Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 17, no. 2, pp. 59–79 (2013)
DOI: 10.7155/jgaa.00285

Universal Line-Sets for Drawing Planar 3-Trees

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Abstract

A set S of lines is universal for drawing planar graphs with n vertices if every planar graph G with n vertices can be drawn on S such that each vertex of G is drawn as a point on a line of S and each edge is drawn as a straight-line segment without any edge crossing. It is known that $\lfloor \frac{2(n-1)}{3} \rfloor$ parallel lines are universal for any planar graph with n vertices. In this paper we show that a set of $\lfloor \frac{n+3}{2} \rfloor$ parallel lines or a set of $\lceil \frac{n+3}{4} \rceil$ concentric circles are universal for drawing planar 3-trees with n vertices. In both cases we give linear-time algorithms to find such drawings. A by-product of our algorithm is the generalization of the known bijection between plane 3-trees and rooted full ternary trees to the bijection between planar 3-trees and unrooted full ternary trees. We also identify some subclasses of planar 3-trees whose drawings are supported by fewer than $\lfloor \frac{n+3}{2} \rfloor$ parallel lines.

Submitted:	Reviewed:	Revised:	Accepted:	Final:
April 2012	August 2012	September 2012	December 2012	December 2012

Published:
January 2013

Article type: Regular paper
Communicated by: S.-i. Nakano

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1 Introduction

Many researchers in the graph drawing community have concentrated their attention on drawing graphs on point-sets [3, 8, 15] and on line-sets [7, 10, 13] due to strong theoretical and practical motivation for such drawings (e.g., computing small-width VLSI layout, approximating pathwidth and data visualization on small form factor). A set S of lines *supports* a drawing of a planar graph G if G has a planar drawing, where each vertex is drawn as a point on a line in S and each edge is drawn as a straight line segment. We say G *has a drawing on* S if S supports a drawing of G. A set of lines that supports the drawing of all n-vertex graphs in some class is called *universal* for that class. In this paper we study the problem of finding universal line sets of smaller size for planar graphs. Given a plane graph G with n vertices, Chrobak and Nakano [5] gave an algorithm to compute a drawing of G on a $\lfloor \frac{2(n-1)}{3} \rfloor \times 4 \lfloor \frac{2(n-1)}{3} \rfloor$ grid. This implies that $\lfloor \frac{2(n-1)}{3} \rfloor$ parallel lines are universal for any planar graph with n vertices. Note that a plane graph is a planar graph with a fixed planar embedding.

Recently, several researchers have studied a *labeled version* of the problem where both the lines in the point set S and vertices of G are labeled from 1 to n and each vertex is drawn on its associated line. Estrella-Balderrama *et al.* [10] showed that no set of n parallel lines supports all n-vertex planar graphs when each vertex is drawn as a point on its associated line. Dujmović *et al.* [7] showed that there exists a set of n lines in general position that does not support all n-vertex planar graphs. An *unlabeled version* of the problem has appeared in the literature as "layered drawing." A *layered drawing* of a plane graph G is a planar drawing of G, where the vertices are drawn on a set of horizontal lines called layers and the edges are drawn as straight line segments. Finding a layered drawing of a graph on the minimum number of layers is a challenging task. Dujmović *et al.* [9] gave a parametrized algorithm to check whether a given planar graph with n vertices admits a layered drawing on h layers or not. Mondal *et al.* [14] gave an $O(n^5)$-time algorithm to compute a layered drawing of a "plane 3-tree" G, where the number of layers is minimum over all possible layered drawings of G.

In this paper we consider the problem of finding a universal line set of smaller size for drawing "planar 3-trees." A *planar* 3*-tree* G_n with $n \geq 3$ vertices is a planar graph for which the following two conditions, (a) and (b) hold: (a) G_n is a maximal planar graph; (b) if $n > 3$, then G_n has a vertex whose deletion gives a planar 3-tree G_{n-1}. Many researchers have shown their interest on planar 3-trees for a long time for their beautiful combinatorial properties which have applications in computational geometry [1, 2, 6, 14, 18]. In this paper we show that a set of $\lfloor \frac{n+3}{2} \rfloor$ parallel lines and a set of $\lceil \frac{n+3}{4} \rceil$ concentric circles are universal for planar 3-trees with n vertices. In both cases we give linear-time algorithms to find such drawings. A by-product of our algorithm is the generalization of the known bijection between plane 3-trees and rooted ternary trees to the bijection between planar 3-trees and unrooted full ternary trees. We also identify some subclasses of planar 3-trees whose drawings are supported by

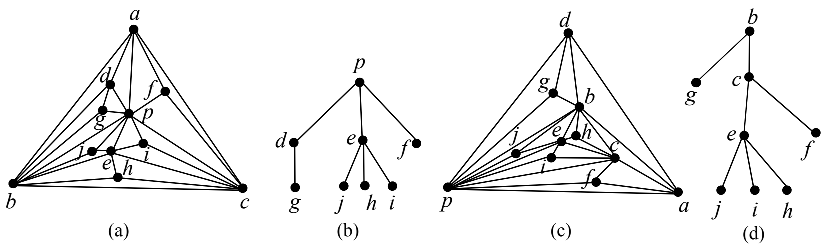

Figure 1: (a) A plane 3-tree G, (b) representative tree T of G, (c) another embedding G' of G and (d) representative tree T of G'.

fewer than $\lfloor\frac{n+3}{2}\rfloor$ parallel lines.

Let G be a plane 3-tree, i.e., a planar 3-tree with a fixed planar embedding. Clearly the outer face of G is a triangle, and let a, b and c be the three outer vertices of G. There is a vertex p in G, which is the common neighbor of a, b and c. The vertex p is called the *representative vertex* of G [14]. The vertex p along with the three outer vertices of G divides the interior region of G into three new regions. It is known that the subgraphs G_1, G_2 and G_3 enclosed by those three regions are also plane 3-trees [14]. G can be represented by a representative tree whose root is the representative vertex p of G and the subtrees rooted at the children of p are the representative trees of G_1, G_2 and G_3. Figure 1(b) illustrates the representative tree of the plane 3-tree in Figure 1(a). The *depth* ρ of a plane 3-tree is the number of vertices that lie on the longest path from the root to a leaf in its representative tree.

We now give an outline of our idea for drawing a planar 3-tree G on $\rho + 2$ parallel lines. One can observe that the depth of different embeddings of a planar 3-tree may differ. Figures 1(a) and (c) illustrate two different planar embeddings of the same planar 3-tree, with depths 3 and 4, respectively. We thus find an embedding of the planar 3-tree with the minimum depth ρ', and find a drawing on $\rho' + 2$ parallel lines. We show that ρ' is at most $\lfloor\frac{n-3}{2}\rfloor + 1$. Thus $\lfloor\frac{n+3}{2}\rfloor$ parallel lines support a drawing of a planar 3-tree with n vertices.

The rest of the paper is organized as follows. Section 2 describes some of the definitions that we have used in our paper. Section 3 deals with drawing plane 3-trees on parallel lines and concentric circles. In section 4 we obtain our bound on universal line set and universal circle set for planar 3-trees, and in Section 5 we consider drawings of some subclasses of planar 3-trees. Finally, Section 6 concludes our paper with discussions. A preliminary version of this paper has been presented at the 6th International Workshop on Algorithms and Computation (WALCOM 2012) [12].

2 Preliminaries

In this section we introduce some definitions and known properties of plane 3-trees. For the graph theoretic definitions not described here, see [17].

A graph is *planar* if it can be embedded in the plane without edge crossing except at the vertices where the edges are incident. A *plane graph* is a planar graph with a fixed planar embedding. A plane graph divides the plane into some connected regions called the *faces*. The unbounded region is called the *outer face* and all the other faces are called the *inner faces*. The vertices on the outer face are called the *outer vertices* and all the other vertices are called *inner vertices*. If all the faces of a plane graph G are triangles, then G is called a triangulated plane graph. We denote by $C_o(G)$ the contour outer face of G. For a cycle C in a plane graph G, we denote by $G(C)$ the plane subgraph of G inside C (including C). A *maximal planar graph* is one to which no edge can be added without losing planarity. Thus in any embedding of a maximal planar graph G with $n \geq 3$ vertices, the boundary of every face of G is a triangle, and hence an embedding of a maximal planar graph is often called a *triangulated plane graph*.

Let G be a plane 3-tree. By a triangle C_{xyz} of G we denote a cycle C of three vertices, where x, y, z are the vertices on the boundary of C in anticlockwise order. The following result is known on plane 3-trees [14].

Lemma 1 [14] *Let G be a plane 3-tree of $n \geq 3$ vertices and let C be any triangle of G. Then the subgraph $G(C)$ is a plane 3-tree.*

Let p be the representative vertex and a, b, c be the outer vertices of G in anticlockwise order. The vertex p, along with the three outer vertices a, b and c, form three triangles C_{abp}, C_{bcp} and C_{cap}. We call these triangles the *nested triangles around p*.

We now define the *representative tree* of a plane 3-tree G of $n > 3$ vertices as an ordered rooted tree T satisfying the following two conditions (a) and (b).

(a) if $n = 4$, then T is a single vertex, which is the representative vertex of G.

(b) if $n > 4$, then the root p of T is the representative vertex of G and the subtrees rooted at the three anticlockwise ordered children q_1, q_2 and q_3 of p in T are the representative trees of $G(C_1), G(C_2)$ and $G(C_3)$, respectively, where C_1, C_2 and C_3 are the nested triangles around p in anticlockwise order.

Figure 1(b) illustrates the representative tree T of the plane 3-tree G of Figure 1(a). We define the *depth* ρ of G as the number of vertices that lie on the longest path from the root to a leaf in its representative tree. The following lemma describes a property of a representative tree.

Lemma 2 ([14]) *Let G be a plane 3-tree and let T be its representative tree. Every vertex v in T corresponds to a unique cycle C of three vertices in G such that $G(C)$ is a plane 3-tree with representative vertex v. Moreover, the subtree rooted at v in T is the representative tree of $G(C)$.*

Let a, b and c be the three outer vertices of a plane 3-tree G. We denote by $\triangle abc$ the drawing of the outer face of G as a triangle. *A line or arc l crosses* *a triangle* $\triangle abc$ if there exists at least one point p on l in the proper interior of the triangle $\triangle abc$. *A line or arc l touches the triangle* $\triangle abc$ if it does not cross the triangle $\triangle abc$ and at least one point among a, b, c lies on l.

The *center* of a tree T is either a single node or an edge, which is obtained by repeatedly deleting all the nodes of degree one, until a single node or an edge is left. Let p and q be two vertices of T. By $d_T(p, q)$ we denote the *distance*, i.e., the length of the unique path, between p and q in T. Two trees T and T' are *isomorphic* if there exists a bijective mapping ϕ from the vertices of T to the vertices of T' such that two vertices u and v are adjacent in T if and only if $\phi(u)$ and $\phi(v)$ are adjacent in T'.

Given a plane graph G with n vertices, Chrobak and Nakano [5] gave an algorithm to compute a straight-line drawing of G on a $\lfloor\frac{2(n-1)}{3}\rfloor \times 4\lfloor\frac{2(n-1)}{3}\rfloor$ grid. We now observe some properties of their drawing algorithm. Let Γ be a triangulated plane graph with n vertices and let x, y be two arbitrary outer vertices of Γ in anticlockwise order. Let $\mathcal{D}$ be the drawing of Γ produced by the algorithm of Chrobak and Nakano [5]. Then $\mathcal{D}$ has the following properties.

(CN1) $\mathcal{D}$ is a drawing on a set of lines $l_0, l_1, \ldots, l_q$, where $q = \lfloor\frac{2(n-1)}{3}\rfloor$.

(CN2) Vertex x and vertex y lie on lines l_0 and l_q in $\mathcal{D}$, respectively. The remaining outer vertex lies on either line l_0 or l_q.

3 Drawings on Parallel Lines and Concentric Circles

In this section we prove that any plane 3-tree of depth ρ has a drawing on $\rho+2$ parallel lines. We first need the following lemma.

Lemma 3 *Let a, b, and c be the three vertices of the outer face $C_o(G)$ of a plane 3-tree G, and let v be the representative vertex of G. Let $\triangle abc$ be a drawing of $C_o(G)$ on a set of $k+2$ parallel lines, for some positive integer k, such that two of the vertices among a, b, c lie on the same or consecutive lines. Assume that k parallel lines $l_1, l_2, ..., l_k$ cross $\triangle abc$. Then there exists a line $l_x, 1 \le x \le k$ such that we can place vertex v on line l_x interior to $\triangle abc$, where at least $k-1$ parallel lines cross each of the triangles $\triangle abv$, $\triangle bcv$ and $\triangle acv$.*

Proof: Without loss of generality assume that a is a top-most and c is the bottom-most vertices in the $\triangle abc$, i.e., vertex a and c lie on the lines l_0 and l_{k+1}, respectively. We now consider the following four cases according to the positions of the vertex b.

Case 1: *Vertex b lies on the line l_{k+1}.*

In this case, vertices b and c lie on the same line l_{k+1}. If we place the representative vertex v on the line l_1 inside the $\triangle abc$, then $k, k-1$ and k lines cross the triangles $\triangle abv$, $\triangle bcv$ and $\triangle acv$, respectively.

Case 2: *Vertex b lies on the line l_0.*

In this case, vertices b and a lie on the same line l_0. If we draw v on the line l_k inside the $\triangle abc$, then $k-1, k$ and k lines cross the triangles $\triangle abv$, $\triangle bcv$ and $\triangle acv$, respectively.

Case 3: *Vertex b lies on the line l_1.*

In this case, vertices a and b lie on consecutive lines. If we draw v on the line l_k inside the $\triangle abc$, then $k-1, k-1$ and k lines cross the triangles $\triangle abv$, $\triangle bcv$ and $\triangle acv$, respectively.

Case 4: *Vertex b lies on the line l_k.*

In this case, vertices b and c lie on consecutive lines. If we draw v on the line l_1 inside the $\triangle abc$, then $k-1, k-1$ and k lines cross the triangles $\triangle abv$, $\triangle bcv$ and $\triangle acv$, respectively. □

We now have the following lemma.

Lemma 4 *Every plane 3-tree G with depth ρ has a drawing on $\rho + 2$ parallel lines.*

Proof: We prove a stronger claim as follows: Given a drawing $\mathcal{D}$ of the outer face of G on $\rho + 2$ lines such that two of its outer vertices lie on the same or consecutive lines, we can extend the given drawing to a drawing $\mathcal{D}'$ of G such that $\mathcal{D}'$ is also a drawing on $\rho + 2$ lines.

The case when $\rho = 0$ is straightforward, since in this case G is a triangle and any given drawing $\mathcal{D}$ of the outer face of G on two lines is itself a drawing of G. We may thus assume that $\rho > 0$ and the claim holds for any plane 3-tree of depth ρ', where $\rho' < \rho$.

Let G be a plane 3-tree of depth ρ and let a, b and c be the three outer vertices of G in anticlockwise order. Let p be the representative vertex of G. We draw $C_o(G)$ on $\rho + 2$ parallel lines by drawing the outer vertex a on Line l_0, and the other two outer vertices b and c on Line l_ρ or on Lines l_ρ and $l_{\rho+1}$, respectively. According to Lemma 3, there is a line $l_x, 1 \leq x \leq \rho + 1$ such that the placement of p on line l_x inside $\triangle abc$ ensures that the triangles $\triangle abp$, $\triangle acp$ and $\triangle cbp$ are crossed by at least $\rho - 1$ parallel lines.

We place p on l_x inside $\triangle abc$. By Lemma 1, $G(C_{abp}), G(C_{bcp})$ and $G(C_{cap})$ are plane 3-trees. Observe that the depth of each of these plane 3-trees is at most $\rho - 1$. By induction hypothesis, each of these plane 3-trees admits a drawing on $\rho + 1$ parallel lines inside the triangles $\triangle abp$, $\triangle bcp$ and $\triangle cap$, respectively. □

Based on the proof of Lemma 4, one can easily develop an $O(n)$-time algorithm for finding a drawing of a plane 3-tree G of n vertices on $\rho + 2$ parallel lines, where ρ is depth of G. Thus the following theorem holds.

Theorem 1 *Let G be a plane 3-tree of n vertices. Then one can find a drawing of G on $\rho + 2$ parallel lines in $O(n)$ time, where ρ is the depth of G.*

We now consider the problem of drawing a plane 3-tree on a concentric circle set. Since a set of $\rho+2$ parallel lines can be formed with $\lceil \frac{\rho+2}{2} \rceil$ infinite concentric circles, each of which contributes two parallel lines, every plane 3-tree admits a drawing on $\lceil \frac{\rho+2}{2} \rceil$ concentric circles. We can observe that Lemma 3 holds even if we consider a set $\mathcal{C}$ of non-crossing concentric circular arcs[1] of finite radii instead of a set of parallel lines, and hence we have the following corollary.

Corollary 1 *Let G be a plane 3-tree of depth ρ. Then G has a drawing on $\lceil \frac{\rho+2}{2} \rceil$ concentric circles. Furthermore, such a drawing can be found in linear-time.*

4 Universal Line Sets for Drawing Planar 3-Trees

In this section we give an algorithm to find an embedding of a planar 3-tree with minimum depth and prove the $\lfloor \frac{n+3}{2} \rfloor$ upper bound on the size of the universal line set for planar 3-trees. For any planar 3-tree the following fact holds.

Fact 1 *Let G be a planar 3-tree and let Γ and Γ' be two planar embeddings of G. Then any face in Γ is a face in Γ' and vice versa.*

We call a triangle, i.e., a cycle of three vertices, in a planar 3-tree G a *facial triangle* if it appears as a face boundary in a planar embedding of G.

Let G be a planar 3-tree of n vertices and let Γ be a planar embedding of G (i.e., Γ is a plane 3-tree). We now define a tree structure that contains the faces of Γ as its leaves. Later, we will prove that such tree structures that correspond to different planar embeddings of G are isomorphic, and consequently, we will be able to find a minimum depth embedding G examining only a single tree structure. A *face-representative tree* of Γ is an ordered rooted tree T_f that satisfies the following conditions.

(a) If $n = 3$, then T_f is a single face-node.

(b) If $n > 3$, then any vertex in T_f is either a *vertex-node*, which corresponds to a vertex of Γ or a *face-node*, which corresponds to a face of Γ. Moreover, the following (i)–(ii) hold.

   (i) The root is a face-node that corresponds to the outer face of Γ. Root has only one child which is the representative vertex p of Γ. Every vertex-node has exactly three children. Every face-node other than the root is a leaf in T_f.

   (ii) The subtrees rooted at the three anticlockwise ordered children q_1, q_2 and q_3 of p in T_f are the face-representative trees of $\Gamma(C_1), \Gamma(C_2)$ and $\Gamma(C_3)$, respectively, where C_1, C_2 and C_3 are the three nested triangles around p in anticlockwise order.

[1]Note that the circular arc segments in $\mathcal{C}$ can be partitioned into two (possibly empty) sets $\mathcal{C}_1$ and $\mathcal{C}_2$ such that two arcs c' and c'' are parallel if they belong to the same set and non-parallel otherwise. The crucial part of the algorithm for drawing G on $\mathcal{C}$ is to draw Δabc carefully.

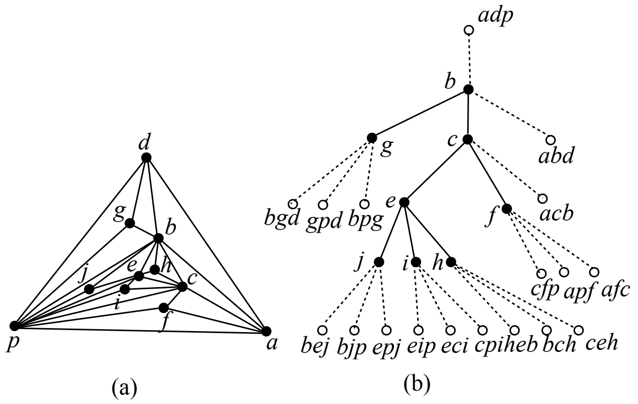

Figure 2: (a)A plane 3-tree Γ and (b) the face-representative tree T_f of Γ.

Figure 2 illustrates a face-representative tree of a plane 3-tree where black nodes are vertex-nodes and white nodes are face-nodes. Observe that every internal node in a face-representative tree has exactly four neighbors. We call such a tree *an unrooted full ternary tree.* A face-representative tree has $2n-4$ face-nodes and $n-3$ vertex-nodes. Deletion of the face-nodes from the face-representative tree yields the representative tree of Γ.

A rooted tree is *semi-labeled* if its internal vertices are unlabeled and the leaves are labeled. Two semi-labeled trees are *isomorphic at root*, if we can assign labels to the unlabeled nodes such that the trees become identical and the labels of the two roots are the same. It is easy to see that if two semi-labeled trees are isomorphic at root, then they are isomorphic. The unordered rooted tree obtained by deleting the labels of the internal nodes of a face-representative tree is a *semi-labeled face-representative tree.* Let T_1 and T_2 be two semi-labeled face representative trees of two different embeddings of a planar 3-tree G. If f is a facial triangle in G, then there is a face-node corresponding to f in T_1 and in T_2, by Fact 1. For convenience, we often denote each of these face-nodes as f.

We now prove that the face-representative trees obtained from different embeddings of a planar 3-tree are isomorphic. In fact, we have a stronger claim in the following lemma.

Lemma 5 *Let G be a planar 3-tree and let Γ', Γ'' be two different planar embeddings of G. Let f be a facial triangle in G, and let T' and T'' be the semi-labeled face-representative trees obtained from the face-representative trees of Γ' and Γ'', respectively, by choosing f as their roots. Then T' and T'' are isomorphic at f.*

Proof: We employ induction on the number of vertices n. The case when $n \leq 4$ is straightforward. We thus assume that $n > 4$ and the claim holds for all planar 3-trees of less than n vertices. Let the outer face of Γ' and Γ'' be C_{abc} and C_{xyz}, respectively. Let the representative vertex of Γ' be v. Then C_{xyz} is a face

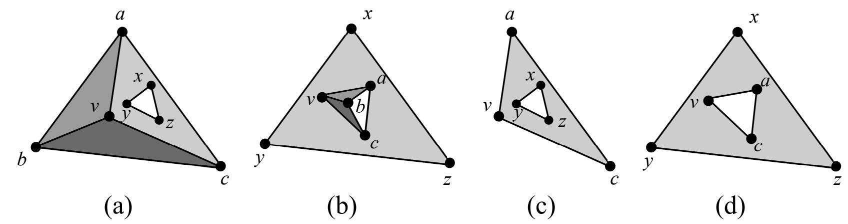

Figure 3: Illustration for the proof of Lemma 5; (a) Γ', (b) Γ'', (c) Γ'_1 and (d) Γ'_2.

(not necessarily an inner face) of $\Gamma'(C_{abv})$, $\Gamma'(C_{bcv})$ or $\Gamma'(C_{cav})$. Without loss of generality assume that C_{xyz} is in $\Gamma'(C_{cav})$. See Figures 3(a)–(b). Observe that the planar 3-trees that correspond to $\Gamma'(C_{bcv})$ and $\Gamma''(C_{cbv})$ are the same. Similarly, the planar 3-trees that correspond to $\Gamma'(C_{abv})$ and $\Gamma''(C_{bav})$ are the same.

Let Γ'_1 be the plane subgraph of Γ' obtained by removing the vertex b and the vertices interior to C_{abv} and C_{bcv}. Let Γ'_2 be the plane subgraph of Γ'' obtained by removing the vertex b and the vertices interior to C_{abv} and C_{bcv}. See Figures 3(c)–(d). Observe that the planar 3-trees that correspond to Γ'_1 and Γ'_2 are the same, which we denote as $\mathcal{G}$. Let T_1 and T_2 be the semi-labeled face-representative trees of Γ'_1 and Γ'_2, respectively. Let f be a facial triangle in $\mathcal{G}$, which is determined by the vertices a, c and v. By induction hypothesis, the semi-labeled face-representative trees, which are obtained from T_1 and T_2 by choosing f as their roots, are isomorphic at f. Similarly, the semi-labeled face-representative trees T'_{f_1} and T''_{f_1} of $\Gamma'(C_{bcv})$ and $\Gamma''(C_{cbv})$ rooted at f_1 are isomorphic at f_1, where f_1 is the face determined by the vertices b, c and v. The semi-labeled face-representative trees T'_{f_2} and T''_{f_2} of $\Gamma'(C_{abv})$ and $\Gamma''(C_{bav})$ rooted at f_2 are isomorphic at f_2, where f_2 is the face determined by the vertices a, b and v. Let T_{f_3} be the face-representative tree of a plane 3-tree of exactly three vertices. Assign the label "abc" to f_3.

We now connect a copy of T'_{f_1}, T'_{f_2} and T_{f_3} with T_1 by adding edges (f, f_1), (f, f_2) and (f, f_3). Remove the label of f and the label of $f_i, i \in \{1, 2\}$, if T'_{f_i} consists of at least two vertices. Let X be the resulting semi-labeled tree. It is now straightforward to observe that the two trees, which are obtained from X and T' by choosing f_3 as their roots, are isomorphic at f_3.

Similarly, we connect a copy of T''_{f_1}, T''_{f_2} and T_{f_3} to T_2 by adding edges (f, f_1), (f, f_2) and (f, f_3). We then remove the label of f and the label of $f_i, i \in \{1, 2\}$, if T''_{f_i} consists of at least two vertices. Let Y be the resulting semi-labeled tree. It is now straightforward to observe that the trees, which are obtained from Y and T'' by choosing f_3 as their roots, are isomorphic.

According to the construction, X and Y are isomorphic at f_3. Therefore, to complete the proof, we show that for any facial triangle f' in G, $f' \neq f_3$, the trees X' and Y' rooted at f', which are obtained respectively from X and Y, are isomorphic at f'. Suppose for a contradiction that X' is not isomorphic to Y' at f'. Since X and Y are isomorphic at f_3, the unlabeled vertices of X and Y can be labeled such that X and Y become identical. Such a labeling can determine an isomorphism for X' and Y' at f', a contradiction. □

Observe that if two semi-labeled face representative trees are isomorphic at their root, then they are isomorphic. Therefore, we have the following corollary.

Corollary 2 *Let G be a planar 3-tree and let Γ', Γ'' be two different planar embeddings of G. Let T' and T'' be the semi-labeled face representative trees of Γ' and Γ'', respectively. Then T' and T'' are isomorphic.*

Let G be a planar 3-tree of n vertices. Since the semi-labeled face-representative trees obtained from different planar embeddings of G are isomorphic, we can choose any leaf of a face-representative tree T_f to obtain another semi-labeled face-representative tree that corresponds to a different planar embedding of G. Observe that T_f has $2n-4$ face-nodes and let x be a face-node in T_f such that the depth of the tree T_x obtained from T_f by choosing x as the root is minimum over all the $2n-4$ possible choices for x. Recall that deletion of the face-nodes from the face-representative tree yields the representative tree of the corresponding embedding. Therefore, deletion of the face-nodes from T_x gives us a representative tree with minimum depth, which in turn corresponds to a minimum-depth embedding of G. The following fact states that x is the nearest face-node from the center of T_f.

Fact 2 *Let T_f be a face-representative tree and let x be a face-node of T_f such that the distance between x and the center of T_f is minimum over all the face nodes of T_f. Then the depth of the tree obtained from T_f by choosing a face-node as the root is greater than or equal to the depth of the tree obtained from T_f by choosing x as the root.*

Proof: First assume that center of T_f is an edge (u, v). Deletion of the edge (u, v) from T_f yields two connected components T_u and T_v that contain u and v, respectively. Let p and q be two vertices of some tree T. Then by $d_T(p, q)$, we denote the distance (i.e., length of the unique path) between p and q in T. Let $k = \min\{d_{T_f}(x, u), d_{T_f}(x, v)\}$. Let y be a leaf such that the distance between y and the center of T_f is maximum over all nodes in T_f. Let $k' = \min\{d_{T_f}(y, u), d_{T_f}(y, v)\}$. Without loss of generality assume that y is in T_u. Then there is a leaf y' in T_v such that $\min\{d_{T_f}(y', u), d_{T_f}(y', v)\} = k'$. Let D_x be the depth of the tree obtained from T_f by choosing x as the root. Since the center of T_f is an edge, $D_x = k + k' + 1$, which is independent of the position of x in T_i, $i \in \{u, v\}$.

Let z be a face-node in T_f, where $l = \min\{d_{T_f}(z, u), d_{T_f}(z, v)\}$ and $l > k$. If there is no such z, then we are done. Otherwise, suppose for a contradiction that the depth of the tree obtained from T_f by choosing z as the root is D_z, where $D_z < D_x$. Observe that $D_z = l + k' + 1$, which is independent of the position of z in T_i. Since $l > k$, therefore $D_z > D_x$, a contradiction.

Now assume that the center is a vertex v. Deletion of the vertex u from T_f yields four connected components T_1, T_2, T_3 and T_4. Let $k = d_{T_f}(x, v)$. Let y be a leaf such that the distance between y and the center of T_f is maximum over all nodes in T_f. Let $k' = d_{T_f}(y, v)$. Assume that y is in some $T_j, 1 \leq j \leq 4$. Then there is a leaf y' not in T_j such that $d_{T_f}(y', v) = k'$. Let the depth of the tree obtained from T_f by choosing x as the root be D_x. Observe that $D_x = k + k'$, which is independent of the position of x in T_j.

Let z be a face-node in T_f, where $l = d_{T_f}(z, v)$ and $l > k$. If there is no such z, then we are done. Otherwise, suppose for a contradiction that the depth of the tree obtained from T_f by choosing z as the root is D_z, where $D_z < D_x$. Observe that $D_z = l + k'$, which is independent of the position of z in T_j. Since $l > k$, therefore $D_z > D_x$, a contradiction. □

The center of a tree is either a single node or an edge, and it is straightforward to find the center of T_f in $O(n)$ time by repeatedly deleting the nodes of degree one, until a single node or an edge is left. We then do a breath-first search to select a nearest node x, which also takes $O(n)$ time. Then by Fact 2, the planar embedding of G that corresponds to the face-representative tree obtained by choosing x as the root is the minimum-depth embedding of G. Thus the following lemma holds.

Lemma 6 *Let G be a planar 3-tree. An embedding Γ of G with the minimum depth can be found in linear time.*

We now have the following lemma on the bound of minimum-depth.

Lemma 7 *The depth of a minimum-depth embedding Γ of a planar 3-tree G with n vertices is at most $\lfloor \frac{n-3}{2} \rfloor + 1$.*

Proof: Let T_x be the face-representative tree of Γ, where x is the root of T_x. Since Γ is a minimum-depth embedding, the length k of the unique path between x and the center of T_x is minimum over all the face nodes of T_x. Let $\mathcal{O}$ be the center of T_x, which may be a node or an edge of T_x. Every internal node of T_x has exactly four neighbors and x is a nearest node from $\mathcal{O}$. Let T be the representative tree of Γ. Note that T is obtained by deleting all face-nodes from T_x. T contains exactly $n-3$ vertices and the distance from the root of T to $\mathcal{O}$ is $k-1$. Since every internal node of T_x has exactly four neighbors and x is a nearest node from the center, the depth of the representative tree T obtained by deleting all the face-nodes from T_x is at most $\lfloor \frac{(n-3)}{2} \rfloor + 1$, when $k = 1$.

We thus assume that $k > 1$. Since every internal node of T_x is a node of degree four and no leaf node of T_x is within the distance $k-1$ from $\mathcal{O}$, every node of T within the distance $k-2$ from $\mathcal{O}$ is a node of degree four. The number of nodes in T which are within the distance $k-1$ from the center is $1 + 4 + 4 \cdot 3^1 + ... + 4 \cdot 3^{k-2} = 1 + 4(3^0 + 3^1 + ... + 3^{k-2}) = 2 \cdot 3^{k-1} - 1$. Figure 4 illustrates an example, where the nodes within the distance $k-1$ from the center lie in the shaded region. The number of nodes of T which are not counted within the distance $k-1$ is $n - 3 - 2 \cdot 3^{k-1} + 1$. Since $\mathcal{O}$ is on the middle of a longest

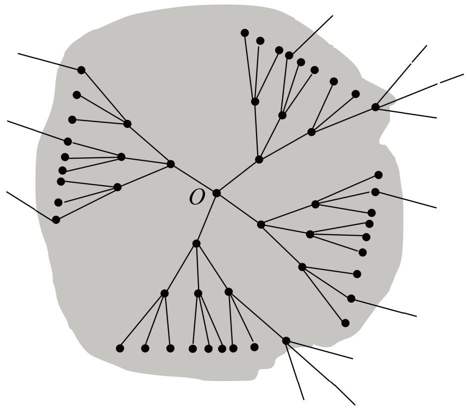

Figure 4: Illustration for the proof of Lemma 7, where $k = 4$.

path in T, these nodes can contribute at most $\lceil \frac{n-3-2\cdot 3^{k-1}+1}{2} \rceil$ to the depth of T. Since the root of T is at a distance $k-1$ from $\mathcal{O}$, the maximum depth of T is at most $\lceil \frac{n-3-2\cdot 3^{k-1}+1}{2} \rceil + 2(k-1) = \lceil (n-3)/2 + 1/2 \rceil - 3^{k-1} + 2(k-1)$ (when $\mathcal{O}$ is a vertex), or $\lceil (n-3)/2 + 1/2 \rceil - 3^{k-1} + 2(k-1) + 1$ (when $\mathcal{O}$ is an edge). Since $3^{k-1} \geq 2(k-1)+1$, the depth of T can be at most $\lfloor \frac{n-3}{2} \rfloor + 1$. □

We now use Theorem 1 and Corollary 1 to obtain the upper bounds on the sizes of universal line set and universal circle set for planar 3-trees, as in the following theorem.

Theorem 2 *A set of $\lfloor \frac{n+3}{2} \rfloor$ parallel lines and a set of $\lceil \frac{n+3}{4} \rceil$ concentric circles are universal for planar 3-trees with n vertices.*

5 Bounds for Special Classes of Planar 3-Trees

In this section we categorize planar 3-trees into three types: Type 0, Type 1 and Type 2. We prove that every planar 3-tree of Type 0 and Type 1 can be embedded on $\lceil \frac{(n+3)}{3} \rceil$ and $\lfloor 4n/9 \rfloor$ parallel lines, respectively. We conjecture that every planar 3-tree of Type 2 admits an embedding on $\lfloor 4n/9 \rfloor$ parallel lines.

Let T be a rooted tree with n vertices. Then there exists a vertex v in T such that the number of vertices in the subtree rooted at v is more than $2n/3$ and the number of vertices in each of the subtrees rooted at the children of v is at most $2n/3$ [16, Theorem 9.1]. If T is a representative tree of some plane 3-tree, then T must have such a vertex. Consequently, we can use Lemma 2 to prove the following lemma.

Lemma 8 *Let Γ be a plane 3-tree. Then there exists a triangle C in Γ satisfying the following. Let r be the representative vertex of $\Gamma(C)$ and let C_1, C_2, C_3 be the three nested triangles around r. Then the number of inner vertices in $\Gamma(C)$ is* *more than* $2(n-3)/3$ *and the number of inner vertices in each* $\Gamma(C_i), 1 \leq i \leq 3$, *is at most* $2(n-3)/3$.

We call C a *heavy triangle* of Γ. Observe that for any heavy triangle C of Γ, one of the following properties hold.

(a) No $\Gamma(C_i)$ contains more than $(n-3)/3$ inner vertices.

(b) The number of inner vertices in exactly one plane 3-tree among $\Gamma(C_1)$, $\Gamma(C_2)$ and $\Gamma(C_3)$ is more than $(n-3)/3$.

(c) The number of inner vertices in exactly two plane 3-trees among $\Gamma(C_1)$, $\Gamma(C_2)$ and $\Gamma(C_3)$ is more than $(n-3)/3$.

Let G be a planar 3-tree. If G admits a plane embedding that contains a heavy triangle satisfying Property (a), then we call G a *planar* 3*-tree of Type* 0. If G is not a planar 3-tree of Type 0, but admits a plane embedding that contains a heavy triangle satisfying Property (b), then we call G a *planar* 3*-tree of Type* 1. If G is not a planar 3-tree of Type 0 or Type 1, but admits a plane embedding that contains a heavy triangle satisfying Property (c), then we call G a *planar* 3*-tree of Type* 2. We now have the following lemma.

Lemma 9 *Given a planar* 3*-tree* G*, one can determine whether it is of Type* 0*, Type* 1 *or Type* 2 *in linear time.*

Proof: Let Γ be any plane embedding of G and let T be its semi-labeled face representative tree. Let v be any vertex-node in T. By N_v we denote the number of vertex-nodes in the subtree rooted at v and by $N(T)$ we denote the number of vertex-nodes in tree T. We now do a postorder traversal on T to find the number N_v for each vertex-node v in T. While traversing T, at each vertex-node v we perform a *Type Test*, as described below. Throughout the computation we maintain five variables $t_1, t_2, \ldots, t_5$ and the value stored in t_1 after the end of the computation indicates the type of G. Initially, we set $t_1 = 2$.

Type Test: Let P, Q, R and S be the subtrees obtained by deleting the vertex v from T. For each subset $\{I, J, K\} \subset \{P, Q, R, S\}$ of three subtrees, we do the following.

(A) If $N(I) + N(J) + N(K) > 2(n-3)/3$ and each of $N(I), N(J), N(K)$ is at most $(n-3)/3$, then we set $t = 0$ and stop the tree traversal.

(B) If $N(I)+N(J)+N(K) > 2(n-3)/3$ and exactly one of $N(I), N(J), N(K)$ exceeds $(n-3)/3$ and $t = 2$, then we set $t = 1$.

Observe that at each step of the traversal we perform only a constant number of tests and numbers $N(.)$ can be computed in constant time with the help of N_v values using the knowledge of the total number of vertex-nodes in T. Therefore, the traversal can be performed in linear time.

In the following we prove that the value stored in t corresponds to the type of G, and such a Type t embedding of G can be constructed in linear time.

Without loss of generality we assume that G is a Type 0 plane 3-tree. The proof for the case when G is a Type 1 or a Type 2 plane 3-tree is similar.

By definition, G has a planar embedding Γ' such that for some vertex v' of its face-representative tree, the condition of Test (A) holds. Observe that by Corollary 2, the face-representative trees obtained from different embeddings of G are isomorphic. Therefore, there exists a vertex v in T and three subtrees $\{I, J, K\} \subset \{P, Q, R, S\}$, such that $N(I)+N(J)+N(K) > 2(n-3)/3$ and each of $N(I), N(J), N(K)$ is at most $(n-3)/3$. Consequently, t must be 0.

We now compute a Type 0 embedding of G. Let $T'' = \{P, Q, R, S\} \setminus \{I, J, K\}$, and let f be a leaf of T''. We claim that the embedding of G taking f as the outer face gives us a Type 0 embedding of G. We distinguish two cases depending on whether v is an ancestor of f or not.

Consider first the case when v is an ancestor of f in T, as shown in Figure 5(a). Let w be the neighbor of v that belongs to T''. By Lemma 2, w corresponds to a unique cycle C_{xyz} of three vertices x, y, z such that $\Gamma(C_{xyz})$ is a plane 3-tree with representative vertex w. We delete all the inner vertices of $\Gamma(C_{xyz})$ from Γ. Let Γ' be the resulting embedding, as shown in Figures 5(b)–(c). Take another embedding Γ'' of Γ' with xyz as the outer face, as shown in Figure 5(d). It is now straightforward to observe that I, J, K correspond to the three nested triangles around the representative vertex of Γ'', and hence xyz is the required heavy triangle. We now extend Γ'' to a Type 0 embedding of G as follows. First take an embedding $\Gamma(C_{xyz})'$ of $\Gamma(C_{xyz})$ with f as the outer face. Then insert Γ'' into the face xyz of $\Gamma(C_{xyz})'$ such that the outer face of Γ'' coincide with the face xyz creating an embedding of G. Figure 5(e) illustrates such a scenario.

The case when f is a descendant of v in T is simpler. By Lemma 2, v corresponds to a unique cycle C_{xyz} of three vertices x, y, z such that $\Gamma(C_{xyz})$ is a plane 3-tree with representative vertex v. Observe that I, J, K correspond to the three nested triangles around v in $\Gamma(C_{xyz})$. Consequently, xyz is the required heavy triangle and Γ itself is a Type 0 embedding of G. $\square$

Before proving the upper bounds for planar 3-trees of Type 0 and Type 1, we need to explain some properties of drawings on line set and some properties of the drawing algorithm of Chrobak and Nakano [5].

Fact 3 *Let G be a plane 3-tree and let x, y, z be the outer vertices of G. Assume that G has a drawing $\mathcal{D}$ on k parallel lines, where x lies on line l_0, y lies on line l_{k-1} and z lies on line $l_i, 0 \le i \le k-1$.*

(a) Let p, q and r be three non-collinear points on lines l_0, l_{k-1} and l_i, respectively. Then G has a drawing $\mathcal{D}'$ on k parallel lines, where the vertices x, y, z lie on points p, q, r, respectively, and for each vertex u, if u lies on line l in $\mathcal{D}$ then u lies on line l in $\mathcal{D}'$. Moreover, $\mathcal{D}'$ respects the combinatorial planar embedding determined by $\mathcal{D}$.

(b) G has a drawing $\mathcal{D}''$ on $k+1$ parallel lines, where y lies on line l_k and for each vertex u of G other than y, if u lies on line l in $\mathcal{D}$ then u lies on

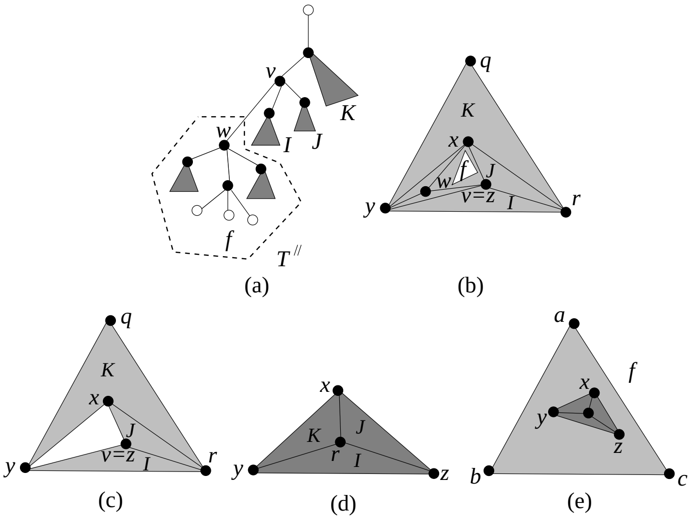

Figure 5: Illustration for the proof of Lemma 9. (a) T, (b) Γ, (c) Γ', (d) Γ'', and (e) a Type 0 embedding of G.

line l in $\mathcal{D}''$. Moreover, $\mathcal{D}''$ respects the combinatorial planar embedding determined by $\mathcal{D}$.

Fact 3 can be easily proved by induction in a fashion similar to the proof of Lemma 8 in [14]. Figure 6(a) illustrates a plane 3-tree Γ, and Figures 6(b), (c) and (d) illustrates examples of $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$.

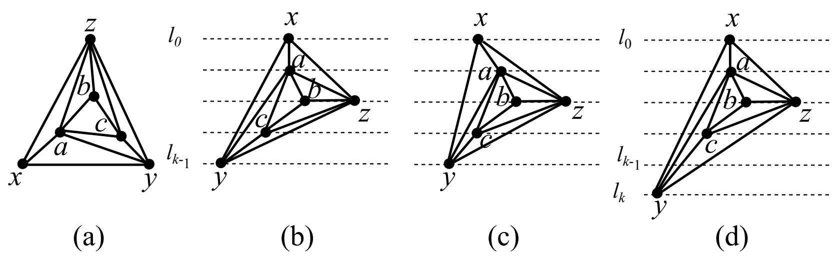

Figure 6: (a)A plane 3-tree Γ. (b) A layered drawing $\mathcal{D}$ of Γ. (c) Illustration for $\mathcal{D}'$. (d) Illustration for $\mathcal{D}''$.

We now have the following theorem.

Theorem 3 *Every planar* 3*-tree of Type* 0 *with n vertices has a drawing on* $\lceil \frac{(n+3)}{3} \rceil$ *parallel lines. Every planar* 3*-tree of Type* 1 *with n vertices has a drawing on* $\lfloor 4n/9 \rfloor$ *parallel lines.*

Proof: Let G be a planar 3-tree with n vertices and let Γ be a plane embedding of G. Let C_{xyz} be a heavy triangle in Γ. Let w be the representative vertex

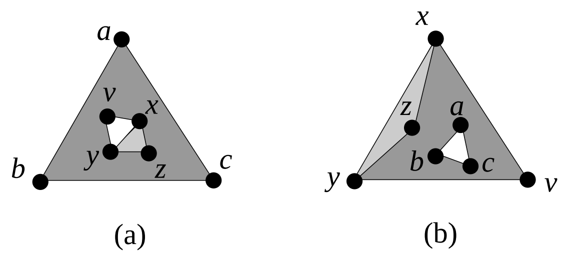

Figure 7: Two different embeddings of G; (a) Γ and (b) Γ'.

of $G(C_{xyz})$. Recall that $C_{xyw}, C_{yzw}, C_{zxw}$ are the three nested triangles around w. We now consider the following two cases.

Case 1. The number of inner vertices in each of the plane 3-trees $\Gamma(C_{xyw})$, $\Gamma(C_{yzw})$ and $\Gamma(C_{zxw})$ is at most $(n-3)/3$ (G is a planar 3-tree of Type 0.)

If (x, y) is an outer edge of Γ, then redefine Γ as Γ'. Otherwise, consider an embedding Γ' of G such that (x, y) is an outer edge of Γ' and the embeddings of $\Gamma'(C_{xyz})$ and $\Gamma(C_{xyz})$ are the same. Observe that any embedding of G taking a face xyv of G as the outer face, where v is not a vertex of $\Gamma(C_{xyz})$, will suffice. An example is illustrated in Figure 7.

Since the number of inner vertices in each of the plane 3-trees $\Gamma'(C_{xyw})$, $\Gamma'(C_{yzw})$ and $\Gamma'(C_{zxw})$ is at most $(n-3)/3$, the depth of the representative tree of $\Gamma'(C_{xyz})$ is at most $(n-3)/3+1$. It is now tempting to claim that the depth of the representative tree of Γ' is bounded by $(n-3)/3+2$ and we can produce a drawing on $(n-3)/3+4$ parallel lines by Theorem 1. However, z might not be the representative vertex in Γ'. Therefore, the depth of the representative tree of Γ' may be very large and hence we compute the drawing in a different technique as described below.

Let $t_0(= z), t_1, t_2, \ldots, t_q(= v)$ be all the vertices of Γ' such that no t_i is interior to $\Gamma'(C_{xyz})$ and each $t_i, 0 \leq i \leq q$ is adjacent to both x and y, and for each $j, 0 \leq j < q$, vertex t_j is interior to the triangle xyt_{j+1}. We claim that $t_0(= z), t_1, t_2, \ldots, t_q(= v)$ is a path in Γ'. Otherwise, assume that t_j and t_{j+1} are not adjacent. By Lemma 1, $\Gamma'(C_{xyt_{j+1}})$ is a plane 3-tree. Let t'_j be the representative vertex of $\Gamma'(C_{xyt_{j+1}})$ which is adjacent to both x and y. If t'_j does not coincide with t_j, then $j' > j+1$, a contradiction to the assumption that t'_j is the representative vertex of $\Gamma'(C_{xyt_{j+1}})$.

We now draw Γ' on $k = \lceil\frac{(n+3)}{3}\rceil$ parallel lines. Place the vertices x and y on lines l_0 and l_{k-1}, respectively, with the same x-coordinate. Place the vertices $t_0(= z), t_1, t_2, \ldots, t_q(= v)$ on lines l_1 and l_{k-2} alternatively with increasing x coordinates such that the triangles xyt_i can be drawn maintaining their nesting order avoiding edge crossings. Then add the edges between t_j and t_{j+1}. See Figure 8 (a). Let the resulting drawing be $\mathcal{D}$. Since $\Gamma'(C_{xyz})$ contains more than $2(n-3)/3$ inner vertices, each plane 3-tree $\Gamma'(C_{xt_jt_{j+1}})$ and $\Gamma'(C_{yt_jt_{j+1}})$ contains less than $(n-3)/3$ vertices. Consequently, the depth of the representative tree of each plane 3-tree $\Gamma'(C_{xt_jt_{j+1}})$ and $\Gamma'(C_{yt_jt_{j+1}})$ is at most $(n-3)/3$. Since

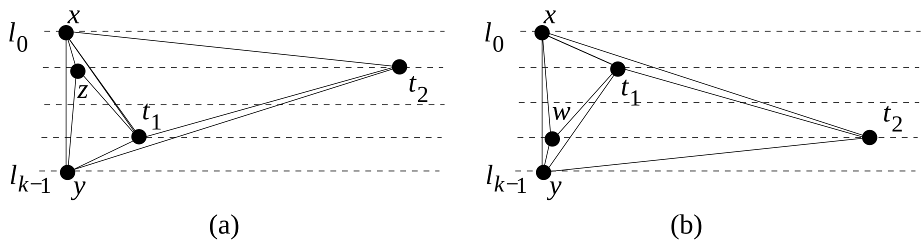

Figure 8: Illustration for the proof of Theorem 3.

each triangle xt_jt_{j+1} and yt_jt_{j+1} in $\mathcal{D}$ is intersected (crossed or touched) by $k-1$ parallel lines and two vertices of the triangle are on consecutive lines, we can draw each plane 3-tree on $k-1$ lines and then insert those drawings into the corresponding triangle in $\mathcal{D}$ using Property (a) of Fact 3. To complete the drawing of Γ', we have to draw $\Gamma'(C_{xyz})$ into triangle xyt_0 in $\mathcal{D}$. Observe that triangle xyt_0 is intersected by k parallel lines and two vertices of the triangle are on consecutive lines. On the other hand, since the number of inner vertices in each of the plane 3-trees $\Gamma'(C_{xyw})$, $\Gamma'(C_{yzw})$, $\Gamma'(C_{zxw})$ is at most $(n-3)/3 = k-3$, the depth of the representative tree of $\Gamma'(C_{xyz})$ is at most $k-2$. It is now straightforward to draw $\Gamma'(C_{xyz})$ on k lines and then insert the drawings into the corresponding triangle in $\mathcal{D}$ using Property (a) of Fact 3.

Case 2. The number of inner vertices in exactly one of the plane 3-trees among $\Gamma(C_{xyw})$, $\Gamma(C_{yzw})$ and $\Gamma(C_{zxw})$ is more than $(n-3)/3$ (G is a planar 3-tree of Type 1.)

Without loss of generality assume that the number of inner vertices in $\Gamma(C_{xyw})$ is more than $(n-3)/3$. If (x, y) is an outer edge of Γ, then redefine Γ as Γ'. Otherwise, consider an embedding Γ' of G such that (x, y) is an outer edge of Γ' and the embeddings of $\Gamma'(C_{xyz})$ and $\Gamma(C_{xyz})$ are the same. As we observed in Case 1, any embedding of G taking a face xyv of Γ as the outer face, where v is not a vertex of $\Gamma(C_{xyz})$, will suffice.

We now draw Γ' on $k = \lfloor 4n/9 \rfloor$ parallel lines as follows. We first place the vertices x and y on lines l_0 and l_{k-2}, respectively, with the same x-coordinate. We then use the algorithm of Chrobak and Nakano [5] to draw $\Gamma'(C_{xyw})$ on lines $l_0, l_1, \ldots, l_{k-2}$ respecting the placement of x and y. Recall the properties (CN1) and (CN2). Since the number of inner vertices in $\Gamma'(C_{xyw})$ is at most $N = 2(n-3)/3$, therefore $k-2 = \lfloor 2(N-1)/3 \rfloor = \lfloor 4n/9 \rfloor - 2$. Without loss of generality assume that w is placed on line l_{k-2}. Modify the drawing using Property (b) of Fact 3 to get an embedding of Γ' on lines $l_0, l_1, \ldots, l_{k-1}$ where x, y, w lies on lines l_0, l_{k-1}, l_{k-2}, respectively. See Figure 8 (b). Let the resulting drawing of $\Gamma'(C_{xyw})$ be $\mathcal{D}$.

We now add the vertices not in $\Gamma'(C_{xyw})$ to $\mathcal{D}$ as follows. Let $t_0(=w), t_1(=z), t_2, \ldots, t_q(=v)$ be all the vertices of Γ' such that no $t_i, 0 \le i \le q$ is interior to $\Gamma'(C_{xyw})$ and each t_i is adjacent to both x and y, and for each $j, 0 \le j < q$, vertex t_j is interior to the triangle xyt_{j+1}. In a similar way as we proved in

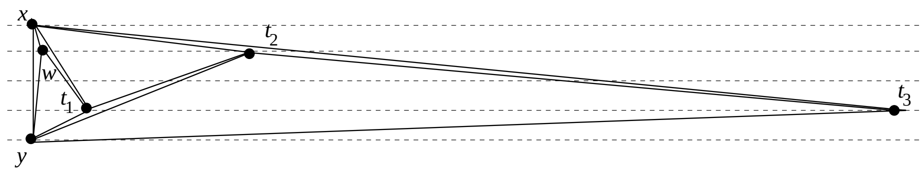

Figure 9: Illustration for a possible drawing of a Type 2 Plane 3-tree.

Case 1, we can show that $t_0, t_1, t_2, \ldots, t_q (= v)$ is a path in Γ'. Now place the vertices $t_1, t_2, \ldots, t_q (= v)$ on lines l_1 and l_{k-2} alternatively with increasing x coordinates such that the triangles xyt_i can be drawn maintaining their nesting order avoiding edge crossings. Then add the edges between t_j and t_{j+1}. See Figure 8 (b). Let the resulting drawing be $\mathcal{D}$. Observe that each of the plane 3-trees $\Gamma'(C_{xt_jt_{j+1}})$ and $\Gamma'(C_{yt_jt_{j+1}})$ contains at most $(n-3)/3$ inner vertices. Therefore, the depth of the representative tree of each of those plane 3-trees is at most $(n-3)/3$. On the other hand, each of the triangles xt_jt_{j+1} and yt_jt_{j+1} contains two vertices on consecutive lines and is crossed by more than $(n-3)/3$ parallel lines. Consequently, we can draw the plane 3-trees $\Gamma'(C_{xt_jt_{j+1}})$ and $\Gamma'(C_{yt_jt_{j+1}})$ on $k-1$ lines and then insert those drawings into the corresponding $\triangle xt_jt_{j+1}$ and $\triangle yt_jt_{j+1}$ in $\mathcal{D}$ using Property (a) of Fact 3. □

The technique we used to draw Type 1 plane 3-trees, uses the property (CN2), i.e., $\Gamma(C_{xyw})$ admits a drawing such that w lies either on line l_{k-2}, or l_1; as shown in Figure 8(b) and Figure 9, respectively. If we could choose the position of w on l_1 or on l_{k-2} arbitrarily, then we could find a drawing of Type 2 plane 3-trees on $k = 4n/9+1$ parallel lines, as follows. Without loss of generality assume that each of $\Gamma(C_{xyw})$ and $\Gamma(C_{wxt_1})$ contains more than $(n-3)/3$ inner vertices. Observe that if we could choose the position of w arbitrarily, then we could compute the drawings of $\Gamma(C_{xyw})$ and $\Gamma(C_{wxt_1})$ inside triangles xyw and wxt_1, respectively. However, it seems very difficult to modify Chrobak and Nakano's algorithm [5] to compute a straight-line drawing respecting a given position for w. Consequently, it would be interesting to examine whether the upper bound of $\lceil (n+3)/2 \rceil$, as proved in Theorem 2, is tight.

6 Conclusion

Let n be a positive integer multiple of six, then there exists a planar 3-tree with n vertices requiring at least $n/3$ parallel lines in any of its drawing on parallel lines [11]. On the other hand, we have proved that $\lfloor \frac{n+3}{2} \rfloor$ parallel lines are universal for planar 3-trees with n vertices. It would be interesting to close the gap between the upper bound and the lower bound of universal line set for planar 3-trees. Finding a universal line set of smaller size for drawing planar 3-trees where the lines are not always parallel is left as an open problem.

Open Problem 1. What is the smallest constant c such that every planar 3-tree with n vertices admits a drawing on cn parallel straight lines?

Observe that we tried to find straight-line drawings with small height. Although in Section 5 we use an algorithm of Chrobak and Nakano [5] that can produce $O(n^2)$-area grid drawings, the drawings we produce can have exponential width because of the scenario depicted in Figure 9. One can decide whether a planar 3-tree admits a straight-line grid drawing on a given area [14], but the only upper bound known is $O(8n^2/9)$ area, which is implied by the algorithm of Brandenburg [4] that can compute an $O(8n^2/9)$-area straight-line grid drawing for arbitrary planar graphs. Since the lower bound on the area requirement of straight-line grid drawings of plane 3-trees is $\Omega(n^2)$, we ask the following question.

Open Problem 2. What is the smallest constant c such that every planar 3-tree with n vertices admits a straight-line grid drawing on cn^2 area?

Acknowledgment

We thank the Ministry of Science and Information & Communication Technology, Government of Bangladesh for supporting the project “Facility Upgradation for Sustainable Research on Graph Drawing & Information Visualization” which facilitated this research. The second author would like to acknowledge the University of Manitoba Graduate Fellowship (UMGF) for supporting his research.

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