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1-Bend Orthogonal Partial Edge Drawing

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Abstract

Recently, a new layout style to avoid edge crossings in straight-line drawings of non-planar graphs received attention. In a *Partial Edge Drawing (PED)*, the middle part of each segment representing an edge is dropped and the two remaining parts, called *stubs*, are not crossed. To help the user inferring the position of the two end-vertices of each edge, additional properties like *symmetry* and *homogeneity* are ensured in a PED. In this paper we explore this approach with respect to orthogonal drawings - a central concept in graph drawing. In particular, we focus on orthogonal drawings with one bend per edge, i.e., *1-bend drawings*, and we define a new model called *1-bend Orthogonal Partial Edge Drawing*, or simply *1-bend OPED*. Similarly to the straight-line case, we study those graphs that admit 1-bend OPEDs when homogeneity and symmetry are required, where these two properties are defined so to support readability and avoid ambiguities. According to this new model, we show that every graph that admits a 1-bend drawing also admits a 1-bend OPED as well as 1-bend homogeneous orthogonal PED, i.e., a *1-bend HOPED*. Furthermore, we prove that all graphs with maximum degree 3 admit a 1-bend symmetric and homogeneous orthogonal PED, i.e., a *1-bend SHOPED*. Concerning graphs with maximum degree 4, we prove that the 2-circulant graphs that admit a 1-bend drawing also admit a 1-bend SHOPED, while there is a graph with maximum degree 4 that does not admit such a representation.

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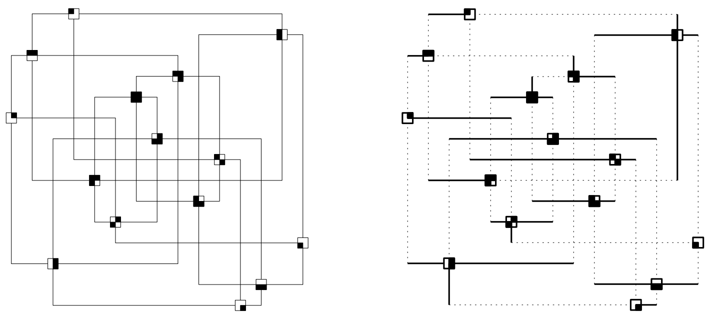
1 Introduction

In the field of graph drawing, one of the main objectives for visualization of graphs is to avoid visual clutter for readability. To achieve this, one aspect is to avoid crossing edges, which seriously affect the comprehensibility of a graph drawing [21, 25]. However, avoiding all crossings is not always possible and a lot of research was focused on minimizing the number of crossings in a drawing [5, 7, 18].



(a) (b)

Figure 1: (a) A 1-bend drawing of the 4-dim. cube where one vertex has been removed, taken from [14]. (b) A 1-bend OPED drawing of the same graph. Omitted segments are drawn by thin dotted lines.

Other papers study which non-planar graphs can be drawn such that the complexity of the edge crossings in the drawing is controlled. In the case of *k-planar drawings*, each edge is crossed at most k times (see, e.g., [9, 11, 16, 20]), while in *k-quasi planar drawings*, no k pairwise crossing edges exist (see, e.g., [1, 8, 23]), and finally in *large angle crossing drawings*, any two crossing edges form a large angle (refer to [12]). Besides this theoretical research, many proposals come from information visualization, like edge bundling (see [26] for a survey) or confluent drawings (see, e.g., [10, 17]).

Previous Work. Recently a new layout style was investigated to avoid crossings in a *straight-line drawing* of a non-planar graph, the resulting drawings were called *Partial Edge Drawings (PEDs)* [3, 4]. The idea of PEDs is to subdivide each segment representing an edge into 3 parts and to drop the middle part, relying on the remaining parts called *stubs*. The beginning of PEDs is dated in 1995, when Becker et al. [2] used stubs the first time to focus on visualizing networked data and not the network itself. They wanted to extract information about overloaded connections between switches and drew the width of edges relative to the severeness of overload, producing a drawing where the underlying network was barely visible. The solution was to draw just a fraction (roughly 10%) of each edge, transforming the drawing into a PED. Later Burch et al. [6] investigated the usefulness of PEDs in a user study regarding directed graphs, observing that certain tasks were answered faster and with less errors when the graphs were drawn as PEDs compared to the straight-line model. A similar approach was re-invented by Rusu et al. [22] by just breaking the edges at their crossing points, inspired by the principles of Gestalt [19], that people intuitively try to complete gaps in drawings.

The first formal definition of the PED model (for straight-line drawings) is given in [4], where different types of PEDs are considered. In its easiest form, a PED only requires the stubs to be crossing free. A PED is *homogeneous (HPED)*, if the ratio of the stub length over the edge length is the same over all the edges, and it is *symmetric (SPED)*, if the two stubs of each edge have the same length. In [3], Bruckdorfer et al. concentrated on SHPEDs, where the length of each stub is a quarter of the total edge length, and proved the existence of SHPEDs for some complete bipartite graphs, k-circulant graphs and graphs with bandwidth k, as well as the non-existence of a SHPED for complete graphs with more than 196 vertices.

Contribution. In this paper we extend PEDs to orthogonal drawings in two dimensions with exactly one bend per edge, called *1-bend drawings*. Indeed, orthogonal drawing is a central concept in graph drawing (see [7, 18] as a reference), and we see as a natural question to ask how the techniques and results for PEDs carry over to orthogonal drawings. Also, restricting to 1-bend drawings is the first step before extending the model to orthogonal drawings with more than one bend per edge. In a 1-bend drawing vertices are represented as points with integer coordinates and edges as chains of two orthogonal (axis-aligned) segments, hence, each vertex can have degree at most 4, ($\Delta_G \leq 4$). A characterization of the graphs that can be drawn orthogonally with one bend per edge has been recently presented by Felsner, Kaufmann and Valtr in [14], where they adopted the *general position model*, i.e., no two points can share a coordinate. Indeed, these drawings are also bend minimal, since each edge is represented exactly by one segment parallel to each coordinate axis. In this paper we also adopt the general position model.

Observe that we cannot directly extend the SHPED model from straight-line drawings to 1-bend drawings, since bent edges might lead to ambiguous interpretations. In particular, it would be unclear how long one should follow a vertical stub or a horizontal stub to reach the bend, see Figure 2. Therefore, we introduce the new model *1-bend Orthogonal Partial Edge Drawing*, or simply *1-bend OPED*, where for every edge in a 1-bend drawing we erase the longer segment (we actually draw it as a thin dotted segment). Figure 1(a) illustrates a 1-bend drawing of the 4-dimensional hypercube, while Figure 1(b) illustrates a 1-bend OPED of the same graph. Similarly as for straight-line PEDs, we define two further properties, *homogeneity* and *symmetry*. Formal definitions for *1-bend homogeneous orthogonal PEDs* (*1-bend HOPED*) and for *1-bend symmetric and homogeneous orthogonal PEDs* (*1-bend SHOPED*) are given in Section 2.

The remainder of the paper is organized as follows. Preliminaries and formal definitions are given in Section 2. We present the main results of the paper in

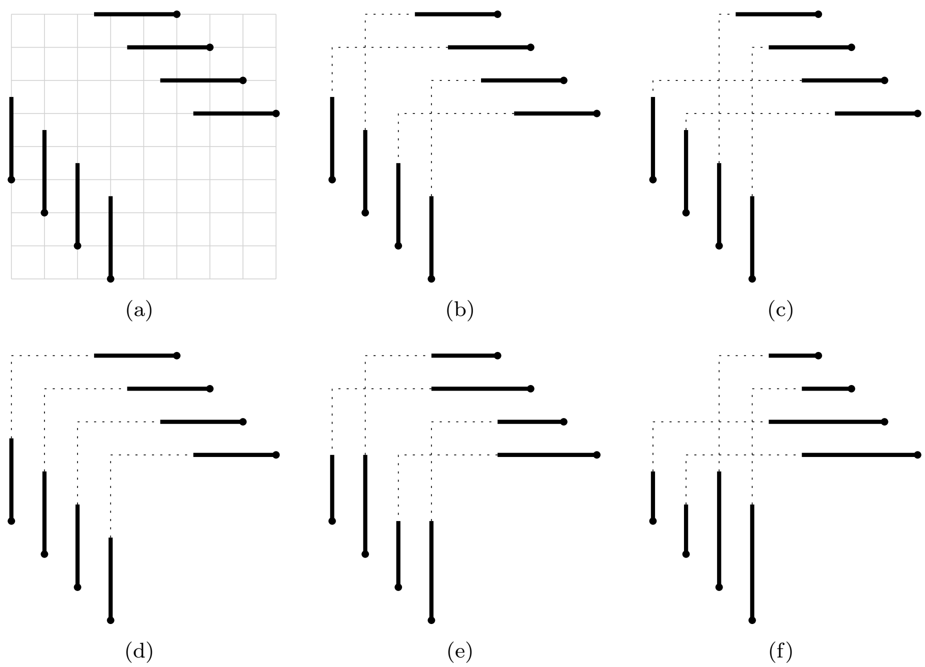


Figure 2: (a) A 1-bend drawing of a graph with four edges when applying the SHPED model of straight-line drawings, i.e. half of the edge is removed in such a way that a quarter of the edge remains incident to an end-vertex and a quarter to the other end-vertex. It is ambiguous which vertices are connected, since (b), (c) and (d) are different graphs that can be extracted from (a) according to the SHPED model of straight-line drawings. Such an ambiguity can be resolved by using the new definitions for 1-bend SHOPEDs introduced in Section 2. Indeed, (d), (e) and (f) are unique 1-bend SHOPEDs of the graphs (d), (b) and (c), resp.

Section 3 to Section 5. In particular, in Section 3 we show that every graph that admits a 1-bend drawing also admits a 1-bend OPED as well as a 1-bend HOPED. Furthermore, in Section 4 we prove that all graphs with maximum degree 3 admit a 1-bend SHOPED. In Section 5 we consider graphs with maximum degree 4 and prove that the 2-circulant graphs that admit a 1-bend drawing also admit a 1-bend SHOPED, while there is a graph with maximum degree 4 that does not admit a 1-bend SHOPED. Conclusions and future work are discussed in Section 6.

2 Preliminaries and Definitions

Felsner et al. [14] showed that any graph $G = (V, E)$ with $\Delta_G \leq 4$ has a 1-bend drawing if and only if $|E(S)| \leq 2|S| - 2$ for all $S \subset V$. The necessity

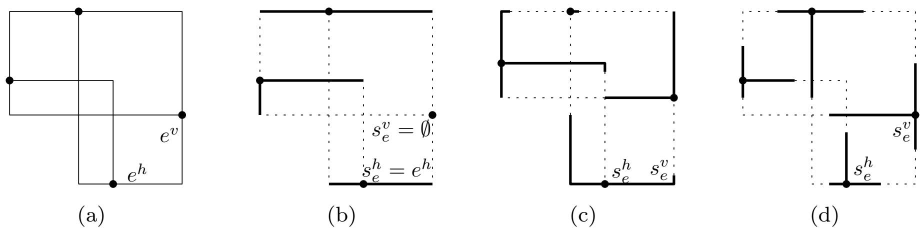


Figure 3: (a) K_4 has no planar 1-bend drawing, (b) but admits a 1-bend OPED, (c) a 1-bend HOPED and also (d) a 1-bend SHOPED. Observe that in (b) and (c) the bend point is part of the stubs, while in (d) it is not.

of this density condition for each induced subgraph of G comes from the clear observation that in any 1-bend drawing there must be a topmost vertex without any vertex towards the top, a bottommost vertex without any vertex towards the bottom, and similarly for the left and right directions. Notice that, 4-regular graphs cannot guarantee such a condition, hence, Felsner et al. [14] placed one vertex at a point at ∞. Since we consider only the Euclidean plane, we will remove this vertex from the graph.

Let Γ be a 1-bend drawing of a graph G. The *bounding box* R of Γ is the minimum axis-aligned rectangle containing the drawing. Let v be a vertex of G. We denote the four possible anchor points for an edge incident to v by *north*, *south*, *west* and *east ports* of v. Also, we denote the x- and y-coordinates of v in Γ by $x(v)$ and $y(v)$, respectively. Let e be an edge of G, the segments parallel to the x-axis and to the y-axis in the chain of segments representing e in Γ are called *horizontal* and *vertical segments*, respectively, and denoted by e^h and e^v, respectively. The length of a segment s is denoted by $|s|$.

Let $s_e^v \subseteq e^v$, respectively $s_e^h \subseteq e^h$ be subsegments (the *stubs* of e), so that $(e^v \cup e^h) - (s_e^v \cup s_e^h)$ is connected. In what follows we introduce three variations of our model for orthogonal partial edge drawings. Each variation is defined as an orthogonal 1-bend drawing (short: *1-bend drawing*) in general position, where for every edge e, the two segments e^v and e^h are replaced by their stubs s_e^v and s_e^h (defined above) and no two stubs cross. Furthermore, instead of completely erasing the non-drawn parts, we draw them by thin dotted segments, which help the user to follow the edge correctly. This is just a visualization tool to support the reader, since the graph extracted from the drawing is already unique.

We define a *1-bend Orthogonal Partial Edge Drawing*, or simply *1-bend OPED*, as a 1-bend drawing in general position where for every edge e, we remove the longer segment and the remaining (shorter) segments do not cross. That is, if $|e^h| > |e^v|$, then we have $s_e^h = \emptyset$ and $s_e^v = e^v$, and otherwise in case of $|e^h| \leq |e^v|$, we have $s_e^h = e^h$ and $s_e^v = \emptyset$. Since we adopt the general position model, each vertical (horizontal) stub uniquely determines a horizontal (vertical) line on which we can find its end-vertex.

An illustration of the 1-bend OPED is given in Figure 3(b).

A *1-bend HOPED* is a 1-bend drawing in general position where half of each edge is dropped, while the shorter segment is always entirely drawn. More precisely $|s_e^v|+|s_e^h| = (|e^v|+|e^h|)/2$ is true for every edge e of G and, in addition, we always draw the shorter segment of an edge completely (as in the 1-bend OPED) and draw on the remaining segment (starting from the bend point or from the end-vertex arbitrarily) as long as we need to reach half of the total edge length (see Figure 3(c)). Therefore, the two stubs may be continuous, forming a unique bent stub.

A *1-bend SHOPED* is a 1-bend drawing in general position where we symmetrically remove half of the horizontal segment and half of the vertical segment for each edge, i.e. $2|s_e^v| = |e^v|$ and $2|s_e^h| = |e^h|$ for every edge e of G. The dropped parts of e^v and e^h are connected by definition, i.e., they meet at the bend point and the stubs s_e^v and s_e^h are incident to the end vertices of e. An illustration of the 1-bend SHOPED is given in Figure 3(d).

3 1-bend OPEDs and 1-bend HOPEDs

First we consider 1-bend OPEDs and begin with a simple observation. Every graph that admits a 1-bend drawing also admits a 1-bend OPED: it is sufficient to drop all the horizontal (vertical) segments of a 1-bend drawing after stretching the drawing horizontally (vertically) by a factor equal to the length of the longest vertical (horizontal) segment. If homogeneity is required, the technique we use is slightly more difficult and it is presented in the following.

In the remainder of this section we consider 1-bend HOPEDs. We prove that every graph that admits a 1-bend drawing also admits a 1-bend HOPED. Namely, let G be a graph and Γ be a 1-bend drawing of G produced by the algorithm described in [14], we modify the x-coordinates of the vertices of Γ so that for each edge e of G, e^v is always shorter than e^h, and therefore $s_e^v = e^v$. Then, we draw s_e^h always from right to left, as much as we need to reach half of the edge length. Observe that, s_e^v and s_e^h might be continuous, forming a unique bent stub. If necessary, we further modify the x-coordinates of the vertices of Γ so that each crossing involving s_e^h is repaired.

Theorem 1 *Every n-vertex graph G with $\Delta_G \leq 4$ that admits a 1-bend drawing also admits a 1-bend HOPED.*

Proof: We start by constructing a 1-bend drawing Γ of $G = (V, E)$ by using the technique in [14]. Recall that we adopt the general position model and consider the total ordering of the edges of G defined as follows: $e = (u, v) \prec e' = (w, z)$, if and only if $\max\{x(u), x(v)\} < \max\{x(w), x(z)\}$ and $v \neq z$. The only incomparable edges are those with common rightmost end-vertex. Then we break the ties as follows. Let $e = (u, v)$ and $e' = (w, v)$ be two edges with common rightmost end-vertex, $e \prec e'$, if and only if $y(u) < y(w)$. We scan the edges following the order $\prec$ described above. Namely, let $e = (u, v) \in E$ and assume v is the end-vertex of e placed at the point with largest x-coordinate. Consider the point p_v (if any) defined by the crossing involving e^h with largest

x-coordinate, denoted as x_{p_v}. Then, we shift v and all the vertices to the right of v in the following way. For all $w \in V$ with $x(w) \geq x(v)$:

$$\begin{aligned} x(w) &= x(w) + \max\{\delta, \delta'\}, (\text{iff} \max\{\delta, \delta'\} > 0) \text{ , where} \\ \delta &= x(u) - x(v) + |e^v| + 1, \\ \delta' &= 2x_{p_v} - x(u) - x(v) - |e^v| + 1, (\delta' = 0, \text{ if } p_v \text{ not exists}). \end{aligned}$$

Clearly $\delta > 0$ if the vertical segment is at least as long as the horizontal segment of an edge. Thus, the shift operation ensures that the vertical segment is always shorter than the horizontal one (i.e., the vertical segment will be drawn entirely). Similarly $\delta' > 0$ if the horizontal distance between u and p_v is at least as large as the horizontal distance between p_v and v plus the length of the vertical segment. In this case, drawing the vertical stub entirely and the horizontal stub from right to left as much as necessary to reach half of the total edge length would cross p_v. Hence, the shift operation ensures that all the crossings involving e^h lie on its non-drawn part. It is easy to see that each crossing has been considered exactly once (due to the total order of the edges) and it has been repaired, i.e., it lies on the non-drawn part of the involved horizontal segment. Also, each shift operation does not affect previously repaired crossings. □

Conversely, in Section 5 we show that, if symmetry is also required, constructing a 1-bend SHOPED is not always possible.

4 1-bend SHOPEDs for graphs with maximum degree 3

We prove that all graphs with maximum degree 3 admit 1-bend SHOPEDs. Namely, we first present an efficient technique to construct 1-bend drawings for biconnected graphs with maximum degree 3, which is of independent interest since it is easy to implement and it has a small time complexity compared to the technique in [14]. Then we show how to turn a 1-bend drawing constructed by this technique into a 1-bend SHOPED. In the end, we extend our results to connected graphs with maximum degree 3.

Lemma 1 *Let G be a biconnected n-vertex graph with $\Delta_G \leq 3$. We can construct a 1-bend drawing Γ of G in $O(n)$ time.*

Proof: Given two vertices of G, s and t connected by an edge in G, an *st-numbering* of G is a bijective function, $V \to \{1, \ldots, n\}$, such that s receives number 1 (i.e., $s = v_1$), t receives number n (i.e., $t = v_n$) and every other vertex, except for s and t, is adjacent to at least one lower-numbered and at least one higher-numbered vertex [13]. Let $\{v_1, \ldots, v_n\}$ be an st-numbering of G (with $s = v_1$ and $t = v_n$).

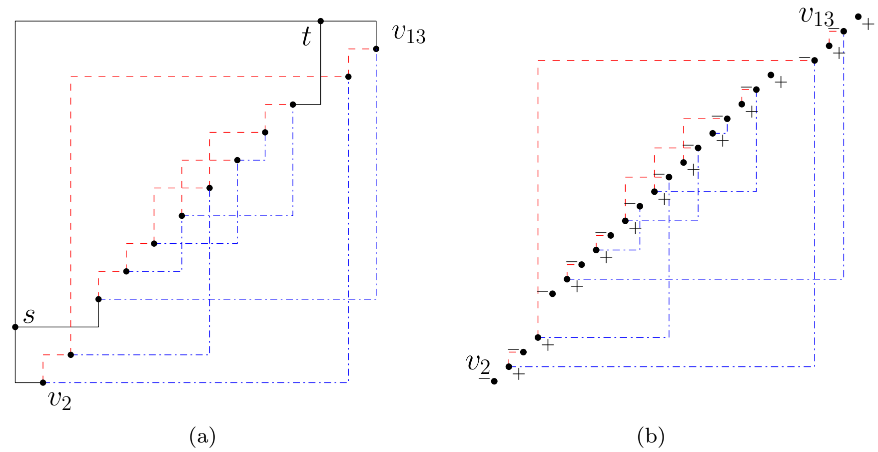


Figure 4: (a) A 1-bend drawing of a 14-vertex biconnected cubic graph G described in Construction 1 and 2. (b) The graph G^* taken from (a) already colored as described in Construction 1 (red edges are represented by dashed segments, while blue edges by dashed-dotted segments).

We first construct a 1-bend drawing Γ' of the subgraph $G' = (V' = V \setminus \{s,t\}, E(V'))$ and then add s and t in a proper way.

Construction 1.

We assign to vertex v_i the coordinates $x(v_i) = i$ and $y(v_i) = i$, $i = 2, \ldots, n-1$. We orient the edges $e = (v_i, v_j)$ so that e goes from v_i to v_j, when $2 \leq i < j \leq n-1$. Notice that there are only two possible shapes for the edges in Γ'. Namely, let $e = (v_i, v_j)$, $2 \leq i < j \leq n-1$, be a directed edge, it can either leave the east port of v_i and enter the south port of v_j or leave the north port of v_i and enter the west port of v_j. In the first case we call e a *blue edge*, while in the second case we say that e is a *red edge*. We denote by E_B (respectively E_R) the set of blue (respectively red) edges. Due to the properties of an st-numbering, each vertex has at most two incoming edges and at most two outgoing edges. Furthermore, v_2 has only one incoming edge from s and v_{n-1} has only one outgoing edge to t. We want to find a 2-coloring of the edges $E(V') = E_R \cup E_B$ such that the following two properties hold:

1. $\forall v \in V'$, the (at most) two incoming edges receive different colors.
2. $\forall v \in V'$, the (at most) two outgoing edges receive different colors.

To this aim, we construct the following undirected graph G^* from G'. For each vertex v in G' there will be two vertices v^- and v^+ in G^*. For each edge $e = (w, z)$ in G' (oriented from w to z) there will be an edge $e^* = (w^+, z^-)$

in G^*. See Figure 4(b) for an illustration. Thus, G^* is a (possibly not connected) bipartite graph (clearly there are no cycles with odd length) with maximum degree 2. Thus, each connected component is either a path or a cycle. It follows that the edges of G^* can be colored with two colors in a straightforward way. Namely, let C be a component of G^* with m_C edges, we define a total ordering of the edges of C, i.e., $e_1 \prec e_2 \prec \cdots \prec e_{m_C}$. If C is a path such an ordering is directly defined by the order of its edges along the path (rooted at an arbitrary end-vertex). If C is a cycle, we simply choose an arbitrary edge to be the first one (e_1) and remove it from C, the rest of the order is defined by the remaining path. Finally, we color the edges as follows, e_i receives color $c_{e_i} = i \mod 2, i = 1, \ldots, m_C$. Since there is a clear one-to-one mapping between edges in G' and edges in G^*, we can directly color the edges of G' with the colors assigned in G^*. Let e' be an edge of G' and let e^* be the related edge in G^*, we assign to e' the red color if $c_{e^*} = 0$ and the blue color if $c_{e^*} = 1$.

We prove now that such a coloring of the edges of G' respects the properties 1 and 2 defined above. Let e' and e'' be two incoming (outgoing) edges with respect to the same vertex v. By construction they will belong to the same component C of G^*. If C is a path, then e' and e'' will always appear consecutive in any possible order of the edges of C, thus, they will be assigned different colors. If C is a cycle, they may not be consecutive only in such an order where e' is the first (last) one and e'' is the last (first) one. In this case, since m_C is even, they will again receive two different colors.

Γ' is now defined and we only need to place s and t to construct Γ.

Construction 2. (adding s and t)

We recall that s and t are connected by an edge, v_2 has coordinates $(2, 2)$ and just one incoming edge from s, as well as v_{n-1} has coordinates $(n-1, n-1)$ and just one outgoing edge to t. Hence, they can be easily connected to s and t, respectively, without causing crossings. Let v_i, $2 < i < n$, be a possible third vertex connected to s and let v_j, $1 < j < n-1$, be a possible third vertex connected to t. We can skip the following consideration for v_i (respectively v_j), if s (respectively t) has degree 2. We consider the two free ports of v_i and we can assume they are two consecutive ports. If not, we can just toggle the color of one of the two edges incident to v_i to match this situation (the colors of the edges in the same component of this edge in G^* must be toggled accordingly). Thus, either the north or the south port of v_i is free, as well as either the west or the east port of v_i is free. We choose which port of v_i to use after considering the free ports of v_j. Consider the two free ports of v_j. Either one between the east or the west port is free or, if both are occupied, then both the north and the south port will be free. In total we have 4 possible cases.

1. If the east port of v_j is free, then we set $x(t) = x(v_j) + 0.5$ and $y(t) = n$. While if the west port of v_j is free, $x(t) = x(v_j) - 0.5$ and $y(t) = n$. Also, we can always assign to s the x-coordinate $x(s) = 1$.

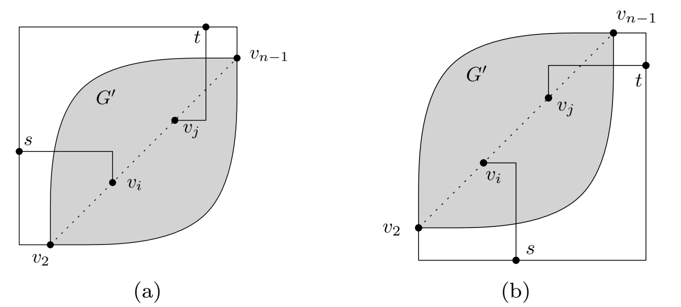


Figure 5: (a) A 1-bend drawing according to the case 1a of the proof of Lemma 1. (b) A 1-bend drawing according to the case 2a of the proof of Lemma 1.

(a) If the north port of v_i is free, then we set $y(s)$=$y(v_i)$ + 0.5, see Figure 5(a).

(b) If the south port of v_i is free, then we set $y(s) = y(v_i) - 0.5$.

2. If the north port of v_j is free, then we set $x(t) = n$ and $y(t) = y(v_j) + 0.5$. While if the south port of v_j is free, $x(t) = n$ and $y(t) = y(v_j) - 0.5$. Also, we can always assign to s the y-coordinate $y(s) = 1$.

(a) If the east port of v_i is free, then we set $x(s)$=$x(v_i)$ + 0.5, see Figure 5(b).

(b) If the west port of v_i is free, then we set $x(s) = x(v_i) - 0.5$.

Notice that, before adding s and t, the vertices were placed in general position, thus there could not be overlaps among edges and vertices. After adding s and t, this property is still maintained due to the introduced fractional coordinates (the grid unit must be halved to get integer coordinates). An example of a 1-bend drawing constructed with this technique is presented in Figure 4(a).

Finally, we observe that constructing an st-numbering of G takes $O(n+m)$ time [13], as well as placing vertices (including s and t), constructing G^* and coloring its edges. Thus, since $m \leq 1.5n$, the algorithm runs in $O(n)$ time. □

Theorem 2 *Every biconnected n-vertex graph G with $\Delta_G \leq 3$ admits a 1-bend SHOPED. Furthermore, such a drawing can be constructed in $O(n)$ time.*

Proof: Let Γ be a 1-bend drawing of G constructed by Construction 1 and Construction 2. We adopt the notation used in the proof of Lemma 1. Consider again the subgraph $G' = (V' = V \setminus \{s,t\}, E(V'))$ and the induced subdrawing Γ'. In Γ' a red edge can be crossed only by red edges, as well as a blue edge can be crossed only by blue edges. Indeed, red edges are all above the diagonal formed by the vertices, while blue edges are all below this diagonal. If a crossing is caused by two red edges, it can be repaired by shifting the rightmost endpoint

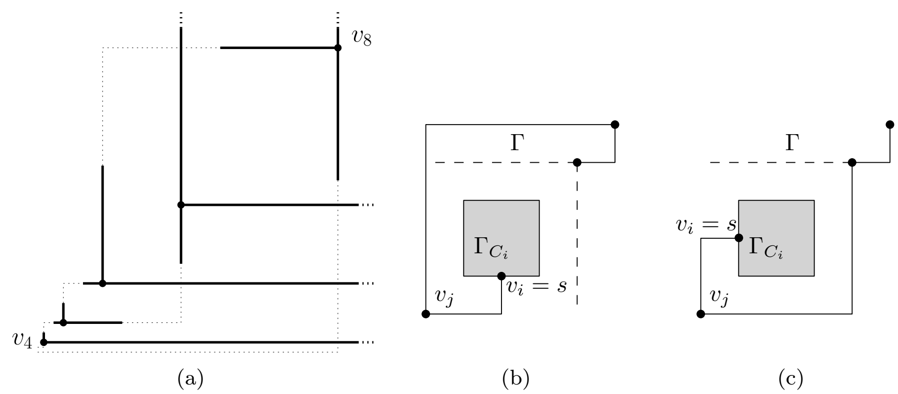


Figure 6: (a) A 1-bend SHOPED constructed from the drawing in Figure 4(a) (for the sake of readability only part of the drawing is shown). (b)-(c) Illustration of the technique described in the proof of Lemma 2 to attach the drawing Γ_{C_i} to Γ when the east port (b) or the north port (c) of v_j is free.

of the horizontal segment involved in the crossing, so that such a crossing will lie (in a SHOPED) on the non-drawn part of this horizontal segment. In a similar way, if the crossing is caused by two blue edges, it can be repaired by shifting the topmost endpoint of the vertical segment involved in the crossing, so that such a crossing will lie (in a SHOPED) on the non-drawn part of this vertical segment. We will repair the crossings by assigning the vertices new coordinates.

Repair 3.
We assign new coordinates to the vertices v_2 to v_{n-1} in the following way: Let v_i, $2 \leq i \leq n-1 : (x(v_i), y(v_i)) = (2^i, 2^i)$ are the new coordinates. After that s, t are again placed according to Construction 2. To prove that all crossings are repaired by the assignment of the new coordinates, we consider the vertical segment of the edge between v_i and $v_j, j < i$. The length of the stub on this segment is

$$\frac{y(v_i) - y(v_j)}{2} \leq \frac{y(v_i) - y(v_2)}{2} = y(v_{i-1}) - 2 = y(v_i) - y(v_{i-1}) - 2.$$

Thus the horizontal segments incident to the vertices $v_j, j \leq i-1$ are never crossed by a vertical stub. To prove that vertical segments are never crossed by horizontal stubs we use the same argument for the x-coordinates.

Finally, we need to repair the crossings caused by the outgoing edges of s and by the incoming edges of t. We observe that edges (s,t), (s, v_2),(v_{n-1}, t) are not crossed due to the placement of s, t, v_2, v_{n-1} on the bounding box of the drawing, see also Figure 5(a) and Figure 5(b). Thus, only the crossings affecting (s, v_i), $2 < i < n$, and (v_j, t), $1 < j < n-1$, must be repaired.

Repair 4.

In case 1 of Construction 2, s is always placed 0.5 grid units above or below v_i, thus the vertical segment of the edge (s, v_i) cannot be crossed. In order to fix the crossings on the horizontal segment of (s, v_i) it is enough to shift s on the left, i.e., $x(s) = x(s) - (x(v_i) - x(v_2)) - 1$. Similarly in case 2 of Construction 2, s is always placed 0.5 grid units to the left or to the right of v_i, thus the horizontal segment of the edge (s, v_i) cannot be crossed. In order to fix the crossings on the vertical segment of (s, v_i) it is enough to shift s to the bottom, i.e., $y(s) = y(s) - (y(v_i) - y(v_2)) - 1$. A symmetric argument can be applied to fix the crossings on the edge (v_j, t).

A 1-bend SHOPED constructed from the 1-bend drawing in Figure 4(a) is shown in Figure 6(a). □

Next we explain how to extend the previous result to any connected graph G with $\Delta_G \leq 3$. Recall that a *cut vertex* is a vertex whose removal disconnects G, while a *bridge* is an edge whose removal disconnects G. We observe for a graph G with $\Delta_G \leq 3$ that cut vertices are absent in G, if and only if bridges are absent in G. Indeed using the fact that the graphs have maximum degree at most 3, any bridge is incident to at least one cut vertex (and vice versa), justifying the observation.

Lemma 2 *Let G be a connected n-vertex graph with $\Delta_G \leq 3$. We can construct a 1-bend drawing Γ of G in $O(n^2)$ time.*

Proof: We start by removing all the bridges of G, obtaining a set of k biconnected components $\mathcal{C} = \{C_1, \ldots, C_k\}$, where each component $C_i \in \mathcal{C}$, $1 \leq i \leq k$, is either a single vertex or a graph such that $\Delta_{C_i} \leq 3$. Next, we define a graph $\mathcal{T}$ having one vertex n_i for each component C_i of G and an edge (n_i, n_j), iff C_i and C_j are connected by a bridge in G. Clearly $\mathcal{T}$ is a tree, since a cycle in $\mathcal{T}$ would imply a biconnected component comprised by the cycle, inferring a contradiction to the decomposition. Also there is at least one vertex n_r in $\mathcal{T}$ that represents a component C_r, which is either a single vertex or a biconnected graph having two adjacent vertices not incident to any bridge. We choose n_r to be the root of $\mathcal{T}$. In the following we describe an algorithm that takes as input G and $\mathcal{T}$ and computes a 1-bend drawing Γ of G. We assume that a 1-bend drawing Γ_{C_i} of a component $C_i \in \mathcal{C}$, $1 \leq i \leq k$, is constructed as follows:

(a) If C_i is a biconnected graph with $\Delta_{C_i} \leq 3$, then Γ_{C_i} is always constructed by Construction 1 and 2, where the two poles of the st-numbering will be defined by this construction.

(b) If C_i is composed by a single vertex v, then Γ_{C_i} is defined by placing v in the origin and the definition of the two poles can be ignored.

We visit $\mathcal{T}$ from the root n_r following a breadth first search (bfs) order as follows.

Construction 5.

Root n_r: Consider C_r, we construct a drawing Γ_{C_r} of C_r, where in case (a) (i.e., C_r is a biconnected graph with $\Delta_{C_r} \leq 3$) the *st*-numbering is defined so that s and t are two adjacent vertices in C_r that are not incident to any bridge. Then we set $\Gamma = \Gamma_{C_r}$.

Node $n_i, i \neq r$: Assume n_i is the next vertex of $\mathcal{T}$ according to the bfs order. Let Γ be the drawing constructed so far. We first compute a 1-bend drawing Γ_{C_i} of C_i, where $s = v_i$ and t is an adjacent vertex of s if C_i is a biconnected graph with $\Delta_{C_r} \leq 3$ (case (a)). If C_i is composed by a single vertex v, then $v = v_i$ (case (b)). Drawing Γ_{C_i} is now attached to drawing Γ as follows.

Attachment: Let n_j be the parent of n_i in $\mathcal{T}$ and let (v_j, v_i) be the bridge in G that corresponds to the edge (n_j, n_i) in $\mathcal{T}$. Consider the vertex v_j of C_j. If C_j is a biconnected graph with $\Delta_{C_j} \leq 3$, then v_j cannot be the s pole in Γ_{C_j}. Furthermore, in this case, since the degree of v_j is 2 in C_j, v_j has either the east port or the north port free in Γ. If C_j consists of a single vertex $v = v_j$, then again v_j has either the east port or the north port free in Γ. We use this free port of v_j to place the bridge $e = (v_j, v_i)$ and connect Γ_{C_i} to Γ. In the first case we rotate Γ_{C_i} such that v_i is the southernmost vertex and it can be connected by its south port, see Figure 6(b), while in the second case we rotate Γ_{C_i} such that v_i is the westernmost vertex and it can be connected by its west port, see Figure 6(c).

Before attaching the drawing Γ_{C_i}, we modify the current drawing Γ in the following way. We assign new x-coordinates to all vertices v with $x(v) > x(v_j)$ by setting $x(v) = x(v)+|C_i|+1$, and we assign new y-coordinates to all vertices v with $y(v) > y(v_j)$ by setting $y(v) = y(v) + |C_i| + 1$. Now we place Γ_{C_i} in this free area, i.e. for each vertex $v_{C_i} \in C_i$, $x(v_{C_i}) = x(v_j)+x_{\Gamma_{C_i}}(v_{C_i})$, and $y(v_{C_i}) = y(v_j) + y_{\Gamma_{C_i}}(v_{C_i})$, where $x_{\Gamma_{C_i}}(v_{C_i})$ and $y_{\Gamma_{C_i}}(v_{C_i})$ are the x- and y-coordinates of v_{C_i} in Γ_{C_i}, respectively. Then we connect v_j with v_i. Notice that, by placing Γ_{C_i} in this free area, no edges of $\Gamma \setminus \Gamma_{C_i}$ can cross edges of Γ_{C_i}.

Finally, we observe that finding the bridges of G can be done in linear time [24]. Furthermore, constructing a 1-bend drawing Γ_{C_i} for each component $C_i \in \mathcal{C}$ takes $O(|C_i|)$ time, thus, constructing Γ takes $\sum_{i=1}^{k} O(|C_i|) = O(n)$ time. However, the time complexity of the technique is dominated by the shift operation required to add the drawing of each component to the current drawing, which takes $O(n^2)$. □

By iteratively applying Repair 3 and 4 to drawings constructed by Construction 5 we can prove the next theorem.

Theorem 3 *Every n-vertex graph G with $\Delta_G \leq 3$ admits a 1-bend SHOPED. Furthermore, such a drawing can be constructed in $O(n^2)$ time.*

Proof: Let Γ be a 1-bend drawing of G constructed by Construction 5. We adopt the notation used in the proof of Lemma 2 and we traverse $\mathcal{T}$ in the reversed bfs order defined in that proof. Also, we assume that a 1-bend drawing Γ_{C_i} of a component $C_i \in \mathcal{C}$, $1 \leq i \leq k$, can be transformed into a 1-bend SHOPED as follows. If C_i is a biconnected graph with $\Delta_{C_i} \leq 3$, then Γ_{C_i} is repaired by Repair 3 and 4. If C_i is composed by a single vertex v, then Γ_{C_i} does not need to be transformed into a 1-bend SHOPED.

Let Γ_{C_i} be the 1-bend drawing of C_i. Let Γ_{C_j} be the 1-bend drawing of C_j, where n_j is the parent of n_i in $\mathcal{T}$. Recall that no edges of C_j can cross edges of C_i in Γ. First, we transform Γ_{C_i} into a 1-bend SHOPED. Then, let $s_i = \max\{width(\Gamma_{C_i}), height(\Gamma_{C_i})\}$, we assign new x-coordinates to all vertices v of Γ_{C_j} with $x(v) > x(v_j)$ by setting $x(v) = x(v)+s_i+1$, and new y-coordinates to all vertices v with $y(v) \geq y(v_j)$ by setting $y(v) = y(v) + s_i + 1$.

The time complexity is dominated by the construction of Γ, which takes $O(n^2)$ time, while repairing the components takes $\sum_{i=1}^{k} O(|C_i|) = O(n)$. □

5 1-bend SHOPEDs for graphs with maximum degree 4

We first present a class of graphs with maximum degree 4, the 2-circulant graphs, that admit a 1-bend SHOPED, and show afterwards that there is a graph that does not admit a 1-bend SHOPED.

Recall that the k-circulant graph C_n^k with $n > 2k$ vertices is the simple graph whose vertex set is $V = \{v_0, \ldots, v_{n-1}\}$ and whose edge set is $E = \{(v_i, v_j) : |j-i| \leq k\}$. The specified index of a vertex is calculated modulo n. Notice that, $\Delta_{C_n^k} = 4$ implies $k = 2$, hence, each vertex has exactly two neighbors with smaller indices and two neighbors with larger indices. Greater values of k are not realizable when $\Delta(C_n^k) = 4$. Extending the techniques in Section 4, we can prove the next theorem.

Theorem 4 *Every 2-circulant n-vertex graph that admit a 1-bend drawing also admits a 1-bend SHOPED. Furthermore, such a drawing can be constructed in $O(n)$ time.*

Proof: We start by observing that one vertex of a 2-circulant n-vertex graph, $v_t \in V$ $(0 \leq t \leq n-1)$, has to be removed in order to match the necessary density condition for 1-bend drawings, i.e. $|E(S)| \leq 2|S| - 2$ for all $S \subset V$. W.l.o.g., let $t = 0$, otherwise the vertices of C_n^2 can be easily renumbered so to match this condition.

We first construct a 1-bend drawing Γ' of $G' = (V' = V \setminus \{v_t{=}v_0, v_1, v_{n-1}\}, E(V'))$ adopting a similar strategy as in Construction 1. Namely, we assign to vertex $v_i \in V'$ coordinates $x(v_i) = i$ and $y(v_i) = i$, $2 \leq i \leq n-2$. Again, there can be only two possible shapes of edges in Γ' according to such a placement of the vertices. Indeed, let $e = (v_i, v_j)$, so that $j = i+1, i = 2, \ldots, n-3$, we call e a red edge, $e \in E_R \subset E(V')$, and we draw it so that it leaves the north port of v_i and enters the west port of v_j. While, if $e = (v_i, v_j)$, so that $j = i+2, i = 2, \ldots, n-4$, we call e a blue edge, $e \in E_B \subset E(V')$, and we draw it so that it leaves the east port of v_i and enters the south port of v_j. Thus, red edges are never crossed, while each blue edge receives at most two crossings (one involving the vertical segment and one involving the horizontal segment).

To construct a 1-bend drawing Γ of G, the addition of v_1 and v_{n-1} to Γ' can be managed by adopting the same strategy used to add s and t in Construction 2 Finally, we can apply Repair 3 and 4 (restricted to blue edges) to turn Γ into a 1-bend SHOPED. $\square$

Now, we show that there exists a graph with maximum degree 4 that admits a 1-bend drawing but not a 1-bend SHOPED. To this end, we need to introduce some additional notation. Consider a 1-bend SHOPED Γ. We say that two horizontal (vertical) stubs s^h_e, $s^h_{e'}$ (s^v_e, $s^v_{e'}$) *overlap*, if there exists a vertical (horizontal) line l such that $s^h_e \cap l \neq \emptyset$ and $s^h_{e'} \cap l \neq \emptyset$ ($s^v_e \cap l \neq \emptyset$ and $s^v_{e'} \cap l \neq \emptyset$). Two horizontal (vertical) stubs overlap by u units if there are two vertical (horizontal) lines l,l' with the above property and such that their horizontal (vertical) distance is u. Also, we call $\prec_x$ and $\prec_y$ the total vertex orderings induced by the projection of the x- and y-coordinates of the vertices in Γ, respectively (we remark that we adopt the general position model).

Recall from [14] (see also Section 2) that that any graph $G = (V, E)$ with $\Delta_G \leq 4$ has a 1-bend drawing if and only if $|E(S)| \leq 2|S|-2$ for all $S \subset V$. This necessary and sufficient condition is satisfied by any of the following three cases: (i) there are four vertices of degree at most three; (ii) there are two vertices of degree at most two; (iii) there are one vertex of degree at most two and two vertices of degree at most three. Since a 4-regular graph cannot guarantee such a condition, one vertex must be removed. Furthermore, if we are in case (i) each side of the bounding box of any possible 1-bend drawing contains one of the four vertices of degree three. In case (ii) the two vertices of degree two must be placed at the opposite corners of the bounding box. While in case (iii) the degree two vertex must be placed at one corner of the bounding box and the other two vertices of degree three must be placed on the two opposite sides. We call these vertices lying on the sides of the bounding box by *external vertices*. All the other vertices must lie in the interior of the bounding box and we refer to them as *internal vertices*.

Consider a graph $G' = (V', E')$ so that $V'=\{v_1, \ldots, v_6\}$ and $E'=\{(v_1, v_2), (v_2, v_3), (v_4, v_5), (v_5, v_6)\}$, see Figure 7(a). We prove that there exist two total vertex orderings $\prec_x, \prec_y$ such that in every 1-bend drawing of G' at least two stubs cross each other. Recall that $\prec_x$ and $\prec_y$ are orders, where the x- and y-order of the vertices coincide with these two orderings.

Lemma 3 *Graph G' does not admit a 1-bend SHOPED so that $v_1 \prec_x v_3 \prec_x v_4 \prec_x v_6 \prec_x v_5 \prec_x v_2$ and $v_5 \prec_y v_2 \prec_y v_1 \prec_y v_3 \prec_y v_4 \prec_y v_6$.*

Proof: Let Γ' be a 1-bend drawing of G' respecting the properties defined in the statement of this lemma (see Figure 7(a)), and consider the stubs (in the 1-bend SHOPED model) of the segments representing the edges of G' in Γ'. First, we observe that the two horizontal stubs $s^h_{v_1v_2}$ and $s^h_{v_2v_3}$ overlap, because

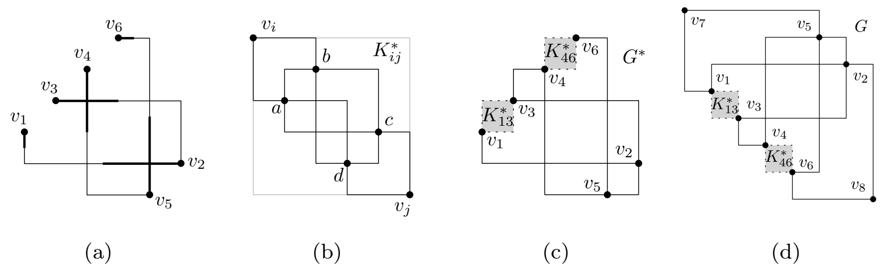


Figure 7: (a) A drawing Γ' of G' that cannot be redrawn as a 1-bend SHOPED if the vertical and horizontal order of the vertices cannot be changed. The relative order of each pair of vertices in G' can be constrained by the gadget in (b). (c) An illustration of the graph G^* as defined in the proof of Theorem 5. (d) A graph G that admits no 1-bend SHOPED.

of $v_1 \prec_x v_3 \prec_x v_2$, that is:

$$|s^h_{v_1v_2}| + |s^h_{v_2v_3}| = |s^h_{v_2v_3}| + |s^h_{v_2v_3}| + \frac{1}{2}[x(v_3) - x(v_1)]$$

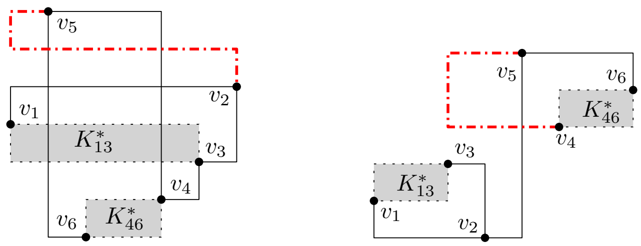
By definition $x(v_2) - x(v_3) = 2|s^h_{v_2v_3}|$, which implies:

$$|s^h_{v_1v_2}| + |s^h_{v_2v_3}| = x(v_2) - x(v_3) + \frac{1}{2}[x(v_3) - x(v_1)]$$

Since $x(v_3) - x(v_1) \geq 1$, the two stubs must overlap by at least half of a grid unit. In a similar way, the two vertical stubs $s^v_{v_4v_5}, s^v_{v_5v_6}$ overlap, because of $v_5 \prec_y v_4 \prec_y v_6$. Assume the stub $s^v_{v_5v_6}$ does not cross the stub $s^h_{v_1v_2}$. Due to the overlap between $s^v_{v_5v_6}$ and $s^v_{v_4v_5}$, and since $v_3 \prec_x v_4 \prec_x v_5$, it follows that $s^v_{v_4v_5}$ will cross either $s^h_{v_1v_2}$ or $s^h_{v_2v_3}$, or $s^v_{v_5v_6}$ will cross $s^h_{v_2v_3}$. □

In order to remove the assumption that the x-order and the y-order of the vertices cannot be changed, we augment G' adding new edges and vertices obtaining a new graph G, such that in every 1-bend drawing of G the x-order and the y-order of the vertices in V' is the same up to rotation. Let K_4 be the complete graph with 4 vertices. Let K_4' be a copy of K_4 and let a, b, c, d (in clockwise order traversing the bounding box) be its 4 vertices. We connect v_1 to a and b and v_3 to c and d. See also Figure 7(b). In a similar way we add one more copy of K_4 between v_4 and v_6. We call the subgraph induced by the pair v_1, v_3 and its copy of K_4 as K^*_{13}, analogously we call K^*_{46} the subgraph induced by the pair v_4, v_6 and its copy of K_4. Furthermore, we add the edge (v_3, v_4) and (v_2, v_5). Finally, we add the vertices v_7 and v_8 and the edges $(v_1, v_7), (v_5, v_7)$ and $(v_2, v_8), (v_6, v_8)$. The final graph G is shown in Figure 7(d), it clearly admits a 1-bend drawing.

Theorem 5 *There exists a graph that does not admit a 1-bend SHOPED.*



(a) (b)

Figure 8: An illustration of the case 1 (a) and 2 (b) in the proof of Theorem 5. The red dash dotted edge indicates an edge, which cannot be drawn with 1 bend.

Proof: Consider the graph G shown in Figure 7(d). We will evaluate all the possible configurations in terms of x-order and y-order for the vertices of G. For every feasible configuration we will then apply Lemma 3 to prove that there is no 1-bend SHOPED.

Let Γ be a 1-bend SHOPED of G. Consider the vertices v_7 and v_8, they must be the external vertices of Γ and they must be placed at the corners of one of the two diagonals of the bounding box R of Γ. Consider the bounding-box R^* of the subdrawing Γ^* induced by the graph $G^* = (V^* = V \setminus \{v_7, v_8\}, E(V^*))$ (see Figure 7(c)). Vertices v_1, v_2, v_5, v_6 must be the external vertices of Γ^* and they must lie one on each side of R^*. For the same reason, vertices v_3, v_4 are internal vertices (as well as all the vertices in the two copies of K_4) of Γ^*. Also, consider the bounding-box R_{13} of the subdrawing Γ_{13} induced by the graph K^*_{13}. Vertices v_1 and v_3 must be the external vertices of Γ_{13} and must be placed at the corners of one of the two diagonals of R_{13}. W.l.o.g., let v_7 be at the top-left corner of R. Hence, either v_1 is the westernmost vertex and v_5 is the northernmost vertex in Γ^* or vice versa. Assume to be in the former case, since the latter case can be proved with symmetric arguments. It follows that v_2 is either the southernmost or the easternmost vertex, while v_6 is either the easternmost or the southernmost vertex in Γ^*.

We start the analysis looking at the vertex v_1. We already know that $v_1 \prec_x v_3$ and consider two possible cases for the relative y-order of these two vertices.

1. Assume $v_3 \prec_y v_1$. In this case, since the only free port of v_1 is the north one, we have $v_1 \prec_y v_2$. This implies that v_2 cannot be the southernmost vertex, but it must be the easternmost vertex and v_6 the southernmost instead. Hence, the edge (v_2, v_3) must leave the south port of v_2 and enter the east port of v_3. It follows that the only free port of v_3 is the south one, and, due to the edge (v_3, v_4), $v_4 \prec_y v_3$. Now consider two further subcases, either $v_4 \prec_x v_3$ or $v_3 \prec_x v_4$. We prove by contradiction that $v_3 \prec_x v_4$.

(a) Let $v_4 \prec_x v_3$ (see Figure 8(a)), this implies that $v_6 \prec_x v_4$. Consider the edge (v_5, v_6). It must leave the west port of v_6 and enter the south port of v_5. Also, consider the edge (v_4, v_5), it must leave the north port of v_4 and enter the east port of v_5. Finally, consider the edge (v_2, v_5), again it must leave the north port of v_2 and enter the west port of v_5, thus, it must be bent at least three times, a contradiction.

(b) Let $v_3 \prec_x v_4$, thus $v_4 \prec_x v_6$. Consider the edge (v_5, v_6), it must leave the east port of v_6 and enter the south port of v_5, thus $v_6 \prec_x v_5$. Also, consider the edge (v_4, v_5), it must leave the north port of v_4 and enter the west port of v_5. Finally, consider the edge (v_2, v_5), it can be placed so that it leaves the east port of v_5 and enters the north port of v_2. Notice that, $v_6 \prec_y v_4 \prec_y v_3 \prec_y v_1 \prec_y v_2 \prec_y v_5$, as well as $v_1 \prec_x v_3 \prec_x v_4 \prec_x v_6 \prec_x v_5 \prec_x v_2$. Thus, by Lemma 3, there is no 1-bend SHOPED for the subdrawing Γ' induced by the subgraph G'.

2. Assume $v_1 \prec_y v_3$, as depicted in Figure 8(b). In this case, since the only free port of v_1 is the south one, we have $v_2 \prec_y v_1$. Now consider the edge (v_2, v_3), it must leave the north port of v_2 and enter the east port of v_3. Also, consider the edge (v_2, v_5), it must leave the east port of v_2 and enter the south port of v_5. This implies that v_2 is the southernmost vertex and v_6 is the easternmost vertex. Thus, consider the edge (v_5, v_6), it must leave the east port of v_5 and enter and north port of v_6. Now, since v_4 must lie on the opposite extreme of the diagonal of R_{46} with respect to v_6, we have that $v_4 \prec_y v_6$. Thus, since the edge (v_4, v_5) must use the west port of v_5 and the west or south port of v_4, it must have at least two bends, which implies that this configuration is not feasible and that $v_3 \prec_y v_1$.

□

6 Conclusion and Future Work

We defined a new layout style for orthogonal drawings with one bend per edge, called 1-bend Orthogonal Partial Edge Drawing, extending the already existent PED model for straight-line drawings. We studied those graphs that admit such a representation when homogeneity or both symmetry and homogeneity are required. In the former case, we proved that every graph that admits a 1-bend drawing also admits a 1-bend HOPED. In the latter case, we proved that all graphs with maximum degree 3 and the 2-circulant graphs that admit a 1-bend drawing, also admit a 1-bend SHOPED. Furthermore we proved that there is a graph with maximum degree 4 that does not admit a 1-bend SHOPED.

The complexity of the decision problem is still open, i.e., deciding whether a graph with maximum degree 4 admits a 1-bend SHOPED. We formalized a related question as an integer linear program (ILP), i.e. a test of a 1-bend drawing, whether it admits a 1-bend SHOPED inside a square of given size without changing the relative horizontal and vertical order of the vertices, but the complexity of this question is also still open. Also, it would be of interest to study 1-bend SHOPEDs where a few crossings among stubs are allowed, so to enlarge the family of graphs that admit these representations. To extend our model, one may consider graphs with degree greater than 4, by following similar approaches as in [15] for representing vertices, or orthogonal drawings with more than one bend per edge.

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