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Computational search of small point sets with small rectilinear crossing number

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Abstract

Let $\overline{\text{cr}}(K_n)$ be the minimum number of crossings over all rectilinear drawings of the complete graph on n vertices in the plane. In this paper we prove that $\overline{\text{cr}}(K_n) < 0.380473\binom{n}{4} + \Theta(n^3)$; improving thus on the previous best known upper bound. This is done by obtaining new rectilinear drawings of K_n for small values of n, and then using known constructions to obtain arbitrarily large good drawings from smaller ones. The "small" sets were found using a simple heuristic detailed in this paper.

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1 Introduction

A *rectilinear drawing* of a graph is a drawing of the graph in the plane in which all the edges are drawn as straight line segments. For a set S of n points in general position in the plane, let $\overline{\text{cr}}(S)$ be the number of (interior) edge crossings in a rectilinear drawing of the complete graph K_n with vertex set S. The *rectilinear crossing number* of K_n, denoted by $\overline{\text{cr}}(K_n)$, is the minimum of $\overline{\text{cr}}(S)$ over all sets of n points in general position in the plane. The problem of bounding the rectilinear crossing number of K_n is an important problem in combinatorial geometry. Most of the progress has been made in the last decade, for a state-of-the-art survey see [4]. Since two edges cross if and only if their endpoints span a convex quadrilateral, $\overline{\text{cr}}(S)$ is equal to the number $\square(S)$, of convex quadrilaterals spanned by S. We use this equality extensively throughout the paper. The current best bounds for $\overline{\text{cr}}(K_n)$ are [3, 1]:

$$0.379972\binom{n}{4} < \overline{\text{cr}}(K_n) < 0.380488\binom{n}{4} + \Theta(n^3)$$

Our main result is the following improvement of the upper bound.

Theorem 1

$$\overline{\text{cr}}(K_n) \leq \frac{9363184}{24609375}\binom{n}{4} + \Theta(n^3) < 0.380473\binom{n}{4} + \Theta(n^3)$$

Although it is a modest improvement, we note that the gap between the lower and upper bound is already quite small and that actually the lower bound is conjectured to be at least $0.380029\binom{n}{4}+\Theta(n^3)$. In [1] the authors conjecture that every optimal set is 3-decomposable[1], and show that every 3-decomposable set contains at least $0.380029\binom{n}{4} + \Theta(n^3)$ crossings. The current general approach to produce rectilinear drawings of K_n with few crossings, is to start with a drawing with few crossings (for a small value of n), and use it to recursively obtain drawings with few number of crossings for arbitrarily large values of n. This approach has been refined and improved over the years [10, 7, 5, 2, 1].

The upper bound provided by the best recursive construction to this date is expressed in Theorem 2.

Theorem 2 (Theorem 3 in [1]) *If S is an m-element point set in general position, with m odd, then*

$$\overline{\text{cr}}(K_n) \leq \frac{24\overline{\text{cr}}(S) + 3m^3 - 7m^2 + (30/7)m}{m^4}\binom{n}{4} + \Theta(n^3)$$

Given these recursive constructions, there is a natural interest in finding sets with few crossings for small values of n. The use of computers to aid this search was initiated in [6].

[1] S is 3-decomposable if there is a triangle T enclosing S, and a balanced partition (A, B, C) of S, such that the orthogonal projections of S onto the sides of T show A between B and C on one side, B between A and C on another side, and C between A and B on the third side.

2 Results

For $n \leq 100$, we improved many of best known point sets of n points with few crossings using the following simple heuristic.

Given a starting set S of n points in general position in the plane, do:

- **Step 1.** Choose randomly a point $p \in S$.
- **Step 2.** Choose a random point q in the plane "close" to p.
- **Step 3.** If $\overline{\text{cr}}(S \setminus \{p\} \cup \{q\}) \leq \overline{\text{cr}}(S)$, then update S to $S := S \setminus \{p\} \cup \{q\}$.
- **Step 4.** Go to Step 1.

For each $n = 3, \ldots, 100$, the starting set was taken from Oswin Aichholzer's homepage. These are available at:
`www.ist.tugraz.at/aichholzer/research/rp/triangulations/crossing/`
Some of the best known examples come from [1], rather than from this page. However, they provide explicit coordinates only for a few of their point sets. In many instances we managed to improve the previous best examples. In many cases we improved the examples from [1], even though we started from a worse point set. For $n = 54, 96$ and 99 we failed to improve upon [1]. Our results are shown in Table 1. Theorem 1 now follows directly from Theorem 2 using the set of 75 points we found with 450492 crossings.

3 The Algorithm

In this section we describe an $O(n^2)$ time algorithm used to compute $\overline{\text{cr}}(S)$ in step 3 of the heuristic. Recall that $\overline{\text{cr}}(S)$ is equal to $\square(S)$. We compute this number instead. Quadratic time algorithms for computing $\square(S)$ have been known for a long time [8, 9]. We learned of these algorithms after we finished the implementation of our algorithm. We present our algorithm nevertheless, since in the process we obtained an equality (Theorem 3) between certain substructures of S and $\overline{\text{cr}}(S)$, which may be of independent interest. We also think that given that the main aim of this paper is to communicate the method by which we obtained these sets, it is important to provide as many details as possible so that an interested reader can obtain similar results.

We compute $\square(S)$ by computing the number of certain subconfigurations of S which determine $\square(S)$. Let (p, q) be an ordered pair of distinct points in S, and let $\{r, s\}$ be a set of two points of $S \setminus \{p, q\}$. We call the tuple $((p, q), \{r, s\})$ a *pattern*. We say that $((p, q), \{r, s\})$ is of *type* A if q lies in the convex cone with apex p and bounded by the rays $\overrightarrow{pr}$ and $\overrightarrow{ps}$, otherwise it is of *type* B. Let $A(S)$ be the number of type A patterns in S, and $B(S)$ the number of its type B patterns. Note that every choice of $((p, q), \{r, s\})$ is either an A pattern or a B pattern. The number of these patterns determine $\square(S)$ as the following theorem shows.

Table 1: Improvements on the starting point sets. Starred numbers come from [1].

n	# of crossings in the best point set obtained	# of crossings in the previous best point set obtained	n	# of crossings in the best point set obtained	# of crossings in the previous best point set obtained
46	59463	59464	76	475793	475849
47	65059	65061	77	502021	502079
49	77428	77430	78	529291	529332*
50	84223	84226	79	557745	557849
52	99169	99170	80	587289	587367
53	107347	107355	81	617958	618018*
54	115979	115977*	82	649900	649983
56	134917	134930	83	682986	683096
57	145174	145176*	84	717280	717360*
58	156049	156058	85	753013	753079
59	167506	167514	86	789960	790038
61	192289	192293	87	828165	828225*
63	219659	219681*	88	867911	868023
64	234464	234470	89	908972	909128
65	249962	249988	90	951418	951459*
66	266151	266181*	91	995486	995678
67	283238	283286	92	1040954	1041165
68	301057	301098	93	1087981	1088055*
69	319691	319731*	94	1136655	1136919
70	339254	339297	95	1187165	1187263
71	359645	359695	96	1238918	1238646*
72	380926	380964*	97	1292796	1292802
73	403180	403234	98	1348070	1348072
74	426419	426466	99	1405096	1404552*
75	450492	450540*			

Theorem 3

$$\square(S) = \frac{3A(S) - B(S)}{4}$$

Proof: Let X be a subset of S, of 4 points. Simple arithmetic shows that if X is not in convex position then it determines 3 patterns of type A and 9 patterns of type B; on the other hand if X is in convex position then it determines 4 patterns of type A and 8 patterns of type B. Assume that we assign a value of 3 to type A patterns and a value of -1 to type B patterns. If X is not in convex position its total contributed value would be zero and if it is convex position it would be 4. Thus $4\square(S) = 3A(S) - B(S)$, and the result follows. $\square$

Note that the total number of patterns is $n(n-1)\binom{n-2}{2}$. Thus by Theorem 3 to compute $\square(S)$ it is sufficient to compute $A(S)$. Let p be a point in S. We now show how to count the number of type A patterns in which p is the apex of the corresponding wedge.

Sort the points in $S \setminus \{p\}$ counterclockwise by angle around p. Let $y_1, y_2, \ldots, y_{n-1}$ be these points in such an order. For $1 \leq i \leq n-1$, starting from y_i and going counterclockwise, let $k(i)$ be the first index (modulo n) such that the angle $\angle y_i p y_{k(i)}$ is more than π. Let $m_i := k(i) - i \mod (n-1)$. Note that for $1 \leq j < m_i$ there are exactly $j-1$ type A patterns of the form $(p,q), \{y_i, y_{i+j}\}$ for some $q \in S$. In total, summing over all such j's, this amounts to $\sum_{j=1}^{m_i-1}(j-1) = \binom{m_i-1}{2}$. Thus the total number of type A patterns in which p is the apex of the corresponding wedge is equal to $\sum_{i=1}^{n-1}\binom{m_i-1}{2}$.

Compute $y_{k(1)}$ and m_1 from scratch in linear time. For $2 \leq i \leq n-1$, to compute $y_{k(i+1)}$ and m_{i+1}, assume that we have computed $y_{k(i)}$ and m_i. Start from $y_{k(i)}$ and go counterclockwise until the first $y_{k(i+1)}$ is found such that the angle $\angle y_{i+1} p y_{k(i+1)}$ is more than π; then $m_{i+1} = k(i+1) - (i+1)$. Since one pass is done over each $y_{k(i)}$, this is done in $O(n)$ total time. Finally, sorting $S \setminus p$ by angle around p, for all $p \in S$, can be done in $O(n^2)$ total time. This is done by dualizing S to a set of n lines. The corresponding line arrangement can be constructed in time $O(n^2)$ with standard algorithms. The orderings around each point can then be extracted from the line arrangement in $O(n^2)$ time.

4 Implementation

In this section we provide relevant information of the implementation of the algorithm described in Section 3 and of the searching heuristic we used to obtain the point sets of Table 1.

Instead of sorting in $O(n^2)$ time the points by angle around each point of S, we used standard sorting functions. This was done because these functions have been quite optimized, and the known algorithms to do it in $O(n^2)$ time are not straightforward to implement. Thus our implementation actually runs in $O(n^2 \log n)$ time.

All our point sets have integer coordinates. This was done to ensure the correctness of the computation. The only geometric primitive involved in the algorithm is to test whether certain angles are greater than π; this can be done with a determinant. Therefore as long as all the points have integer coordinates, the result is an integer as well. We did two implementations of our algorithm, one in Python and the other in C. In Python, integers have arbitrarily large precision, so the Python implementation is always correct. In the C implementation we used 128-bit integers. Here, we have to establish a safety margin—as long as the absolute value of the coordinates is at most 2^{62}, the C implementation will produce a correct answer. Empirically we observed a 30× speed up of the C implementation over the Python implementation. At each step of the heuristic we checked if it was safe to use the (faster) C implementation.

To find the point q replacing $p = (x,y)$ in **Step 2**, we first chose two natural numbers t_x and t_y. These number were distributed exponentially with a prespecified mean M and rounded to the nearest integer. Afterwards with probability 1/2 they were replaced by their negative. Point q was then set to $(x + t_x, y + t_y)$. We should note that the exponential distribution was chosen only to ensure that q can be arbitrarily far away from p. It is possible that other distributions yield better results.

After choosing an initial mean, the heuristic was left to run for some time, if no improvement was found by then, the mean was halved (or rather the point set was doubled by multiplying each of its points by two). Many attempts varying the amount of time spent waiting for an improvement were done; we kept the best point sets we found. This was done over the course of several months. We also focused our computing resources on those points sets with a better chance of improving the upper bound. As a result some sets were processed for a far longer time. We also mention that the computational resources used were quite modest—only 3 personal computers were used in total.

All the code used in this paper is available upon request from the first author. The point sets obtained can be downloaded from the sources of the arXiv version of this paper.

Set of 75 points with 450492 crossings

(4473587539, 8674070321),	(2195118038, 12138376393),	(3359570710, 10389672946)
(2067188794, 12364750532),	(3798074340, 9176659177),	(−495951185, 16620108498)
(1133302705, 13923635114),	(1044611367, 14069644578),	(−311149395, 16314077753)
(2027617952, 3459524378),	(4601468259, 7662169961),	(4601078091, 7662133857)
(4113182393, 7619250691),	(4116054424, 7605654413),	(3570685582, 9808713565)
(3722340414, 9316231785),	(4112078622, 7625130881),	(4107912992, 7542476726)
(4106745227, 7535480343),	(3189483730, 5743999450),	(3168421193, 5701152359)
(8944839519, 7965414411),	(3955068845, 6639763085),	(4012346331, 6733970340)
(3648786718, 6305728855),	(3653540692, 6310524663),	(3253433517, 5873175144)
(2113073755, 12281280867),	(−1364755153, −2899618565),	(1679455404, 2812631891)
(1549775961, 2575359287),	(2154725117, 3676030999),	(2297590336, 3930708704)
(1474528964, 2436685704),	(1293365372, 2095165431),	(5207789612, 7710691788)
(1889666524, 3220648103),	(1902363904, 3245131307),	(4899124137, 8128629846)
(4897948559, 8128714256),	(5216754785, 7718023020),	(1683153691, 13003463181)
(5202684700, 7706307614),	(5277878757, 7741749531),	(5279252153, 7742686707)
(7370957968, 7863465953),	(7493305742, 7871610457),	(3571434484, 9806112525)
(6168237700, 8065376268),	(6032867454, 8070589271),	(5981198967, 8072572208)
(6888712646, 7936512772),	(6851478487, 7943849321),	(3214935430, 10605538217)
(7338699912, 7861922951),	(9000883017, 7965096231),	(4059850707, 6811671897)
(8806696260, 7963533399),	(3839573186, 9100031657),	(4471841261, 8674882284)
(15041590733, 8118237065),	(10588618608, 8002947798),	(10174892708, 7993197449)
(1902291407, 12661152660),	(1811935937, 12802330604),	(11185824774, 8018462436)
(10634751909, 8004278071),	(9630596054, 7968154616),	(9350903224, 7955792213)
(4338851382, 8157414467),	(4338568456, 8157953847),	(4520171724, 8637506721)
(4532317105, 8633237970),	(4538689274, 8630906861),	(3400009645, 10327277784)

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