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On the Upward Planarity of Mixed Plane Graphs

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Abstract

A *mixed plane graph* is a plane graph whose edge set is partitioned into a set of directed edges and a set of undirected edges. An *orientation* of a mixed plane graph G is an assignment of directions to the undirected edges of G resulting in a directed plane graph $\vec{G}$. In this paper, we study the computational complexity of testing whether a given mixed plane graph G is *upward planar*, i.e., whether it can be oriented to obtain a directed plane graph $\vec{G}$ such that $\vec{G}$ admits a planar drawing in which each edge is represented by a y-monotone curve.

Our contribution is threefold. First, we show that upward planarity can be tested in cubic time for *mixed outerplane graphs*. Second, we show that the problem of testing the upward planarity of mixed plane graphs reduces in quadratic time to the problem of testing the upward planarity of *mixed plane triangulations*. Third, we design linear-time testing algorithms for two classes of mixed plane triangulations, namely *mixed plane 3-trees* and mixed plane triangulations in which *the undirected edges induce a forest*.

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1 Introduction

Upward planarity is the natural extension of planarity to directed graphs. When visualizing a directed graph, one usually requires an *upward drawing*, that is, a drawing in which the directed edges are curves with monotonically increasing y-coordinates. A drawing is *upward planar* if it is planar and upward. Testing the upward planarity of a directed graph is $\mathcal{NP}$-hard [9]; however, the upward planarity of a directed graph $\vec{G}$ can be tested in polynomial time if $\vec{G}$ has a *fixed planar (combinatorial) embedding* [3], if it has a *single source* [2, 13], if it is *outerplanar* [15], or if it is a *series-parallel graph* [7]. *Exponential-time algorithms* [1] and *FPT algorithms* [12] for testing upward planarity are known.

In this paper we deal with *mixed graphs*. A mixed graph is a graph whose edge set is partitioned into a set of directed edges and a set of undirected edges. Mixed graphs unify the expressive power of directed and undirected graphs, as they allow one to simultaneously represent hierarchical and non-hierarchical relationships. A number of problems on mixed graphs have been studied, e.g., *coloring mixed graphs* [11, 17] and *orienting mixed graphs to satisfy connectivity requirements* [5, 6].

Upward planarity generalizes to mixed graphs as follows. A drawing of a mixed graph is *upward planar* if it is planar, if the directed edges are curves with monotonically increasing y-coordinates, and if each undirected edge is a curve with monotonically increasing or monotonically decreasing y-coordinates. Hence, testing the upward planarity of a mixed graph is equivalent to testing whether its undirected edges can be oriented to produce an upward planar directed graph. Since testing the upward planarity of directed graphs is $\mathcal{NP}$-hard [9], testing the upward planarity of mixed graphs is $\mathcal{NP}$-hard as well. However, the question was raised by Binucci and Didimo [4] of determining the time complexity of testing the upward planarity of *mixed plane graphs*, that is, of mixed graphs with a prescribed planar (combinatorial) embedding in the plane. Binucci and Didimo describe an ILP formulation for the problem and present experiments showing the efficiency of their solution. Some other graph drawing questions on mixed graphs (related to crossing and bend minimization) have been studied in [8, 10].

We show the following results.

- In Section 3 we show that upward planarity can be tested in $O(n^3)$ time for n-vertex *mixed outerplane graphs*. A mixed outerplane graph is a mixed plane graph such that all its vertices are incident to the outer face. Our algorithm uses a characterization for the upward planarity of directed plane graphs due to Bertolazzi et al. [3] in order to decide the upward planarity of a mixed outerplane graph G based on the upward planarity of two subgraphs of G.

- In Section 4 we show how to construct, for every n-vertex mixed plane graph G, an $O(n^2)$-vertex *mixed plane triangulation* G' such that G is upward planar if and only if G' is upward planar. A mixed plane triangulation is a mixed plane graph such that all its faces are delimited by

3-cycles. As a consequence, the problem of testing the upward planarity of mixed plane graphs is polynomial-time solvable ($\mathcal{NP}$-hard) if and only if the problem of testing the upward planarity of mixed plane triangulations is polynomial-time solvable (resp., $\mathcal{NP}$-hard).

- In Section 5, motivated by the previous result, we present algorithms to test the upward planarity for two classes of mixed plane triangulations, namely *mixed plane* 3*-trees* and mixed plane triangulations in which *the undirected edges induce a forest*. Both algorithms are recursive and work in linear time.

2 Preliminaries

A planar drawing of a graph determines a circular ordering of the edges incident to each vertex. Two planar drawings of the same graph are *equivalent* if they determine the same circular orderings around each vertex. A *planar (combinatorial) embedding* is an equivalence class of planar drawings. A planar drawing subdivides the plane into topologically connected regions, called *faces*. The unbounded face is the *outer face* and the bounded faces are the *internal faces*. An edge of G incident to the outer face (not incident to the outer face) is called *external* (resp., *internal*). Two planar drawings with the same planar (combinatorial) embedding have the same faces. However, they could still differ in their outer faces. A *plane embedding* is a planar (combinatorial) embedding together with a choice for the outer face. A *plane graph* is a graph with a given plane embedding. An *outerplane graph* is a plane graph whose vertices are all incident to the outer face. A *plane triangulation* is a plane graph such that all its faces are delimited by 3-cycles. An *outerplane triangulation* is an outerplane graph whose internal faces are delimited by 3-cycles.

A planar drawing of a directed graph is *upward* if every directed edge is represented by a Jordan arc $\gamma : [0,1] \rightarrow \mathbb{R}^2$ such that the y-coordinate of γ is a monotonically increasing function. A directed graph is *upward planar* if it admits an upward planar drawing. A mixed graph is *upward planar* if its undirected edges can be oriented to produce an upward planar directed graph.

A graph is *connected* if there is a path between every pair of vertices. A k*-connected* graph G is such that removing any $k-1$ vertices leaves G connected. A *cutvertex* is a vertex whose removal disconnects the graph. A *bridge* is an edge whose removal disconnects the graph. A *separating* 3*-cycle* is a 3-cycle whose removal disconnects the graph. A *block* of a graph G is a maximal (both in terms of vertices and in terms of edges) 2-connected subgraph of G; in particular, a bridge of G is considered as a block of G. In this paper, when talking about the connectivity of mixed graphs or directed graphs, we always refer to the connectivity of their underlying undirected graphs.

In the remainder of the section we present some preliminaries about directed plane graphs (Subsection 2.1) and about mixed plane graphs (Subsection 2.2).

2.1 Upward Planarity of Directed Plane Graphs

Bertolazzi et al. [3] characterized the directed plane graphs that are upward planar. In the first part of this section we briefly present this characterization; then, we state two simplified versions of it, specifically dealing with directed outerplane triangulations and directed plane triangulations; finally, we present one further simplified version of the characterization, again dealing with directed plane triangulations.

A vertex v in a directed graph is a *sink* (*source*) if every edge incident to v is directed to v (resp., directed from v). A vertex v in a directed plane graph is *bimodal* if the edges directed to v are consecutive in the cyclic ordering of the edges incident to v (which implies that the edges directed from v are also consecutive). A directed plane graph is *bimodal* if every vertex is bimodal.

Consider a 2-connected directed plane graph $\vec{G}$, consider a face f of $\vec{G}$, and denote by C_f the simple cycle delimiting f. A vertex v of C_f is a *sink-switch* (*source-switch*) for f if the edges of C_f incident to v are both directed to v (resp., directed from v). Observe that the number of sink-switches for f and the number of source-switches for f coincide. The *demand* of f is defined as follows: If f is an internal face, then its demand is equal to the number of sink-switches for f minus 1; if f is the outer face, then its demand is equal to the number of sink-switches for f plus one. An *upward assignment* for $\vec{G}$ maps each source s of $\vec{G}$ to one of the faces s is incident to and each sink t of $\vec{G}$ to one of the faces t is incident to. An upward assignment for $\vec{G}$ is *consistent* if it maps to each face f a number of sources and sinks equal to the demand of f.

The following characterization has been proved by Bertolazzi et al. [3].

Theorem 1 ([3]) *A 2-connected directed plane graph is upward planar if and only if it is acyclic, bimodal, and it admits a consistent upward assignment.*

In this paper, we will use the characterization in Theorem 1 when dealing with two specific classes of directed plane graphs, namely directed outerplane triangulations and directed plane triangulations. For this reason, we state such a characterization directly for such graph classes.

We start from directed outerplane triangulations. When dealing with such graphs, by *sink-switch* (*source-switch*) we always mean a sink-switch (resp., source-switch) for the outer face. Thus, a vertex v in a directed outerplane graph $\vec{G}$ is a sink-switch (resp., source-switch) if the two external edges incident to v in $\vec{G}$ are both directed to v (resp., directed from v). Also, we call *upward outerplane triangulation* a directed outerplane triangulation that is upward planar.

Theorem 2 *A 2-connected directed outerplane triangulation $\vec{G}$ is upward planar if and only if it is acyclic, it is bimodal, and the number of sources plus the number of sinks in $\vec{G}$ equals the number of its sink-switches (or source-switches) plus one.*

Proof: By Theorem 1, it suffices to prove that $\vec{G}$ admits a consistent upward assignment if and only if the number of sources plus the number of sinks in $\vec{G}$ equals the number of sink-switches (or source-switches) plus one.

First, observe that every internal face f of $\vec{G}$ is delimited by a 3-cycle, hence there is exactly one sink-switch for f in $\vec{G}$, provided that $\vec{G}$ is acyclic. It follows that the demand of every internal face is 0.

For the necessity, since the demand of each internal face is 0, a consistent upward assignment maps all the sources and sinks of $\vec{G}$ to the outer face. Since the demand of the outer face equals the number of sink-switches (or source-switches) plus one, it follows that the number of sources plus the number of sinks in $\vec{G}$ equals the number of sink-switches (or source-switches) plus one.

For the sufficiency, since the demand of each internal face is 0 and since the demand of the outer face is equal to the number of sink-switches (or source-switches) plus one, it follows that assigning all the sources and all the sinks of $\vec{G}$ to the outer face results in a consistent upward assignment. □

We next deal with directed plane triangulations.

Theorem 3 *A directed plane triangulation is upward planar if and only if it is acyclic, it is bimodal, and it has exactly one source and one sink that are incident to the outer face.*

Proof: Let $\vec{G}$ be a directed plane triangulation. By Theorem 1, it suffices to prove that $\vec{G}$ admits a consistent upward assignment if and only if it has exactly one source and one sink that are incident to its outer face.

First, observe that every face f of $\vec{G}$ is delimited by a 3-cycle, hence there is exactly one sink-switch for f in $\vec{G}$, provided that $\vec{G}$ is acyclic. It follows that the demand of every internal face is 0 and the demand of the outer face is 2.

For the necessity, since the demand of the outer face is 2, it follows that, for the assignment to be consistent, $\vec{G}$ has one source s and one sink t incident to the outer face of $\vec{G}$. Moreover, since the demand of each internal face is 0, no vertex is assigned to any internal face, hence no internal vertex of $\vec{G}$ is a source or a sink.

For the sufficiency, since the demand of each internal face is 0 and since the demand of the outer face is 2, it follows that assigning the unique source and sink of $\vec{G}$ to the outer face results in a consistent upward assignment. □

To the best of our knowledge, it has gone unnoticed in the literature that the bimodality condition in Theorem 3 is redundant, as proved in the following.

Theorem 4 *A directed plane triangulation is upward planar if and only if it is acyclic and it has a single source and a single sink incident to the outer face.*

Proof: Consider a directed acyclic plane triangulation T with a single source s and a single sink t incident to the outer face. By Theorem 3, in order to prove the statement it suffices to prove that T is bimodal. Suppose, for a contradiction, that T is not bimodal. Let b be a vertex that is not bimodal, i.e., there exist directed edges (b, x), (u, b), (b, y), and (z, b) in clockwise order around it. Consider two monotone paths $P(x) = (x = x_1, x_2, \ldots, t)$ and $P(y) = (y = y_1, y_2, \ldots, t)$ from x and y to t, respectively. Such paths exist given that t

is the only sink of T and given that T is acyclic. Denote by t' the first vertex shared by $P(x)$ and $P(y)$ (possibly $t' = t$). Denote by T' the subgraph of T composed of those vertices, edges, and faces that are inside or on the boundary of cycle $(b, x = x_1, x_2, \ldots, t') \cup (b, y = y_1, y_2, \ldots, t')$. No internal vertex of T' is a source in T', as otherwise it would be a source in T as well. Further, every vertex of $P(x)$ has an edge directed to it (in fact the edge of path $(b, x) \cup P(x)$), hence it is not a source in T'. Analogously, no vertex of $P(y)$ is a source in T'. Moreover, by assumption b is not a source in T', given that either edge (u, b) or edge (z, b) belongs to T'. It follows that T' has no source, hence it must contain a directed cycle. Thus, T also contains a directed cycle, a contradiction. It follows that T is bimodal. By Theorem 3, T is upward planar. □

2.2 Upward Planarity of Mixed Plane Graphs

An *orientation* $\vec{G}$ of an undirected graph G or of a mixed graph G is an assignment of directions to the undirected edges of G. With a slight abuse of notation we denote by $\vec{G}$ both the orientation of G and the resulting directed graph. An orientation of a (plane) graph G is *upward planar* if the resulting directed (plane) graph is upward planar. Testing the upward planarity of a mixed graph G is equivalent to testing whether G admits an upward planar orientation. The orientation of an edge in a mixed graph G is *prescribed* if the edge is directed in G.

A mixed plane graph is upward planar if and only if each of its connected components is upward planar. Thus, without loss of generality, we only consider connected mixed plane graphs. In the following, we show that a stronger condition can in fact be assumed for each considered plane graph G, namely that G is 2-*connected*. Before proving such a claim, we present the following auxiliary lemma.

Lemma 1 *Let G be a mixed plane graph, let f be any face of G, and let (v, x) and (v, y) be two edges that appear consecutively along the border of f. Let G^* be the graph obtained from G by inserting a vertex w and undirected edges (w, x) and (w, y) inside f in such a way that cycle (v, x, w, y) delimits a face of G^*. We have that G is upward planar if and only if G^* is.*

Proof: One implication is trivial, namely G is a subgraph of G^*, hence if G^* is upward planar, then G is upward planar as well. Assume that G is upward planar. We prove that G^* is upward planar.

Let $\vec{G}$ be an arbitrary upward planar orientation of G. We define an orientation $\vec{G}^*$ of G^* as follows. Every edge of G^* that is also an edge of G is oriented as in $\vec{G}$; further, edge (x, w) is directed from x in $\vec{G}^*$ if and only if edge (x, v) is directed from x in $\vec{G}$; also, edge (y, w) is directed from y in $\vec{G}^*$ if and only if edge (y, v) is directed from y in $\vec{G}$. Consider an arbitrary upward planar drawing Γ of G with orientation $\vec{G}$. We are going to construct an upward planar drawing Γ^* of G^* with orientation $\vec{G}^*$ so that Γ^* coincides with Γ when restricted to G. For this sake, draw edges (x, w) and (y, w) inside f in Γ as y-monotone curves

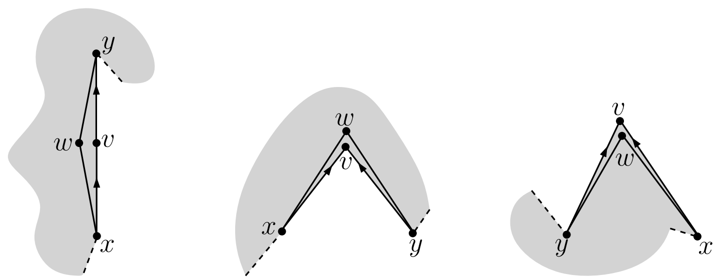

Figure 1: Cases that might arise when inserting vertex w and its incident edges (x, w) and (y, w) inside f in Γ. The gray region is part of f. (a) v is neither a source nor a sink. (b) v is a sink and there exist points in Γ that have y-coordinates larger than the one of v, that are arbitrarily close to v, and that belong to f. (c) v is a sink and there exists no point in Γ that has y-coordinate larger than the one of v, that is arbitrarily close to v, and that belongs to f.

arbitrarily close to edges (x, v) and (y, v), respectively. Observe that this is always possible given that (x, w) and (x, v) are both directed from x or both directed to x, and given that (y, w) and (y, v) are both directed from y or both directed to y. See Fig. 1. The resulting drawing Γ^* of G^* is upward because Γ is upward and because edges (x, w) and (y, w) are represented by y-monotone curves; further, Γ^* is planar because Γ is planar and because edges (x, w) and (y, w) do not cross any edge of G, given that they are drawn inside f. It follows that $\vec{G}^*$ is an upward planar orientation of G^*, and hence that G^* is upward planar. This concludes the proof of the lemma. □

We are now ready to prove the following.

Lemma 2 *Every n-vertex mixed plane graph G can be augmented with new edges and vertices to a 2-connected mixed plane graph G' with $O(n)$ vertices such that G is upward planar if and only if G' is. If G is outerplane, then G' is also outerplane. Moreover, G' can be constructed from G in $O(n)$ time.*

Proof: Let G be an n-vertex mixed plane graph and let f be any face of G. Let C_f be the (possibly non-simple) cycle delimiting f and let c be any vertex of C_f. If c is a cutvertex of G that is incident to at least two blocks of G containing edges incident to f, then define $k(c, f, G)$ to be the number of blocks of G incident to c and containing edges incident to f; otherwise, define $k(c, f, G) = 0$. Observe that, if G is outerplane, then $k(u, g, G) = 0$ for every internal face g of G and every vertex u incident to g.

Consider an arbitrary face f of G and an arbitrary cutvertex c of G such that $k(c, f, G) > 0$. Refer to Fig. 2. Denote by $B_1, B_2, \ldots, B_{k(c,f,G)}$ the blocks of G incident to c and containing edges incident to f in clockwise order. For each block B_i that is not a single edge, let C_i be the simple cycle in B_i whose edges

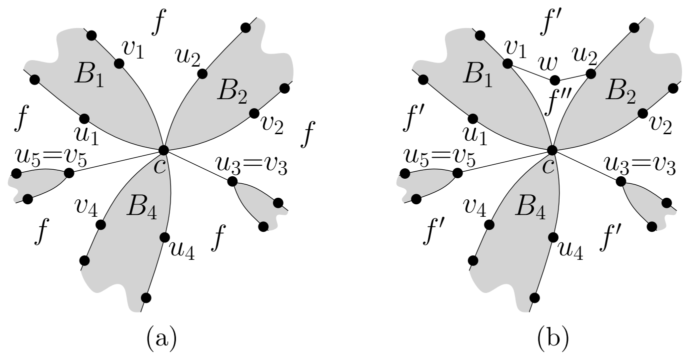

Figure 2: Mixed plane graph G before (a) and after (b) inserting vertex w and edges (v_1, w) and (u_2, w) inside f.

are all incident to f; orient C_i clockwise and let u_i and v_i be the vertices that follow and precede c in C_i, respectively. For each block B_i that is a single edge, let $u_i = v_i$ be the vertex of B_i adjacent to c. Insert a vertex w and undirected edges (w, v_1) and (w, u_2) in G, in such a way that vertex w and edges (w, v_1) and (w, u_2) lie inside f. Denote by G^* the resulting mixed plane graph. Also, denote by $C_{f'}$ the cycle obtained from C_f by replacing path $P_c = (v_1, c, u_2)$ with path $P_w = (v_1, w, u_2)$. Denote by f' the face of G^* delimited by $C_{f'}$ and denote by f'' the face of G^* delimited by path P_w and P_c.

We have that the following statements hold:

(i) G^* has one vertex more than G;

(ii) $k(c, f', G^*) < k(c, f, G)$, $k(u, f'', G^*) = 0$ for every vertex u incident to f'', and $k(u, g, G^*) = k(u, g, G)$ for every face g of G different from f and for every vertex u incident to g;

(iii) G^* is upward planar if and only if G is;

(iv) $C_{f'}$ contains all the vertices of C_f plus w; and

(v) G^* can be constructed from G in constant time.

Before proving the statements, we prove that they imply the lemma. While G has a face f and a vertex c incident to f such that $k(c, f, G) > 0$, insert a vertex w and undirected edges (w, v_1) and (w, u_2) inside f. Denote by G' the graph obtained at the end of this process. For every face g of G' and every vertex u incident to g it holds $k(u, g, G') = 0$ (by repeated applications of statement (ii)), that is G' is 2-connected. Moreover, G' is upward planar if and only if G is (by repeated applications of statement (iii)). Also, G' has $m \in O(n)$ vertices (by repeated applications of statement (i)). Further, if G is outerplane, then G' is outerplane (by repeated applications of statement (iv)). Finally, G' can be constructed in $O(n)$ time. Namely, G has $O(n)$ cutvertices, which can be detected at once in $O(n)$ time; further, each augmentation decreases the number of cutvertices by statement (ii) and takes constant time by statement (v).

We now prove statements (i)-(v). Statements (i) and (v) easily follow from the construction of G^*. Statement (ii) is a consequence of the following observations: $k(c, f', G^*)$ is either equal to $k(c, f, G) - 1$ or to $k(c, f, G) - 2$, depending on whether c is a cutvertex of G^* incident to at least two 2-connected components of G^* containing edges incident to f' or not; further, $k(u, f'', G^*) = 0$, given that f'' is delimited by simple cycle (v_1, c, u_2, w); moreover, every face g of G different from f is delimited by the same cycle in G and in G^*, thus $k(u, g, G^*) = k(u, g, G)$ holds for every vertex u incident to g. Statement (iv) is proved as follows: $C_{f'}$ contains w, by definition, and it contains all the vertices of C_f. In particular, it contains c given that c occurs at least twice in C_f and given that $C_{f'}$ has exactly one fewer occurrence of c than C_f. Finally, statement (iii) directly follows from Lemma 1. □

3 Upward Planarity Testing for Mixed Outerplane Graphs

This section is devoted to the proof of the following theorem.

Theorem 5 *The upward planarity of an n-vertex mixed outerplane graph can be tested in $O(n^3)$ time.*

Let G be any n-vertex mixed outerplane graph. By Lemma 2, an $O(n)$-vertex 2-connected mixed outerplane graph G^* can be constructed in $O(n)$ time such that G is upward planar if and only if G^* is.

We introduce some notation and terminology. Let u and v be distinct vertices of G^*. We denote by $G^* + (u, v)$ the graph obtained from G^* by adding edge (u, v) if it is not already in G^*, and by $G^* - u$ the graph obtained from G^* by deleting u and its incident edges. Consider an orientation $\vec{G}^*$ of G^*. A vertex is *sinky* (*sourcey*) in $\vec{G}^*$ if it is a sink-switch but not a sink (if it is a source-switch but not a source, resp.). A vertex that is neither a sink, a source, sinky, nor sourcey is *ordinary*; that is, v is ordinary if the two external edges incident to v are one directed to v and one directed from v in $\vec{G}^*$. We say the *status* of a vertex of G^* in $\vec{G}^*$ is sink, source, sinky, sourcey, or ordinary. See Fig. 3.

First note that G^* is upward planar if and only if there is an upward outerplane triangulation T of G^*, that is, if and only if G^* can be augmented to a mixed outerplane triangulation, and the undirected edges of such a triangulation can be oriented in such a way that the resulting directed outerplane triangulation T is upward planar. The approach of our algorithm is to determine if there is such a T using recursion. The algorithm can be easily modified to produce T if it exists.

We observe that a directed outerplane triangulation T is acyclic if and only if every 3-cycle in T is acyclic. One direction is trivial. Conversely, suppose

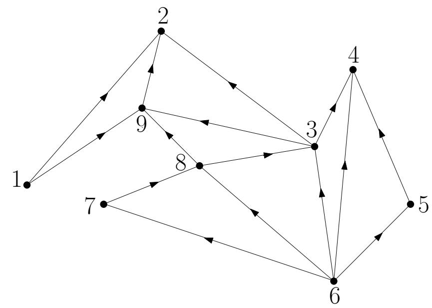

Figure 3: A directed outerplane triangulation T. Vertices 1 and 6 are sources in T, vertices 2 and 4 are sinks in T, vertices 5, 7, and 8 are ordinary in T, vertex 3 is sourcey in T, and vertex 9 is sinky in T.

that T contains a directed cycle. Let C be a shortest directed cycle of T. If C is a 3-cycle, then we are done. Otherwise, an edge $(x, y) \notin C$ exists in T between two vertices x and y both in C. Thus, $C + (x, y)$ contains two shorter cycles, one of which is a directed cycle, contradicting the choice of C. Hence, to ensure the acyclicity of a directed outerplane triangulation, it suffices to ensure that its internal faces are acyclic.

A *potential edge* of G^* is a pair of distinct vertices x and y in G^* such that $G^* + (x, y)$ is outerplane, which is equivalent to saying that x and y are incident to a common internal face of G^* (notice that an edge of G^* is a potential edge of G^*). Fix some external edge r of G^*, called the *root edge*. Let $e = \{x, y\}$ be an internal potential edge of G^*. See Fig. 4. Then $\{x, y\}$ separates G^*, that is, G^* contains two subgraphs G_1^* and G_2^* such that $G^* = G_1^* \cup G_2^*$ and $V(G_1^* \cap G_2^*) = \{x, y\}$. (Thus, there is no edge between $G_1^* - x - y$ and $G_2^* - x - y$.) Without loss of generality, $r \in E(G_1^*)$. Let $G_e^* := G_2^* + (x, y)$. Observe that G_e^* is a 2-connected mixed outerplane graph with e incident to the outer face. Also, let $e = \{x, y\} \neq r$ be an external potential edge of G^*. Then, we define G_e^* to be the 2-vertex graph containing the single edge (x, y). Further, let $G_r^* := G^*$. For each (internal or external) potential edge $e = \{x, y\}$ of G^* and for each orientation $\overrightarrow{xy}$ of e, let $G^*_{\overrightarrow{xy}}$ be G_e^* with e oriented $\overrightarrow{xy}$. Define a partial order $\prec$ on the potential edges of G^* as follows. For distinct potential edges e and f of G^*, say $e \prec f$ if both end-vertices of f are in G_e^*. Loosely speaking, $e \prec f$ if $G^* + e + f$ is outerplane and e is "between" r and f.

A *potential arc* of G^* is a potential edge that is assigned an orientation preserving its orientation in G^*. So if e is an undirected edge of G^* or a potential edge not in G^*, then there are two potential arcs associated with e, while if e is a directed edge of G^*, then there is one potential arc associated with e. If a potential arc $\overrightarrow{xy}$ is part of a triangulation T of G^*, then x is a source, sourcey, or ordinary, and y is a sink, sinky, or ordinary in $G^*_{\overrightarrow{xy}}$. We define the *status* of $\overrightarrow{xy}$ in $G^*_{\overrightarrow{xy}}$ as an ordered pair S of $S(x) \in$ {source, sourcey, ordinary} and

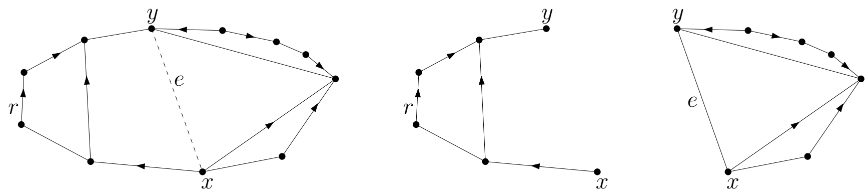

Figure 4: (a) A mixed plane graph G^* with a root edge r and an internal potential edge $e = \{x, y\}$ of G^*. Edge e is dashed since it does not belong to G^*. (b) Graph G_1^*. (c) Graph $G_e^* := G_2^* + (x, y)$.

$S(y) \in \{\text{sink, sinky, ordinary}\}$.

We now define a function $\mathrm{UP}(\overrightarrow{xy}, S)$, that takes as an input a potential arc $\overrightarrow{xy}$ and a status S of $\overrightarrow{xy}$, and has value "true" if and only if there is an upward outerplane triangulation $T_{\overrightarrow{xy}}$ of $G^*_{\overrightarrow{xy}}$ that respects $S(x)$ and $S(y)$; notice that, if $\overrightarrow{xy}$ is external and does not correspond to r, then $T_{\overrightarrow{xy}}$ is a single edge.

First, the values of $\mathrm{UP}(\overrightarrow{xy}, S)$ can be computed in total $O(n)$ time for all the external potential arcs $\overrightarrow{xy}$ of G^* not corresponding to r and for all statuses of $\overrightarrow{xy}$. Indeed, $\mathrm{UP}(\overrightarrow{xy}, S)$ is true if and only if $S(x) = \text{source}$ and $S(y) = \text{sink}$.

We show below that, for each potential arc $\overrightarrow{xy}$ in G^* that is internal or that is external and corresponds to r, and for each status S of $\overrightarrow{xy}$, the value of $\mathrm{UP}(\overrightarrow{xy}, S)$ can be computed in $O(n)$ time from values associated to potential arcs corresponding to potential edges e with $\{x, y\} \prec e$. Since there are at most $n(n+1)$ potential arcs and nine statuses for each potential arc, all the values of $\mathrm{UP}(\overrightarrow{xy}, S)$ can be computed in $O(n^3)$ time by dynamic programming in reverse order to a linear extension of $\prec$. Then, there is an upward outerplane triangulation of G^* if and only if $\mathrm{UP}(\overrightarrow{xy}, S)$ is true for some orientation $\overrightarrow{xy}$ of r and some status S of $\overrightarrow{xy}$.

Let $\overrightarrow{xy}$ be a potential arc that is internal to G^* or that corresponds to r. Let S be a status of $\overrightarrow{xy}$. Suppose that $\mathrm{UP}(\overrightarrow{xy}, S)$ is true. Then, there is an upward outerplane triangulation $T_{\overrightarrow{xy}}$ of $G^*_{\overrightarrow{xy}}$ that respects $S(x)$ and $S(y)$. Such a triangulation contains a vertex $z \in V(G^*_{xy}) - x - y$ such that (x, y, z) is an internal face of $T_{\overrightarrow{xy}}$. Since $T_{\overrightarrow{xy}}$ has edge (x, y) oriented from x to y, then edges (x, z) and (y, z) cannot be simultaneously directed to x and directed from y, respectively, as otherwise $T_{\overrightarrow{xy}}$ would contain a directed cycle, which is not possible by Theorem 2. Hence, edges (x, z) and (y, z) in $T_{\overrightarrow{xy}}$ are either directed from x and directed to y, or directed from x and directed from y, or directed to x and directed to y, respectively.

Now, for each status S of $\overrightarrow{xy}$ and for a particular vertex $z \in V(G^*_{xy}) - x - y$, we characterize the conditions for which an upward outerplane triangulation $T_{\overrightarrow{xy}}$ exists that respects $S(x)$ and $S(y)$ and that contains edges (x, z) and (y, z) oriented according to each of the three orientations described above.

Lemma 3 *There is an upward outerplane triangulation $T_{\overrightarrow{xy}}$ that respects $S(x)$ and $S(y)$, that contains edge (x, z) directed from x, and that contains edge (z, y) directed to y, if and only if $\overrightarrow{xz}$ and $\overrightarrow{zy}$ are potential arcs of G^* and there are statuses S_1 of $\overrightarrow{xz}$ and S_2 of $\overrightarrow{zy}$ such that the following conditions hold:*

(a) $S_1(x) = S(x) \in \{$*source, sourcey, ordinary*$\}$,

(b) $S_2(y) = S(y) \in \{$*sink, sinky, ordinary*$\}$,

(c) $S_1(z) \in \{$*sink, ordinary*$\}$,

(d) $S_2(z) \in \{$*source, ordinary*$\}$,

(e) $S_1(z) =$ *sink or* $S_2(z) =$ *source, and*

(f) both UP$(\overrightarrow{xz}, S_1)$ *and UP*$(\overrightarrow{zy}, S_2)$ *are true.*

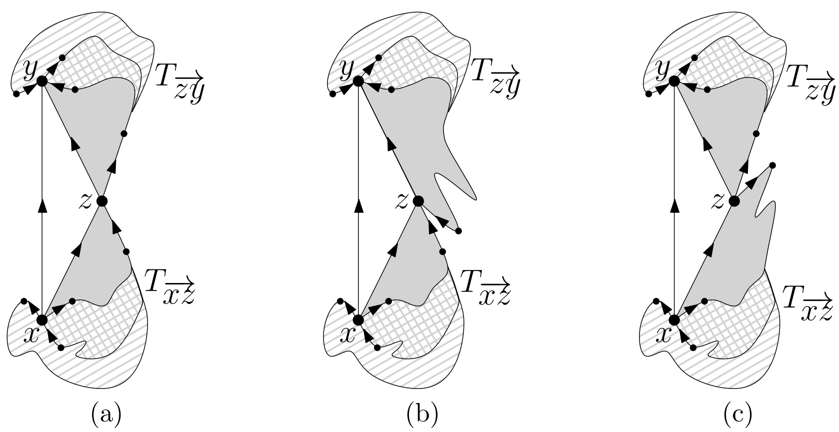

Figure 5: Illustration for the proof of Lemma 3. The gray filled regions represent $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{zy}}$ if $S(x) = S_1(x) =$ source and $S(y) = S_2(y) =$ sink; the gray filled regions plus the regions striped in two directions represent $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{zy}}$ if $S(x) = S_1(x) =$ ordinary and $S(y) = S_2(y) =$ ordinary; the gray filled regions plus the regions striped in two directions plus the regions striped in one direction represent $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{zy}}$ if $S(x) = S_1(x) =$ sourcey and $S(y) = S_2(y) =$ sinky. (a) $S_1(z) =$ sink and $S_2(z) =$ source. (b) $S_1(z) =$ sink and $S_2(z) =$ ordinary. (c) $S_1(z) =$ ordinary and $S_2(z) =$ source.

Proof: Refer to Fig. 5.

($\Longrightarrow$) Let $T_{\overrightarrow{xy}}$ be an upward outerplane triangulation of $G^*_{\overrightarrow{xy}}$ that respects $S(x)$ and $S(y)$, that contains edge (x, z) directed from x, and that contains edge (z, y) directed to y. Then, $\overrightarrow{xz}$ and $\overrightarrow{zy}$ are potential arcs of G^*. Further, $T_{\overrightarrow{xy}}$ determines upward outerplane triangulations $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{zy}}$ respectively of $G^*_{\overrightarrow{xz}}$ and $G^*_{\overrightarrow{zy}}$ (where $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{zy}}$ are single edges if $\overrightarrow{xz}$ and $\overrightarrow{zy}$ are external, respectively), as well as statuses S_1 and S_2 of $\overrightarrow{xz}$ and $\overrightarrow{zy}$, respectively, such that (f) both $\text{UP}(\overrightarrow{xz}, S_1)$ and $\text{UP}(\overrightarrow{zy}, S_2)$ are true. Since $\overrightarrow{xy}$ and $\overrightarrow{xz}$ are consecutive arcs directed from x, we have (a) $S_1(x) = S(x) \in \{\text{source, sourcey, ordinary}\}$. Similarly, (b) $S_2(y) = S(y) \in \{\text{sink, sinky, ordinary}\}$. Since $\overrightarrow{xz}$ is directed to z, we have $S_1(z) \in \{\text{sink, ordinary, sinky}\}$. However, if $S_1(z) = \text{sinky}$, then z is not bimodal in $T_{\overrightarrow{xy}}$. Thus (c) $S_1(z) \in \{\text{sink, ordinary}\}$. Similarly, (d) $S_2(z) \in \{\text{source, ordinary}\}$. Finally, if z is ordinary in both $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{zy}}$, then z is not bimodal in $T_{\overrightarrow{xy}}$. Thus (e) $S_1(z) = \text{sink}$ or $S_2(z) = \text{source}$.

($\Longleftarrow$) Let $T_{\overrightarrow{xz}}$ be an upward outerplane triangulation of $G^*_{\overrightarrow{xz}}$ respecting S_1 ($T_{\overrightarrow{xz}}$ is a single edge if $\overrightarrow{xz}$ is external). Let $T_{\overrightarrow{zy}}$ be an upward outerplane triangulation of $G^*_{\overrightarrow{zy}}$ respecting S_2 ($T_{\overrightarrow{zy}}$ is a single edge if $\overrightarrow{zy}$ is external). Such triangulations exist because $\text{UP}(\overrightarrow{xz}, S_1)$ and $\text{UP}(\overrightarrow{zy}, S_2)$ are true. Let $T_{\overrightarrow{xy}}$ be the triangulation of $G^*_{\overrightarrow{xy}}$ determined from $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{zy}}$ by adding the arc $\overrightarrow{xy}$. Since $T_{\overrightarrow{xz}}$, $T_{\overrightarrow{zy}}$, and (x, y, z) are acyclic, $T_{\overrightarrow{xy}}$ is acyclic. Since x is bimodal in $T_{\overrightarrow{xz}}$, it is bimodal in $T_{\overrightarrow{xy}}$. Similarly, y is bimodal in $T_{\overrightarrow{xy}}$. As described above, the conditions on $S_1(z)$ and $S_2(z)$ imply that z is bimodal in $T_{\overrightarrow{xy}}$. Every other vertex is bimodal in $T_{\overrightarrow{xy}}$ because it is bimodal in $T_{\overrightarrow{xz}}$ or in $T_{\overrightarrow{zy}}$. Hence, $T_{\overrightarrow{xy}}$ is bimodal.

Let s_1, t_1 and w_1 be the number of sources, sinks, and source-switches in $T_{\overrightarrow{xz}}$, respectively. Let s_2, t_2 and w_2 be the number of sources, sinks, and source-switches in $T_{\overrightarrow{zy}}$, respectively. By Theorem 2, $s_i + t_i = w_i + 1$ for $i \in \{1, 2\}$. Let s, t and w be the number of sources, sinks, and source-switches in $T_{\overrightarrow{xy}}$, respectively.

If z is a sink in $T_{\overrightarrow{xz}}$ and ordinary in $T_{\overrightarrow{zy}}$, then $s = s_1 + s_2$, $t = t_1 + t_2 - 1$ (for z), and $w = w_1 + w_2$. If z is a source in $T_{\overrightarrow{zy}}$ and ordinary in $T_{\overrightarrow{xz}}$, then $s = s_1 + s_2 - 1$ (for z), $t = t_1 + t_2$, and $w = w_1 + w_2$. If z is a sink in $T_{\overrightarrow{xz}}$ and a source in $T_{\overrightarrow{zy}}$, then $s = s_1 + s_2 - 1$ (for z) and $t = t_1 + t_2 - 1$ (for z) and $w = w_1 + w_2 - 1$ (for z). In all three cases, it follows that $s + t = w + 1$.

By Theorem 2, $T_{\overrightarrow{xy}}$ is upward planar. By construction, $T_{\overrightarrow{xy}}$ respects $S(x)$ and $S(y)$ and contains edge (x, z) directed from x and edge (z, y) directed to y. □

Lemma 4 *There is an upward outerplane triangulation $T_{\overrightarrow{xy}}$ that respects $S(x)$ and $S(y)$ and that contains edges (x, z) and (y, z) directed to z if and only if $\overrightarrow{xz}$ and $\overrightarrow{yz}$ are potential arcs of G^* and there are statuses S_1 of $\overrightarrow{xz}$ and S_2 of $\overrightarrow{yz}$ such that the following conditions hold:*

- *(a)* $S_1(x) = S(x) \in \{$*source, sourcey, ordinary*$\}$,
- *(b)* $S(y) \in \{$*sinky, ordinary*$\}$,
- *(c)* $S_2(y) \in \{$*source, ordinary*$\}$,
- *(d)* $S(y) =$ *ordinary if and only if* $S_2(y) =$ *source,*
- *(e)* $S(y) =$ *sinky if and only if* $S_2(y) =$ *ordinary,*
- *(f)* $S_1(z) \in \{$*sink, sinky, ordinary*$\}$,

(g) $S_2(z) \in \{sink,\ sinky,\ ordinary\}$,

(h) $S_1(z) \in \{sink,\ ordinary\}$ *or* $S_2(z) = sink$,

(i) $S_2(z) \in \{sink,\ ordinary\}$ *or* $S_1(z) = sink$, *and*

(j) both $UP(\overrightarrow{xz}, S_1)$ *and* $UP(\overrightarrow{yz}, S_2)$ *are true.*

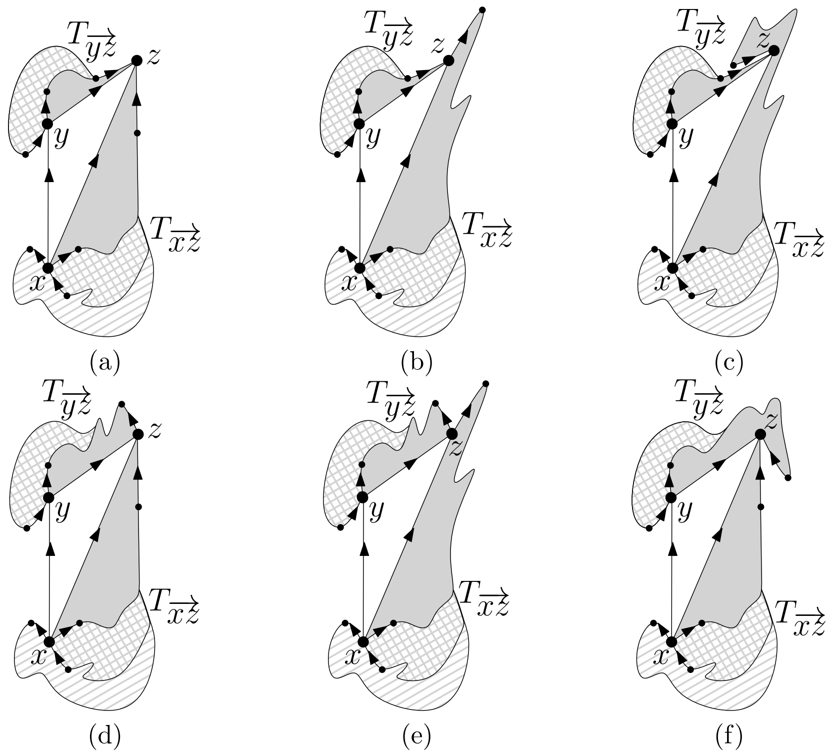

Figure 6: Illustration for the proof of Lemma 4. The gray filled regions represent $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{yz}}$ if $S(x) = S_1(x) =$ source and $S_2(y) =$ source (and $S(y) =$ ordinary); the gray filled regions plus the regions striped in two directions represent $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{yz}}$ if $S(x) = S_1(x) =$ ordinary and $S_2(y) =$ ordinary (and $S(y) =$ sinky); the gray filled region plus the region striped in two directions plus the region striped in one direction represents $T_{\overrightarrow{xz}}$ if $S(x) = S_1(x) =$ sourcey. (a) $S_1(z) =$ sink and $S_2(z) =$ sink. (b) $S_1(z) =$ ordinary and $S_2(z) =$ sink. (c) $S_1(z) =$ sinky and $S_2(z) =$ sink. (d) $S_1(z) =$ sink and $S_2(z) =$ ordinary. (e) $S_1(z) =$ ordinary and $S_2(z) =$ ordinary. (f) $S_1(z) =$ sink and $S_2(z) =$ sinky.

Proof: Refer to Fig. 6.

($\Longrightarrow$) Let $T_{\overrightarrow{xy}}$ be an upward outerplane triangulation of $G^*_{\overrightarrow{xy}}$ that respects $S(x)$ and $S(y)$, and that contains edge (x, z) and (y, z) directed to z. Then

$\overrightarrow{xz}$ and $\overrightarrow{yz}$ are potential arcs of G^*. $T_{\overrightarrow{xy}}$ determines upward outerplane triangulations $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{yz}}$ respectively of $G^*_{\overrightarrow{xz}}$ and $G^*_{\overrightarrow{yz}}$ (where $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{yz}}$ are single edges if $\overrightarrow{xz}$ and $\overrightarrow{yz}$ are external, respectively), as well as statuses S_1 and S_2 of $\overrightarrow{xz}$ and $\overrightarrow{yz}$, respectively, such that (j) both $\mathrm{UP}(\overrightarrow{xz}, S_1)$ and $\mathrm{UP}(\overrightarrow{yz}, S_2)$ are true. Since $\overrightarrow{xy}$ and $\overrightarrow{xz}$ are consecutive arcs directed from x, we have (a) $S_1(x) = S(x) \in \{\text{source, sourcey, ordinary}\}$. Since $\overrightarrow{xy}$ and $\overrightarrow{yz}$ are incident to y, we have (b) $S(y) \in \{\text{sinky, ordinary}\}$. Since $\overrightarrow{yz}$ in $T_{\overrightarrow{yz}}$, we have $S_2(y) \in \{\text{source, sourcey, ordinary}\}$. Moreover, if y is sourcey in $T_{\overrightarrow{yz}}$, then y is not bimodal in $T_{\overrightarrow{xy}}$. Thus (c) $S_2(y) \in \{\text{source, ordinary}\}$. Observe that (d) $S(y) = \text{ordinary}$ if and only if $S_2(y) = \text{source}$ (otherwise y is not bimodal in $T_{\overrightarrow{xy}}$). Similarly, (e) $S(y) = \text{sinky}$ if and only if $S_2(y) = \text{ordinary}$. Since $\overrightarrow{xz}$ in $T_{\overrightarrow{xz}}$, we have (f) $S_1(z) \in \{\text{sink, ordinary, sinky}\}$. Analogously, we have (g) $S_2(z) \in \{\text{sink, ordinary, sinky}\}$. Moreover, (h) if z is sinky in $T_{\overrightarrow{xz}}$, then z is a sink in $T_{\overrightarrow{yz}}$, as otherwise z is not bimodal in $T_{\overrightarrow{xy}}$. Analogously, (i) if z is sinky in $T_{\overrightarrow{yz}}$, then z is a sink in $T_{\overrightarrow{xz}}$.

($\Longleftarrow$) Let $T_{\overrightarrow{xz}}$ be an upward outerplane triangulation of $G^*_{\overrightarrow{xz}}$ respecting S_1 ($T_{\overrightarrow{xz}}$ is a single edge if $\overrightarrow{xz}$ is external). Let $T_{\overrightarrow{yz}}$ be an upward outerplane triangulation of $G^*_{\overrightarrow{yz}}$ respecting S_2 ($T_{\overrightarrow{yz}}$ is a single edge if $\overrightarrow{yz}$ is external). Such triangulations exist because $\mathrm{UP}(\overrightarrow{xz}, S_1)$ and $\mathrm{UP}(\overrightarrow{yz}, S_2)$ are true. Let $T_{\overrightarrow{xy}}$ be the triangulation of $G^*_{\overrightarrow{xy}}$ determined from $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{yz}}$ by adding the arc $\overrightarrow{xy}$. Since $T_{\overrightarrow{xz}}$, $T_{\overrightarrow{yz}}$, and (x, y, z) are acyclic, $T_{\overrightarrow{xy}}$ is acyclic.

Since x is bimodal in $T_{\overrightarrow{xz}}$, it is bimodal in $T_{\overrightarrow{xy}}$. Since y is not sourcey in $T_{\overrightarrow{yz}}$, it is bimodal in $T_{\overrightarrow{xy}}$. If z is not bimodal in $T_{\overrightarrow{xy}}$, then z is sinky in $T_{\overrightarrow{xz}}$ and is not a sink in $T_{\overrightarrow{yz}}$ (or vice versa, switching the role of $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{yz}}$). However, this is not possible by conditions (h) and (i), hence z is bimodal in $T_{\overrightarrow{xy}}$. Every other vertex is bimodal in $T_{\overrightarrow{xy}}$ because it is bimodal in $T_{\overrightarrow{xz}}$ or in $T_{\overrightarrow{yz}}$. Hence, $T_{\overrightarrow{xy}}$ is bimodal.

Let s_1, t_1 and w_1 be the number of sources, sinks, and source-switches in $T_{\overrightarrow{xz}}$, respectively. Let s_2, t_2 and w_2 be the number of sources, sinks, and source-switches in $T_{\overrightarrow{yz}}$, respectively. By Theorem 2, $s_i + t_i = w_i + 1$ for $i \in \{1, 2\}$. Let s, t and w be the number of sources, sinks, and source-switches in $T_{\overrightarrow{xy}}$, respectively.

We distinguish the following cases: y is a source or is ordinary in $T_{\overrightarrow{yz}}$. And z is a sink in $T_{\overrightarrow{xz}}$ or $T_{\overrightarrow{yz}}$, or z is in ordinary in both $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{yz}}$.

If y is a source in $T_{\overrightarrow{yz}}$ and z is a sink in $T_{\overrightarrow{xz}}$ or $T_{\overrightarrow{yz}}$ (possibly both), then $s = s_1 + s_2 - 1$ (for y) and $t = t_1 + t_2 - 1$ (for z) and $w = w_1 + w_2 - 1$ (for y). (Here, if z is a sink in both $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{yz}}$, then z is a sink in $T_{\overrightarrow{xy}}$, but still $t = t_1 + t_2 - 1$.) If y is a source in $T_{\overrightarrow{yz}}$ and z is ordinary in both $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{yz}}$, then $s = s_1 + s_2 - 1$ (for y) and $t = t_1 + t_2$ and $w = w_1 + w_2 - 1 + 1$ (for y and z). If y is ordinary in $T_{\overrightarrow{yz}}$ and z is a sink in $T_{\overrightarrow{xz}}$ or in $T_{\overrightarrow{yz}}$, then $s = s_1 + s_2$ and $t = t_1 + t_2 - 1$ (for z) and $w = w_1 + w_2$. If y is ordinary in $T_{\overrightarrow{yz}}$ and z is ordinary in $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{yz}}$, then $s = s_1 + s_2$ and $t = t_1 + t_2$ and $w = w_1 + w_2 + 1$ (for z). In all cases, it follows that $s + t = w + 1$.

By Theorem 2, $T_{\overrightarrow{xy}}$ is upward planar. By construction, $T_{\overrightarrow{xy}}$ respects $S(x)$ and $S(y)$ and contains edge (x, z) and (y, z) directed to z. □

Lemma 5 *There is an upward outerplane triangulation $T_{\overrightarrow{xy}}$ that respects $S(x)$ and $S(y)$ and that contains edges (z, x) and (z, y) directed from z if and only if $\overrightarrow{zx}$ and $\overrightarrow{zy}$ are potential arcs of G^* and there are statuses S_1 of $\overrightarrow{zx}$ and S_2 of $\overrightarrow{zy}$ such that the following conditions hold:*

(a) $S_2(y) = S(y) \in \{$*sink, sinky, ordinary*$\}$,

(b) $S(x) \in \{$*sourcey, ordinary*$\}$,

(c) $S_1(x) \in \{$*sink, ordinary*$\}$,

(d) $S(x) =$ *ordinary if and only if* $S_1(x) =$ *sink,*

(e) $S(x) =$ *sourcey if and only if* $S_1(x) =$ *ordinary,*

(f) $S_1(z) \in \{$*source, sourcey, ordinary*$\}$,

(g) $S_2(z) \in \{$*source, sourcey, ordinary*$\}$,

(h) $S_1(z) \in \{$*source, ordinary*$\}$ *or* $S_2(z) =$ *source,*

(i) $S_2(z) \in \{$*source, ordinary*$\}$ *or* $S_1(z)$=*source, and*

(j) both UP$(\overrightarrow{zx}, S_1)$ *and UP*$(\overrightarrow{zy}, S_2)$ *are true.*

Proof: The proof is symmetric to the proof of Lemma 4. □

For each status S of $\overrightarrow{xy}$ and for a particular vertex $z \in V(G^*_{xy}) - x - y$, it can be checked in $O(1)$ time whether an upward outerplane triangulation $T_{\overrightarrow{xy}}$ exists that respects $S(x)$ and $S(y)$ and that contains edges (x, z) and (y, z) by checking whether the conditions in at least one of Lemmata 3-5 are satisfied. Further, UP$(\overrightarrow{xy}, S)$ is true if and only if there exists a vertex $z \in V(G^*_{xy}) - x - y$ such that an upward outerplane triangulation $T_{\overrightarrow{xy}}$ exists that respects $S(x)$ and $S(y)$ and that contains edges (x, z) and (y, z). Thus, we can determine UP$(\overrightarrow{xy}, S)$ in $O(n)$ time since there are less than n possible choices for z.

This completes the proof of Theorem 5. The time complexity analysis can be strengthened as follows. Suppose that every internal face of G^* has at most t vertices. Then each vertex v is incident to less than $t \cdot \deg_{G^*}(v)$ potential edges and the total number of potential arcs is less than $2\sum_v t \cdot \deg_{G^*}(v) \leq 8tn$. Since each potential arc has nine statuses, and since there are less than t choices for z, the time complexity is $O(t^2 n)$. In particular, if G^* is an outerplane triangulation, then the time complexity is $O(n)$.

4 From Mixed Plane Graphs To Mixed Plane Triangulations

This section is devoted to the proof of the following theorem.

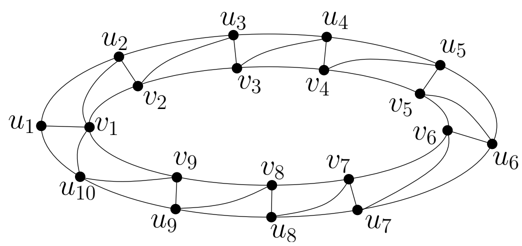

Figure 7: Augmentation of a face f with $n_f = 10$ vertices.

Theorem 6 *Let G be an n-vertex mixed plane graph. There exists an $O(n^2)$-vertex mixed plane triangulation G' such that G is upward planar if and only if G' is. Moreover, G' can be constructed from G in $O(n^2)$ time.*

Proof: By Lemma 2, an $O(n)$-vertex 2-connected mixed plane graph G^* can be constructed in $O(n)$ time such that G is upward planar if and only if G^* is.

We show how to construct a graph G' satisfying the statement of the theorem. In order to construct G', we augment G^* in several steps. At each step, vertices and edges are inserted inside a face f of G^* delimited by a cycle C_f with $n_f \geq 4$ vertices. The new edges and vertices subdivide f into one face with $n_f - 1$ vertices and $2n_f - 1$ faces with three vertices each. The repetition of such an augmentation yields the desired graph G'.

We now describe how to augment G^*. Consider an arbitrary face f of G^* delimited by a cycle C_f with $n_f \geq 4$ vertices. Let $(u_1, u_2, \dots, u_{n_f})$ be the clockwise order of the vertices along C_f starting at an arbitrary vertex. Insert a cycle C'_f inside f with $n_f - 1$ vertices $v_1, v_2, \dots, v_{n_f-1}$ in this clockwise order along C'_f. For $1 \leq i \leq n_f - 1$, insert edges (v_i, u_i) and (v_i, u_{i+1}) inside C_f and outside C'_f; also, insert edge (v_1, u_{n_f}) inside cycle $(u_{n_f}, u_1, v_1, v_{n_f-1})$. All the edges inserted in f are undirected. See Fig. 7. Denote by G'_f the graph consisting of cycle C_f together with the vertices and edges inserted in f. Observe that the face of G'_f delimited by C'_f has $n_f - 1$ vertices, while all the other faces into which f is split by the insertion of C'_f and of its incident edges have three vertices.

We now show that G^* before the augmentation is upward planar if and only if G^* after the augmentation is upward planar. One implication is trivial, given that G^* before the augmentation is a subgraph of G^* after the augmentation. For the other implication, it suffices to prove that, for an arbitrary upward planar orientation $\vec{C}_f$ of C_f, there exists an upward planar orientation $\vec{G}'_f$ of G'_f that coincides with $\vec{C}_f$ when restricted to C_f.

Consider an upward planar drawing Γ_f of C_f with orientation $\vec{C}_f$ (see Fig. 8(a)). We describe how to place vertices and edges inside f to obtain an upward planar drawing Γ'_f of $\vec{G}'_f$.

Pach and Tóth [14] proved that every planar drawing of a graph G in which all the edges are y-monotone can be triangulated by the insertion of y-monotone edges inside the faces of G (the result in [14] states that the addition of a vertex might be needed to triangulate the outer face of G, which however is not the case if the outer face is bounded by a simple cycle, as in our case). In every triangulation of a cycle C_f with $n_f \geq 4$ vertices, at least two vertices have degree two. Hence there is an index j, $1 < j \leq n_f$, such that u_{j-1} and u_{j+1} can be connected by a y-monotone curve inside f.

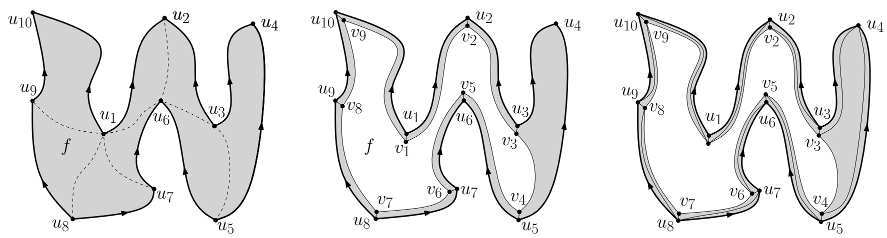

Figure 8: (a) An upward planar drawing Γ_f of C_f with orientation $\vec{C}_f$, and a y-monotone triangulation (dashed). Vertices u_4 and u_{10} have degree two, so we can take $j = 4$. (b) We place the points $v_1, \ldots, v_3$ in small neighborhoods of $u_1, \ldots, u_3$; and the points $v_4, \ldots, v_9$ in small neighborhoods of $u_5, \ldots, u_{10}$; and then draw the cycle $(v_1, \ldots, v_9)$ with y-monotone edges. (c) The edges between the cycles $(u_1, \ldots, u_{10})$ and $(v_1, \ldots, v_9)$ can be realized with y-monotone curves.

Let $\delta > 0$ be such that the δ-neighborhood $\delta(u_i)$ of every vertex u_i, $i = 1, \ldots, n_f$, is disjoint from all nonincident edges in the drawing Γ_f, and the projections of these δ-neighborhoods into the y-axis are pairwise disjoint. We call the subset of $\delta(u_i)$ whose points have y-coordinates smaller (larger) the one of u_i, the δ^--*neighborhood*, $\delta^-(u_i)$, of u_i (resp. the δ^+-*neighborhood*, $\delta^+(u_i)$, of u_i). We further define $\delta_f(u_i)$ as follows:

$$\delta_f(u_i) = \begin{cases} \delta^-(u_i) & \text{if } u_i \text{ is a source in } \vec{C}_f \text{ and } \delta^-(u_i) \subset f. \\ \delta^+(u_i) \cap f & \text{if } u_i \text{ is a source in } \vec{C}_f \text{ and } \delta^-(u_i) \cap f = \emptyset. \\ \delta^+(u_i) & \text{if } u_i \text{ is a sink in } \vec{C}_f \text{ and } \delta^+(u_i) \subset f. \\ \delta^-(u_i) \cap f & \text{if } u_i \text{ is a sink in } \vec{C}_f \text{ and } \delta^+(u_i) \cap f = \emptyset. \\ \delta^-(u_i) \cap f & \text{if } u_i \text{ is ordinary in } \vec{C}_f. \end{cases}$$

We augment the drawing Γ_f with the new vertices and edges as follows:

- First, for $1 \leq i < j$, we place vertex v_i in $\delta_f(u_i)$ and we connect v_i to u_i by a y-monotone curve. Further, for $j \leq i < n_f$, we place vertex v_i in $\delta_f(u_{i+1})$ and we connect v_i to u_{i+1} by a y-monotone curve. See Fig. 8(b).
- Second, we realize the cycle $(v_1, \ldots, v_{n_f-1})$ with y-monotone edges as follows. For $0 < i < j - 1$, the edge (v_i, v_{i+1}) closely follows (u_i, u_{i+1}) inside f. The edge (v_{j-1}, v_j) closely follows a y-monotone curve between

u_{j-1} and u_{j+1}, which is an edge of the triangulation of C_f mentioned above. For $j < i < n_f$, the edge (v_{i-1}, v_i) closely follows (u_i, u_{i+1}) inside f. Edge (v_{n_f-1}, v_1) closely follows (u_{n_f}, u_1). See Fig. 8(b).

- Third, we realize all remaining edges between the cycles $(u_1, \dots, u_{n_f})$ and $(v_1, \dots, v_{n_f-1})$. For $0 < i < j-1$, the edge (v_i, u_{i+1}) threads between (u_i, u_{i+1}) and (v_i, v_{i+1}). The edge (v_{j-1}, u_j) closely follows edge (u_{j-1}, u_j) of C_f. The edge (u_j, v_j) closely follows edge (u_j, u_{j+1}) of C_f. For $j < i < n_f$, the edge (u_i, v_i) threads between (u_i, u_{i+1}) and (v_{i-1}, v_i). Finally, the edge (u_{n_f}, v_1) threads between (u_{n_f}, u_1) and (v_{n_f-1}, v_1). See Fig. 8(c).

The number of vertices of the mixed plane triangulation G' resulting from the augmentation is $O(n^2)$. Namely, the number of vertices inserted inside a face f of G^* with n_f vertices is $(n_f - 1) + (n_f - 2) + \cdots + 3$, hence the number of vertices of G' is $\sum_f (n_f(n_f - 1)/2 - 3) = O(n^2)$, given that $\sum_f n_f \in O(n)$ (where the sums are over all the faces of G^*). Finally, the augmentation of G^* to G' can be easily performed in a time that is linear in the size of G', hence quadratic in the size of the input graph. □

Corollary 1 *The problem of testing the upward planarity of mixed plane graphs is polynomial-time solvable ($\mathcal{NP}$-hard) if and only if it is polynomial-time solvable (respectively, $\mathcal{NP}$-hard) for mixed plane triangulations.*

5 Upward Planarity Testing of Mixed Plane Triangulations

In this section we show how to test in linear time the upward planarity of two classes of mixed plane triangulations.

A *plane* 3*-tree* is a plane triangulation that can be constructed as follows. Denote by H_{abc} a plane 3-tree whose outer face is delimited by a cycle (a, b, c), with vertices a, b, and c in this clockwise order along the cycle. A cycle (a, b, c) is the only plane 3-tree H_{abc} with three vertices. Every plane 3-tree H_{abc} with $n > 3$ vertices can be constructed from three plane 3-trees H_{abd}, H_{bcd}, and H_{cad} by identifying the vertices incident to their outer faces with the same label. See Fig. 9(a).

Theorem 7 *The upward planarity of an n-vertex mixed plane* 3*-tree can be tested in $O(n)$ time.*

Consider a mixed plane 3-tree H_{uvw} with n vertices. We define a function $\text{UP}(xy, H_{abc})$ as follows. For each graph H_{abc} in the construction of H_{uvw} and for every distinct $x, y \in \{a, b, c\}$ we have that $\text{UP}(xy, H_{abc})$ is true if and only if there exists an upward planar orientation of H_{abc} in which cycle (a, b, c) has x as a source and y as a sink.

Observe that H_{uvw} is upward planar if and only if $\text{UP}(xy, H_{uvw})$ is true for some $x, y \in \{u, v, w\}$ with $x \neq y$. The necessity comes from the fact that, in

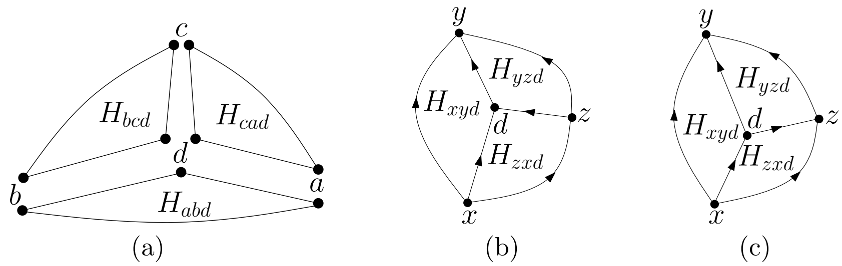

Figure 9: (a) Construction of a plane 3-tree H_{abc} with $n > 3$ vertices. (b)-(c) Distinct orientations of edge (z, d) in two upward planar orientations of H_{abc}.

every upward planar orientation of H_{uvw}, the cycle delimiting the outer face of H_{uvw} has exactly one source x and one sink y, by Theorem 4. The sufficiency is trivial.

We show how to compute the value of $\text{UP}(xy, H_{abc})$, for each graph H_{abc} recursively.

If $|H_{abc}| = 3$, then let $x, y, z \in \{a, b, c\}$ with $x \neq y$, $x \neq z$, and $y \neq z$. Then, $\text{UP}(xy, H_{abc})$ is true if and only if edges (x, y), (x, z), and (z, y) are not prescribed to be directed from y, directed from z, and directed from y, respectively. Hence, if $|H_{abc}| = 3$ the value of $\text{UP}(xy, H_{abc})$ can be computed in $O(1)$ time.

Second, if $|H_{abc}| > 3$, denote by H_{abd}, H_{bcd}, and H_{cad} the three graphs that compose H. We have the following:

Lemma 6 *For any distinct $x, y, z \in \{a, b, c\}$, $UP(xy, H_{abc})$ is true if and only if:*

(1) $UP(xy, H_{xyd})$, $UP(xd, H_{zxd})$, and $UP(zy, H_{yzd})$ are all true; or

(2) $UP(xy, H_{xyd})$, $UP(xz, H_{zxd})$, and $UP(dy, H_{yzd})$ are all true.

Proof: For the necessity, assume that H_{abc} has an upward planar orientation $\vec{H}_{abc}$. Then, by Theorem 4, cycle (a, b, c) contains exactly one source-switch and one sink-switch for the outer face in this orientation. Denote the source-switch and the sink-switch for the outer face by x and y, respectively. Again by Theorem 4, we have that x and y are a source and a sink for $\vec{H}_{abc}$. Hence, edges (x, d) and (d, y) are directed from x and directed to y in $\vec{H}_{abc}$, respectively. On the other hand, edge (z, d) might be directed from or to z. Refer to Figs. 9(b) and 9(c), respectively. In the first case, H_{xyd}, H_{yzd}, H_{zxd} admit upward planar orientations with x and y, with z and y, and with x and d as a source and sink, respectively, namely $\vec{H}_{abc}$ restricted to H_{xyd}, H_{yzd}, H_{zxd} provides us with such orientations. Hence, $\text{UP}(\overrightarrow{xy}, H_{xyd})$, $\text{UP}(\overrightarrow{zy}, H_{yzd})$, and $\text{UP}(\overrightarrow{xd}, H_{zxd})$ are all true. In the second case, H_{xyd}, H_{yzd}, and H_{zxd} admit upward planar orientations with x and y, with d and y, and with x and z as a source and sink, respectively, namely $\vec{H}_{abc}$ restricted to H_{xyd}, H_{yzd}, and H_{zxd} provides such orientations. Hence, $\mathrm{UP}(\overrightarrow{xy}, H_{xyd})$, $\mathrm{UP}(\overrightarrow{dy}, H_{yzd})$, and $\mathrm{UP}(\overrightarrow{xz}, H_{zxd})$ are all true.

For the sufficiency, consider the case in which $\mathrm{UP}(\overrightarrow{xy}, H_{xyd})$, $\mathrm{UP}(\overrightarrow{xd}, H_{zxd})$, and $\mathrm{UP}(\overrightarrow{zy}, H_{yzd})$ are all true, the other case being analogous. Then, there exist upward planar orientations $\vec{H}_{xyd}$, $\vec{H}_{zxd}$, and $\vec{H}_{yzd}$ of H_{xyd}, H_{zxd}, and H_{yzd} in which the outer face of $\vec{H}_{xyd}$ has x and y as a source and sink, in which the outer face of $\vec{H}_{zxd}$ has x and d as a source and sink, and in which the outer face of $\vec{H}_{yzd}$ has z and y as a source and sink, respectively. Orientations $\vec{H}_{xyd}$, $\vec{H}_{zxd}$, and $\vec{H}_{yzd}$ coincide on the common edges, hence altogether they yield an orientation $\vec{H}_{xyz}$ of H_{xyz}. We confirm that $\vec{H}_{xyz}$ is upward planar (and hence that $\mathrm{UP}(\overrightarrow{xy}, H_{xyz})$ is true) using Theorem 4. Namely, $\vec{H}_{xyz}$ has a single source and a single sink, namely x and y respectively, that are incident to the outer face of $\vec{H}_{xyz}$. Also, suppose for a contradiction that $\vec{H}_{xyz}$ has a directed cycle C. Assume without loss of generality that C is minimal, i.e., no directed cycle C' exists whose vertices are a subset of the vertices of C. Since $\vec{H}_{xyd}$, $\vec{H}_{zxd}$, and $\vec{H}_{yzd}$ are acyclic and since the orientation of the subgraph of H_{xyz} induced by x, y, z, and d is acyclic in $\vec{H}_{xyz}$, it follows that C passes through an internal vertex of one of H_{xyd}, H_{zxd}, and H_{yzd}, say H_{xyd}. Then C contains a path internal to H_{xyd} and connecting two vertices out of x, y, and d, say x and d. However, either such a path is directed from d to x, thus contradicting the acyclicity of $\vec{H}_{xyd}$, or it is directed from x to d, hence it can be replaced by edge (x, d), thus contradicting the minimality of C. □

For each graph H_{abc} in the construction of H_{uvw} and for any distinct $x, y \in \{a, b, c\}$, the conditions in Lemma 6 can be computed in $O(1)$ time by dynamic programming. Thus, the running time of the algorithm is $O(n)$. This concludes the proof of Theorem 7.

We now deal with mixed plane triangulations with no cycle of undirected edges.

Theorem 8 *The upward planarity of an n-vertex mixed plane triangulation in which the undirected edges induce a forest can be tested in $O(n)$ time.*

Proof: Let G be an n-vertex mixed plane triangulation. Let F be the set of undirected edges of G. We assume that F contains no external edge of G. Indeed, F contains at most two external edges: We can guess the orientation of all the external edges in F, and for each of the four possibilities, independently, test the upward planarity for the mixed graph G in which only the internal edges of F are undirected.

We prove the statement by induction, primarily on the size of F and secondarily on the number of vertices of G.

If $|F| = 0$, then G is a directed plane triangulation and its upward planarity can be tested in linear time by checking whether G satisfies the conditions in Theorem 4.

If $|F| > 0$, consider a leaf v in the forest whose edge set is F and let $(v, w) \in F$ be the undirected edge incident to v. By the assumptions, (v, w) is an internal

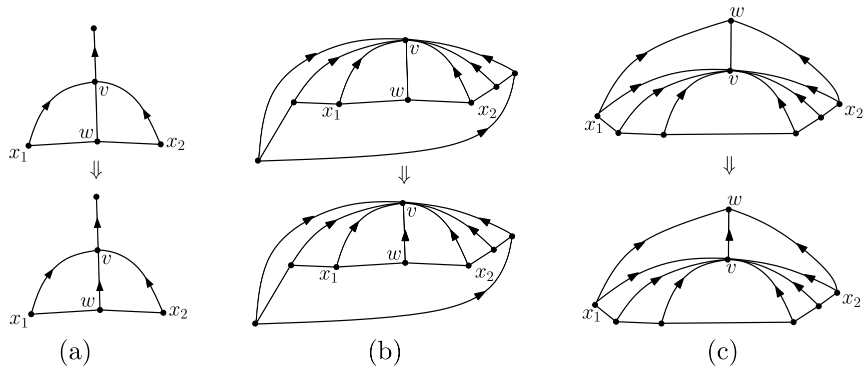

Figure 10: Deciding the orientation of edge (v, w) if edges (x_1, v) and (x_2, v) are directed to v. (a) v has an edge directed from it. (b) v has no edge directed from it and it is the sink of G. (c) v has no edge directed from it and it is not the sink of G.

edge of G. Let (v, w, x_1) and (v, w, x_2) be the internal faces of G incident to edge (v, w).

Suppose that both edges (x_1, v) and (x_2, v) are directed to v. We show that the orientation of edge (v, w) can be decided without loss of generality. If v has an edge directed from it, then by the bimodality condition in Theorem 3, edge (v, w) is directed to v in every upward planar orientation of G (see Fig. 10(a)). Suppose that v has no edge directed from it. If v is the sink of G (recall that the edges incident to the outer face of G are directed), then edge (v, w) is directed to v in every upward planar orientation of G, by the single sink condition in Theorem 3 (see Fig. 10(b)). Otherwise, edge (v, w) is directed from v in every upward planar orientation of G, again by the single sink condition in Theorem 3 (see Fig. 10(c)).

Analogously, if both (x_1, v) and (x_2, v) are directed from v, the orientation of edge (v, w) can be decided without loss of generality.

Assume, without loss of generality, that (x_1, v) and (x_2, v) are directed to v and directed from v, respectively. We distinguish between two cases.

Case 1: (x_1, x_2) is an edge of G. By the acyclicity condition in Theorem 3, edge (x_1, x_2) is directed from x_1 in every upward planar orientation of G.

If $\deg(v) = 3$ (Fig. 11(a)), then remove v and its incident edges from G, obtaining a mixed plane triangulation G' with one fewer undirected edge than G. Observe that the cycle delimiting the outer face of G' coincides with the cycle delimiting the outer face of G, hence such a cycle contains no undirected edge. Test recursively whether G' admits an upward planar orientation. If not, then G does not admit any upward planar orientation as well. If G' admits an upward planar orientation $\vec{G}'$, then construct an upward drawing Γ' of $\vec{G}'$; insert v in Γ' inside cycle (w, x_1, x_2), so that $y(v) > y(x_1)$, $y(v) < y(x_2)$, and $y(v) \neq y(w)$. Draw y-monotone curves connecting v with each of w, x_1, and x_2. The resulting drawing Γ of G provides us with an orientation $\vec{G}$ of G, which

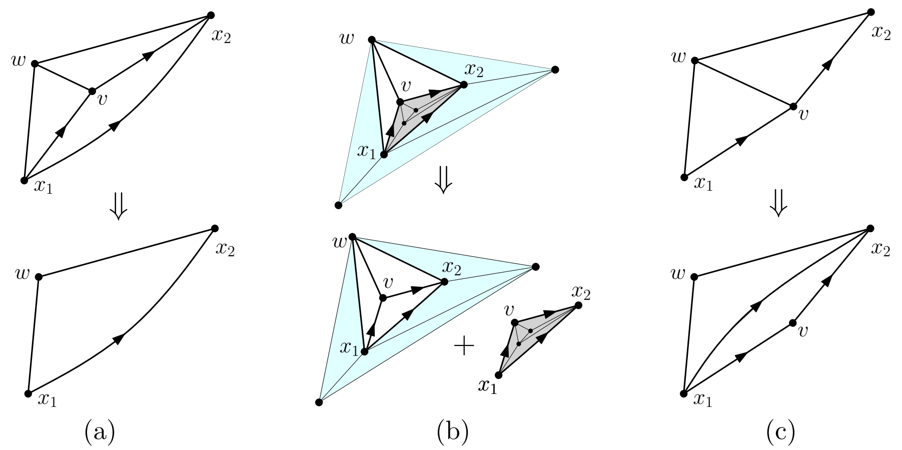

Figure 11: The induction step in the proof of Theorem 8. (a) (x_1, x_2) is an edge and $\deg(v) = 3$. (b) (x_1, x_2) is an edge and $\deg(v) > 3$. (c) (x_1, x_2) is not an edge.

is upward planar, given that it coincides with $\vec{G}'$ when restricted to G', given that edges (x_1, v) and (x_2, v) are drawn as y-monotone curves according to their orientations, and given that edge (v, w) is drawn as a y-monotone curve.

If $\deg(v) > 3$ (Fig. 11(b)), then (v, x_1, x_2) is a separating 3-cycle in G; namely, such a cycle contains w and any other neighbor of v different from w, x_1, and x_2 (such a neighbor exists because $\deg(v) > 3$) on different sides, given that (v, w, x_1) and (v, w, x_2) are internal faces of G. Denote by V' and V'' the non-empty sets of vertices in the interior and in the exterior of (v, x_1, x_2), respectively; also, denote by G' the subgraph of G induced by $V' \cup \{v, x_1, x_2\}$ and by G'' the subgraph of G induced by $V'' \cup \{v, x_1, x_2\}$. Observe that the cycle delimiting the outer face of G'' coincides with the cycle delimiting the outer face of G, hence such a cycle contains no undirected edge; further, the cycle delimiting the outer face of G' consists of edges (x_1, v), (x_2, v), and (x_1, x_2), which are all directed by assumption. Moreover, both G' and G'' have fewer vertices than G, given that (v, x_1, x_2) is a separating 3-cycle in G. Then, we test recursively the upward planarity for G' and for G''. If one of the tests fails, then G admits no upward planar orientation. Otherwise, upward planar orientations $\vec{G}'$ of G' and $\vec{G}''$ of G'' together provide an upward planar orientation $\vec{G}$ of G, given that each edge of (v, x_1, x_2) has the same orientation in $\vec{G}'$ and in $\vec{G}''$.

Case 2: (x_1, x_2) is not an edge of G (Fig. 11(c)). Remove (v, w) from G and insert a directed edge (x_1, x_2) directed from x_1 inside face (x_1, v, x_2, w). This results in a graph G' with one fewer undirected edge than G. Observe that the cycle delimiting the outer face of G' coincides with the cycle delimiting the outer face of G, hence such a cycle contains no undirected edge. We show that G is upward planar if and only if G' is.

Suppose that G admits an upward planar orientation $\vec{G}$. Let Γ be an upward planar drawing of $\vec{G}$. Remove edge (v, w) from G in Γ. Draw edge (x_1, x_2) inside cycle $C_f = (x_1, v, x_2, w)$, thus ensuring the planarity of the resulting drawing Γ' of G', following closely the drawing of path (x_1, v, x_2), thus ensuring the upwardness of Γ'.

Suppose that G' admits an upward planar orientation $\vec{G}'$. Let Γ' be an upward planar drawing of $\vec{G}'$. Remove (x_1, x_2) from Γ'. Since $\vec{G}'$ is acyclic, C_f has three possible orientations in $\vec{G}'$. In Orientation 1, w is its source and x_2 its sink; in Orientation 2, x_1 is its source and w its sink; finally, in Orientation 3, x_1 is its source and x_2 its sink. If C_f is oriented in $\vec{G}'$ as in Orientation 1 (as in Orientation 2), then draw edge (v, w) inside C_f in Γ', thus ensuring the planarity of the resulting drawing Γ of G, following closely the drawing of path (w, x_1, v) (resp., of path (v, x_2, w)), thus ensuring the upwardness of Γ. If C_f is oriented in $\vec{G}'$ as in Orientation 3, slightly perturb the position of the vertices in Γ' so that $y(v) \neq y(w)$. Draw edge (v, w) in Γ' as follows. Suppose that $y(v) < y(w)$, the other case being analogous. Draw a line segment inside C_f starting at v and slightly increasing in the y-direction, until reaching path (x_1, w, x_2). Then, follow such a path to reach w. This results in an upward drawing of edge (v, w) inside C_f, hence in an upward planar drawing of G.

Finally, the running time of the described algorithm is clearly $O(n)$. $\square$

6 Conclusions

We considered the problem of testing the upward planarity of mixed plane graphs.

We proved that upward planarity can be tested in cubic time for mixed outerplane graphs. It would be interesting to investigate whether our techniques can be strengthened to deal with larger classes of mixed plane graphs, e.g. series-parallel plane graphs. Also, since upward planarity can be tested in polynomial time for directed outerplanar graphs [15], it might be tested in polynomial time for mixed outerplanar graphs without a prescribed plane embedding as well.

We proved that the upward planarity testing problem for mixed plane graphs is polynomial-time solvable ($\mathcal{NP}$-hard) if and only if it is polynomial-time solvable (respectively, $\mathcal{NP}$-hard) for mixed plane triangulations (and showed two classes of mixed plane triangulations for which the problem can be solved efficiently). Further, we proved that a directed plane triangulation is upward planar if and only if it is acyclic and it has a single source and a single sink incident to the outer face. The combination of these results proves that the problem of testing the upward planarity of general mixed plane graphs is polynomial-time solvable ($\mathcal{NP}$-hard) if and only if the problem of deciding whether a mixed plane triangulation admits an acyclic orientation with a single source and single sink incident to the outer face is polynomial-time solvable (respectively, $\mathcal{NP}$-hard).

This might indicate that a polynomial-time algorithm for testing the upward planarity of mixed plane triangulations should be pursued. On the other hand, we remark that Patrignani [16] proved that testing the existence of an acyclic and bimodal orientation for a general mixed plane graph is $\mathcal{NP}$-hard.

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