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Crossing Angles of Geometric Graphs

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Abstract

We study the crossing angles of geometric graphs in the plane. We introduce the *crossing angle number* of a graph G, denoted $\text{can}(G)$, which is the minimum number of angles between crossing edges in a straight-line drawing of G. We show that an n-vertex graph G with $\text{can}(G) = O(1)$ has $O(n)$ edges, but there are graphs G with bounded degree and arbitrarily large $\text{can}(G)$. We also initiate the study of *global crossing angle rigidity* for geometric graphs. We construct bounded degree graphs $G = (V, E)$ such that for any two straight-line drawings of G with the same crossing angle pattern, there is a subset $V' \subset V$ of $|V'| \geq |V|/2$ vertices that are embedded into similar point sets in the two drawings.

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1 Introduction

A *straight-line drawing* of a graph $G = (V, E)$ is a representation of G in which the vertices are mapped into distinct points in the plane, and the edges are drawn as straight-line segments between the corresponding vertices that do not pass through any other vertex. A *geometric graph* is a graph $G = (V, E)$ together with a straight-line drawing.

The *rectilinear crossing number* of a graph G (i.e., the minimum number of crossings pairs of edges in any straight-line drawing of G) has been studied intensely for decades [1]. However, angle conditions for the crossing edges have only been recently considered. The motivation comes from cognitive experiments showing that having small crossing angles is negatively correlated to path-tracking ability in a graph drawing [17].

In this paper, we consider two combinatorial aspects of crossing angles in the straight-line drawings of graphs: (1) What is the maximum size of a graph that admits a straight-line drawing with a certain number of different angles between crossing edges? (2) Do the angles between the crossing edges determine the straight-line drawing of a graph (completely or at least partially) up to similarity?

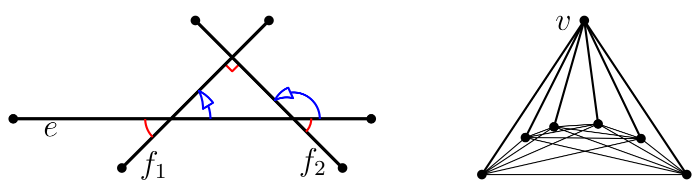

Figure 1: Left: The undirected angles $\angle\{e, f_1\}$ and $\angle\{e, f_2\}$ are both $\pi/4$, with $\angle\{f_1, f_2\} = \pi/2$. However, the directed crossing angles between these edges are different: we have $\angle(e, f_1) = \pi/4$ and $\angle(e, f_2) = 3\pi/4$. Right: A straight-line drawing of K_7. The edges incident to v are crossing free, hence v can be relocated without changing the crossing angles in the drawing.

We define both directed and undirected crossing angles. The (undirected) *crossing angle* between two crossing edges e and f in a straight-line drawing of a graph is the minimum angle $\angle\{e, f\} \in (0, \frac{\pi}{2}]$ between the supporting lines of the two edges. The *directed crossing angle* of two crossing edges e and f is the angle $\angle(e, f) \in (0, \pi)$ such that a counterclockwise rotation through this angle carries the supporting line of e to that of f. Refer to Fig. 1(left). Given a straight-line drawing of an edge e and the (undirected) crossing angle $\angle\{e, f\} \neq \frac{\pi}{2}$, then the edge f may have two possible directions. The drawing of e and the directed crossing angle $\angle(e, f)$ uniquely determine the direction of f.

Crossing Angle Number. The *crossing angle number* of a graph G, denoted $\text{can}(G)$, is the minimum number of crossing angles in any straight-line drawing of G. In Section 2, we show that every n-vertex graph G has less than $(6\,\text{can}(G) + 3)n$ edges. We also show that for every $\varepsilon > 0$, there are n-vertex graphs of maximum degree $O(1/\varepsilon)$ such that $\text{can}(G) = \Omega(n^{1/2-\varepsilon})$.

Global crossing angle rigidity. Motivated by analogous results in rigidity theory, we ask whether the (directed) crossing angles determine a straight-line drawing of a graph uniquely up to similarity. Recall that a straight-line drawing $D = D(G)$ of a graph $G = (V, E)$ is *rigid* if every straight-line drawing of G with the same (Euclidean) edge lengths as in D is congruent to D (i.e., the drawing D is unique up to congruence). A graph $G = (V, E)$ is *globally rigid* in the plane if for every function $\ell : E \to \mathbb{R}^+$, any two straight-line drawings of G in which the Euclidean length of each edge $e \in E$ is $\ell(e)$ are congruent. In other words, the edge lengths determine at most one straight-line drawing up to congruence. For instance, it is not difficult to see that every complete graph with 3 or more vertices is globally rigid.

Consider a straight-line drawing D of a graph $G = (V, E)$, and assume we have complete information about the directed crossing angles between the edges: we know which pairs of edges cross, and we also know the directed crossing angle of every two crossing edges. Is this information enough to reconstruct D up to similarity? (Since similarities preserve both angles and the intersection pattern of the edges, the best we can hope for is uniqueness up to similarity.) The answer to this question is negative. If no two edges cross (hence D is a plane drawing), then D is clearly not unique up to similarity. It is not difficult to see that even the complete graphs K_n have straight-line drawings that are not uniquely determined by the crossing angles: It is possible that no edge incident to a vertex v crosses any other edges in D (Fig. 1, right), and then v can be relocated without changing the crossing angles. This motivates the definition of global crossing angle rigidity (see below), where we require that at least a constant fraction of the vertices be uniquely determined up to similarity.

Let $G = (V, E)$ be a graph. A *directed crossing angle pattern* is a function $\alpha : E^2 \to [0, \pi) \cup \{\star\}$. We say that a straight-line drawing $D(G)$ is *compatible* with α if for every two *crossing* edges, e and f, the directed crossing angle is $\angle(e, f) = \alpha(e, f)$; and for every two *noncrossing* edges, we have $\alpha(e, f) = \star$. We can now define the analogues of rigidity and global rigidity for crossing angles: A straight-line drawing D_1 of a graph G is *crossing angle rigid* if in every straight-line drawing D_2 of G, if D_1 and D_2 have the same directed crossing angle pattern, then at least $|V|/2$ vertices are mapped to similar point sets in the two drawings. A graph $G = (V, E)$ is *globally crossing angle rigid* if for every function $\alpha : E^2 \to [0, \pi) \cup \{\star\}$, and for every two straight-line drawings, D_1 and D_2, compatible with α, there is a vertex set $V'(\alpha) \subset V$ of size $|V'(\alpha)| \geq |V|/2$ that are mapped to similar point sets in the two drawings.

In Section 3, we prove that the complete graph K_n is globally crossing angle rigid for $n \geq 24$ (Theorem 2), and we also construct an infinite family of globally crossing angle rigid graphs with maximum degree 47, diameter $O(\log n)$, and $n \geq 24$ vertices (Theorem 3).

Related Work. Previous research on crossing angles focused on the *crossing resolution* [8, 16] of straight-line (or polyline) drawings, that is, the *minimum* angle at which crossing edges meet. Didimo et al. [10] consider graphs that admit straight-line drawings where crossing edges meet at a right angle. Such drawings are called *right angle crossing* (for short *RAC*) drawings. They prove that every n-vertex graph that admits a RAC drawing has at most $4n-10$ edges, and this bound is best possible. Argyriou et al. [2] show that it is NP-hard to decide whether a given graph admits a RAC drawing. Refer to [7, 9, 14] for recent results on RAC drawings. Note that if a graph G admits a RAC drawing, then its crossing angle number is $\mathrm{can}(G) \leq 1$ (all edge crossings are at angle $\pi/2$). By contrast, we show (Observation 1) that a graph G with $\mathrm{can}(G) = 1$ may have $4.5n - O(\sqrt{n})$ edges. Dujmović et al. [13] generalize RAC-drawings and consider so-called α *angle crossing* (αAC) graphs, for $0 < \alpha \leq \frac{\pi}{2}$. These are straight-line drawings where crossing edges meet at an angle *at least* α. They prove that an n-vertex αAC graph has at most $(\pi/\alpha)(3n-6)$ edges for $0 < \alpha < \pi/2$ and at most $6n-12$ edges for $2\pi/5 < \alpha < \pi/2$.

The rectilinear crossing number has been studied for decades, but its exact value is not even known for the complete graph [1]. It is known, however, that there are families of bounded degree graphs (even cubic graphs [25]) for which the rectilinear crossing number is unbounded [6]. Our results imply (Corollary 2) that there exist families of bounded degree graphs for which the crossing angle number is unbounded.

The crossing angle number is also related to the *slope number* of a graph G, introduced by Wade and Chu [29]. It is the smallest integer $s(G)$ such that G has a straight-line drawing in which the edges have $s(G)$ distinct slopes. The slope number gives an easy upper bound for the crossing angle number, $\mathrm{can}(G) \leq \binom{s(G)}{2}$, since the slopes of two crossing edges in a straight-line drawing determine the crossing angle of the two edges. Mukkamala and Pálvölgyi [23] show that every cubic graph has slope number at most 4. On the other hand, Pach et al. [24] show that there are graphs of maximum degree $d \geq 5$ with arbitrarily large slope numbers. Dujmović et al. [12] improve the lower bound on slope number for $d \geq 9$ and showed that for every $\varepsilon > 0$, there are Δ-regular graphs with slope number at least $n^{1-(8+\varepsilon)/(\Delta+4)}$. For *planar* graphs, Keszegh et al. [22] show that for every $d \in \mathbb{N}$, there is a constant $f(d) = 2^{O(d)}$ such that every planar graph with maximum degree at most d admits a straight-line drawing with at most $f(d)$ slopes.

For background information on rigidity theory, refer to the excellent survey by Whiteley [30]. Saxe [27] showed that it is strongly NP-hard to decide whether a graph is globally rigid. Jackson and Jordan [18, 20, 21] gave a simple combinatorial characterization of *generic* global rigidity, where the edge lengths determine at most one straight-line drawing (up to congruence) if the vertices are in general position. They also extended this notion to a so-called *length-direction* rigidity [19], where each edge has either a prescribed length or a prescribed direction vector. Angle constraints between intersecting circles are considered by Saliola and Whiteley [26], in the context of computer aided design (CAD), and are modeled by distance constraints between points in Euclidean 3-space.

2 Graphs with bounded crossing angle numbers

The *geometric thickness* [11] of a graph, denoted $\mathrm{gth}(G)$, is the smallest number of layers such that one can draw G in the plane with straight-line edges and assign each edge to a layer so that no two edges on the same layer cross. We establish a relation between the geometric thickness and the crossing angle number.

Theorem 1 *For every graph G, we have* $\mathrm{gth}(G) \le 2\,\mathrm{can}(G) + 1$.

Proof: Consider a straight-line drawing D of a graph $G = (V, E)$ with $\mathrm{can}(G)$ crossing angles. We begin by partitioning the edges of G as follows. We define a binary relation on the edges of G where two edges are related if and only if they cross in the drawing D. The transitive closure of this relation is an equivalence relation. The set of edges in an equivalence class is a called *block* of G with respect to D. (See Fig. 2 for examples.) We partition the edges of G into blocks.

Let $k = \mathrm{can}(G)$. We partition each block of G into the union of at most $2k+1$ subsets, each of which is crossing-free in D. Let $A = \{\alpha_1, \ldots, \alpha_k\}$ denote the set of crossing angles in the drawing D. We construct a (possibly infinite) graph H whose vertices are the elements in $A' = \langle \alpha_1, \ldots, \alpha_k \rangle$, the Abelian group generated by the angles α_i where addition is performed modulo π. Two vertices of H are adjacent if and only if their difference is $\pm\alpha_i$ for some $\alpha_i \in A$.

For a fixed $\alpha \in A$ and $\beta \in A'$, there exists a unique $\beta' \in A'$ such that $\beta - \beta' = \alpha$. Hence, the degree of each $\beta \in H$ is at most $2k$ and there exists a proper coloring of the vertices of H with at most $2k + 1$ colors. Moreover, each color class is an independent set in H. We use the color classes to partition each block of G into planar subgraphs.

Consider a block of G and assume without loss of generality that one edge has slope zero in D. Thus, every edge in the block has a *direction* in $\langle \alpha_1, \ldots, \alpha_k \rangle$. (An edge has *direction* $\alpha \in [0, \pi)$ if it can be rotated clockwise through angle α to horizontal position.) Thus, if β_i and β_j are in a color class of $V(H)$, then edges with directions β_i and β_j do not cross. We may partition the edges in each block independently to obtain a partition of all edges of G into subgraphs, each of which is planar in the drawing D. □

We show that every n-vertex graph with bounded crossing angle number has $O(n)$ edges. Recall that a planar graph with $n \ge 3$ vertices has at most $3n - 6$ edges.

Corollary 1 *A graph G with $n \ge 3$ vertices has at most* $(2\,\mathrm{can}(G) + 1)(3n - 6)$ *edges.*

Barát et al. [5] proved that for every $\Delta \ge 9$, $\varepsilon > 0$ and $n \in \mathbb{N}$, there is a Δ-regular graph with at least n vertices and geometric thickness $\Omega(\sqrt{\Delta} n^{1/2-\Delta/4-\varepsilon})$.

Corollary 2 *For every $\Delta \ge 9$, $\varepsilon > 0$ and $n \in \mathbb{N}$, there is a Δ-regular graph G with at least n vertices and* $\mathrm{can}(G) = \Omega(\sqrt{\Delta} n^{1/2-\Delta/4-\varepsilon})$.

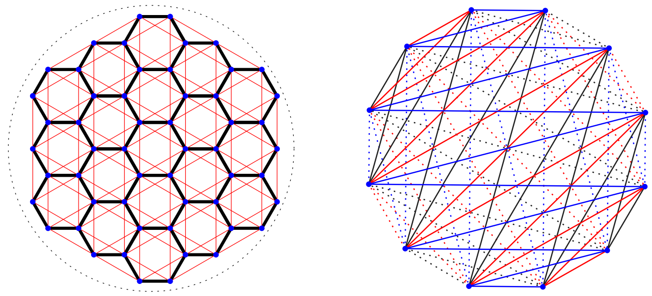

Figure 2: Left: A section of a hexagonal lattice with 54 vertices. The 6 diagonals in each hexagon form a *block* (even though not all pairs of diagonals cross). Right: A straight-line drawing of K_{12} where the vertices form a regular polygon.

We note here that the class of graphs G with $\mathrm{can}(G) = 1$ is strictly larger than the class of graphs that admit RAC drawings (i.e., straight-line drawings in which crossing edges meet at a right angle).

Observation 1 *For every $n \in \mathbb{N}$, there exist an n-vertex graph G_n with $4.5n - O(\sqrt{n})$ edges and $\mathrm{can}(G) = 1$.*

Proof: We construct an infinite graph $G = (V, E)$. Refer to Fig. 2(left). Let V be the set of points in a hexagonal lattice, and let E consist of the edges of all hexagons, and all 6 diagonals between second neighbors in each hexagon. The degree of every vertex is $3 + 3 \cdot 2 = 9$ in G. The edges of the hexagons form a plane graph; the diagonals of distinct hexagons do not cross; and two diagonals of the same (regular) hexagon are either parallel or meet at an angle of $\pi/3$.

Let o be a point in general position with respect to V. For every $n \in \mathbb{N}$, let O_n be a the smallest disk centered at o that contains exactly n point in V; and let G_n be the subgraph of G induced by the n vertices in O_n. The vertex degree is 9 for all but $O(\sqrt{n})$ vertices around the boundary of O_n. Hence G_n has n vertices and $4.5n - O(\sqrt{n})$ edges. $\square$

We also note that the crossing angle number of the n-vertex complete graph K_n is less than $\frac{n}{2}$.

Observation 2 *For $n \geq 2$, we have $\mathrm{can}(K_n) \leq \lfloor n/2 \rfloor - 1$.*

Proof: We say that an edge of G has *direction* $\alpha \in [0, \pi)$ if it can be rotated clockwise through angle α to horizontal position. Consider the straight-line drawing of K_n such that the vertices are represented by the vertices of a regular n-gon with a horizontal side (Fig. 2, right). The set of directions of the edges in this drawing is $\{i\pi/n : 0 \leq i < n\}$. Consequently, the (undirected) angle between the supporting lines of any two edges is in the set $U = \{i\pi/n : 0 \leq i \leq n/2\}$, where $|U| = \lfloor n/2 \rfloor + 1$. However, if the angle between two edges is 0, then the two edges are parallel; and if the angle is π/n, then they do not cross. Hence, this drawing of K_n has only $|U| - 2$ crossing angles, as claimed. $\square$

The bound in Observation 2 is tight for $n = 2, 3$, and 5, but $\text{can}(K_4) = 0$ and $\text{can}(K_6) = 1$. Determining $\text{can}(K_n)$ for $n \geq 7$ is left for future research. For comparison, the slope number of K_n, $n \geq 3$, is known to be exactly n [29].

3 Globally angle-rigid graphs

Assume that we are given a graph $G = (V, E)$ together with a directed crossing angle pattern $\alpha : E^2 \to [0, \pi) \cup \{\star\}$, and we consider straight-line drawings $D(G)$ compatible with α. That is, whenever $\alpha(e, f) = \star$, then the relative interiors of the edges e and f are disjoint (although, they may share a common endpoint); and whenever $\alpha(e, f) \in (0, \pi/2]$, then e and f cross at angle $\alpha(e, f)$. We assume that the position of some vertices is given and we wish to determine the position of additional vertices based on α. We start with a simple observation.

Proposition 1 *Let $G = (V, E)$ be a graph with edges $pp_1, pp_2, p_1p_2, e_1, e_2 \in E$, and let α be a crossing angle pattern such that $\alpha(pp_1, e_1) \neq \star$ and $\alpha(pp_2, e_2) \neq \star$ (Fig. 3). Then in any straight-line drawing compatible with α, the position of p is determined by the slopes of e_1 and e_2, and the position of p_1 and p_2.*

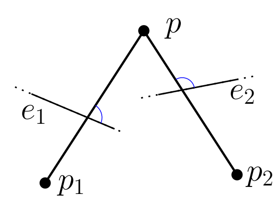

Figure 3: Edges pp_1 and pp_2 each cross some other edges of the graph.

Proof: The slope of e_1 and the crossing angle $\angle(pp_1, e_1)$ determine the slope of edge pp_1. This slope together with the location of p_1 determines the supporting line of pp_1. Similarly, $\angle(pp_2, e_2)$, the slope of e_2, and the location of p_2 determine the supporting line of pp_2. Note that pp_1 and pp_2 cannot be collinear, otherwise an edge would pass through a vertex in the straight-line drawing of G. Hence, p is the unique intersection point of the supporting lines of pp_1 and pp_2. □

3.1 Tools from Combinatorial Geometry

Assuming that we already know the position of some of the vertices in a straight-line drawing of the complete graph (resp., a complete bipartite graph), in this subsection, we develop basic tools for determining how many points can evade Proposition 1.

Crossing Free and Almost Crossing Free Vertices. We say that a vertex p in a straight-line drawing of a graph is *crossing free* if no edge incident to p

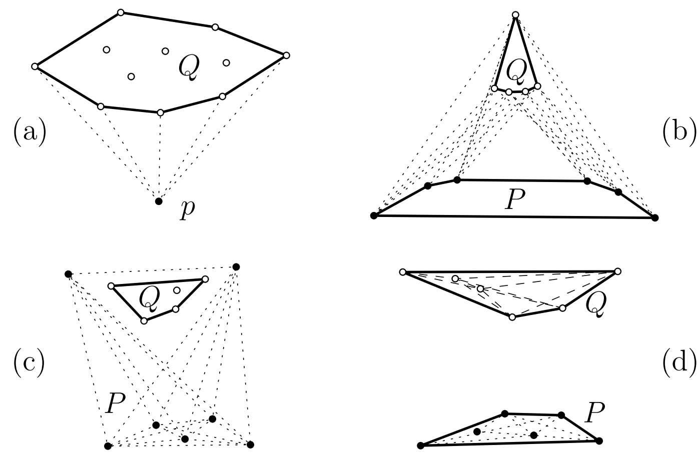

Figure 4: (a) Point p *sees* 5 vertices of the convex hull conv(Q). (b) P and Q are *convexly avoiding* (but not mutually avoiding). (c) P *avoids* Q (but Q does not avoid P). (d) P and Q are *mutually avoiding* (but not convexly avoiding).

crosses any other edge; and p is *almost crossing free* if exactly one edge incident to p crosses other edges. A point p *sees* a vertex q_i of a convex polygon $(q_1, \dots q_k)$ if the line segment pq_i is disjoint from the interior of $(q_1, \dots q_k)$. Two point sets, P and Q, are *convexly avoiding* if for every $p \in P$ and $q \in Q$, the segment pq is disjoint from the interior of both convex hulls conv(P) and conv(Q). See Fig. 4(a)–(b). Note that if P and Q are convexly avoiding, then P and Q are each in convex position. We show that in any drawing of a complete graph with 5 or more vertices, at most one vertex is crossing free or almost crossing free.

Proposition 2 *Consider a straight-line drawing of the complete graph $K_n = (V, E)$ with $n \geq 5$ vertices. If $p \in V$ is a crossing free vertex in this drawing, then p is a vertex of the convex hull* conv(V)*, the set $V \setminus \{p\}$ is in convex position, and p sees all vertices of* conv$(V \setminus \{p\})$.

Proof: Suppose that p lies in the interior of conv(V). By Carathéodory's theorem, there are vertices $q_1, q_2, q_3 \in V$ on the boundary of conv(V) such that p lies in the interior of the triangle conv$(\{q_1, q_2, q_3\})$. Since $|V| \geq 5$, there is another vertex $r \in V$. The segments pq_1, pq_2 and pq_3 decompose conv(V) into 3 sectors, and we may assume without loss of generality that r lies in the sector bounded by pq_1 and pq_2. The path $pq_1 \cup pq_2$ decomposes conv(V) into two regions, and points r and q_3 are in different regions by construction. Hence the segment rq_3 crosses either pq_1 or pq_2. This contradicts our assumption that p is crossing free. Therefore, p is a vertex of the convex hull conv(V).

Now suppose that p does not convexly avoid $V \setminus \{p\}$. Then there is a vertex r_1 on the boundary of conv$(V \setminus \{p\})$ such that pr_1 intersect the interior of conv$(V \setminus \{p\})$, and so pr_1 crosses some edge r_2r_3 on the boundary of conv$(V \setminus \{p\})$. This contradicts our assumption that p is crossing free. □

Proposition 3 *Consider a straight-line drawing of the complete graph $K_n = (V, E)$ with $n \geq 6$ vertices. If $p \in V$ is an almost crossing free vertex where edge pq crosses some other edges of the graph, then p is a vertex of* conv(V), *$V \setminus \{p, q\}$ is in convex position, and p sees all vertices of* conv$(V \setminus \{p, q\})$.

Proof: By definition, there is exactly one edge incident to p, namely pq, that crosses other edges. If we delete vertex q, we obtain a straight-line embedding of K_{n-1}, $n - 1 \geq 5$, where p is crossing free. Proposition 2 completes the proof. □

Mutually Avoiding Sets. Let P and Q be two point sets in the plane. We say that P *avoids* Q if the supporting line of any two points in P is disjoint from conv(Q); and P and Q are *mutually avoiding* if P avoids Q and Q avoids P. See Fig. 4(c)–(d). Aronov et al. [4] proved that any two point sets, P and Q, of size $|P| = |Q| = n/2$ contain two mutually avoiding subsets $P' \subseteq P$ and $Q' \subseteq Q$ of size $|P'|, |Q'| \geq \sqrt{n/24}$. We strengthen their results when P or Q is in convex position.

Proposition 4 *Let P and Q be disjoint point sets in the plane such that $|P| \geq 5$ and every $q \in Q$ sees at least $|P| - 1$ vertices of* conv(P). *Then,*

(i) *there is a subset $P' \subseteq P$ of size $|P'| \geq |P| - 3$ in convex position such that P' avoids Q, and every point $q \in Q$ sees all vertices of* conv(P')*; and*

(ii) *there is a subset $P'' \subseteq P$ of size $|P''| \geq |P| - 1$ in convex position and a subset $Q'' \subseteq Q$ of size $|Q''| \geq \lceil |Q|/3 \rceil$ such that P'' avoids Q'' and every point $q \in Q''$ sees all vertices of* conv(P'').

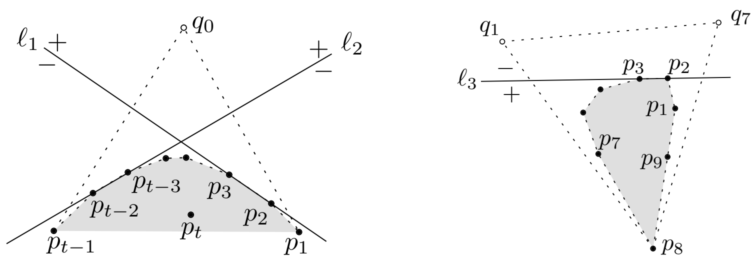

Figure 5: Case 1 (left) and Case 2 (right) in the proof of Proposition 4.

Proof: Let $t = |P| \geq 5$. Since every $q \in Q$ sees at least $|P| - 1$ vertices of conv(P), the convex hull conv(P) has either $t - 1$ or t vertices. We consider these two cases separately.

Case 1: conv(P) **has** $t - 1$ **vertices.** In this case, every point $q \in Q$ sees all vertices of conv(P). Pick an arbitrary point $q_0 \in Q$. Label the points in P such that conv$(P) = (p_1, \ldots, p_{t-1})$ where q_0p_1 and q_0p_{t-1} are tangent to conv(P). Refer to Fig. 5 (left). Let $P' = \{p_2, \ldots, p_{t-2}\}$. It is clear that P' is in convex position and every point $q \in Q$ sees all vertices of $\text{conv}(P')$. It remains to show that P' avoids Q.

First assume that $t = 5$. In this case, $P' = \{p_2, p_3\}$, and $\text{conv}(P')$ is the line segment p_2p_3. Let ℓ_1 be a directed line spanned by p_2p_3 such that $q_0 \in \ell^+$ and $p_1, p_2 \in \ell^-$. For every point $q' \in \ell^-$, the segment p_2q or p_3q intersects the interior of $\text{conv}(P)$. Consequently, every point $q \in Q$ lies in ℓ^+, and so P' avoids Q.

Assume now that $t \geq 6$. Let ℓ_1 and ℓ_2 be the supporting lines of p_2p_3 and $p_{t-3}p_{t-2}$, respectively. Lines ℓ_1 and ℓ_2 subdivide the plane into 4 wedges. Orient the lines ℓ_1 and ℓ_2 such that q lies in the wedge $\ell_1^+ \cap \ell_2^+$. Every point q' in $\ell_1^+ \cap \ell_2^+$ sees the vertices of $\text{conv}(P')$ in the same circular order. If $q' \in \ell_1^+ \cap \ell_2^-$ (resp., $q' \in \ell_1^- \cap \ell_2^+$) then $p_{t-2}q'$ (resp., p_1q') intersects the interior of $\text{conv}(P)$. For every point $q' \in \ell_1^- \cap \ell_2^-$, segment p_2q' intersects the interior of $\text{conv}(P)$. Hence, $Q \subset \ell_1^+ \cap \ell_2^+$. Therefore, every point $q' \in Q$ sees the vertices of $\text{conv}(P')$ in the same circular order, and so P' avoids Q. This proves part (i).

For part (ii), we set $P'' = \{p_1, \ldots, p_{t-1}\}$. For every $q \in Q$, consider the counterclockwise order of the points in P'' between two tangent lines from q to $\text{conv}(P'')$. Since $Q \subseteq \ell_1^+ \cap \ell_2^+$, this order is $(p_1, \ldots, p_{t-1})$, $(p_2, \ldots, p_{t-1}, p_1)$ or $(p_{t-1}, p_1, \ldots, p_{t-2})$. Partition Q into three subsets based on the order in which they see the vertices of $\text{conv}(P'')$. Let Q'' be the largest subset of Q, of size at least $\lceil |Q|/3 \rceil$. Now the point set P'' avoids Q'', as required.

Case 2: $\text{conv}(P)$ has t vertices. Let $\text{conv}(P) = (p_1, \ldots, p_t)$ and let $B \subseteq P$ be the set of points $p_i \in P$ such that p_iq intersects the interior of $\text{conv}(P)$ for some $q \in Q$. If $B = \emptyset$, we can take $P' = P$, and then P' avoids Q. If $|B| = 1$, then the proof of part (i) is analogous to *Case 1* with $B = \{p_t\}$.

Assume that $|B| \geq 2$. We show that for any two points in B, the hop distance along $\text{conv}(P)$ is at most 2. It will follow that $|B| \leq 3$. Suppose, to the contrary, that $p_1, p_i \in B$, where $4 \leq i \leq t-2$. Then there exist points $q_1, q_i \in Q$ such that both p_1q_1 and p_iq_i intersect the interior of $\text{conv}(P)$. Let ℓ_3 be the supporting line of p_2p_3 such that $p_1 \in \ell_3^+$. Refer to Fig. 5 (right). We have $q_1, q_i \in \ell_3^-$, since the line segments between $\{q_1, q_i\}$ and $\{p_2, p_3\}$ cannot cross the interior of $\text{conv}(P)$ by our assumptions. Let p_j be the vertex of the convex chain $(p_{i+1}, \ldots, p_t)$ that lies farthest from the line ℓ_3. If $j = t$, then the segment $p_{j-1}q_i$ intersects the interior of $\text{conv}(P)$. If $j < t$, then $p_{j+1}q_1$ intersects the interior of $\text{conv}(P)$. That is, the point q_1 or q_i sees at most $t-1$ vertices of $\text{conv}(P)$, contradicting our assumptions. It follows that $|B| \leq 3$ since the hop distance between any two points in B is at most two, and $t \geq 5$. Let $P' = P \setminus B$ of size $|P'| \geq |P| - 3$. Now P' avoids Q by the definition of B. This completes the proof of part (i).

For part (ii), notice that every point $q \in Q$ sees at least $|P| - 1$ vertices of $\text{conv}(P)$, namely is sees the vertices in $P \setminus \{b\}$ for some $b \in B$. Since $|B| \leq 3$, we can partition the points in Q into at most three subsets based on which vertices of $\text{conv}(P)$ they see. Let Q'' be a largest subset of Q, and P'' the corresponding set $P \setminus \{b\}$, $b \in B$, that they all see. Then the $P'' \subseteq P$ has size $t-1$ and it avoids the set Q'' of size at least $\lceil |Q|/3 \rceil$. □

Proposition 5 *Let P and Q be disjoint point sets in the plane such that P is in convex position, every point $q \in Q$ sees all vertices of* $\mathrm{conv}(P)$, *and P avoids Q. Then, there exist subsets $P' \subset P$ and $Q' \subset Q$ of size $|P'| \geq \lceil |P|/2 \rceil$ and $|Q'| \geq \lceil \sqrt{|Q|} \rceil$ such that P' and Q' are mutually avoiding.*

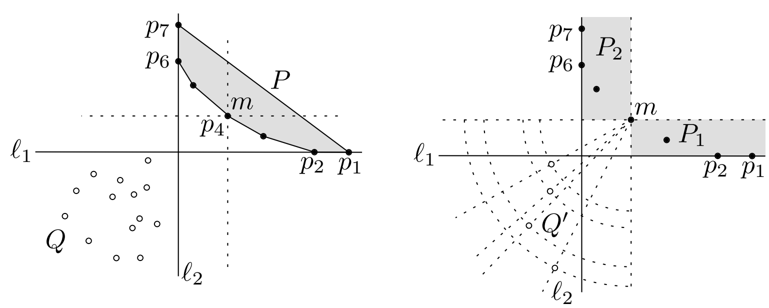

Figure 6: Left: Every point $q \in Q$ sees the same convex arc $(p_1, \ldots, p_7)$ on the boundary of $\mathrm{conv}(P)$. The supporting lines of p_1p_2 and p_6p_7 are ℓ_1 and ℓ_2, respectively. Right: A subsequence $Q' = (q_{k_i})$ whose angles θ_{k_i} about m are monotonically increasing.

Proof: Let $t = |P|$. Since P avoids Q, every point $q \in Q$ sees the same convex arc on the boundary of $\mathrm{conv}(P)$, say $(p_1, \ldots, p_t)$ in counterclockwise order. Let ℓ_1 and ℓ_2 be the supporting lines of p_1p_2 and $p_{t-1}p_t$, respectively. Refer to Fig. 6. The points in Q must lie in the wedge between ℓ_1 and ℓ_2 not containing points of P. We may assume (after performing an appropriate affine transformation) that ℓ_1 and ℓ_2 are orthogonal and parallel to the coordinate axes, P lies in the 1st quadrant and Q lies in the 3rd quadrant.

The midpoint m of the (possibly degenerate) line segment $p_{\lceil t/2 \rceil} p_{\lceil (t+1)/2 \rceil}$ decomposes the convex arc $(p_1, \ldots, p_t)$ into two arcs: $P_1 = (p_1, \ldots, p_{\lceil t/2 \rceil})$ and $P_2 = (p_{\lceil (t+1)/2 \rceil}, \ldots, p_t)$. Consider the points in Q written in polar coordinates (r, θ) where m is the origin and θ is measured counterclockwise. Order the points $q_i = (r_i, \theta_i) \in Q$ in decreasing distance r_i from m. By the Erdős-Szekeres theorem [15], there is a subsequence $Q' = (q_{k_i})$ of length at least $\lceil \sqrt{|Q|} \rceil$ whose angles θ_{k_i} are in either increasing or decreasing order. If they are increasing, we show that Q' avoids P_1. (Analogously, one can show that Q' avoids P_2 if the angles are decreasing.) Consider two points $q_{k_i}, q_{k_j} \in Q'$ with $i < j$. Then q_{k_j} lies below the supporting line of mq_{k_i} farther from m than q_{k_i}. Therefore, the supporting line of $q_{k_i}q_{k_j}$ intersects the vertical line through m either above m or below the x-axis ℓ_1. Since all points in P_1 lie to the right of m, in a horizontal strip between m and ℓ_1, the supporting line of $q_{k_i}q_{k_j}$ avoids $\mathrm{conv}(P_1)$, as claimed. □

Proposition 6 *Let P and Q be two convexly avoiding sets of size $|P| \geq 3$ and $|Q| \geq 3$. Then there exist subsets $P' \subseteq P$ and $Q' \subset Q$ such that $|P'| + |Q'| \geq |P| + |Q| - 1$, and P' and Q' are mutually avoiding.*

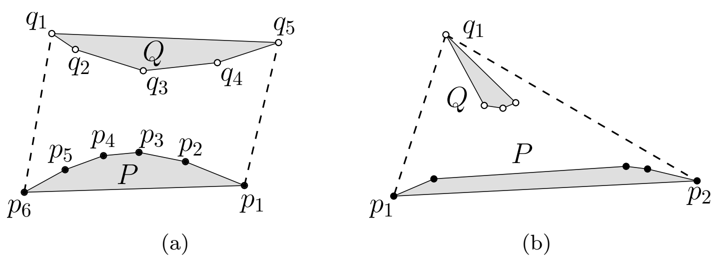

Figure 7: (a) conv($P \cup Q$) is a quadrilateral. (b) conv($P \cup Q$) is a triangle.

Proof: Since P and Q are convexly avoiding, the convex hulls conv(P) and conv(Q) are disjoint, and so conv(P) and conv(Q) have exactly two common external tangents. No three consecutive vertices of conv($P \cup Q$) are in P or Q, otherwise the middle vertex would be incident to an edge p_iq_i that intersects the interior of conv(P) or conv(Q). Therefore, conv($P \cup Q$) is either a triangle or a quadrilateral.

Assume first that conv($P \cup Q$) is a quadrilateral (Fig. 7(a)). In this case, we show that P and Q are mutually avoiding. Let conv(P) = conv($p_1, \ldots, p_t$) in counterclockwise order such that p_tp_1 lies on the boundary of conv($P \cup Q$). Suppose, to the contrary, that there are two points $p_i, p_j \in P$, $1 \le i < j \le t$, such that the supporting line of p_ip_j intersects conv(Q). Clearly, $p_ip_j \ne p_1p_t$, and we may assume (by applying a reflection if necessary) that $j < t$. The line p_ip_j intersects two edges of conv(Q). One of them is an edge of conv($P \cup Q$)), and we denote the other one by q_kq_{k+1}. Then the segments p_ip_t, p_iq_k, p_ip_j, and p_iq_{k+1} appear in this counterclockwise order around p_i. Therefore, segment p_iq_k intersects the interior of conv(P), contradicting our assumption that P and Q are convexly avoiding.

Assume now that conv($P \cup Q$) is a triangle. Without loss of generality, we may assume conv($P \cup Q$) = (p_1, p_2, q_1). In this case, conv($P \cup (Q \setminus \{q_1\})$) is a quadrilateral (Fig. 7(b)). The previous argument readily implies that P and $Q \setminus \{q_1\}$ are mutually avoiding. □

3.2 From One Triangle to Another

In this subsection, we show that if we already know the position of 8 vertices of a complete graph, then the crossing angle pattern uniquely determines the position of all but at most 4 remaining vertices. It is enough to argue about the position of one new vertex at a time.

Proposition 7 *Let $G = K_n$ be a complete graph on the vertex set $V = P \cup Q$, with $|P| \ge 3$ and $|Q| \ge 3$, and let α be a crossing angle pattern. Consider a straight-line drawing compatible with α such that P is in convex position, every point $q \in Q$ sees all vertices of* conv(P)*, and P and Q are mutually avoiding. Then the position of the vertices in P uniquely determines the position of at least one vertex in Q.*

Proof: Since P and Q are mutually avoiding, every point $q \in Q$ sees the vertices of conv(P) in the same (counterclockwise) order. Let $p_1, p_2, p_3 \in P$ be three arbitrary points in P, labeled such that every $q \in Q$ sees p_1, p_2, and p_3 in this (counterclockwise) order. Similarly, let $q_1, q_2, q_3 \in Q$ be three arbitrary points in Q such that every $p \in P$ sees them in this (counterclockwise) order. Refer to Fig. 8(a). The edge p_1q_1 crosses edges q_2p_2 and q_2p_3, and these three edges bound a triangular region in the plane. Since the crossing angles are known, $\angle p_2q_2p_3$ is uniquely determined. Similarly, using edge p_3q_3, angle $\angle p_1q_2p_2$ is uniquely determined.

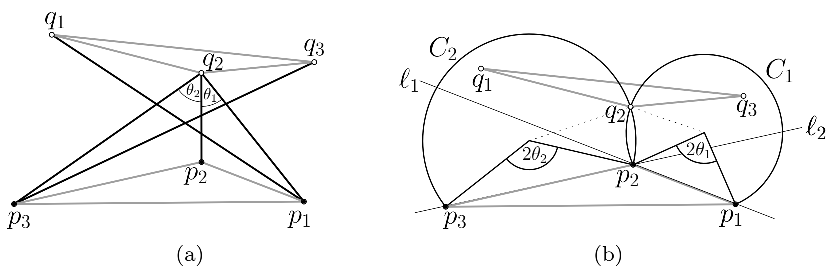

Figure 8: (a) Determining $\angle p_2q_2p_3$. (b) Inscribed angles.

Let ℓ_1 and ℓ_2 be the supporting lines of p_1p_2 and p_2p_3, respectively. The lines ℓ_1 and ℓ_2 subdivide the plane into 4 wedges. Direct the lines ℓ_1 and ℓ_2 such that Q lies in $\ell_1^+ \cap \ell_2^+$ (Fig. 8(b)). We use a fact from elementary geometry: given two points a and b and an angle $\theta \in (0, \pi)$, the locus of points c with $\angle abc = \theta$ is the union of two circular arcs that lie on opposite sides of the supporting line of ab. We invoke this result with $c = q_2$ twice: once for $ab = p_1p_2$ and once for $ab = p_2p_3$. Since $q_2 \in \ell_1^+ \cap \ell_2^+$, it is enough to use one of the two circular arcs in each invocations. Thus, q_2 lies on the intersection of two circles: C_1 defined by $p_1, p_2, \angle p_1q_2p_2$, and C_2 defined by $p_2, p_3, \angle p_2q_2p_3$. Two distinct circles intersect in at most two points. Since $p_2 \in C_1 \cap C_2$, the position of q_2 is uniquely determined unless $C_1 = C_2$. If $C_1 = C_2$, then p_1, p_2, p_3, and q_2 all lie on a common circle. This case, however, cannot occur, since it would contradict our assumption that P and Q are convexly avoiding. □

Lemma 1 *Let G be the complete graph on the vertex set $V = P \cup Q$ such that $|P| \geq 8$ and $|Q| \geq 5$. Let α be a crossing angle pattern. In a straight-line drawing compatible with α, the position of the vertices in P uniquely determines the position of at least one vertex in Q.*

Proof: The vertex set of G is $P \cup Q$, with $|P| \geq 8$ and $|Q| \geq 5$, where the position of the vertices in P are known. If there is a vertex $q \in Q$ such that edges p_1q, p_2q cross some edges induced by P for some $p_1, p_2 \in P$, then the position of q is determined by Proposition 1. Otherwise, every vertex $q \in Q$ is crossing free or almost crossing free in the subgraph induced by $P \cup \{q\}$. By Proposition 2 or 3, every $q \in Q$ convexly avoids a subset of at least $|P| - 1$ points in P. By

Propositions 4(i), there is a subset $P' \subset P$ of size $8 - 3 = 5$ such that every $q \in Q$ sees all vertices of $\text{conv}(P')$ and P' avoids Q. By Propositions 5 there are mutually avoiding subsets $P'' \subseteq P$ and $Q'' \subseteq Q$, with $|P''| \geq \lceil 5/2 \rceil = 3$ and $|Q''| \geq \lceil \sqrt{5} \rceil = 3$, such that every $q \in Q''$ convexly avoids P''. By Proposition 7, the position of at least one point $q \in Q''$ is uniquely determined by the position of $P'' \subset P$ and the crossing angle pattern α. □

3.3 Complete Graphs Are Globally Crossing Angle Rigid

In Proposition 1 and in Subsection 3.2, we determined the position of a new vertex assuming that we already know the position of some other vertices. We can now drop this assumption and show (Theorem 2) that in complete graph K_n, $n \geq 24$, the directed crossing angles always determine the position of at least $n - 4$ vertices up to similarity.

Proposition 8 *Let $G = K_n$ be the complete graph on n vertices, and let α be a crossing angle pattern. If the n vertices form a convex polygon P in a straight-line drawing compatible with α, then the slope of any diagonal of P determines the slopes of all diagonals of P.*

Proof: Assume that we are given the slope of a diagonal e of P, and all directed crossing angles between diagonals. Let e' be another diagonal of P. If e crosses e', then their crossing angle determines the slope of e'. If e and e' do not cross (Fig. 9(a)), then there is a third diagonal f that crosses both e and e', and the slope of e' is determined by the angles $\angle(e, f)$ and $\angle(f, e')$. □

Szekeres and Peters [28] proved, by an exhaustive computer search, that every set of 17 points in the plane, no three of which are collinear, contains 6 points in convex position. Note that it is easy to test whether a set of vertices is in convex position based on the directed crossing angle pattern $\alpha : E^2 \to (0, \pi) \cup \{\star\}$. Indeed, $(p_1, \ldots, p_k)$ is a convex polygon in a straight-line embedding compatible with α if and only if $\alpha(p_i p_j, p_{i'} p_{j'}) \neq \star$ whenever the indices (i, j, i', j') cross combinatorially in the cyclic sequence $(1, \ldots, k)$.

Theorem 2 *Let $G = K_n$ be a complete graph with $n \geq 24$ vertices, and let α be directed crossing angle pattern. Then in every straight-line drawing compatible with α, the position of at least $n - 4$ vertices of G are uniquely determined up to similarity.*

Proof: Denote by V the vertex set of G. Since all vertex pairs are adjacent, no three vertices are collinear in a straight-line drawing of G. Since $|V| \geq 17+6$, we can successively choose two sets, $P \subset V$ and $Q \subseteq (V \setminus P)$, each consisting of 6 points in convex position using [28]. Let $\text{conv}(P) = (p_1, \ldots, p_6)$ and $\text{conv}(Q) = (q_1, \ldots, q_6)$.

By Proposition 8, the directed crossing angles determine the slopes of all diagonals of a convex hexagon up to similarity. For instance, if we fix two arbitrary vertices in either $\Delta(p_1, p_3, p_5)$ or $\Delta(p_2, p_4, p_6)$, then the third vertex

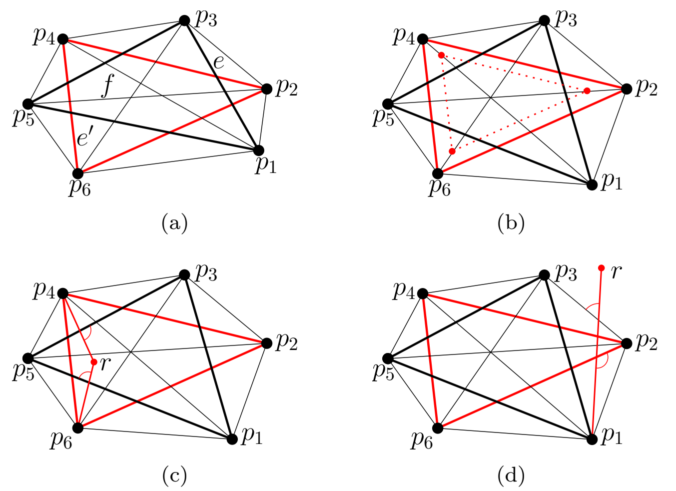

Figure 9: (a) Diagonals of a convex hexagon. (b) If $\Delta p_1p_3p_5$ is fixed, $\Delta p_2p_4p_6$ has one degree of freedom. (c) A point r lies in the interior of $\text{conv}(P)$. (d) A point r lies in the exterior $\text{conv}(P)$ but does not see all vertices of $\text{conv}(P)$.

of the triangle is uniquely determined. However, if we fix $\Delta(p_1, p_3, p_5)$, then the vertices of $\Delta(p_2, p_4, p_6)$ are not necessarily determined (Fig. 9(b)). We first show that the directed crossing angles determine the position of six points from $P \cup Q$ up to similarity. We distinguish between three cases.

Case 1: There is a vertex $r \in V$ lying in the interior of $\text{conv}(P)$ *or* $\text{conv}(Q)$. Assume without loss of generality that r lies in the interior of $\text{conv}(P)$ (Fig. 9(c)). Fix the position of p_1 and p_3. This immediately determines the position of p_5. Note that at least two edges in $\{p_1r, p_3r, p_5r\}$ cross some edges of the triangle $\Delta(p_2, p_4, p_6)$. Therefore, by Proposition 1, the position of r is uniquely determined. The position of p_1, p_3, p_5, and r uniquely determine p_2, p_4, and p_6, by repeatedly applying Proposition 1.

Case 2: There is a vertex $r \in V$ such that r lies in the exterior of $\text{conv}(P)$ *and an edge from r to P intersects the interior of* $\text{conv}(P)$*, or r lies in the exterior of* $\text{conv}(Q)$ *and an edge from r to Q intersects the interior of* $\text{conv}(Q)$. Assume without loss of generality that p_1r crosses the interior of $\text{conv}(P)$ (Fig. 9(d)). Since p_1 enters the interior of $\text{conv}(P)$, it crosses some diagonals of $\text{conv}(P)$. Fix the position of p_1 and p_3. This immediately determines the position of p_5. The location of p_1 and the directed crossing angles with a diagonal of $\text{conv}(P)$ determine the supporting line of p_1r. The supporting line of p_1r crosses some edge of $\text{conv}(P)$, which is incident to a vertex p_j, $j \in \{2, 4, 6\}$. Now the position of p_j is determined by Proposition 1. By repeatedly applying Proposition 1, we determine the position of p_2, p_4, and p_6.

Case 3: Both $\text{conv}(P)$ *and* $\text{conv}(Q)$ *have empty interiors, every vertex in the exterior of* $\text{conv}(P)$ *sees all vertices of* $\text{conv}(P)$*, and every vertex in the exterior* *of* conv(Q) *sees all vertices of* conv(Q). In this case, P and Q are convexly avoiding. By Proposition 6, P and Q have mutually avoiding subsets $P' \subset P$ and $Q' \subset Q$ of total size $|P'| + |Q'| \geq 11$. Without loss of generality, $P' = \{p_1, \ldots, p_5\}$ and $Q' = \{q_1, \ldots, q_6\}$ are mutually avoiding. Fix the position of p_1 and p_3, which immediately determines the position of p_5. By Proposition 7, the position of $\{p_1, p_3, p_5\}$ determines the position of at least one vertex in each 3-element subset of Q'. Hence, $\{p_1, p_3, p_5\}$ determines the position of at least 3 vertices in Q'.

In all three cases, the position of at least 6 vertices are determined up to similarity. Use Proposition 1 successively to determine as many more vertices as possible. Let $A \subset V$ be the set of vertices whose position is already determined up to similarity, and let $B = V \setminus A$. If we already know the position of 8 vertices in V, then we can use Lemma 1 to determine the position of all but at most 4 vertices in V. It remains to consider the case where $6 \leq |A| \leq 7$.

If $6 \leq |A| \leq 7$, then $|B| \geq 24 - 7 \geq 17$. Since Proposition 1 is not applicable for the points in B, they all see at least $|A| - 1 \geq 5$ vertices of conv(A) by Propositions 2 and 3. By Proposition 4(ii), there is a subset $A' \subseteq A$ of size $|A'| \geq |A| - 1 \geq 5$ in convex position and a subset $B' \subseteq B$ of size $|B'| \geq \lceil 17/3 \rceil = 6$ such that A' avoids B' and every point $b \in B'$ sees all vertices of conv(A''). By Propositions 5 there are mutually avoiding subsets $A'' \subseteq A'$ and $B'' \subseteq B'$, with $|A''| \geq \lceil 5/2 \rceil = 3$ and $|B''| \geq \lceil \sqrt{6} \rceil = 3$, such that every $b \in B''$ sees all vertices of conv(A''). By Proposition 7, the position of some point $b \in B''$ is uniquely determined. Thus A can be incremented until $|A| \geq 8$. □

3.4 Globally Angle-Rigid Graphs of Bounded Degree

For every $n \geq 24$, we construct a globally crossing angle rigid graph $F = (V, E)$ with n vertices, bounded vertex degree, and $O(\log n)$ diameter. Refer to Fig. 10. We start with an auxiliary graph F_0 on a vertex set $V_0 = \{v_1, \ldots, v_{n_0}\}$, for some fixed $n_0 \geq 2$. Let F_0 be a binary tree of diameter $O(\log n_0)$. The graph F is obtained from F_0 by replacing each vertex in V_0 with a clique K_{12}, and replacing each edge of F_0 with a biclique $K_{12,12}$ between the corresponding cliques. (See Fig. 10 for an illustration.) The vertex set of F is $V = \bigcup_{i=1}^{n_0} V_i$, where V_i is a set of 12 vertices corresponding to v_i. Hence F has $n = 12n_0$ vertices, and its maximum degree is $4 \cdot 12 - 1 = 47$.

Theorem 3 *Let D_1 be a straight-line drawing of $F = (V, E)$ compatible with a directed crossing angle pattern $\alpha : E^2 \to [0, \pi) \cup \{\star\}$. Then there is a subset $V'(\alpha) \subset V$ of vertices such that (i) $V'(\alpha)$ contains at least 8 vertices from each V_i, $1 \leq i \leq k$, and (ii) for every straight-line drawing D_2 of F compatible with α, a similarity transformation carries all vertices in $V'(\alpha)$ to the corresponding vertices in D_1.*

Proof: Applying Theorem 2 for $V_1 \cup V_2$, we find a subset $V_1' \subset V_1$ of size $|V_1'| = 8$ such that the position of all vertices in V_1' is determined up to similarity by the

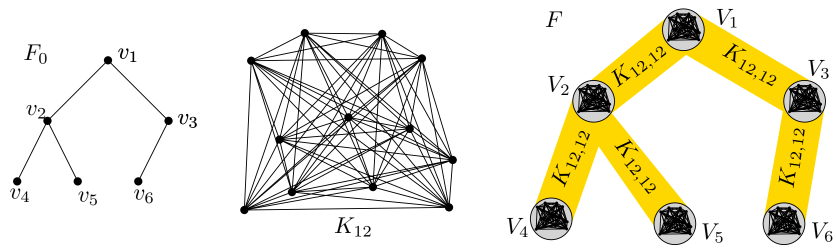

Figure 10: Left: A binary tree F_0 with $n_0 = 9$ vertices. Middle: A straight-line drawing of K_{12}. Right: The graph F constructed from F_0.

directed crossing angles. We incrementally choose a vertex set $V_i' \subset V_i$ of size $|V_i'| = 8$ for all $j = 2, \ldots, n_0$. Assume that we have already chosen a subset $V_i' \subset V_i$, $|V_i'| = 8$, and $v_i v_j$ is an edge of the auxiliary graph F_0. Then F contains a complete graph on the vertex set $V_i' \cup V_j$, where $|V_i'| = 8$ and $|V_j| = 12$. By Lemma 1, we can successively choose 8 elements from V_j whose positions are each determined by the position of the vertices in V_i' and the crossing angles. Denote by V_j' the set of these elements, with $V_j' \subset V_j$ and $|V_j'| = 8$. Finally, let $V'(\alpha) = \bigcup_{i=1}^{n_0} V_i'$. Note that the crossing angle pattern α determines the position of all vertices in $V'(\alpha)$ up to similarity, as claimed. □

4 Conclusions

Note that the function $\alpha : E^2 \to [0, \pi) \cup \{\star\}$ defined above encodes the *directed* crossing angle pattern. It is natural to consider the *undirected* crossing angle pattern $\beta : \binom{E}{2} \to (0, \pi/2) \cup \{\star\}$ for unordered pairs $\{e, f\} \in \binom{E}{2}$. Our method is likely to extend to this variant of the problem, and produce results analogous to Theorems 2 and 3), with higher vertex degrees. We have not pursued this direction in this paper.

Our results represent a first step towards a possible combinatorial characterization of globally crossing angle rigid graphs. In our definition of globally crossing angle rigid graphs, a directed crossing angle pattern α determines at least *half* of the vertices up to similarity. In fact, Theorem 3 holds with factor $\frac{2}{3}$ in place of one half. For every constant $c \in (0, 1)$, there is a threshold $\Delta(c)$ such that there exist infinitely many graphs $G = (V, E)$ of maximum degree $\Delta(c)$ where a directed crossing angle pattern α determines at least $c|V|$ vertices up to similarity. It is left for future research to find the minimum value of the threshold $\Delta(c)$. Several related problems remain open. Is it NP-hard to decide whether a given straight-line drawing of a graph $G = (V, E)$ is angle-rigid? Is it NP-hard to find the crossing angle number $\text{can}(G)$ of a given graph G? We do not even know whether the bound $\text{can}(K_n) \leq \lfloor n/2 \rfloor - 1$ in Observation 2 is tight for $n \geq 7$.

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