1-Bend 3-D Orthogonal Box-Drawings: Two Open Problems Solved

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Abstract

This paper studies three-dimensional orthogonal box-drawings where edge-routes have at most one bend. Two open problems for such drawings are: (1) Does every drawing of K_n have volume $\Omega(n^3)$? (2) Is there a drawing of K_n for which additionally the vertices are represented by cubes with surface $O(n)$? This paper answers both questions in the negative, and provides related results concerning volume bounds as well.

1   Background

A *3-D orthogonal box-drawing* of a graph is a drawing of the graph where vertices are represented by disjoint axis-parallel boxes and edges are represented by disjoint routes along an underlying three-dimensional rectangular grid. (Since no other type of drawings will be studied here, the term *drawing* is used to mean a 3-D orthogonal box-drawing from now on.)

The route of each edge thus consists of a sequence of contiguous *grid segments*, i.e., axis-parallel line segments for which the fixed coordinates are integers. The transition from one grid segment to another is called a *bend*. A drawing is called a *k-bend drawing* if all edge routes have at most k bends.

Every vertex is represented by an axis-parallel box with integral boundaries; such a box is called a *grid box*. An *X-plane* is a plane that is perpendicular to the X-axis. It is called an X-grid plane if its fixed coordinate is integral. *Y-planes* and *Z-planes* are defined similarly. For any vertex v, let $X(v)$ be the number of X-grid planes that intersect the box of v; $Y(v)$ and $Z(v)$ are defined similarly. The *surface of v* is $2(X(v)Y(v) + Y(v)Z(v) + Z(v)X(v))$. The *volume of v* is $X(v)Y(v)Z(v)$.

When no confusion arises, we will use graph-theoretic terms, such as "vertex" and "edge", to also mean the representation in a fixed drawing.

Given a drawing, denote by $X \times Y \times Z$ the size of the smallest enclosing rectangular box of the drawing. The *volume* of the drawing is $X \cdot Y \cdot Z$.

This paper studies bounds on the volume of drawings with very few bends per edge. Since not all graphs have a 0-bend drawing (also known as visibility representation) [BSWW99, FM99], the smallest applicable number of bends per edge is one.

1.1   Existing results for 1-bend drawings

In [BSWW99], it was shown that the complete graph K_n has a 1-bend drawing with $O(n^3)$ volume (more precisely, in an $n/2 \times n/2 \times n/2$-grid.) In the same paper, it was also shown that any drawing of K_n has volume $\Omega(n^{2.5})$. However, the lower bound does not take restrictions on the number of bends into account, and in particular, it was left as an open problem whether any 1-bend drawing of K_n needs $\Omega(n^3)$ volume.

One criticism of the drawings in [BSWW99] is that vertex boxes resemble "sticks", i.e., one dimension is very large while the other two dimensions are one unit each, hence there is no bound on the aspect ratio. A drawing is said to have *aspect-ratios at most r*, for some constants $r \geq 1$, if any vertex box has aspect ratio at most r. If $r = 1$, then the drawing is called a *cube-drawing*.

The construction in [BSWW99] can be modified to obtain a cube-drawing of K_n by "blowing up" every vertex (see also Figure 2). However, this increases the volume of the drawing to $O(n^4)$. Also, the surface of each vertex box then becomes $O(n^2)$, which seems excessive since every vertex has only $O(n)$ incident edges. A drawing is said to be *degree-restricted* if the surface of a vertex v is at most $\alpha \deg(v)$, for some constant $\alpha \geq 1$. The construction in [BSWW99] is

degree-restricted for K_n, but when converted to a cube-drawing, it is no longer degree-restricted. Hence, the question was posed whether K_n has a degree-restricted 1-bend cube-drawing.

In [BTW01], the lower bounds of [BSWW99] were extended to graphs other than the complete graph. More precisely, it was shown that there exist graphs with n vertices and m edges that have volume $\Omega(mn^{1/2})$ in any drawing. This lower bound also does not take restrictions on the number of bends into account.

Finally, in [Woo00], it was shown that every n-vertex m-edge graph with genus g has a 1-bend drawing of volume $O(nm\sqrt{g})$, which is $O(nm^{3/2})$ in the worst case.

1.2   Contributions of this paper

This paper settles the two open problems mentioned above, and provides other results for 1-bend drawings of *simple graphs*, i.e., graphs without loops and multiple edges. Specific results are as follows:

- Any 1-bend cube-drawing of a simple graph G with $\Omega(\Delta n)$ edges represents $\Omega(n)$ many vertices with an $\Omega(\Delta) \times \Omega(\Delta) \times \Omega(\Delta)$-box, where Δ is the maximum degree of G.[1]

  This has the following consequences:

    – Any such graph does not have a 1-bend degree-restricted cube-drawing. In particular, K_n does not have a 1-bend degree-restricted cube-drawing. (This settles the second open problem mentioned above.)

    – Any 1-bend cube-drawing of such a graph has volume $\Omega(\Delta^3 n)$. In particular, since K_n is $(n-1)$-regular, any 1-bend cube-drawing of K_n has volume $\Omega(n^4)$. (This bound is matched by a construction.)

- Other lower bounds are obtained using a so-called *Ramanujan-graph $G_{n,d}$*, which is a simple d-regular n-vertex graph with special cut-properties which will be reviewed in Section 3.1:

    – Any 1-bend drawing of $G_{n,d}$, for n and d sufficiently big, has a grid plane that intersects at least $\frac{1}{8}n$ vertices.

    – Any 1-bend drawing of $G_{n,d}$ has volume $\Omega(n^2 d)$.
      Since $K_n = G_{n,n-1}$, any 1-bend drawing of K_n has volume $\Omega(n^3)$, which answers the first open problem mentioned above.

2   Cube-drawings

This section proves that K_n (or more generally, any graph with $\Omega(\Delta n)$ edges) does not have a degree-restricted 1-bend cube-drawing. As a preliminary result,

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[1] Note that a graph with $\Omega(\Delta n)$ edges has asymptotically the maximum number of edges, since all graphs have at most $\frac{1}{2}\Delta n$ edges. However, there are graphs with $o(\Delta n)$ edges.

we first show that in any 1-bend drawing of such a graph many (i.e., $\Omega(n)$) vertices are intersected by many (i.e., $\Omega(\Delta)$) grid planes each.

Lemma 2.1 *If G is a simple graph with at least $\kappa\Delta n$ edges, for some $0 < \kappa \leq \frac{1}{2}$, then at least $\frac{1}{6}\kappa n$ vertices intersect at least $\frac{1}{6}\kappa\Delta$ grid planes each.*

Proof: Fix an arbitrary 1-bend drawing of G. For any edge e, the route of e has at most one bend, and hence is entirely contained within one grid plane P. We say that edge e *belongs* to P and P *owns* e. (If the route of e has no bend, then it is contained in two grid planes. Arbitrarily choose one of them to own e, so that each edge belongs to exactly one grid plane.)

Let $P_1, \ldots, P_l$ be the grid planes that own at least one edge. For $i = 1, \ldots, l$, let $n(P_i)$ be the number of vertices for which an incident edge belongs to P_i. See also Figure 1.

The crucial observation is that edges do not cross, hence the graph formed by the edges owned by P_i is planar. In particular, by simplicity of G at most $3n(P_i)$ edges can be owned by P_i. Since each of the m edges of G belongs to a grid plane,

$$\sum_{i=1}^{l} 3n(P_i) \geq m \geq \kappa\Delta n. \tag{1}$$

Now count $\sum_{i=1}^{l} n(P_i)$ in another way. For every vertex v, denote by $p(v)$ the number of grid planes that own an incident edge of v; see also Figure 1. Observe that $p(v) \leq X(v) + Y(v) + Z(v)$ because any grid plane that contributes to $p(v)$ must also intersect the box of v. Also, $\sum_{i=1}^{l} n(P_i) = \sum_{v \in V} p(v)$, because both sums count the incidences between a vertex v and a grid plane that owns an edge incident to v.

Let V_b be the set of vertices v with $p(v) \geq \frac{1}{6}\kappa\Delta$. The lemma holds if $|V_b| \geq \frac{1}{6}\kappa n$, because $X(v) + Y(v) + Z(v) \geq p(v) \geq \frac{1}{6}\kappa\Delta$ for every vertex $v \in V_b$. So assume for contradiction that fewer than $\frac{1}{6}\kappa n$ vertices belong to V_b. Observe that $p(v) \leq \Delta$ for all vertices (because for each grid plane there is at least one incident edge of v), and that at most n vertices could be in $V - V_b$. Therefore

$$\sum_{i=1}^{l} n(P_i) = \sum_{v \in V} p(v) = \sum_{v \notin V_b} p(v) + \sum_{v \in V_b} p(v)$$

$$< |V - V_b| \cdot \frac{1}{6}\kappa\Delta + |V_b| \cdot \Delta < n \cdot \frac{1}{6}\kappa\Delta + \frac{1}{6}\kappa n \cdot \Delta = \frac{1}{3}\kappa\Delta n.$$

This contradicts inequality (1), therefore $|V_b| \geq \frac{1}{6}\kappa\Delta$ and the lemma holds. $\square$

Note that the constants in the above lemma could be improved for bipartite graphs, because then at most $2n(P)$ edges could be owned by grid plane P. While this would improve some of the lower bounds to follow by a small fraction, we will not pursue this detail for simplicity's sake. Also note that the proof relies only on that any edge is routed entirely within one grid plane. While this
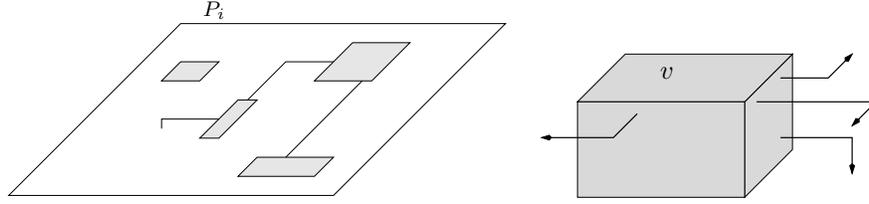
Figure 1: Illustration of $n(P_i)$ and $p(v)$. In the left picture $n(P_i) = 3$, because while four vertices intersect P_i, only three of them are incident to an edge that belongs to P_i. In the right picture, $p(v) = 2$.

certainly holds for any 1-bend drawing, it also holds for many other constructions (e.g., the ones of [BSWW99]). Hence, the lemma and its corollaries could be generalized to any so-called *co-planar* drawing in which each edge is routed within a grid plane.

Lemma 2.1 implies that any 1-bend cube-drawing contains many big vertex boxes. In fact, this result holds for any drawing with bounded aspect ratios.

Lemma 2.2 *Let G be a graph with $\Omega(\Delta n)$ edges. Then any 1-bend drawing of G with aspect ratios at most r contains $\Omega(n)$ vertices whose box has minimum dimension $\Omega(\Delta/r)$, surface $\Omega(\Delta^2/r)$ and volume $\Omega(\Delta^3/r^2)$.*

Proof: Assume that G has at least $\kappa\Delta n$ edges for some constant $0 < \kappa \leq \frac{1}{2}$. Fix an arbitrary drawing of G and let v be one of the at least $\frac{1}{6}\kappa n$ vertices whose box is intersected by at least $\frac{1}{6}\kappa\Delta$ grid planes; these exist by Lemma 2.1. Let the box representing v be an $X \times Y \times Z$-box; without loss of generality assume that $X \leq Y \leq Z$. The box of v intersects $X + Y + Z$ grid planes, so $X + Y + Z \geq \frac{1}{6}\kappa\Delta$ by assumption on v. Also, $Z \leq rX$ because the aspect ratio of v is at most r.

Minimizing the minimum dimension X of v under the constraints $X \leq Y \leq Z$, $X + Y + Z \leq \frac{1}{6}\kappa\Delta$ and $Z \leq rX$ yields $X \geq \frac{1}{6}\kappa\Delta/(1 + 2r) \in \Omega(\Delta/r)$. The surface of v is $2(XY + YZ + XZ)$ and the volume is XYZ. Minimizing each expression, subject to the above constraints, one obtains (for both of them) the solution $X = \frac{1}{6}\kappa\Delta/(r+2) = Y$ and $Z = rX = \frac{1}{6}\kappa r\Delta/(r+2)$. Hence the surface of v is

$$2(XY + YZ + XZ) \geq \frac{2}{36}\kappa^2\Delta^2(1 + 2r)/(r + 2)^2 \in \Omega(\Delta^2/r),$$

and the volume is

$$XYZ \geq \frac{1}{216}\kappa^3\Delta^3 r/(r + 2)^3 \in \Omega(\Delta^3/r^2).$$

$\square$

This lemma implies the answer for the open problem: K_n does not have a degree-restricted 1-bend cube-drawing.

Theorem 1 *Any simple graph G with $\Omega(\Delta n)$ edges does not have a degree-restricted 1-bend drawing with aspect ratios $o(\Delta)$.*

Proof: In any 1-bend drawing of G with aspect ratios at most r, there are $\Omega(n)$ vertices with surface $\Omega(\Delta^2/r)$ by Lemma 2.2. Unless $r \in \Omega(\Delta)$, the surface of these vertices is not proportional to their degrees, which is at most Δ.   □

This lemma can also be used for lower bounds on the volume of drawings with bounded aspect ratios.

Theorem 2 *If a simple graph G has $\Omega(\Delta n)$ edges, then any 1-bend drawing with aspect ratios at most r has volume $\Omega(n\Delta^3/r^2)$.*

Proof: In any 1-bend drawing of G with aspect ratios at most r, there are $\Omega(n)$ vertices with volume $\Omega(\Delta^3/r^2)$ by Lemma 2.2. Since vertex boxes are disjoint, these $\Omega(n)$ vertices together occupy an area of volume $\Omega(n\Delta^3/r^2)$. □

Depending on the values of Δ and r, this theorem improves in some cases on the lower bound of $\Omega(m^{3/2}/\sqrt{r})$ for such drawings known from [BTW01].

The above lower bound is optimal for cube-drawings of K_n, because the lower bound states $\Omega(n^4)$ for K_n, and a construction with volume $O(n^4)$ can be obtained easily by "blowing up" the vertex boxes of the construction of [BSWW99]. See Figure 2.
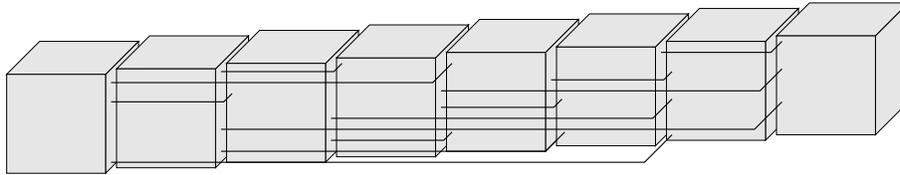
Figure 2: A cube-drawing of K_n with volume $O(n^4)$. Only half of the edges are shown; the other half is routed behind the cubes.

3 Lower Bounds

This section provides lower bounds on the volume of 1-bend drawings, and proves that the $O(n^3)$ volume drawing for K_n in [BSWW99] is asymptotically optimal.

The lower bound proof follows a scheme developed in [BSWW99] and also used in [BTW01]. For a given drawing there are three cases: (1) One grid line intersects "many" vertices; (2) one grid plane, but no grid line, intersects "many" vertices; (3) neither of the above is the case. In [BSWW99], it was shown that the volume of K_n is $\Omega(n^3)$ in the first and third case, but in the second case, only a bound of $\Omega(n^{2.5})$ was achieved. In [BTW01], it was shown

that the volume for so-called Ramanujan-graphs is $\Omega(\Delta n^2)$ in the first case, $\Omega(\Delta n^{1.5})$ in the second case and $\Omega((\Delta n)^{1.5})$ in the third case.

This paper shows a lower bound of $\Omega(\Delta n^2)$ for all 1-bend drawings of Ramanujan-graphs. If the drawing is in the first case, then this is proved exactly as in [BTW01] (the proof is repeated here for completeness). The proof in the second case uses the observation that every edge has at most one bend, and hence the two endpoints must "see" each other in some sense. Finally one can show that the third case cannot happen for sufficiently large n when edges have at most one bend.

This section is structured as follows: We first review the Ramanujan-graphs. Then we prove that the third case cannot happen. Finally, we proceed to prove lower bounds for all drawings.

3.1   Ramanujan-graphs

Ramanujan-graphs were introduced in [LPS88] and have already been used in [BTW01] for lower bounds for orthogonal graph drawing. They have the useful property that for any two disjoint subsets of size $\Omega(n)$, there are $\Omega(m)$ edges between the two subsets. This was first reported in [BTW01], we repeat and slightly modify their proof to obtain the statement for an arbitrary constant μ.

Lemma 3.1 *[BTW01] Let $0 < \mu < 1$ be a constant. If $p \neq q$ are primes, $p \equiv 1 \bmod 4$, $q \equiv 1 \bmod 4$, $p+1 \geq 16/\mu^2$, then there exists a simple graph $G_{n,d}$ (called a* Ramanujan-graph*) with the following properties:*

- *$G_{n,d}$ is d-regular for $d = p + 1$,*

- *the number n of vertices of $G_{n,d}$ is at least $q(q-1)/2$ and at most $q(q-1)$.*

- *for any disjoint vertex sets S, T of $G_{n,d}$ with $|S| \geq \mu n$, $|T| \geq \mu n$, there are at least $\frac{1}{2}\mu^2 \cdot dn$ edges between S and T.*

Proof: Let $G_{n,d}$ be the graph $X^{p,q}$ defined in [LPS88]; the first two properties of the graph were shown in this paper. It was also shown that $\lambda \leq 2\sqrt{d-1}$, where λ denotes the second-largest eigenvalue of $G_{n,d}$. Assume S and T are as specified above. As shown in [AS92], the number of edges between S and T is at least $\frac{d|S||T|}{n} - \lambda\sqrt{|S||T|}$. Now,

$$\lambda\sqrt{|S||T|} \leq 2\sqrt{d-1}\sqrt{|S||T|} \cdot \underbrace{\sqrt{|S||T|/\mu^2 n^2}}_{\geq 1} \cdot \underbrace{\sqrt{d\mu^2/16}}_{\geq 1} \leq \frac{1}{2}d|S||T|/n.$$

Hence, the number of edges between S and T is at least $(1 - \frac{1}{2}) \cdot d|S||T|/n \geq \frac{1}{2}\mu^2 \cdot dn$. $\square$

It suffices to state lower bounds only for Ramanujan-graphs, because as was shown in [BTW01], graphs containing Ramanujan-graphs can be constructed for almost all values of m and n.

Lemma 3.2 *[BTW01] There exist constants n_0 and d_0 such that for any $n \geq n_0$ and any $m \geq d_0 n$ there exists a graph with n vertices and m edges that has a Ramanujan-graph with $\theta(n)$ vertices and $\theta(m)$ edges as a subgraph.*

In particular, using these graphs, the lower bounds can be transferred from Ramanujan-graphs to all values of n and m without affecting the order of magnitude, similarly as done in [BTW01].

3.2   Vertices in one grid plane

Now we prove that the "third case" mentioned above cannot happen for 1-bend drawings of Ramanujan-graphs, i.e., there always exists a grid plane intersecting $\Omega(n)$ vertices. For this and the lower bound proofs to come, we will often refer to positions of vertices relative to grid planes. A vertex is said to be *left* (*right*) of an $(X = X_0)$-plane if all the points in its box have X-coordinates less than X_0 (greater than X_0). A vertex is said to be *before* (*behind*) a $(Y = Y_0)$-plane if all the points in its box have Y-coordinates less than Y_0 (greater than Z_0). A vertex is said to be *below* (*above*) a $(Z = Z_0)$-plane if all the points in its box have Z-coordinates less than Z_0 (greater than Z_0).

Also, for the proofs to come, for ease of notation we neglect rounding issues, and assume that n is divisible as needed. This has no effect on the order of magnitude of the lower bounds, since for example in the next theorem, one could show a bound of $\frac{1}{8}n - o(n)$ vertices for all values of n.

Theorem 3 *Let $G_{n,d}$ be a Ramanujan-graph with $d \geq 2^{16}$ and n divisible by 8. Then any 1-bend drawing of $G_{n,d}$ has a grid plane that intersects at least $\frac{1}{8}n$ vertices.*

Proof: Assume to the contrary that no grid plane intersects as many as $\frac{1}{8}n$ vertices.

Informally, this leads to a contradiction because the drawing can be split into non-empty octants. Two of these octants have no grid-plane in common, and hence cannot have an edge with 0 or 1 bends between them. See Figure 3 for an illustration. The precise proof is as follows:

As an $(X = X_0)$-plane is swept from smaller to larger values, we encounter an integer X' where, for the last time, there are at most $\frac{7}{16}n$ vertices to the left of the $(X = X')$-plane. Thus, there are at least $\frac{7}{16}n$ vertices to the left of the $(X = X' + 1)$-plane. All these vertices, call them V_-, are also to the left of the $(X = X' + \frac{1}{2})$-plane. Also, since the $(X = X')$-plane intersects at most $\frac{1}{8}n$ vertices, and at most $\frac{7}{16}n$ vertices are to the left of it, there are at least $n - \frac{1}{8}n - \frac{7}{16}n = \frac{7}{16}n$ vertices to the right of the $(X = X')$-plane. All these vertices, call them V_+, are also to the right of the $(X = X' + \frac{1}{2})$-plane. Denote $X^* = X' + \frac{1}{2}$.

Note that no X-plane intersects both a vertex in V_+ and a vertex in V_-.

Apply the same argument to a sweep with a $(Y = Y_0)$-plane, considering only the vertices in V_-; recall that $|V_-| \geq \frac{7}{16}n$. Thus there is a value Y_-^* such that at least $\frac{5}{32}n$ vertices of V_- are before the $(Y = Y_-^*)$-plane, and at least
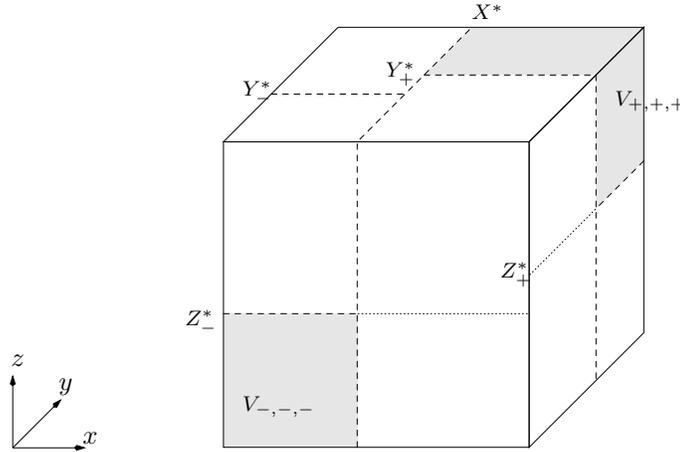
Figure 3: Two diagonally opposite octants yield two non-empty sets of vertices that cannot have an edge with 0 or 1 bends connecting them.

$\frac{7}{16}n - \frac{1}{8}n - \frac{5}{32}n = \frac{5}{32}n$ vertices of V_- are behind the $(Y = Y_-^*)$-plane. Denote these two sets of vertices as $V_{-,-}$ and $V_{-,+}$.

Apply the same argument to a sweep with a $(Y = Y_0)$-plane, considering only the vertices in V_+. Thus there is a value Y_+^* such that at least $\frac{5}{32}n$ vertices of V_+ are before the $(Y = Y_+^*)$-plane, and at least $\frac{5}{32}n$ vertices of V_+ are behind the $(Y = Y_+^*)$-plane. Denote these two sets of vertices as $V_{+,-}$ and $V_{+,+}$.

Without loss of generality, assume that $Y_-^* \leq Y_+^*$. In particular therefore, no Y-plane intersects both a vertex in $V_{-,-}$ and a vertex in $V_{+,+}$.

Apply the same argument to a sweep with a $(Z = Z_0)$-plane, considering only the vertices in $V_{-,-}$; recall that $|V_{-,-}| \geq \frac{5}{32}n$. Thus there is a value Z_-^* such that at least $\frac{1}{64}n$ vertices of $V_{-,-}$ are below the $(Z = Z_-^*)$-plane, and at least $\frac{5}{32}n - \frac{1}{8}n - \frac{1}{64}n = \frac{1}{64}n$ vertices of $V_{-,-}$ are above the $(Z = Z_-^*)$-plane. Denote these two sets of vertices as $V_{-,-,-}$ and $V_{-,-,+}$.

Apply the same argument to a sweep with a $(Z = Z_0)$-plane, considering only the vertices in $V_{+,+}$. Thus there is a value Z_+^* such that at least $\frac{1}{64}n$ vertices of $V_{+,+}$ are below the $(Z = Z_+^*)$-plane, and at least $\frac{1}{64}n$ vertices of $V_{+,+}$ are above the $(Z = Z_+^*)$-plane. Denote these two sets of vertices as $V_{+,+,-}$ and $V_{+,+,+}$.

Without loss of generality, assume that $Z_-^* \leq Z_+^*$. In particular therefore, no Z-plane intersects both a vertex in $V_{-,-,-}$ and a vertex in $V_{+,+,+}$.

Hence no grid plane intersects both a vertex in $V_{-,-,-}$ and $V_{+,+,+}$. These sets each contain at least $\frac{1}{64}n$ vertices. Since $G_{n,d}$ is a Ramanujan-graph with $d \geq 2^{16} = 16 \cdot 64^2$, there are edges between these two vertex sets. These edges cannot be drawn with at most one bend, a contradiction. □

Remark: Any constant smaller than $\frac{1}{7}$ could take the role of $\frac{1}{8}$ in the theorem; the smaller the constant, the smaller also the lower bound on d. For example,

a bound $d \geq 82$ would suffice after replacing $\frac{1}{8}$ by $\frac{1}{407}$.

Also note that the above proof did not use that the drawing had no crossings, and hence would hold even if crossings were allowed.

3.3   1-bend drawings

Now we prove that any 1-bend drawing must have a large volume. The constants in the proof to follow are rather small and chosen for the convenience of a simple proof; they could be improved with a more detailed analysis.

Theorem 4 *Let $G_{n,d}$ be a Ramanujan-graph with $d \geq 2^{22}$ and n divisible by 512. Then any 1-bend drawing of $G_{n,d}$ has volume at least $2^{-27}dn^2$.*

Proof: There are two cases:

Case 1: There exists a grid line that intersects at least $\frac{1}{256}n = 2^{-8}n$ many vertices. Assume that this grid line is an X-*line*, i.e., a line parallel to the X-axis; the other two directions are similar.

The argument in this case is exactly the same (except for a change of constants and directions) as in [BTW01]. Namely, let $v_1, \dots, v_t$ be the vertices intersected by the X-line, listed in order of occurrence along the line. Let X_0 be a not necessarily integer X-coordinate such that the $(X = X_0)$-plane intersects none of these t vertices and separates the first $2^{-9}n$ of them from the remaining ones, of which there are at least $2^{-9}n$ many.
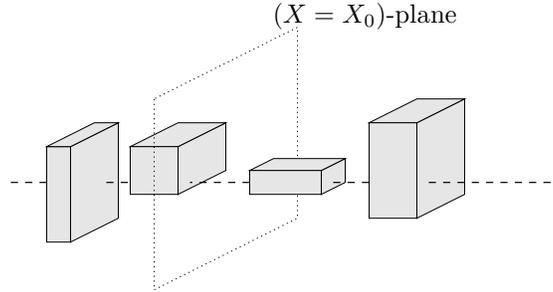
Figure 4: Illustration of case (1).

Because $G_{n,d}$ is a Ramanujan-graph and $d \geq 16 \cdot 2^{18}$, at least $2^{-19} \cdot dn$ edges connect these two vertex sets. Their edge routes cross the $(X = X_0)$-plane, which thus must contain at least $2^{-19} \cdot dn$ points having integer Y- and Z-coordinates. Hence $YZ \geq 2^{-19} \cdot dn$. Since the X-line intersects at least $2^{-8}n$ vertices, also $X \geq 2^{-8}n$, so $XYZ \geq 2^{-27} \cdot dn^2$.

Case 2: No grid line intersects many vertices.

By Theorem 3, there exists a grid plane, say the $(Z = Z')$-plane, that intersects at least $\frac{1}{8}n$ vertices; denote these vertices as V'. In all of the following argument, only vertices of V' are used.

As an $(X = X_0)$-plane is being swept from smaller to larger values, the intersection of the $(X = X_0)$-plane with the $(Z = Z')$-plane is a Y-line, which by assumption intersects at most $\frac{1}{256}n$ vertices at any one time. With an argument similar as in the proof of Theorem 3, we can thus obtain a value X^* such that at least $\frac{15}{256}n$ vertices of V' are to the left of the $(X = X^*)$-plane, and at least $\frac{1}{8}n - \frac{1}{256}n - \frac{15}{256}n \geq \frac{15}{256}n$ vertices of V' are to the right of the $(X = X^*)$-plane. Denote these two sets of vertices as V_- and V_+. Note that no X-plane intersects both a vertex in V_+ and a vertex in V_-.

Apply the same argument to a sweep with a $(Y = Y_0)$-plane, considering only the vertices in V_-. Thus there is a value Y_-^* such that at least $\frac{7}{256}n$ vertices of V_- are before the $(Y = Y_-^*)$-plane, and at least $\frac{15}{256}n - \frac{1}{256}n - \frac{7}{256}n = \frac{7}{256}n$ vertices of V_- are behind the $(Y = Y_-^*)$-plane. Denote these two sets of vertices as $V_{-,-}$ and $V_{-,+}$.

Apply the same argument to a sweep with a $(Y = Y_0)$-plane, considering only the vertices in V_+. Thus there is a value Y_+^* such that at least $\frac{7}{256}n$ vertices of V_+ are before the $(Y = Y_+^*)$-plane, and at least $\frac{7}{256}n$ vertices of V_+ are behind the $(Y = Y_+^*)$-plane. Denote these two sets of vertices as $V_{+,-}$ and $V_{+,+}$.
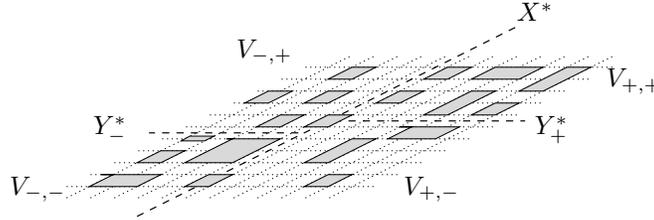
Figure 5: Illustration of case (2).

Without loss of generality, assume that $Y_-^* \leq Y_+^*$. In particular therefore, no Y-plane intersects both a vertex in $V_{-,-}$ and a vertex in $V_{+,+}$.

Since the graph is a Ramanujan-graph, there must be edges between $V_{-,-}$ and $V_{+,+}$. None of these edges can be routed within an X-plane or a Y-plane as observed above, hence they are all routed within a Z-plane.

Now we use the fact that every edge is drawn with one bend. Namely, let (v, w) be an edge with $v \in V_{-,-}$ and $w \in V_{+,+}$, and assume that it is routed in the $(Z = z)$-plane. The route of (v, w) consists of one X-segment and one Y-segment. If (say) its X-segment is incident to v, then no other vertex can be placed on the grid segment between v and the $(X = X^*)$-plane. This motivates the following definition illustrated in Figure 6.

Definition 1 *A vertex $v \in V_{-,-}$ ($v \in V_{+,+}$) is said to be* exposed *at level z if*

- *there exists an X-grid line in the $(Z = z)$-plane that intersects v and does not intersect any other vertex in $V_{-,-}$ ($V_{+,+}$) between v and the $(X = X^*)$-plane, or*

- *there exists a Y-grid line in the $(Z = z)$-plane that intersects v and does not intersect any other vertex in $V_{-,-}$ ($V_{+,+}$) between v and the $(Y = Y_-^*)$-plane $((Y = Y_+^*)$-plane).*

A vertex is called hidden *at level z if it is not exposed.*
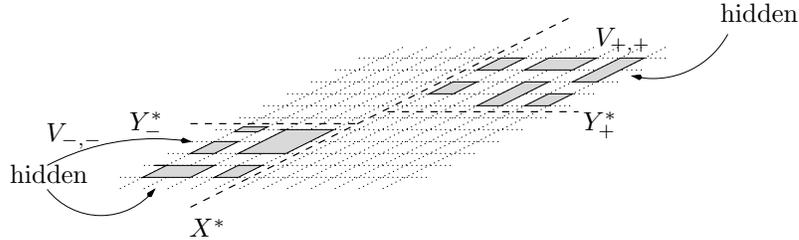
Figure 6: Examples of hidden vertices (we only show the cross-section with one Z-plane). All vertices not marked otherwise are exposed. Note that the top right vertex is hidden even though there is an X-line from it not intersecting other vertices, because this X-line is not a grid-line.

Hence, any edge (v, w) between $V_{-,-}$ and $V_{+,+}$ must be routed in a $(Z = z)$-plane such that both v and w are exposed at level z.

The crucial observation is now that if X and Y (the dimensions of the drawing) are small, then not very many vertices are exposed at any one level. This leads to a contradiction, because then not all edges can be routed. More precisely:

Claim: $X + Y > 2^{-8}n$.

To prove this claim, assume to the contrary that $X + Y \leq 2^{-8}n$. In particular therefore, at most $2^{-8}n$ vertices of $V_{-,-}$ can be exposed at any one given level, simply because there are at most $2^{-8}n$ possible grid lines, each of which can only intersect at most one vertex.

A vertex v is called *active* at level z if the $(Z = z)$-plane intersects the box of v, and *inactive* otherwise. Recall that all vertices in $V_{-,-}$ intersect the $(Z = Z')$-plane, so all vertex in $V_{-,-}$ are active on level Z'. If a vertex $v \in V_{-,-}$ is hidden on level $z - 1 \geq Z'$, but exposed on level z, then some other vertex $w \in V_{-,-}$ was "blocking" v at level $z - 1$, but not on level z, so w must have disappeared, i.e., w became inactive at level z. Hence, every time one vertex becomes exposed, another vertex must become inactive.

The precise argument is now as follows. Sweep a $(Z = Z_0)$-plane from smaller to larger values of Z, starting at $Z = Z'$. Initially, all vertices in $V_{-,-}$ are active (there are at least $\frac{7}{256}n$ many of them), and at most $2^{-8}n = \frac{1}{256}n$ of them are exposed.

During the sweep, more and more vertices become inactive, and hence more and more vertices become exposed. At some point, an integer $Z_{-,+}^*$ is encountered where for the first time at least $\frac{3}{256}n$ vertices of $V_{-,-}$ are inactive. Denote these vertices by $V_{-,-,-}$; hence none of them is exposed on any level $z \geq Z_{-,+}^*$.

At most $\frac{1}{256}n$ vertices were exposed on level Z', and at most $\frac{3}{256}n$ vertices became exposed on level $Z' + 1, \ldots, Z^*_{-,+} - 1$, because at most $\frac{3}{256}n$ vertices became inactive on these levels. Hence there are at least $|V_{-,-}| - \frac{1}{256}n - \frac{3}{256}n \geq \frac{3}{256}n$ vertices that are not exposed on any level between Z' and $Z^*_{-,+} - 1$. Denote these vertices as $V_{-,-,+}$.

With a similar argument, find an integer $Z^*_{+,+}$ such that at least $\frac{3}{256}n$ vertices of $V_{+,+}$ are inactive on any level $z \geq Z^*_{+,+}$ (call them $V_{+,+,-}$), and at least $\frac{3}{256}n$ vertices of $V_{+,+}$ are not exposed on any level between Z' and $Z^*_{+,+} - 1$ (call them $V_{+,+,+}$). See Figure 7.
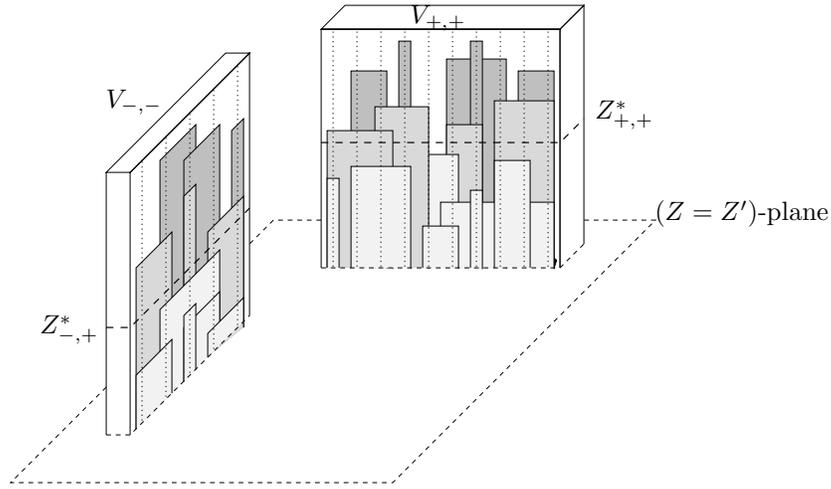
Figure 7: The darker vertices are hidden on all levels between Z' and $Z^*_{-,+}$ ($Z^*_{+,+}$), whereas the lighter vertices are inactive on all levels $z \geq Z^*_{-,+}$ ($z \geq Z^*_{+,+}$). (Only the projection of the vertices onto the bounding box is shown. Also, the picture is simplified in that only one of two possible directions of exposure is considered.)

Without loss of generality, assume that $Z_{-,+} \leq Z_{+,+}$. Therefore, there exists no level $z \geq Z'$ on which both a vertex in $V_{-,-,-}$ is active and a vertex in $V_{+,+,+}$ is exposed. Hence no edge between $V_{+,+,+}$ and $V_{-,-,-}$ can be routed on a $(Z = z)$-plane with $z \geq Z'$.

Now repeat the argument for the layers below Z', applied only to vertices in $V_{-,-,-}$ and $V_{+,+,+}$, respectively. Since $|V_{-,-,-}| \geq \frac{3}{256}n$, there is an integer $Z^*_{-,-} < Z'$ such that at least $\frac{1}{256}n$ vertices of $V_{-,-,-}$ are inactive on any level $z \leq Z^*_{-,-}$ (call them $V_{-,-,-,+}$), and at least $\frac{3}{256}n - \frac{1}{256}n - \frac{1}{256}n = \frac{1}{256}n$ vertices of $V_{-,-,-}$ (call them $V_{-,-,-,-}$) are hidden on any level between Z' and $Z^*_{-,-} + 1$.

Also, there is a value $Z^*_{+,-} < Z'$ such that at least $\frac{1}{256}n$ vertices of $V_{+,+,+}$ are inactive on any level $z \leq Z^*_{+,-}$ (call them $V_{+,+,+,+}$), and at least $\frac{3}{256}n - \frac{1}{256}n - \frac{1}{256}n = \frac{1}{256}n$ vertices of $V_{+,+,+}$ (call them $V_{+,+,+,-}$) are hidden on any level between Z' and $Z^*_{+,-} + 1$.

Without loss of generality, assume that $Z^*_{-,-} \leq Z^*_{+,-}$. Therefore, there exists no level $z \leq Z'$ on which both a vertex in $V_{-,-,-,-}$ is active and a vertex in $V_{+,+,+,+}$ is exposed. Hence none of the edges between $V_{+,+,+,+}$ and $V_{-,-,-,-}$ can be routed on a $(Z = z)$-plane with $z \leq Z'$. But as shown before, none of these edges can be routed in a $(Z = z)$-plane with $z \geq Z'$, and not in an X-plane or a Y-plane either. So if $X + Y \leq 2^{-8}n$, then the edges between $V_{+,+,+,+}$ and $V_{-,-,-,-}$ (which must exist because the graph is a Ramanujan-graph and $d \geq 16(256)^2 = 2^{20}$) cannot be routed with 0 or 1 bends, a contradiction.

Thus $X + Y > 2^{-8}n$, and if, say, $X = \max\{X, Y\}$, then $X > 2^{-9}n$. There are at least $\frac{1}{2}(\frac{7}{256})^2 \cdot dn$ edges between $V_{-,-}$ and $V_{+,+}$, since each of these sets contains at least $\frac{7}{256}n$ vertices. All their edges routes intersect the $(X = X^*)$-plane in a point with integer coordinates, therefore $YZ \geq \frac{1}{2}(\frac{7}{256})^2 dn = 49 \cdot 2^{-17}dn$. Combining this with $X > 2^{-9}n$, we obtain $XYZ > 49 \cdot 2^{-26}dn^2$, which gives the result. □

4 Conclusion and open problems

This paper solved two open problems regarding three-dimensional orthogonal 1-bend drawings, namely, that any 1-bend drawing of K_n has volume $\Omega(n^3)$ and degree-restricted 1-bend cube-drawings are impossible for K_n, or more generally, for simple graphs with $\Omega(\Delta n)$ edges. Lower bounds for 1-bend cube-drawings were also established and hold for any graph for which any cut contains many edges, in particular for Ramanujan-graphs.

A number of open problems remain to be studied:

- Does every graph have a 1-bend drawing of volume $O(\Delta n^2)$? It is easy to construct such a drawing if crossings are allowed, by splitting the edges into $\Delta + 1$ matchings, and assigning a separate Z-plane to each matching, similarly as in [BSWW99]. Can a graph be split in $O(\Delta)$ matchings such that for a suitable vertex order all matchings are without crossing?

  If the answer is yes, does every graph have a 1-bend drawing of volume $O(mn)$?

- Does every graph have a 1-bend cube-drawing of volume $O(\Delta^3 n)$? If the answer is yes, does every graph have a 1-bend drawing of volume $O(\Delta^2 m)$?

- What is the correct lower bound for 2-bend drawings? There are drawings of size $O(n^3)$ for K_n [BSWW99] as well as $O(\Delta n^2)$ for all graphs [Bie98]. Is the lower bound $\Omega(\Delta n^2)$, as for the 1-bend case?

- What is the correct lower bound for 3-bend drawings? There are drawings of size $O(n^{2.5})$ for K_n [BSWW99], and this is optimal [BSWW99]. For Ramanujan-graphs, the lower bound is $\Omega(\Delta n^{1.5})$ [BTW01], but it is not known whether every graph has a 3-bend drawing of volume $O(\Delta n^{1.5})$. (Such drawings exist with 4 bends per edge [BTW01]; 3-bend drawings can be constructed with similar techniques if crossings are allowed.)

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