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The Complexity of Bendless Three-Dimensional Orthogonal Graph Drawing

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Abstract

We consider embeddings of 3-regular graphs into 3-dimensional Cartesian coordinates, in such a way that two vertices are adjacent if and only if two of their three coordinates are equal (that is, if they lie on an axis-parallel line) and such that no three points lie on the same axis-parallel line; we call a graph with such an embedding an *xyz graph*. We show that planar xyz graphs can be recognized in linear time, but that it is NP-complete to determine whether an arbitrary graph is an xyz graph. We also describe an algorithm with running time $O(n2^{n/2})$ for testing whether a given n-vertex graph is an xyz graph.

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1 Introduction

Consider a finite point set V in $\mathbb{R}^3$ with the following property: every axis-parallel line in $\mathbb{R}^3$ contains either zero or two points of V. For instance, the vertices of an axis-aligned cube have this property. Then V defines the vertex set of a cubic (that is, 3-regular) graph, in which two vertices are adjacent if and only if two of their three coordinates are equal; each vertex v is connected to the three other points of V that lie on the three axis-parallel lines through v. We call such a graph an *xyz graph*. Figure 1 depicts three possible xyz graphs with coordinates in $\{0, 1, 2\}^3$.

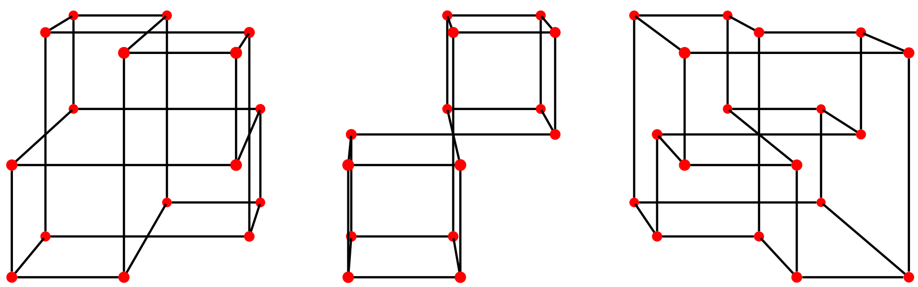

Figure 1: Three possible xyz graphs with coordinates in $\{0, 1, 2\}^3$.

From the point of view of graph drawing, three-dimensional orthogonal drawings [2–4, 8, 9, 19, 22–24] are significant objects of study. However, past work on three-dimensional orthogonal drawing generally requires *bends*, in which edges follow axis-aligned polygonal paths rather than simple line segments. An xyz graph provides a particularly simple and well-characterized form of bendless three-dimensional orthogonal drawing. In xyz graphs, unlike much other work on three-dimensional drawing, we allow edges to intersect each other at crossing points, but this raises no ambiguities in interpreting the drawing, because there are no bends allowed on edges; edge crossings can be distinguished visually from vertices by whether the line segments of the drawing terminate at that point. In this sense, our work is closely related to work on the *slope number* of a graph, the minimum number of different edge slopes needed to draw the graph without bends in the plane [6, 14, 16–18]; indeed, axonometric projection of the drawings described here can be used to generate planar drawings with slope number three.

As we show, xyz graphs also have strong and unexpected connections to topological graph theory and graph coloring. We provide the following new results:

- We prove an equivalence between the graphs that can be embedded in $\mathbb{R}^3$ as xyz graphs and the graphs that can be embedded on two-dimensional surfaces as the boundaries of certain 3-face-colored cell complexes, which we call xyz surfaces. Using this equivalence, we may use topological tools to study the existence of xyz graph embeddings; for instance we show that an xyz graph is bipartite if and only if the corresponding xyz surface is orientable.

- We show that it is NP-complete to determine whether a given graph can be embedded in $\mathbb{R}^3$ as an xyz graph.
- We provide an algorithm with running time $O(n2^{n/2})$ for determining whether a given graph with n vertices can be embedded in $\mathbb{R}^3$ as an xyz graph.

A preliminary version of these results was presented at Graph Drawing 2008. An extended version of this paper, containing also many additional examples of xyz graphs, is online as arXiv:0709.4087.

2 Topology of xyz graphs

If G is an undirected graph, and C is a multiset of simple cycles in G, we may define a *cell complex* as a disjoint union of points, line segments, and disks: one point for each vertex, one line segment for each edge, and one disk for each cycle, glued together topologically according to the connection pattern given by G. For instance, if G is a cube graph (having eight vertices and twelve edges), and C is the set of four-cycles in G, the surface we get can be realized as the set of vertices, edges, and facets of a geometric cube. However, complexes of this type may be defined in an abstract way, independently of any embedding of the whole complex into three-dimensional space. If, further, every point on the cell complex has a neighborhood that is topologically equivalent to an open disk, it is called a *2-manifold* (without boundary). In graph theoretic terms, the cell complex is a manifold whenever two conditions are satisfied:

1. Each edge of G must belong to exactly two cycles of C.
2. At each vertex v of G, each cycle of C that contains v must contain exactly two edges incident to v, and the adjacencies between edges incident to v and cycles of C containing v must have the structure of a single chordless cycle.

The second condition prevents nonmanifold complexes such as those formed by two polyhedra that meet at a single vertex. If G is a cubic graph, the second condition is satisfied automatically, and only the first condition is needed (otherwise, the first follows from the second). We call the cell complex defined from G and C an *embedding* of G onto a manifold, and we call the cycles of C the *faces* of the embedding.

We define an *xyz surface* to be an embedding of a cubic graph G onto a manifold, defined by a collection of faces C, with the following additional properties.

1. Any two faces intersect in either a single edge of G or the empty set.

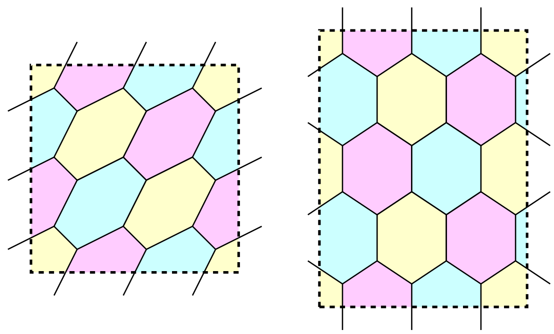

Figure 2: Three xyz surfaces, each with the topology of the torus. In each case, the torus is depicted as cut and unrolled into a rectangle; the corresponding topological surface is formed by gluing opposite pairs of rectangle edges.

2. The faces of C can be assigned three colors such that no two faces sharing an edge have the same color.

The first of these two properties is commonly referred to in the topological graph theory literature by saying that the embedding must be *polyhedral* (see., e.g., [15]). For non-cubic graphs, polyhedral embeddings may also include pairs of faces that intersect in a single vertex, but this cannot happen in a cubic graph. Craft and White [5] have studied a similar 3-coloring condition on orientable maps without the polyhedral condition.

Figure 2 depicts three xyz surfaces, all topologically equivalent to tori. The leftmost is an embedding of the Pappus graph onto a torus, with nine faces and 18 vertices; the same graph is shown as the rightmost graph of Figure 1. The middle surface shows the embedding of a graph with twelve faces, 24 vertices, and 36 edges. The right surface is a torus embedding of the 64-vertex four-dimensional cube-connected cycles network [20].

Theorem 1 *A cubic graph G is an xyz graph if and only if G can be embedded as an xyz surface.*

Proof: Suppose that G is an xyz graph; we must show that G can be embedded as an xyz surface. Fix a particular xyz graph representation of G. We let C consist of the cycles in G that use only two of the three orientations in the xyz graph representation; that is, each such cycle lies in a plane parallel to two coordinate axes. Each edge of G belongs to two such cycles, one for each of the coordinate planes to which it is parallel; therefore, since G is cubic, C forms an embedding of G onto a manifold. The cycles of C can be colored according to the coordinate planes they are parallel to. Since the cycles of G in any single coordinate plane are disjoint, two cycles can have a nonempty intersection only if they belong to different planes; in that case, the intersection must lie on the axis-parallel line formed by the intersection of the two planes containing the cycles, and consists of the edge of G that lies on that same line. Thus, these

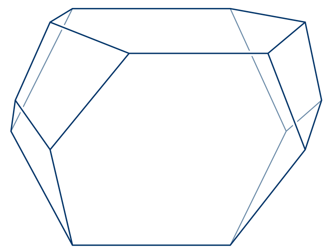

Figure 3: Nine-sided polyhedron with the same face structure as the xyz surface formed from the graph in the center of Figure 1.

cycles form a three-colored polyhedral embedding of G, so they form an xyz surface.

Conversely, suppose that G is embedded as an xyz surface, with cycle set C; we must find an xyz graph representation for G. Let X, Y, and Z be the three monochromatic subsets of C, and let the faces in C be numbered $f_0, f_1, \ldots$. Each vertex v in G is incident to exactly three faces: f_i in X, f_j in Y, and f_k in Z for some i, j, k. We assign v the three-dimensional coordinates (i, j, k). If two vertices u and v are adjacent, they share the two coordinates determined by the two faces containing edge uv, and therefore lie on an axis-parallel line of the embedding of G into $\mathbb{R}^3$. Conversely, if two vertices u and v are not adjacent, they can lie on at most one face of C (else the faces they lie on would intersect in more than one vertex) and therefore have at most one coordinate in common, so they do not lie on an axis-parallel line. Thus, the three axis-parallel lines through each embedded vertex v each contain exactly two points, one of which is v and the other of which is a neighbors of v, so the embedding forms an xyz graph representation of G. □

For instance, we can conclude from this result and the rightmost xyz surface in Figure 2 that the four-dimensional cube-connected cycle graph is an xyz graph. In the other direction, the three xyz graphs depicted in Figure 1 form xyz surfaces that are (in left to right order) a surface on the projective plane, a spherical surface combinatorially equivalent to a polyhedron with three hexagonal facets and six quadrilateral facets (Figure 3), and an embedding of the Pappus graph on the torus combinatorially equivalent to the leftmost xyz surface of Figure 2.

The construction in the proof of Theorem 1 assigns integer coordinates to the vertices of the graph, which may be assumed to lie within an $\frac{n}{4} \times \frac{n}{4} \times \frac{n}{4}$ grid. To see this, observe that each face of an xyz graph or of the corresponding xyz surface must have even length, because it alternates between edges parallel to two different coordinate axes in the case of an xyz graph, and between edges incident to pairs of faces with two different sets of colors in the case of an xyz surface. Since the minimum length of a face in an embedding of a simple graph, three, is not an even number, all faces must have length at least four. Thus, the number of faces in any color class of an xyz surface coloring is at most $n/4$:

each vertex belongs to one face of that color, but each face contains at least four vertices. Since each face provides a value for one of the coordinates in the grid embedding, the number of distinct values for each coordinate is therefore also at most $n/4$. However this $n/4$ bound on the coordinate size is tight only for the cube, because the cube is the only xyz surface in which all faces are quadrilaterals. Any other xyz surface has at least one face with more than four vertices, and therefore it has a color class with fewer than $n/4$ faces, leading to an embedding on a grid with fewer than $n/4$ distinct values in at least one of the coordinates.

For many of the subsequent results, we will find it more convenient to work topologically, in terms of xyz surfaces, and less convenient to work geometrically, in terms of an explicit xyz graph representation. However, in situations where an xyz graph representation is desired (e.g. as the output of a graph drawing algorithm) it may easily be reconstructed using the face-numbering technique of the proof above.

3 Properties of xyz graphs

In this section we prove some basic properties of xyz graphs that will be of use in our subsequent algorithms.

Theorem 2 *Every xyz graph is triangle-free.*

Proof: The edges of a triangle must be mutually perpendicular, as each pair of edges meets at a vertex. But then, if one follows a path around the triangle starting from one of its vertices, in an xyz graph embedding, each step of the path changes one of the three coordinates, and each coordinate is changed exactly once, so the path cannot return to its starting point, a contradiction. □

Lemma 1 *Let abc be a path of three vertices in an xyz graph G, and let d be a vertex of G distinct from a, b, and c. Then there exists a path in G that starts at a, ends at c, and does not pass through either b or d.*

Proof: If d does not belong to the face of G that contains edges ab and bc, we may choose as our path the complementary set of edges in the same face. Otherwise, let e be the third neighbor of b in G, and form a path by concatenating the complementary set of edges to edges ab and be in the face shared by those two edges, together with the complementary set of edges to edges bc and be in the face shared by those two edges. Neither of these two faces can contain d, as then they would intersect the face containing ab and bc in more than a single edge. □

Theorem 3 *Every connected xyz graph is 3-vertex-connected.*

Proof: Let s and t be two vertices in an xyz graph G from which two other vertices u and v have been deleted. We must show that there still remains a path from s to t. Let P be a path in G that connects s to t. If P contains u, we may apply Lemma 1 to the segment of P formed by the two edges incident to u, replacing it by another path that avoids u. Similarly, if P contains v, we may apply Lemma 1 again, replacing it by another path that avoids v without adding any additional component of path through u. Thus, since we can connect any two vertices after the removal of any other two vertices, the graph is 3-vertex-connected. □

We conclude this section with an interesting connection between bipartiteness and topology. An *orientation* of a graph embedding onto a surface, described by a set of cycles C, can be described as a choice of orientation for each edge in each cycle, such that each cycle of C is given a consistent cyclic orientation, and such that the two cycles shared by any edge e assign opposite orientations to e. A surface is *orientable* if graphs embedded on it may be oriented in this way; for instance the sphere and torus are orientable, while the projective plane is not.

Theorem 4 *Let G be a graph embedded onto an xyz surface. Then G is bipartite if and only if the surface is orientable.*

Proof: Let G be a bipartite xyz graph; we must show that the corresponding xyz surface is orientable. Color the vertices of G black and white, then we can orient the faces in xy-parallel planes from black to white in the x direction and from white to black in the y direction, in xz-parallel planes from white to black in the x direction and from black to white in the z direction, and in yz-parallel planes from black to white in the y direction and from white to black in the z direction; the result is a consistent orientation of all faces of the graph.

Conversely, suppose that G is embedded as an xyz graph on an orientable surface; that is, the embedding is polyhedral and its faces are colored red, yellow, and blue. Choose a consistent orientation for the surface; color a vertex white if the clockwise ordering of faces around the vertex (according to this orientation) is in the cyclic order red–yellow–blue and black if the clockwise ordering of the faces around it is red–blue–yellow. Then this gives a two-coloring of G, so G is bipartite. □

A standard result in topology is that 2-manifolds may be classified by their orientability and their *Euler characteristic* $|V| - |E| + |C|$, so from Theorem 4 it is straightforward to determine the topological type of any xyz embedding by computing the Euler characteristic of the embedding and testing the bipartiteness of the graph.

4 Algorithms for xyz embedding

As we now show, there exist efficient algorithms to determine whether an embedded surface is an xyz surface, or whether a partition of the edges of a graph into three perfect matchings can be used as the three parallel classes of edges in an xyz graph. However, it is not so easy to find an xyz graph representation for an initially unlabeled graph.

Theorem 5 *Let G be a connected undirected n-vertex graph, and let C be a collection of cycles in G. Then in time $O(n)$ we may determine whether C is the set of cycles of an xyz surface embedding of G, and if so construct an xyz graph representation of G.*

Proof: We first check that G is a cubic graph and that C covers each edge of G exactly twice. Next, we assign arbitrary index numbers to the cycles in C. Each edge has an associated pair of index numbers, which we order lexicographically. We may sort the edges of G according to this lexicographic ordering in linear time, by two passes of bucket sorting; in the resulting sorted order, if some pairs of edges both belong to the same two cycles, at least one such pair will appear consecutively. Thus, by performing this sorting algorithm and then testing adjacent pairs of edges in the sorted order, we may verify in linear time that each pair of cycles intersects in at most a single edge.

To test 3-colorability of the cycles in C, we apply the following algorithm. We store a set of the available colors for each cycle (initially, all three colors are available for each cycle), and a list L of cycles that have only one remaining color. We will color the cycles in some order; whenever we color a cycle we remove that color from the available colors of all cycles that share an edge with it, and update L whenever that removal causes an adjacent cycle to have only one remaining available color. We begin this sequence of color choices by choosing arbitrarily two cycles that share an edge, and assigning arbitrarily two different colors to those two cycles. Then, as long as L remains nonempty, we remove a cycle from L, and assign it the one color that is available to it.

If this process terminates with a 3-coloring of all faces in C, we have found an xyz surface representation for G. Conversely, suppose that G has an xyz surface representation: we argue that this process will necessarily find a correct 3-coloring of all faces. To show this, permute the colors of the coloring if necessary so that they match the colors chosen for these faces at the start of the algorithm. Clearly, every color choice subsequent to that is forced, so the algorithm can never choose an incorrect color for a face, and therefore also can never eliminate the correct color for any face; the only way it could fail to 3-color all faces would be if it terminated with L empty before coloring all faces. But if f is any face of C, let p be any path connecting a vertex of the shared edge of the first two colored faces with any vertex of f. At any stage in the algorithm until f has been colored, let v be the vertex of p that is closest along the path to the first two colored faces, and that is incident to an uncolored face f'; then the two differently-colored neighboring faces of f' at v would force f' to belong to L. Thus, L cannot be empty until f is colored, but since this is true for any face f the algorithm cannot terminate when given as input a 3-colorable surface embedding until all faces are colored. □

Corollary 1 *Let G be a connected undirected n-vertex graph, and let E_1, E_2, E_3 be a partition of the edges of G into three matchings. Then in time $O(n)$*

we may determine whether there is an xyz graph representation of G in which each set E_i is the set of edges parallel to the ith coordinate axis.

Proof: For each pair E_i and E_j, $E_i \cup E_j$ is a disjoint union of cycles; we let C be the set of cycles formed in this way for all three pairs of matchings, and apply Theorem 5 to this set of cycles. □

Lemma 2 *Let G be a biconnected cubic graph. Then there are at most $2^{(n-2)/2}$ partitions of the edges of G into three perfect matchings, and these partitions may be listed in time $O(2^{n/2})$.*

Proof: We compute an *st*-numbering of G [11]; that is, an ordering of the vertices of G in which each vertex, except for the ones at the start and the end of the sequence, has a neighbor that occurs earlier in the sequence and a neighbor that occurs later in the sequence. We define a *split vertex* to be one with one previous neighbor and two later neighbors, and a *merge vertex* to be one with two previous neighbors and one later neighbor. If there are k split vertices there would be $3 + 2k + (n - k - 2)$ edges, as the first vertex in the *st*-numbering is the earlier endpoint of three edges, the split vertices are each the earlier endpoint of two edges, the $n - k - 2$ merge vertices are each the earlier endpoint of only one edge, and the final vertex in the *st*-numbering is the earlier endpoint of no edges. Observing that the graph has $3n/2$ edges total and solving for k, we find that there must be exactly $(n - 2)/2$ split vertices.

To list all partitions, we then perform a backtracking algorithm in which we assign the edges to partitions in order by their earlier endpoints in the *st*-numbering; once we make an assignment for an edge e we recursively list all partitions for edges occurring later in this ordering before backtracking and trying an alternative assignment for e (if an alternative exists). If this backtracking process ever reaches a contradictory state in which no possible assignment is available from an edge, it backtracks without recursing.

At the initial vertex of the *st*-numbering, the backtracking algorithm has no choices to make: it can partition the incident edges into three disjoint subsets in only one way. At the final vertex, there is again no choice to make, because all incident edges must already have been partitioned. And at each merge vertex, there is no choice to make, because there are two incident edges which must already have been placed into two sets of the partition, and the third incident edge can only go in the third set of the partition. Thus, the only branch points of this backtracking algorithm are the split vertices, at which the two edges for which the vertex is the earlier endpoint must be assigned to the two remaining partition sets, in either of two different ways.

Since the algorithm makes a binary choice at each of $(n - 2)/2$ levels of its recursion, its total time is $O(2^{n/2})$. The number of partitions listed is at most the number of leaves in a binary tree of height $(n - 2)/2$, which is $2^{(n-2)/2}$. □

Greg Kuperberg (personal communication) has pointed out that the prisms over $n/2$-gons form biconnected cubic graphs with $\Omega(2^{n/2})$ partitions into three perfect matchings, showing that this bound is tight to within a constant factor.

Theorem 6 *We can test whether a given unlabeled graph is an xyz graph, and if so find an xyz graph representation of it, in time* $O(n2^{n/2})$.

Proof: We list all partitions into matchings using Lemma 2, and test whether any of them can be used to define an xyz graph representation using Corollary 1. □

We have implemented our algorithms for listing all partitions of a cubic graph into perfect matchings and for testing whether a given graph is an xyz graph, using the Python programming language. The implementation is available online at `http://www.ics.uci.edu/~eppstein/PADS/xyzGraph.py`.

5 Planar graphs

We may exactly characterize the planar xyz graphs.

Lemma 3 *Let G be a planar cubic graph, and C be the family of face cycles of a planar embedding of G. Then any polyhedral embedding of G must have C as its face set.*

Proof: Suppose for a contradiction that there is a polyhedral embedding of G with a different face set C'. Since every edge of G must remain covered twice by cycles in C', C' must contain a face f that does not belong to C. In the planar embedding defined by C, f forms a simple cycle that is not a face of the embedding, so the interior and exterior of this cycle both contain at least one edge or vertex of G. Each vertex of f is incident to two edges of f, and to a third edge, which must be either interior to or exterior to f. Since both the interior and exterior of f are nonempty, there must exist an edge e of f with the property that the third edge at one endpoint of e is interior and the third edge at the other endpoint of e is exterior.

Now consider the cycle f' in C' that contains e but is not f. Removing e from f' leaves a path that connects the endpoints of e; because f' and f cannot share any edges other than e, the terminal edges of this path are the interior and exterior edges incident to the endpoints of e. But in the planar drawing given by C, the Jordan curve theorem implies that this path must cross f. At the point where it has this crossing, it must do so by containing at least one other edge of f, violating the requirement of polyhedral embeddings that cycles f and f' may share at most a single edge and contradicting the assumption that C' is the face set of a polyhedral embedding of G. □

Theorem 7 *Let G be a planar graph. Then G is an xyz graph if and only if G is bipartite, cubic, and 3-connected. If it is an xyz graph it has a unique representation as an xyz surface, up to permutation of the face colors of the surface.*

Proof: If G is a planar xyz graph, it must be cubic, and by Theorem 3 it must be 3-connected. By Lemma 3 its corresponding xyz surface is unique and must be topologically a sphere, so by Theorem 4 it must be bipartite.

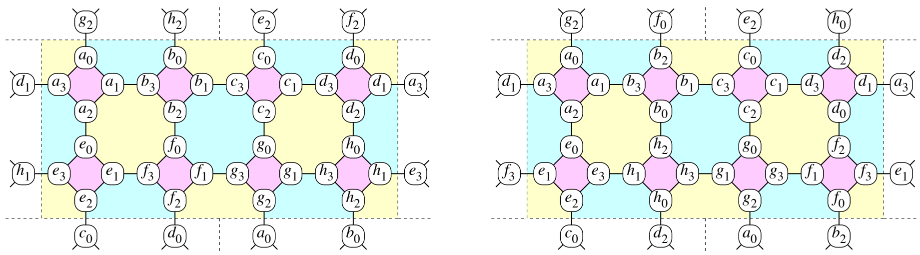

Figure 4: A graph that can be embedded on the torus as an xyz surface in two different ways.

Conversely, suppose that G is a bipartite cubic 3-connected planar graph. By Steinitz' theorem G can be represented as the skeleton of a convex polyhedron; by convexity, every pair of faces meets in at most a single edge. Heawood [12] proved that the faces of any bipartite convex polyhedron can be 3-colored and this coloring provides an xyz surface representation of G. By Theorem 1, G is an xyz graph. □

Corollary 2 *We may test in linear time whether a given planar graph G is an xyz graph.*

Proof: Planarity, bipartiteness, 3-regularity, and 3-connectivity are all known to be testable in linear time. □

6 Complexity of xyz graph recognition

We will show that recognizing xyz graphs is NP-complete, via a reduction from graph 3-colorability. Throughout this section, when we refer to the *orientation* of an edge, we mean the choice of which coordinate axis to make the edge parallel to in an xyz graph representation; this choice will correspond to the choice of a color in our 3-colorability reduction. Our reduction will connect together gadgets (certain cubic graphs) via connected sums. We explain below the behavior of the key gadgets for this reduction, an ambiguously embeddable torus shown in Figure 4, and another part of a cubic graph shown in Figure 5, which we call the *connector gadget*.

The 32-vertex graph shown in Figure 4 consists of eight "diamonds" of four vertices each, connected in pairs to form a cubic graph. The central colored rectangle of the graph denotes a surface that may be glued to itself along its boundary to form a torus; the left and right ends of the rectangle are connected to each other in the usual way, but the top and bottom ends are connected after a shift to form a brick wall pattern. Several vertices are drawn twice, both inside the rectangle and a second time outside it, to show the edges that cross these glued rectangle boundaries. The faces of the surface (eight diamonds and eight octagons) are colored, showing that it forms an xyz surface. However, the same graph may be drawn in two different ways, as shown in the figure. The left and right sides of the figure show two embeddings that are isomorphic as unlabeled graphs but nonisomorphic when the graph vertices are labeled, as they are in the figure, although they have edges and vertices forming isomorphic underlying graphs. The octagonal faces on the right-hand embedding form zigzag paths through the left-hand embedding: the blue faces on the right correspond to cycles in the left embedding that zigzag from top left to bottom right, while the yellow faces in the right drawing zigzag from the bottom left to the top right of the left drawing. Each 4-cycle of the graph forms a face (drawn as a pink diamond) in both embedding, but for every pair of 4-cycles connected by an edge in the first embedding, one of the two 4-cycles has the opposite orientation in the second embedding. Thus, the graph in the figure is an xyz graph in two different ways, showing that the uniqueness of xyz surface representations for planar graphs does not directly generalize to other surfaces.

The connector gadget, as drawn in Figure 5, can be viewed as part of a cylindrical piece of surface, in which the gadget partitions the surface of the cylinder into rings of three curved hexagonal faces; each such ring can be 3-colored. As we now show, that is the only possible way in which this gadget can be part of an xyz surface.

Lemma 4 *Suppose that the connector gadget shown in Figure 5 is a subgraph of some larger xyz graph G. Then in any xyz graph representation of G, the three edges entering the gadget from the left of the figure must be mutually perpendicular and lie on three lines that all meet in a single point. Similarly, the three edges entering the gadget from the right of the figure must be mutually perpendicular and lie on lines that all meet at a single point. If the left three lines are A, B, and C (as shown top-down in the figure) and the right three lines are D, E, and F (again, top-down), then A and F must be parallel, B and E must be parallel, and C and D must be parallel.*

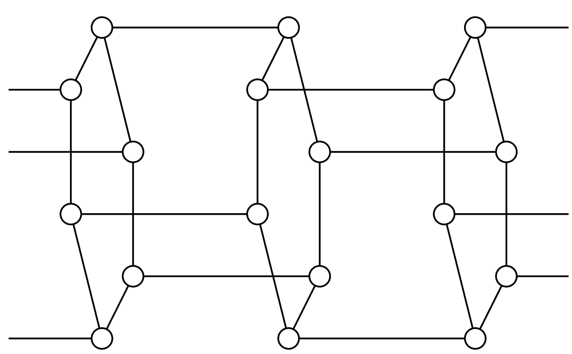

Figure 5: The connector gadget, part of an NP-completeness proof for xyz graph recognition.

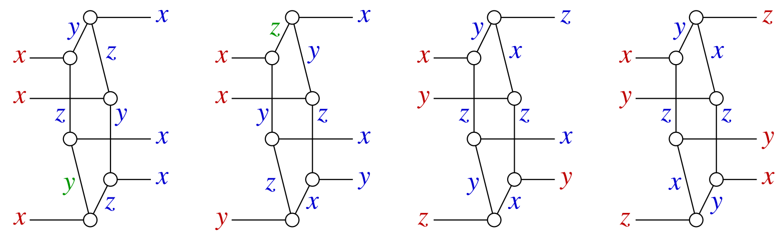

Figure 6: Case analysis for Lemma 4. In each case the red letters denote the edge orientations that specify that case, the green letter (if present) denotes an arbitrary choice of orientation that may be made once the red orientations are set, and the blue letters denote edge orientations that are forced by the red and green orientations and the requirement that incident edges be perpendicular.

Proof: We examine cases, showing that in each case other than the one described the axis-parallel polygons of the drawing could not form an xyz surface.

First, suppose that A, B, and C are all parallel to each other; by symmetry we may assume that they are parallel to the x axis, as shown by the red labels in Figure 6(a). Then the hexagon to which these three edges attaches must have edges that alternate $yzyzyz$ in one of two ways (the green and blue labels in the same figure), and the next three edges drawn as horizontal in the drawing must also be parallel to the x axis. Repeating this reasoning, all twelve horizontal edges in the drawing of Figure 5 must be parallel to the x axis. There would be six paths through this drawing labeled $xyxyxyx$ or $xzxzxzx$, which must be part of axis-parallel polygons in any xyz graph representation for this drawing. But these six paths share 12 x-parallel edges, while they can be at most part of three xy-parallel and three xz-parallel polygons, which can together share at most 9 edges in any xyz graph representation. Therefore, if A, B, and C are drawn parallel, the drawing cannot form an xyz graph.

Next, suppose that two of A, B, and C are parallel to each other, while the third is not. We may assume by symmetry that two (A and B) are parallel to the x axis while the third (C) is parallel to the y axis, as shown in Figure 6(b). Then, if we pick arbitrarily the orientation of one of the two edges connecting A and B, and then orient the remaining edges of the hexagon to which A, B, and C attach by choosing the only remaining orientation for an edge whenever two adjacent edges have already been oriented in perpendicular directions, we find that the edges around this hexagon must be oriented in the pattern (clockwise starting from the endpoint of C) $xzyzyz$ or its reverse. This forces the orientation of the next three horizontal edges of the gadget to again have two edges parallel to the x axis and the third parallel to the y axis, and so forth from left to right across the connector gadget. There must then be two paths labeled $xyxyxyx$ that extend from left to right across the gadget, and one path labeled $xzxzxzx$ that also extends from left to right all the way across the gadget; these three paths together share four x-parallel edges, but they belong to at most one xz-parallel polygon and at most two xy-parallel polygons that can only share a total of two edges in any xyz graph representation. Therefore, in this case, we again fail to get an xyz graph.

In the third case, A, B, and C are labeled differently from each other. Without loss of generality they are labeled x, y, and z respectively, but the horizontal edge in the next layer of the gadget to the right that is farthest from A is not labeled x; without loss of generality its label is y, as shown in Figure 6(c). In this case, when we assign orientations to edges that are forced by having two perpendicular adjacent edges, as shown in the figure, we get two paths through the part of the gadget depicted in Figure 6(c) that start from C, that are labeled $zxzxz$ and $zyzyz$, and that share both their starting and their ending edges. This would cause the corresponding two faces of any surface embedding to share two edges, impossible in an xyz surface.

Finally, the only remaining case is that A, B, and C are labeled distinctly, and that each label matches the label of the opposite edge in the next layer of horizontal edges, as shown in Figure 6(d). A, B, and C are mutually perpendicular, and form the boundaries of three mutually adjacent faces of the xyz surface. These faces include paths through the part of the gadget shown in Figure 6(d), labeled $xyxy$, $yzyz$, and $zxzx$. Since the three lines through A, B, and C lie on the intersections of the pairs of planes through these three faces, the three lines must meet at the point where these three planes meet, as stated in the lemma. The concurrence of the three lines through D, E, and F follows by a symmetric argument, and the parallelism between pairs (A, F), (B, E), and (C, D) follows immediately from the labeling. □

We will use this gadget in our NP-completeness reduction by attaching it to other gadgets via a connected sum construction: if G is a (possibly disconnected) cubic graph with designated vertices u and v, define a graph $G_{u,v}$ by deleting vertices u and v from G, reconnecting the edges that were incident to u to the left side of a connector gadget, and reconnecting the edges that were incident to v to the right side of a connector gadget. The connector gadget may be connected to u and v in multiple non-isomorphic ways, by choosing different permutations of the neighbors of u and v to attach to edges A, B, C, D, E, and F of the gadget. Given a particular choice of how to attach the gadget to form $G_{u,v}$, we say that an orientation of G is *consistent* if the edges of G corresponding to edges A and F of the gadget have the same orientation as each other, as do the edges of G corresponding to B and E, and the edges of G corresponding to C and D.

Lemma 5 *The orientations of edges of $G_{u,v}$ that come from valid xyz graph representations are in one-to-one correspondence with the orientations of edges of G that come from valid xyz graph representations and that assign consistent orientations to the edges of G at u and at v. Thus, $G_{u,v}$ is an xyz graph if and only if G has an xyz graph representation in which u and v are consistently oriented.*

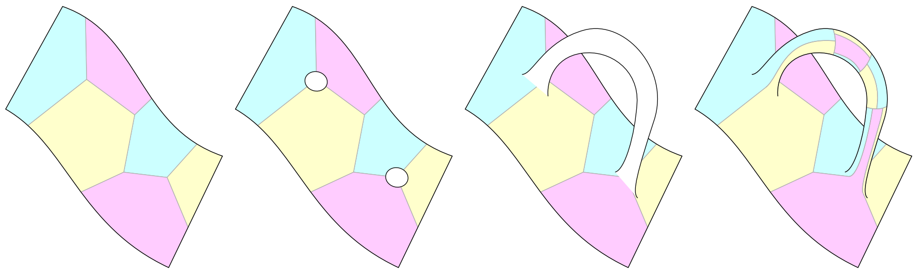

Figure 7: Creating an xyz surface for $G_{u,v}$ from an xyz surface for G.

Proof: From any xyz graph representation of $G_{u,v}$, we may return to an xyz graph representation of G by deleting the connector gadget and replacing the deleted vertices of G at the points of incidence of the lines that the connector gadget edges lie on; these points of incidence are guaranteed to exist by Lemma 4. By Lemma 4, the edges of the connector gadget in $G_{u,v}$ will be consistently oriented at u and at v, and this consistent orientation remains when we transform the xyz graph representation of $G_{u,v}$ into an xyz graph representation of G.

In the other direction, suppose that G has an xyz graph representation that is consistently oriented. Form the corresponding xyz surface, and transform it into an xyz surface representation of $G_{u,v}$ by cutting out a small disk from the surface surrounding u, cutting out another small disk surrounding v, and connecting these two disks by a cylinder with six colored faces, as shown in Figure 7. Since $G_{u,v}$ has a valid xyz surface representation, by Theorem 1, it corresponds to a valid xyz graph representation of $G_{u,v}$. □

Theorem 8 *It is NP-complete, given an undirected graph G, to determine whether G can be represented as an xyz graph.*

Proof: Membership in NP follows from Theorem 5. To prove NP-hardness, we reduce from the standard NP-complete problem of graph 3-coloring.

Thus, given a graph H, we wish to construct from it in polynomial time a graph G, such that G is an xyz graph if and only if H is 3-colorable. We do so by connecting together subgraphs of various types using the connector gadget described above. In our graph G, colors of vertices will be represented by the orientations of edges on connector gadgets. Specifically, if we label the three edges on one side of a connector gadget A, B, and C, then orientations with A parallel to the x axis will correspond to one color, orientations with A parallel to the y axis will correspond to a second color, and orientations with A parallel to the z axis will correspond to a third color. Thus, each color can be represented by two possible orientations, as we do not care which orientation is assigned to B and which to C.

To represent a vertex in H with d incident edges, we use any cubic bipartite 3-connected planar graph with at least d vertices, for instance a $d/2$-prism. By

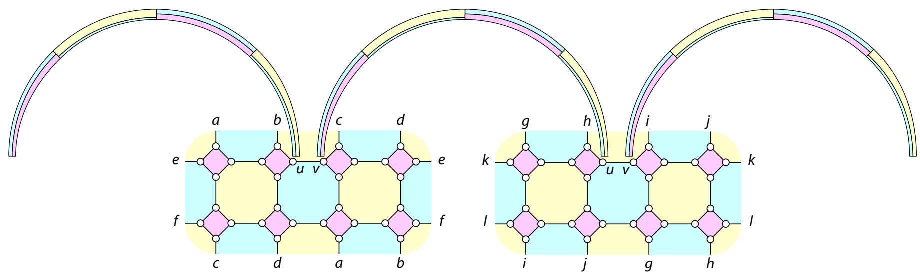

Figure 8: Schematic view of edge gadget: three connector gadgets connected sequentially via two flip gadgets.

Theorem 7 each such graph has a representation as an xyz graph that uniquely orients the edges up to permutation of the coordinates. Let M be a matching in this vertex gadget consisting of edges that have the same orientation in the xyz graph representation of the gadget. We will attach connector gadgets to this vertex gadget in such a way that all connector gadgets have their A and F edges attached to edges of M. Therefore, all attached connector gadgets must represent the same color, which may be any of the three available colors.

The next gadget we will use is the 32-vertex graph with two different toroidal xyz surfaces described in the previous section. Let u and v be any two vertices of this graph that are endpoints of an edge uv but do not belong to a 4-cycle in the graph. Then among the xyz graph representations of this gadget, if u has the triple of orientations (a, b, c) (where a is the orientation of edge uv), v may have either of the triples of orientations (a, b, c) or (a, c, b). We call this a *flip gadget* as it allows us to flip the orientations of two of the edges without allowing the orientation of the third to change; we will attach connector gadgets at u and v.

Our *edge gadget* is formed by the sequential attachment of a connector, flip gadget, connector, flip gadget, and connector (Figure 8). The A, B, and C edges of the first connector and the D, E, and F edges of the third connector will be attached to vertex gadgets; as described above, the A edge of the first connector and the F edge of the third connector will be connected to the chosen matching within their vertex gadgets, so their orientations will correspond to the vertex colors at the endpoints of the edge. The F edge of the first connector and the A edge of the second connector are connected to the uv edge of the first flip gadget; this flip gadget constrains the A edges of the first two connectors to be parallel while allowing the B and C edges of the second connector to be oriented independently of the B and C edges of the first connector. Next, the E edge of the second connector is connected to the uv edge of its flip gadget, as is the A edge of the third connector, forcing the A edge of the third connector to have a different orientation from the F edge of the second connector. For each orientation of the A, B, and C edges of the first connector, there are four possible orientations of the D, E, and F edges of the last connector for which the A of the first connector differs from the F of the last connector; the two binary choices allowed by the two flip gadgets within the edge gadget allow all four of these orientations to be used, while preventing orientations in which the A of the first connector and the F of the last connector are parallel. Thus, this connection pattern allows the unattached edges of the edge gadget's connectors to be oriented in all ways that represent pairs of different colors, but not to be oriented in ways for which both endpoints represent the same color.

Our overall construction of G from H is to create a vertex gadget for each vertex of H, an edge gadget for each edge e of H, and attach the free ends of the edge gadgets for e to the vertex gadgets for the endpoints of e. If H is three-colorable, we may form an xyz surface that corresponds to the coloring within each vertex gadget, and extend it to an xyz surface for all of G by repeatedly applying Lemma 5; therefore, in this case, G has an xyz surface and an xyz graph representation. Conversely, if G has an xyz surface, repeated application of Lemma 5 shows that each vertex gadget must be colored as if it were unattached, so we may determine from the xyz surface a color assignment by examining the orientation of the edges within each vertex gadget. Lemma 5 and the properties of the edge gadget we have constructed force this color assignment to be a valid coloring of H.

Thus, we can reduce 3-coloring to testing whether a graph is an xyz graph, in polynomial time. Since testing for being an xyz graph is in NP, and 3-coloring is NP-complete, testing for being an xyz graph is also NP-complete. □

The reduction given in this proof produces graphs of high genus: if the graph to be colored has n vertices and m edges, the resulting xyz surface (if it exists) has genus $3m - n + 1$. The use of high genus surfaces is a necessary feature of this proof, because it is possible to find xyz surface embeddings of low genus in polynomial time. More specifically, for any fixed constant g, an algorithm of Kawarabayashi and Mohar [13] takes as input a 3-connected graph and in linear time produces as output a list of all of the polyhedral embeddings of the graph that have genus at most g; at most a constant number of distinct embeddings are output by this algorithm. By applying Corollary 1 to each embedding generated by this algorithm, it is possible to test in linear time whether a given graph has an xyz graph representation of genus at most g.

7 Conclusions and further work

We have studied algorithms and complexity of xyz graph drawing. In work performed subsequently to the results reported here, we extended these results in two directions. With E. Mumford [10], we showed that the planar xyz graphs can always be represented geometrically, as the graphs of (non-convex) polyhedra in which three perpendicular edges meet at each vertex, and we extended this representation to a complete characterization of the planar graphs that have representations of this type (relaxing the restrictions of xyz graphs to allow multiple collinear edges). And with M. Bannister and J. Simons [1], we studied compaction problems for orthogonal drawing programs, showing that

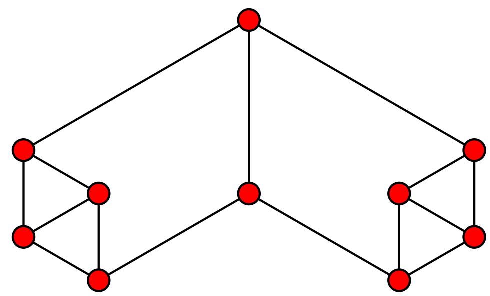

Figure 9: A point set such that lines in three parallel families each contain zero or exactly two points, and the cubic graph derived from it. This graph is not an xyz graph, as it contains triangles.

it is hard to approximate the smallest bounding box into which an orthogonal drawing such as an xyz drawing may be compacted.

Any xyz graph can (after a suitable perturbation of its coordinate values) be projected isometrically onto the plane in such a way that all edges are parallel to the sides of an equilateral triangle, and no line through an edge contains any other vertex. More generally, if we have any point set in the plane such that any line parallel to the sides of an equilateral triangle contains either zero or two points, we may define a cubic graph analogously to the three-dimensional definition of xyz graphs. However, these two-dimensional three-orientation graphs are somewhat more general than xyz graphs; for instance Figure 9 shows a graph that may be drawn in this way that contains triangles and is therefore not an xyz graph. Richter [21] has studied two-dimensional three-orientation drawings, from a similar perspective to the work reported here.

Similarly to the NP-completeness proof presented here, Eades et al. proved that it is NP-complete to determine whether a graph has a bendless orthogonal drawing in the plane [7]. However, their reduction allows parallel edges leaving a vertex in opposite directions, and uses vertices of variable degree, unlike our constraints that all edges at a vertex be perpendicular and all vertices have degree three.

Our investigation opens up several additional avenues for further research:

- We saw that testing xyz graph representability is easy in the case of planar graphs, because in this case there cannot exist a nonplanar xyz surface, so we need only test whether the unique planar embedding gives an xyz surface. Additionally, as discussed in Section 6, an algorithm of Kawarabayashi and Mohar [13] for listing all bounded-genus polyhedral embeddings of a graph allows us to test whether a given graph has an xyz representation of bounded genus in linear time. However, this does not yet give us a fixed-parameter algorithm for finding xyz graph representations, parameterized by the minimum genus of the graph, because as far as we know it might be possible for graphs of bounded minimum genus to have

xyz surface embeddings with unbounded genus. Is this possible?

- Kuperberg's example of the prism shows that our algorithm for testing xyz graph representability using all partitions of the graph into three matchings cannot be improved, unless we avoid some partitions. However, for the prism itself, there are many partitions that can safely be avoided: for an xyz graph representation, we cannot use any partition into three matchings that uses three different orientations in a single quadrilateral. One can also devise similar conditions that restrict the matchings in hexagons and other short cycles of a given graph. Can one take advantage of these forbidden configurations to eliminate some partitions into matchings earlier in the algorithm and reduce its running time?

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