountered when rotating e_θ around s counter-clockwise and clockwise, respectively. Then e_1 or e_2 is a θ-edge, as the angle spanned by a clockwise rotation bringing e_1 to coincide with e_2 is smaller than $90°$, given that G is a Gabriel triangulation, and e_θ is encountered during such a rotation. If e_θ is outside $\mathcal{H}_P$, which might happen if s on the boundary of $\mathcal{H}_P$, then assume that the slope of e_θ is in the interval $[0°; 45°]$ (the case in which the slope of e_θ is in the interval $[-45°; 0°]$ is analogous). Then the angle spanned by a clockwise rotation bringing e_θ to coincide with $\overline{st}$ is at most $45°$. Since $\overline{st}$ is in interior or on the boundary of $\mathcal{H}_P$, an edge e_1 of G is encountered during such a rotation, hence e_1 is a θ-edge. An analogous proof shows that, for *every* θ in the interval $[-45°; 45°]$, there exists a maximal $(\theta + 180°)$-path starting at t that is low or high.

Second, we prove that, for *some* $\theta \in [-45°; 45°]$, there exist a maximal low θ-path *and* a maximal high θ-path both starting at s. All the maximal $(-45°)$-paths (all the maximal $(45°)$-paths) starting at s are low (resp. high), given that every edge on these paths has slope in the interval $[-90°; 0°]$ (resp. $[0°; 90°]$). Thus, let θ be the smallest constant in the interval $[-45°; 45°]$ such that a maximal high θ-path exists. We prove that there also exists a maximal low θ-path starting at s. Consider an arbitrarily small $\epsilon > 0$. By assumption, there exists no high $(\theta - \epsilon)$-path. Hence, from the previous argument there exists a low $(\theta - \epsilon)$-path $\mathcal{P}$. If ϵ is sufficiently small, then no edge of $\mathcal{P}$ has slope in the interval $[\theta - 45° - \epsilon; \theta - 45°)$. Thus every edge of $\mathcal{P}$ has slope in the interval $[\theta - 45°; \theta + 45° - \epsilon)$, hence $\mathcal{P}$ is a maximal low θ-path starting at s.

Since there exist a maximal high θ-path starting at s, a maximal low θ-path starting at s, and a maximal $(\theta + 180°)$-path starting at t that is low or high, it follows that there exist a maximal θ-path $\mathcal{P}_s$ starting at s and a maximal $(\theta + 180°)$-path $\mathcal{P}_t$ starting at t that are both high or both low. This proves Claim B and hence the lemma. □

Lemma 3 and Lemma 4 immediately imply the following.

Corollary 1 *Any Gabriel triangulation is increasing-chord.*

We are now ready to state the main result of this section.

Theorem 1 *Let P be a point set with n points. One can construct in $O(n \log n)$ time an increasing-chord planar graph $G(P', S)$ such that $P \subseteq P'$ and $|P'| \in O(n)$.*

Proof: Bern, Eppstein, and Gilbert [4] proved that, for any point set P, there exists a point set P' with $P \subseteq P'$ and $|P'| \in O(n)$ such that P' admits a Gabriel triangulation G. Both P' and G can be computed in $O(n \log n)$ time [4]. By Corollary 1, G is increasing-chord, which concludes the proof. □

We remark that $o(|P|)$ Steiner points are not always enough to augment a point set P to a point set that admits a Gabriel triangulation. Namely, consider any point set B with $O(1)$ points that admits no Gabriel triangulation. Construct a point set P out of $|P|/|B|$ copies of B placed "far apart" from each other, so that any triangle with two points in different copies of B is obtuse. Then a Steiner point has to be added inside the convex hull of each copy of B to obtain a point set that admits a Gabriel triangulation.

4 Increasing-Chord Graphs with Few Edges Spanning Convex Point Sets

In this section we prove the following theorem;

Theorem 2 *For every convex point set P with n points, there exists an increasing-chord geometric graph $G(P,S)$ such that $|S| \in O(n \log n)$.*

The main idea behind the proof of Theorem 2 is that any convex point set P can be decomposed into some one-sided convex point sets $P_1, \dots, P_k$ (which by Lemma 1 admit increasing-chord spanning graphs with linearly many edges) in such a way that every two points of P are part of some P_i and that $\sum |P_i|$ is small. In order to perform such a decomposition, we introduce the concept of *balanced $(\vec{d_1}, \vec{d_2})$-partition.*

Let P be a convex point set and let $\vec{d}$ be a directed straight line not orthogonal to any line through two points of P. See Figure 6. Let $p_{\min}(\vec{d})$ and $p_{\max}(\vec{d})$ be the minimum and maximum point of P with respect to $\vec{d}$, respectively. Let $P_1(\vec{d})$ be composed of those points in P that are encountered when walking clockwise along the boundary of $\mathcal{H}_P$ from $p_{\min}(\vec{d})$ to $p_{\max}(\vec{d})$, where $p_{\min}(\vec{d}) \in P_1(\vec{d})$ and $p_{\max}(\vec{d}) \notin P_1(\vec{d})$. Analogously, let $P_2(\vec{d})$ be composed of those points in P that are encountered when walking clockwise along the boundary of $\mathcal{H}_P$ from $p_{\max}(\vec{d})$ to $p_{\min}(\vec{d})$, where $p_{\max}(\vec{d}) \in P_2(\vec{d})$ and $p_{\min}(\vec{d}) \notin P_2(\vec{d})$.

Let $\vec{d_1}$ and $\vec{d_2}$ be two directed straight lines not orthogonal to any line through two points of P, where the clockwise rotation that brings $\vec{d_1}$ to coincide with $\vec{d_2}$ is at most 180°. The *$(\vec{d_1}, \vec{d_2})$-partition* of P partitions P into subsets $P_a = P_1(\vec{d_1}) \cap P_1(\vec{d_2})$, $P_b = P_1(\vec{d_1}) \cap P_2(\vec{d_2})$, $P_c = P_2(\vec{d_1}) \cap P_1(\vec{d_2})$, and $P_d = P_2(\vec{d_1}) \cap P_2(\vec{d_2})$. Note that every point in P is contained in one of P_a, P_b, P_c, and P_d. A *$(\vec{d_1}, \vec{d_2})$-partition* of P is *balanced* if $|P_a| + |P_d| \leq \frac{|P|}{2} + 1$ and $|P_b| + |P_c| \leq \frac{|P|}{2} + 1$. We now argue that, for every point set P, a balanced $(\vec{d_1}, \vec{d_2})$-partition of P always exists, even if $\vec{d_1}$ is arbitrarily prescribed.

Lemma 5 *Let P be a convex point set and let $\vec{d_1}$ be a directed straight line not orthogonal to any line through two points of P. Then there exists a directed straight line $\vec{d_2}$ that is not orthogonal to any line through two points of P such that the $(\vec{d_1}, \vec{d_2})$-partition of P is balanced.*

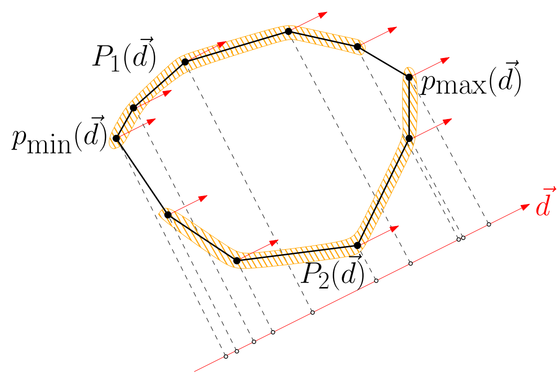

Figure 6: Subsets $P_1(\vec{d})$ and $P_2(\vec{d})$ of a point set P determined by a directed straight line $\vec{d}$.

Proof: Denote by $q_1 = p_{\min}(\vec{d_1}), q_2, \dots, q_l, q_{l+1} = p_{\max}(\vec{d_1})$ the points of P encountered when walking clockwise on the boundary of $\mathcal{H}_P$ from $p_{\min}(\vec{d_1})$ to $p_{\max}(\vec{d_1})$. Also, denote by $r_1 = p_{\max}(\vec{d_1}), r_2, \dots, r_m, r_{m+1} = p_{\min}(\vec{d_1})$ the points of P encountered when walking clockwise on the boundary of $\mathcal{H}_P$ from $p_{\max}(\vec{d_1})$ to $p_{\min}(\vec{d_1})$.

Initialize $\vec{d_2}$ to be a directed straight line coincident with $\vec{d_1}$. When $\vec{d_2} = \vec{d_1}$, we have $P_a = \{q_1, q_2, \dots, q_l\}$, $P_d = \{r_1, r_2, \dots, r_m\}$, $P_b = \emptyset$, and $P_c = \emptyset$. We now rotate $\vec{d_2}$ clockwise until it is opposite to $\vec{d_1}$ (that is, parallel and pointing in the opposite direction). As we rotate $\vec{d_2}$, sets $P_1(\vec{d_2})$ and $P_2(\vec{d_2})$ change, hence sets P_a, P_b, P_c, and P_d change as well. When $\vec{d_2}$ is opposite to $\vec{d_1}$, we have $P_a = \emptyset$, $P_d = \emptyset$, $P_b = \{q_1, q_2, \dots, q_l\}$, and $P_c = \{r_1, r_2, \dots, r_m\}$. We will argue that there is a moment during such a rotation of $\vec{d_2}$ in which the corresponding $(\vec{d_1}, \vec{d_2})$-partition of P is balanced. Assume that at any time instant during the rotation of $\vec{d_2}$ the following hold (see Figs. 7(a)–(b)):

- $P_b = \{q_1, q_2, \dots, q_j\}$ (possibly P_b is empty);
- $P_a = \{q_{j+1}, q_{j+2}, \dots, q_l\}$ (possibly P_a is empty);
- $P_c = \{r_1, r_2, \dots, r_k\}$ (possibly P_c is empty);
- $P_d = \{r_{k+1}, r_{k+2}, \dots, r_m\}$ (possibly P_d is empty); and
- q_{j+1} and r_{k+1} are the minimum and maximum point of P with respect to $\vec{d_2}$, respectively.

The assumption is indeed true when $\vec{d_2}$ starts moving, with $j = 0$ and $k = 0$.

As we keep on rotating $\vec{d_2}$ clockwise, at a certain moment $\vec{d_2}$ becomes orthogonal to $\overline{q_{j+1}q_{j+2}}$ or to $\overline{r_{k+1}r_{k+2}}$ (or to both if $\overline{q_{j+1}q_{j+2}}$ and $\overline{r_{k+1}r_{k+2}}$ are

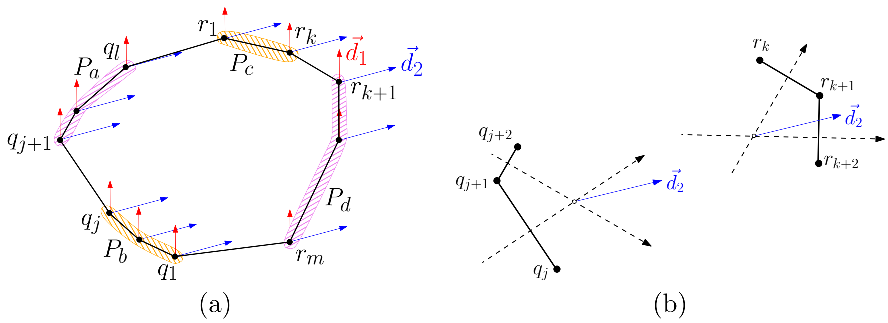

Figure 7: (a) Sets P_a, P_b, P_c, and P_d at a certain time instant during the rotation of $\vec{d}_2$. (b) The slope of $\vec{d}_2$ with respect to the slopes of the lines orthogonal to $\overline{q_j q_{j+1}}$, to $\overline{q_{j+1} q_{j+2}}$, to $\overline{r_k r_{k+1}}$, and to $\overline{r_{k+1} r_{k+2}}$.

parallel). Thus, as we keep on rotating $\vec{d}_2$ clockwise, sets P_a, P_b, P_c, and P_d change. Namely:

If $\vec{d}_2$ becomes orthogonal first to $\overline{q_{j+1} q_{j+2}}$ and then to $\overline{r_{k+1} r_{k+2}}$, then as $\vec{d}_2$ rotates clockwise after the position in which it is orthogonal to $\overline{q_{j+1} q_{j+2}}$, we have

- $P_b = \{q_1, q_2, \dots, q_j, q_{j+1}\}$;
- $P_a = \{q_{j+2}, q_{j+3}, \dots, q_l\}$ (possibly P_a is empty);
- $P_c = \{r_1, r_2, \dots, r_k\}$ (possibly P_c is empty);
- $P_d = \{r_{k+1}, r_{k+2}, \dots, r_m\}$ (possibly P_d is empty); and
- q_{j+2} and r_{k+1} are the minimum and maximum point of P with respect to $\vec{d}_2$, respectively.

If $\vec{d}_2$ becomes orthogonal first to $\overline{r_{k+1} r_{k+2}}$ and then to $\overline{q_{j+1} q_{j+2}}$, then as $\vec{d}_2$ rotates clockwise after the position in which it is orthogonal to $\overline{r_{k+1} r_{k+2}}$, we have that P_a and P_b stay unchanged, that r_{k+1} passes from P_d to P_c, and that q_{j+1} and r_{k+2} are the minimum and maximum point of P with respect to $\vec{d}_2$, respectively.

If $\vec{d}_2$ becomes orthogonal to $\overline{q_{j+1} q_{j+2}}$ and $\overline{r_{k+1} r_{k+2}}$ simultaneously, then as $\vec{d}_2$ rotates clockwise after the position in which it is orthogonal to $\overline{q_{j+1} q_{j+2}}$, we have that q_{j+1} passes from P_a to P_b, that r_{k+1} passes from P_d to P_c, and that q_{j+2} and r_{k+2} are the minimum and maximum point of P with respect to $\vec{d}_2$, respectively.

Observe that:

1. whenever sets P_a, P_b, P_c, and P_d change, we have that $|P_a| + |P_d|$ and $|P_b| + |P_c|$ change at most by two;

2. when $\vec{d_2}$ starts rotating we have that $|P_a|+|P_d|=|P|$, and when $\vec{d_2}$ stops rotating we have that $|P_a|+|P_d|=0$;
3. when $\vec{d_2}$ starts rotating we have that $|P_b|+|P_c|=0$, and when $\vec{d_2}$ stops rotating we have that $|P_b|+|P_c|=|P|$; and
4. $|P_a|+|P_b|+|P_c|+|P_d|=|P|$ holds at any time instant.

By continuity, there is a time instant in which $|P_a|+|P_d|=\lfloor |P|/2\rfloor$ and $|P_b|+|P_c|=\lceil |P|/2\rceil$, or in which $|P_a|+|P_d|=\lfloor |P|/2\rfloor+1$ and $|P_b|+|P_c|=\lceil |P|/2\rceil-1$. This completes the proof of the lemma. □

We now show how to use Lemma 5 in order to prove Theorem 2.

Let P be any point set. Assume that no two points of P have the same y-coordinate. Such a condition is easily met after rotating the Cartesian axes. Denote by $\vec{l}$ a vertical straight line directed toward increasing y-coordinates. Each of $P_1(\vec{l})$ and $P_2(\vec{l})$ is convex and one-sided with respect to $\vec{l}$. By Lemma 1, there exist increasing-chord graphs $G_1=(P_1(\vec{l}),S_1)$ and $G_2=(P_2(\vec{l}),S_2)$ with $|S_1|<2|P_1(\vec{l})|$ and $|S_2|<2|P_2(\vec{l})|$. Then graph $G(P,S_1\cup S_2)$ has less than $2(|P_1(\vec{l})|+|P_2(\vec{l})|)=2|P|$ edges and contains an increasing-chord path between every pair of vertices in $P_1(\vec{l})$ and between every pair of vertices in $P_2(\vec{l})$. However, G does not have increasing-chord paths between any pair (a,b) of vertices such that $a\in P_1(\vec{l})$ and $b\in P_2(\vec{l})$.

We now present and prove the following claim.

Claim 1 *Consider a convex point set Q and a directed straight line $\vec{d_1}$ not orthogonal to any line through two points of Q. Then there exists a geometric graph $H(Q,R)$ that contains an increasing-chord path between every point in $Q_1(\vec{d_1})$ and every point in $Q_2(\vec{d_1})$, such that $|R|\in O(|Q|\log|Q|)$.*

The application of the claim with $Q=P$ and $\vec{d_1}=\vec{l}$ provides a graph $H(P,R)$ that contains an increasing-chord path between every pair (a,b) of vertices such that $a\in P_1(\vec{l})$ and $b\in P_2(\vec{l})$. Thus, the union of G and H is an increasing-chord graph with $O(|P|\log|P|)$ edges spanning P. Therefore, the above claim implies Theorem 2.

We give an inductive algorithm to construct H. Let $f(Q,\vec{d_1})$ be the number of edges that H has as a result of the application of our algorithm on a point set Q and a directed straight-line $\vec{d_1}$. Also, let $f(n)=\max\{f(Q,\vec{d_1})\}$, where the maximum is among all point sets Q with $n=|Q|$ points and among all the directed straight-lines $\vec{d_1}$ that are not orthogonal to any line through two points of Q.

Let Q be any convex point set with n points and let $\vec{d_1}$ be any directed straight line not orthogonal to any line through two points of Q. By Lemma 5, there exists a directed straight line not orthogonal to any line through two points of Q and such that the $(\vec{d_1},\vec{d_2})$-partition of Q is balanced.

Let $Q_a=Q_1(\vec{d_1})\cap Q_1(\vec{d_2})$, let $Q_b=Q_1(\vec{d_1})\cap Q_2(\vec{d_2})$, let $Q_c=Q_2(\vec{d_1})\cap Q_1(\vec{d_2})$, and let $Q_d=Q_2(\vec{d_1})\cap Q_2(\vec{d_2})$.

Point set $Q_a \cup Q_c$ is convex and one-sided with respect to $\vec{d_2}$. By Lemma 1 there exists an increasing-chord graph $H_1(Q_a \cup Q_c, R_1)$ with $|R_1| < 2(|Q_a| + |Q_c|)$ edges. Analogously, by Lemma 1 there exists an increasing-chord graph $H_2(Q_b \cup Q_d, R_2)$ with $|R_2| < 2(|Q_b| + |Q_d|)$ edges.

Hence, there exists a graph $H_3(Q, R_1 \cup R_2)$ with $|R_1 \cup R_2| < 2(|Q_a| + |Q_c| + |Q_b| + |Q_d|) = 2|Q| = 2n$ edges containing an increasing-chord path between every point in Q_a and every point in Q_c, and between every point in Q_b and every point in Q_d. However, G does not have an increasing-chord path between any point in Q_a and any point in Q_d, and does not have an increasing-chord path between any point in Q_b and any point in Q_c.

By Lemma 5, we have $|Q_a| + |Q_d| \leq \frac{n}{2} + 1$ and $|Q_b| + |Q_c| \leq \frac{n}{2} + 1$. By definition, we have $f(Q_a \cup Q_d, \vec{d_1}) \leq f(|Q_a| + |Q_d|) \leq f(\frac{n}{2} + 1)$. Analogously, we have $f(Q_b \cup Q_c, \vec{d_1}) \leq f(|Q_b| + |Q_c|) \leq f(\frac{n}{2} + 1)$. Hence, $f(n) \leq 2n + 2f(\frac{n}{2} + 1) \in O(n \log n)$. This proves the claim and hence Theorem 2.

5 Conclusions

We considered the problem of constructing increasing-chord graphs spanning point sets. We proved that, for every point set P, there exists a planar increasing-chord graph $G(P', S)$ with $P \subseteq P'$ and $|P'| \in O(|P|)$. We also proved that, for every convex point set P, there exists an increasing-chord graph $G(P, S)$ with $|S| \in O(|P| \log |P|)$.

Despite our research efforts, the main question on this topic remains open:

Open Problem 1 *Is it true that, for every (convex) point set P, there exists an increasing-chord planar graph $G(P, S)$?*

One of the directions we took in order to tackle this problem is to assume that the points in P lie on a constant number of straight lines. While a simple modification of the proof of Lemma 1 allows us to prove that an increasing-chord planar graph always exists spanning a set of points lying on two straight lines, it is surprising and disheartening that we could not prove a similar result for sets of points lying on three straight lines. The main difficulty seems to lie in the construction of planar increasing-chord graphs spanning sets of points lying on the boundary of an acute triangle.

Open Problem 2 *Is it true that, for every set P of points lying on the boundary of an acute triangle, there exists an increasing-chord planar graph $G(P, S)$?*

Gabriel graphs naturally generalize to higher dimensions, where empty balls replace empty disks. In Section 3 we showed that, for points in $\mathbb{R}^2$, every Gabriel triangulation is increasing-chord. Can this result be generalized to higher dimensions?

Open Problem 3 *Is it true that, for every point set P in $\mathbb{R}^d$, any Gabriel triangulation of P is increasing-chord?*

Finally, it would be interesting to understand if increasing-chord graphs with few edges can be constructed for any (possibly non-convex) point set:

Open Problem 4 *Is it true that, for every point set P, there exists an increasing-chord graph $G(P,S)$ with $|S| \in o(|P|^2)$?*

References

[1] O. Aichholzer, F. Aurenhammer, C. Icking, R. Klein, E. Langetepe, and G. Rote. Generalized self-approaching curves. *Discr. Appl. Math.*, 109(1-2):3–24, 2001. doi:10.1016/S0166-218X(00)00233-X.

[2] S. Alamdari, T. M. Chan, E. Grant, A. Lubiw, and V. Pathak. Self-approaching graphs. In W. Didimo and M. Patrignani, editors, *GD '12*, volume 7704 of *LNCS*, pages 260–271, 2013. doi:10.1007/978-3-642-36763-2_23.

[3] P. Angelini, F. Frati, and L. Grilli. An algorithm to construct greedy drawings of triangulations. *J. Graph Algorithms Appl.*, 14(1):19–51, 2010. doi:10.7155/jgaa.00197.

[4] M. W. Bern, D. Eppstein, and J. R. Gilbert. Provably good mesh generation. *J. Comput. Syst. Sci.*, 48(3):384–409, 1994. doi:10.1016/S0022-0000(05)80059-5.

[5] M. de Berg, O. Cheong, M. van Kreveld, and M. Overmars. *Computational Geometry: Algorithms and Applications*. Springer, Heidelberg, 3rd edition, 2008. doi:10.1007/978-3-540-77974-2.

[6] H. R. Dehkordi, F. Frati, and J. Gudmundsson. Increasing-chord graphs on point sets. In C. Duncan and A. Symvonis, editors, *International Symposium on Graph Drawing (GD '14)*, volume 8871, pages 464–475. Springer, 2014. doi:10.1007/978-3-662-45803-7_39.

[7] G. Di Battista and L. Vismara. Angles of planar triangular graphs. *SIAM J. Discrete Math.*, 9(3):349–359, 1996. doi:10.1137/S0895480194264010.

[8] M. B. Dillencourt and W. D. Smith. Graph-theoretical conditions for inscribability and Delaunay realizability. *Discrete Mathematics*, 161(1-3):63–77, 1996. doi:10.1016/0012-365X(95)00276-3.

[9] P. Eades and S. Whitesides. The realization problem for Euclidean minimum spanning trees is NP-hard. *Algorithmica*, 16(1):60–82, 1996. doi:10.1007/BF02086608.

[10] K. R. Gabriel and R. R. Sokal. A new statistical approach to geographic variation analysis. *Systematic Biology*, 18:259–278, 1969.

[11] X. He and H. Zhang. On succinct greedy drawings of plane triangulations and 3-connected plane graphs. *Algorithmica*, 68(2):531–544, 2014. doi:10.1007/s00453-012-9682-y.

[12] C. Icking, R. Klein, and E. Langetepe. Self-approaching curves. *Math. Proc. Camb. Phil. Soc.*, 125(3):441–453, 1999.

[13] T. Leighton and A. Moitra. Some results on greedy embeddings in metric spaces. *Discrete & Computational Geometry*, 44(3):686–705, 2010. doi:10.1007/s00454-009-9227-6.

[14] G. Liotta. *Chapter 4 of Handbook of Graph Drawing.* R. Tamassia, ed. CRC press, 2014.

[15] D. W. Matula and R. R. Sokal. Properties of Gabriel graphs relevant to geographic variation research and clustering of points in the plane. *Geographical Analysis*, 12(3):205–222, 1980. doi:10.1111/j.1538-4632.1980.tb00031.x.

[16] C. L. Monma and S. Suri. Transitions in geometric minimum spanning trees. *Discrete & Computational Geometry*, 8:265–293, 1992. doi:10.1007/BF02293049.

[17] M. Nöllenburg, R. Prutkin, and I. Rutter. On self-approaching and increasing-chord drawings of 3-connected planar graphs. In C. Duncan and A. Symvonis, editors, *International Symposium on Graph Drawing (GD '14)*, volume 8871, pages 476–487. Springer, 2014. doi:10.1007/BF02293049.

[18] C. H. Papadimitriou and D. Ratajczak. On a conjecture related to geometric routing. *Theoretical Computer Science*, 344(1):3–14, 2005. doi:10.1016/j.tcs.2005.06.022.

[19] A. Rao, C. H. Papadimitriou, S. Shenker, and I. Stoica. Geographic routing without location information. In D. Johnson, A. Joseph, and N. Vaidya, editors, *MOBICOM '03*, pages 96–108, 2003. doi:10.1145/938985.938996.

[20] G. Rote. Curves with increasing chords. *Math. Proc. Camb. Phil. Soc.*, 115(1):1–12, 1994.