



Hamilton Cycles in Restricted and Incomplete Rotator Graphs

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Abstract

The nodes of a rotator graph are the permutations of n , and an arc is directed from \mathbf{u} to \mathbf{v} if the first r symbols of \mathbf{u} can be rotated one position to the left to obtain \mathbf{v} . Restricted rotator graphs restrict the allowable rotations to $r \in R$ for some $R \subseteq \{2, 3, \dots, n\}$. Incomplete rotator graphs only include nodes whose final symbol is $i \leq m$ for a fixed maximum value $m \in \{1, 2, \dots, n\}$. Restricted rotator graphs are directed Cayley graphs, whereas incomplete rotator graphs are not Cayley graphs. Hamilton cycles exist for rotator graphs (Corbett 1992), restricted rotator graphs with $R = \{n-1, n\}$ (Ruskey and Williams 2010), and incomplete rotator graphs for all m (Ponnuswamy and Chaudhary 1994). These previous results are based on sequence building operations that we name ‘reusing’, ‘recycling’, and ‘rewinding’. In this article, we combine these operations to create Hamilton cycles in rotator graphs that are (1) restricted by $R = \{2, 3, n\}$, (2) restricted by $R = \{2, 3, n-1, n\}$ and incomplete for any m , and (3) restricted by $R = \{n-2, n-1, n\}$ and incomplete for any m . Result (1) is ‘optimal’ since restricted rotator graphs are not strongly connected for $R = \{3, n\}$ when n is odd, and do not have Hamilton cycles for $R = \{2, n\}$ when n is even (Rankin 1944, Swan 1999). Similarly, we prove (3) is ‘optimal’. Our Hamilton cycles can be easily implemented for potential applications, and we provide $O(1)$ -time algorithms that generate successive rotations for (1)–(3).

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1 Introduction

The *rotator graph* \mathcal{G}_n is the directed graph whose nodes are the permutations of n symbols written as strings, and whose arcs travel between nodes whose strings differ by the rotation of a prefix. More precisely, the nodes are the strings in Π_n , which is the set of strings representing the permutations of $\{1, 2, \dots, n\}$ written in one-line notation. For example, $\Pi_3 = \{123, 132, 213, 231, 312, 321\}$. The *prefix-rotation of length r* cyclically moves the first r symbols of a string one position to the left and is denoted σ_r . For example, $541362 \sigma_4 = 413562$ since 413 is moved one position to the left and 5 “wraps around” into the fourth position. We frequently refer to prefix-rotations simply as *rotations*. The arcs in \mathcal{G}_n are directed from $\alpha \in \Pi_n$ to $\beta \in \Pi_n$ when $\beta = \alpha \sigma_r$ for some $r \geq 2$, and such an arc is labeled σ_r . Figure 1 (a) shows the rotator graph for $n = 3$ (with arc labels omitted). The rotator graph \mathcal{G}_n can also be described as $\overrightarrow{\text{Cay}}(\{\sigma_2, \sigma_3, \dots, \sigma_n\}, \mathbb{S}_n)$, the directed Cayley graph on the symmetric group generated by prefix-rotations.

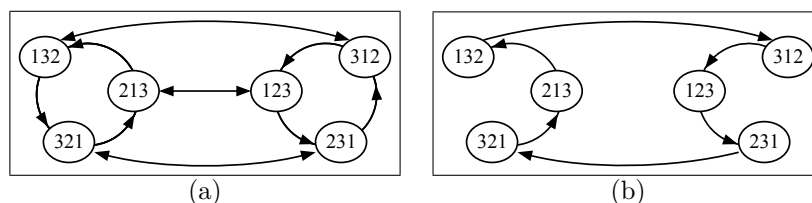


Figure 1: (a) The rotator graph \mathcal{G}_3 , and (b) a Hamilton cycle in \mathcal{G}_3 . The bidirectional arcs are two σ_2 arcs, and the unidirectional arcs are σ_3 arcs.

The literature has considered two natural subgraphs of the rotator graph. First, a *restricted rotator graph* $\mathcal{R}_n(R)$ for $R \subseteq \{2, 3, \dots, n\}$ is obtained by only including arcs labelled σ_r for $r \in R$. Figure 2 shows the restricted rotator graph for $n = 4$ and $R = \{3, 4\}$. Second, an *incomplete rotator graph* $\mathcal{M}_n(m)$ for some fixed maximum value $m \in \{1, 2, \dots, n\}$ is the induced subgraph using the nodes whose final symbol is i for $i \leq m$. In particular, $\mathcal{M}_n(1)$ is isomorphic to \mathcal{G}_{n-1} , and $\mathcal{M}_n(n) = \mathcal{G}_n$. Figure 3 shows the incomplete rotator graph for $n = 4$ and $m = 3$. The restricted rotator graph $\mathcal{R}_n(R)$ with $R = \{r_1, r_2, \dots, r_k\}$ is the directed Cayley graph $\overrightarrow{\text{Cay}}(\{\sigma_{r_1}, \sigma_{r_2}, \dots, \sigma_{r_k}\}, \mathbb{S}_n)$, whereas the incomplete rotator graph $\mathcal{M}_n(m)$ is not isomorphic to a Cayley graph when $1 < m < n$.

We define a *restricted incomplete rotator graph* $\mathcal{B}_n(R, m)$ as the subgraph of \mathcal{G}_n that takes into account both of the prior generalizations. That is, $\mathcal{B}_n(R, m)$ only includes arcs labelled σ_r for $r \in R$ and nodes endings with $i \leq m$. In particular, $\mathcal{B}_n(\{2, 3, \dots, n\}, n) = \mathcal{G}_n$. Figure 5 shows the restricted incomplete rotator graph for $n = 4$, $R = \{3, 4\}$, and $m = 3$.

Rotator graphs are discussed by Corbett [3], restricted rotator graphs with $R = \{n - 1, n\}$ are discussed by Ruskey and Williams [18], and incomplete rotator graphs for all m are discussed by Ponnuswamy and Chaudhary [15]. In each case, the authors construct Hamilton cycles and discuss their applications.

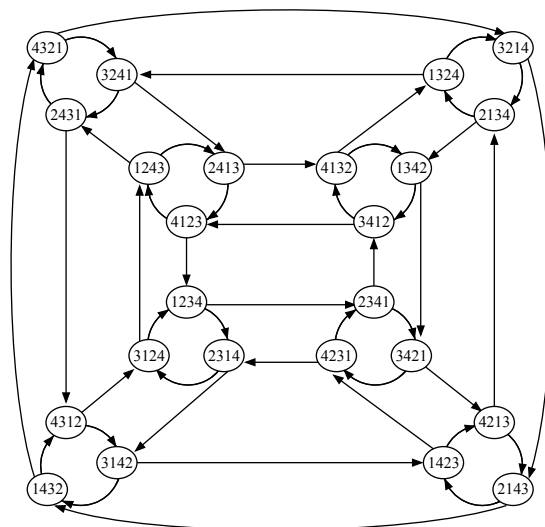


Figure 2: The restricted rotator graph $\mathcal{R}_n(R)$ for $R = \{3, 4\}$. The circular arcs are σ_3 arcs, while the remaining arcs are σ_4 arcs.

For example, Corbett proposed rotator graphs as a topology for point-to-point multiprocessor networks, where Hamilton cycles establish indexing schemes for sorting and for mapping rings and linear arrays. In this context, Ponnuswamy and Chaudhary generalize the size of the networks from $n!$ to $m \cdot (n - 1)!$ for all $m \in \{1, 2, \dots, n\}$ at the expense of node-transitivity when $m \neq n$. On the other hand, the result by Ruskey and Williams provides a construction for “shorthand universal cycles for permutations”, which answered a challenge posed by Knuth in *The Art of Computer Programming* [11] (exercise 112 of section 7.2.1.2). Holroyd, Ruskey, and Williams [8, 9] extended this work by maximizing the number of σ_n used by Hamilton cycles of $\mathcal{R}_n(\{n - 1, n\})$. Other applications of rotator graphs include fault-tolerant file transmission by Hamada et al [7] and parallel sorting by Corbett and Scherson [4]. Additional properties of rotator graphs have been examined including minimum feedback sets by Kuo et al [12] and node-disjoint paths by Yasuto, Ken’Ichi, and Mario [25]. Additional variations of rotator graphs include bi-rotator graphs introduced by Lin and Hsu [13].

In these aforementioned articles, the authors provide efficient algorithms for generating a *Hamilton sequence* r_1, r_2, \dots , meaning that $\sigma_{r_1}, \sigma_{r_2}, \dots$ are the arcs of a Hamilton cycle. For example, 3, 3, 2, 3, 3, 2 is Corbett’s Hamilton sequence in \mathcal{G}_3 as seen in Figure 1 (b). We express this Hamilton cycle as follows

$$321 \sigma_3 213 \sigma_3 132 \sigma_2 312 \sigma_3 123 \sigma_3 231 \sigma_2 \tag{1}$$

where the values represent successive nodes and arcs, and the final node is omitted since it equals the first.

Our motivation for targeting these special cases is simple: In applications, there is often special significance to the rotations σ_r where r is as small or as large as possible. For example, if the sequences are used to generate successive permutations stored in an array, then σ_2 and σ_3 are the two most efficient operations. On the other hand, in the setting of universal cycles, σ_n is the most efficient operation, as discussed in [9].

Our notation is described in Section 2. The ‘optimality’ of our main results is discussed in Section 3 by giving non-existence results for Hamilton cycles in various rotator graphs. Section 4 describes the three sequence building operations, and then we combine these operations to obtain our three main results in Section 5. Efficient algorithms for generating these Hamilton sequences appear in Section 6. Section 7 concludes with open problems.

A preliminary version of this article appeared at IWOCA 2011 [20]. The earlier version focused only on Hamilton cycles in restricted rotator graphs, while the results on incomplete rotator graphs and ‘rewinding’ are new to this version. The discussion of non-existence results is also new. We have changed our notation to accommodate the increased breadth. We have also streamlined our presentation by omitting the results on recycling Corbett’s reuse sequence in Theorems 2 and 5 of [20], and by using an alternative to Algorithm 1 of [20].

2 Notation

Throughout this article we will be applying permutations to strings. There are different conventions for this in the literature, and in this article we follow those used in [9]. For example, suppose $\mathbf{a} = a_1 a_2 \cdots a_n$ and $\mathbf{b} = b_1 b_2 \cdots b_n$ are strings, and $\pi = (\pi_1 \pi_2 \cdots \pi_k)$ is a permutation in cycle notation. The result of applying π to \mathbf{a} is denoted $\mathbf{a} \pi$. If $\mathbf{b} = \mathbf{a} \pi$, then $b_{\pi_i} = a_{i+1}$ for all $1 \leq i \leq k$ (where the index $k + 1$ is treated as 1). In other words, the permutation $(\pi_1 \pi_2 \cdots \pi_k)$ causes the π_i th symbol of \mathbf{b} to be assigned the π_{i+1} st symbol of \mathbf{a} . Given this convention, the prefix-rotation of length r is $\sigma_r = (1 \ 2 \ \cdots \ r)$ since $\mathbf{b} = \mathbf{a} \sigma_r$ implies that $b_i = a_{i+1}$ for $1 \leq i \leq r - 1$ and $b_r = a_1$. For convenience, the specific permutations used in this article are summarized below:

- *inverse prefix-rotation* $\sigma_i^{-1} = (i \ i-1 \ \cdots \ 1)$ (see Sections 3 and 4.3);
- *suffix-rotation* $\varsigma_i = (n \ n-1 \ \cdots \ n-i+1)$ (see Section 4.1);
- *modified prefix-rotation* $\sigma'_i = (2 \ 3 \ \cdots \ i+1)$ (see Section 4.2).
- *adjacent-transposition* $\tau_i = (i-1 \ i)$ (see Section 4.3);

In each case, exponentiation denotes repeated multiplication of permutations, so $\pi^2 = \pi\pi$. When applying multiple permutations to a string, we proceed from left-to-right. For example,

$$1234567 \sigma_5 \sigma_4 = 2345167 \sigma_4 = 3452167. \tag{2}$$

We use three pieces of notation to refer to multiple prefix-rotations applied to a string. In each case, our terminology is based on the associated rotator graphs. Given sequence $S = r_1, r_2, \dots, r_t$ and $\mathbf{a} \in \Pi_n$, the entire path is $\text{path}(\mathbf{a}, S)$ (where the final string is omitted if the path is a cycle), the set of nodes visited on the path is $\text{nodes}(\mathbf{a}, S)$, and the final node visited on the path is $\text{node}(\mathbf{a}, S)$. For example, with respect to (2) we have $S = 5, 4$ and $\mathbf{a} = 1234567$ and

$$\begin{aligned} \text{path}(1234567, 5, 4) &= 1234567 \sigma_5 2345167 \sigma_4 3452167 \\ \text{nodes}(1234567, 5, 4) &= \{1234567, 2345167, 3452167\} \\ \text{node}(1234567, 5, 4) &= 3452167. \end{aligned}$$

In keeping with our graph-based terminology, we refer to nodes and strings interchangeably, and to arcs and rotations interchangeably.

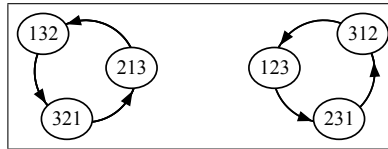
3 Non-Existence Results

In this section we prove several results on the non-existence of Hamilton cycles in certain restricted and restricted incomplete rotator graphs. In each case we relate these non-existence results to the new results we prove in Section 5. The non-existence results also add merit to the previous positive results on incomplete rotator graphs [15] and restricted rotator graphs [18].

We begin with a simple remark. Given a string $\mathbf{a} = a_1 a_2 \dots a_n \in \Pi_n$, an *inversion* is a pair of indices (i, j) such that $i < j$ and $p_i > p_j$. The *parity* of \mathbf{a} is *even* or *odd* when its number of inversions is even or odd, respectively.

Remark 3 *If $\mathbf{a} \in \Pi_n$, then $\mathbf{b} = \mathbf{a} \sigma_i$ has the same parity as \mathbf{a} when i is odd.*

If R consists only of odd values, then Remark 3 implies that there are no directed paths between permutations of opposite parity in $\mathcal{R}_n(R)$. For example, this is verified below for the trivial case of $n = 3$ and $R = \{3\}$, where the odd and even permutations appear on the left and right, respectively.



Remark 4 *The restricted rotator graph $\mathcal{R}_n(R)$ is not strongly connected when R only contains odd values. In particular, $\mathcal{R}_n(R)$ is not strongly connected when n is odd and $R = \{3, n\}$ or $R = \{n - 2, n\}$.*

Remark 4 proves that the restricted rotator graphs do not always have Hamilton cycles when they allow only the rotation σ_n and the “second smallest” rotation σ_3 . Similarly, there is not always a Hamilton cycle when allowing σ_n and the ‘smallest’ rotation σ_2 . This follows from a general result by Rankin [16] whose proof was simplified by Swan [21] (also see Theorem R in [11]).

Remark 5 ([16, 21]) *The restricted rotator graph $\mathcal{R}_n(R)$ does not have a Hamilton cycle when n is even and $R = \{2, n\}$.*

The restricted rotator graph $\mathcal{R}_n(R)$ for $R = \{2, n\}$ is known in the literature as the *directed σ - τ graph*, where $\sigma = \sigma_n$ and $\tau = \tau_2 = \sigma_2$. Remark 5 implies that the directed σ - τ graph does not have a Hamilton cycle for even n . On the other hand, it may have a Hamilton path, as illustrated for $n = 4$ by Figure 4. Determining the existence of Hamilton paths and cycles in the directed σ - τ graph is considered an interesting and difficult open problem. In particular, Knuth rates the difficulty of finding a Hamilton cycle in $\mathcal{R}_n(2, n)$ (for odd n) as 48 out of 50 [11] (see exercise 71 in section 7.2.1.2). Ruskey, Jiang, and Weston [17] investigated this problem using backtracking algorithms and were able to find all five non-isomorphic Hamilton cycles for $n = 5$, as well as Hamilton paths for $n = 4$ and $n = 6$.

The *undirected σ - τ graph* also includes arcs for the inverse operations $\tau_2^{-1} = \tau_2$ and $\sigma_n^{-1} = \sigma_n$. Compton and Williamson provide the only known Hamilton cycle in the undirected σ - τ graph. The special structure of their solution also provides a “doubly-adjacent Gray code” for Π_n but requires 50 pages to prove [2]. The undirected σ - τ graph can also be described as the undirected Cayley graph $\text{Cay}(\{\sigma_2, \sigma_n\}, \mathbb{S}_n)$, where \mathbb{S}_n is the symmetric group corresponding to Π_n . A well-known conjecture is that a Hamilton cycle exists in every connected undirected Cayley graph. This conjecture is often referred to as the *Lovász conjecture*, although the reader should refer to the survey by Pak and Radoičić for a more detailed historical account [14]. Additional surveys on the Hamiltonicity of Cayley graphs are due to Witte and Gallian [24], and Curran and Gallian [5]. In this setting, the rotator graph \mathcal{G}_n is the directed Cayley graph $\overrightarrow{\text{Cay}}(\{\sigma_2, \sigma_3, \dots, \sigma_n\}, \mathbb{S}_n)$ with generators σ_r for $r \in \{2, 3, \dots, n\}$, and the restricted rotator graph $\mathcal{R}_n(R)$ limits the generators to σ_r for $r \in R$.

Remarks 4 and 5 indicate that our first main result — a Hamilton cycle for the restricted rotator graph $\mathcal{R}_n(R)$ with $R = \{2, 3, n\}$ — is ‘optimal’ in the sense that the result is not possible when any rotation from R is removed.

Next we consider the ‘optimality’ of our third main result: A Hamilton cycle for the restricted incomplete rotator graph $\mathcal{B}_n(R, m)$ with $R = \{n-2, n-1, n\}$. From Remark 4, the restricted rotator graph $\mathcal{R}_n(R)$ is not strongly connected for $R = \{n-2, n\}$ when n is odd. Thus, we need to consider $R = \{n-1, n\}$ to prove our third main result is ‘optimal’. Although the restricted rotator graph $\mathcal{R}_n(R)$ with $R = \{n-1, n\}$ has a Hamilton cycle [18], Proposition 1 proves that there is no Hamilton cycle when these rotations are used in restricted incomplete rotator graphs. In fact, these directed graphs are not strongly connected unless $m \in \{n-1, n\}$. This is illustrated for $n = 4$ and $m = 2$ by Figure 6 (a).

To simplify the proof of Proposition 1 it is helpful to consider a Hamilton cycle in the restricted incomplete rotator graph $\mathcal{B}_n(R, m)$ where $n = 4$, $R = \{n-1, n\} = \{3, 4\}$ and $m = n-1 = 3$. This graph appears in Figure 5, and it

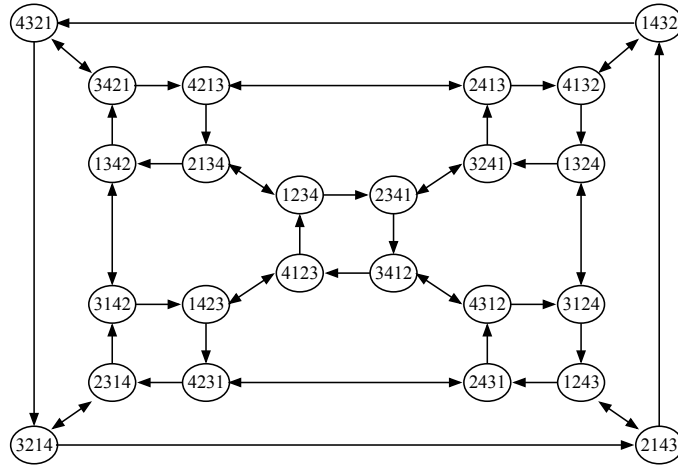


Figure 4: The restricted rotator graph $\mathcal{R}_n(R)$ for $R = \{2, 4\}$. The bidirectional arcs represent two σ_2 arcs, and the unidirectional arcs represent σ_4 arcs. This graph does not have a Hamilton cycle, but it does have a Hamilton path.

has the following Hamilton cycle

$$\begin{aligned}
 &\underline{4}321 \sigma_3 \underline{3}2\underline{4}1 \sigma_3 \underline{2}4\underline{3}1 \sigma_4 \underline{4}3\underline{1}2 \sigma_3 \underline{3}1\underline{4}2 \sigma_4 \underline{1}4\underline{2}3 \sigma_4 \\
 &\underline{4}231 \sigma_3 \underline{2}3\underline{4}1 \sigma_4 \underline{3}4\underline{1}2 \sigma_4 \underline{4}1\underline{2}3 \sigma_3 \underline{1}2\underline{4}3 \sigma_3 \underline{2}4\underline{1}3 \sigma_4 \\
 &\underline{4}1\underline{3}2 \sigma_3 \underline{1}3\underline{4}2 \sigma_4 \underline{3}4\underline{2}1 \sigma_4 \underline{4}2\underline{1}3 \sigma_3 \underline{2}1\underline{4}3 \sigma_4 \underline{1}4\underline{3}2 \sigma_4 .
 \end{aligned}$$

Observe that the underlined symbol 4 has *periodicity* 3 in the above Hamilton cycle, since it appears cyclically in positions 3, 2, 1, 3, 2, 1, ... (starting at position 1 in $\underline{4}321$). This is due to the fact that strings beginning with 4 are followed by applying σ_3 , and otherwise 4 is moved once to the left regardless of whether σ_3 or σ_4 is applied. The proof of Proposition 1 uses the fact that exhaustive lists of $\Pi_{n,m}$ cannot have two symbols with periodicity $n-1$ when $n \geq 4$.

Proposition 1 *The restricted incomplete rotator graph $\mathcal{B}_n(R, m)$ is not strongly connected when $R = \{n-1, n\}$ and $m \in \{2, 3, \dots, n-2\}$.*

Proof: Assume the values of R , m , and n satisfy the stated conditions. We assume $n \geq 4$ since otherwise the proposition is vacuously true by $m \in \{2, 3, \dots, n-2\}$. Therefore, there is a node $\mathbf{a} = a_1 a_2 \dots a_n \in \Pi_{n,m}$ with prefix $a_1 a_2 a_3 a_4 = n \ n-2 \ n-1 \ n-3$. Observe that the symbols n and $n-1$ have periodicity $n-1$ in any directed path of $\mathcal{B}_n(R, m)$. Therefore, there is no directed path from \mathbf{a} to any node $\mathbf{b} = b_1 b_2 \dots b_n \in \Pi_{n,m}$ with prefix $b_1 b_2 = n \ n-1$. Therefore, the graph is not strongly connected. \square

Our Hamilton cycle in the restricted incomplete rotator graph $\mathcal{B}_n(R, m)$ with $R = \{n-2, n-1, n\}$ avoids the obstruction in Proposition 1 by including

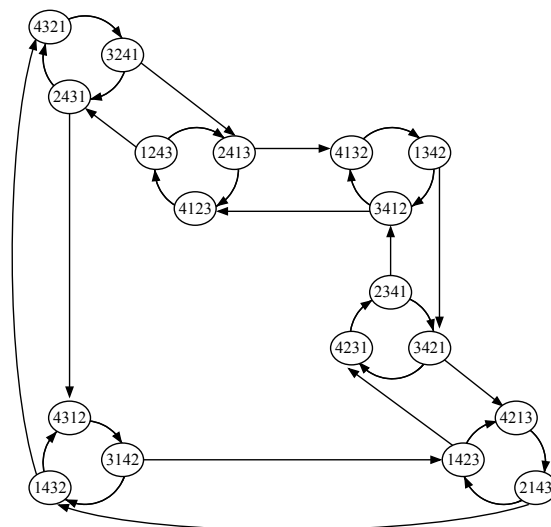


Figure 5: The restricted incomplete rotator graph $\mathcal{B}_4(\{3, 4\}, 3)$. The symbol 4 has periodicity 3 in every Hamilton cycle of this graph.

the next ‘largest’ rotation σ_{n-2} . On the other hand, our second main result — a Hamilton cycle in $\mathcal{B}_n(R, m)$ with $R = \{2, 3, n-1, n\}$ — includes the two ‘smallest’ rotations. We now prove that the inclusion of σ_{n-1} is necessary for Hamiltonicity in our second main result. In fact, we underscore the importance of σ_{n-1} by showing that $\mathcal{B}_n(R, m)$ is not strongly connected when $m = 2$ and $R = \{2, 3, \dots, n-2\} \cup \{n\}$, as illustrated by Figure 6 (b) for $n = 4$. More generally, Proposition 2 considers the absence of many ‘larger’ rotations. (Our second main result may not be ‘optimal’ since the Hamiltonicity of $\mathcal{B}_n(R, m)$ with $R = \{2, n-1, n\}$ and $R = \{3, n-1, n\}$ is unknown.)

Proposition 2 *The restricted incomplete rotator graph $\mathcal{B}_n(R, m)$ is not strongly connected when $R = \{2, 3, \dots, n-m\} \cup \{n\}$ and $m \in \{2, 3, \dots, n-1\}$.*

Proof: Assume the values of R and m satisfy the stated conditions. Let X be the subset of strings in Π_n that have suffix $1\ 2\ \dots\ m$. Notice that X is a strict subset of Π_n . Fix an arbitrary string $\mathbf{a} = a_1 a_2 \dots a_n \in X$. Observe that $a_1 \notin \{1, 2, \dots, m\}$ since $m < n$. Now consider arcs of the form $\mathbf{a}\mathbf{b}$ in $\mathcal{B}_n(R, m)$. Notice that $\mathbf{a}\mathbf{b}$ is not an arc for $\mathbf{b} = \mathbf{a}\ \sigma_n$ since \mathbf{b} is not a node in $\mathcal{B}_n(R, m)$. Thus, if $\mathbf{a}\mathbf{b}$ is an arc, then $\mathbf{b} = \mathbf{a}\ \sigma_i$ where $i \in \{2, 3, \dots, n-m\}$. However, this implies that \mathbf{b} also has suffix $1\ 2\ \dots\ m$. More generally, there is no arc from a node inside X to a node outside X . Therefore, $\mathcal{B}_n(R, m)$ is not strongly connected. \square

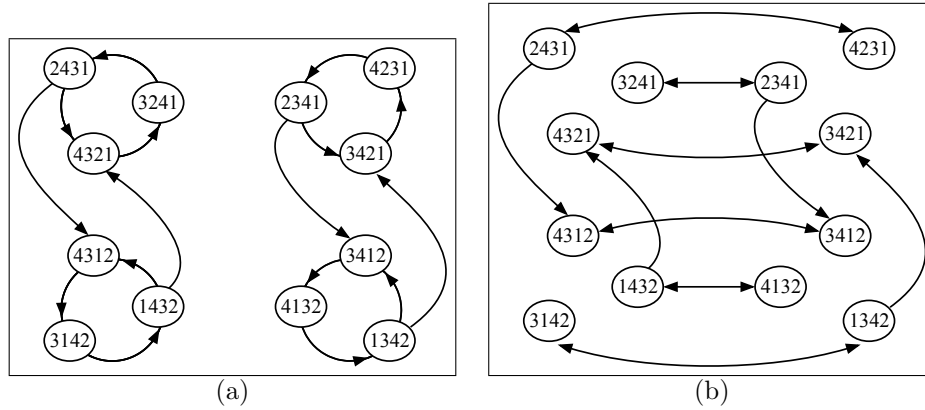


Figure 6: The restricted incomplete rotator graph $\mathcal{B}_n(R, m)$ for $n = 4$ and $m = 2$ with (a) $R = \{3, 4\}$, and (b) $R = \{2, 4\}$. These directed graphs illustrate Propositions 1 and 2.

4 Sequence Building

This section describes the three sequence building operations: reuse, recycle, and rewind. For each operation we state the original theorems using these operations, and discuss generalizations.

4.1 Reusing and Rotator Graphs

The *reuse operation* takes each integer r in a sequence and replaces it by $n, \dots, n, n + 1 - r$, where n is repeated $n - 1$ times. We visualize this operation below.

$$\begin{array}{c}
 \xrightarrow{S=r_1, r_2, \dots} \boxed{\text{reuse}(S)} \xrightarrow{\overbrace{n, \dots, n, n+1-r_1}^{n-1 \text{ copies}}, \overbrace{n, \dots, n, n+1-r_2, \dots}^{n-1 \text{ copies}}} \\
 \hspace{10em} (n = \max(S) + 1)
 \end{array}$$

The value of n is in parentheses since if it is not explicitly provided, then it is implicitly set to the largest value in S plus one. We subscript the operation as $\text{reuse}_n(S)$ when n is explicitly provided. To describe repeated applications of the operation, let $U_1 = 1$ and $U_k = \text{reuse}(U_{k-1})$ for $k > 1$. For example, the first few *reuse sequences* appear below.

$$\begin{aligned}
 U_2 &= \text{reuse}(U_1) = \text{reuse}_2(1) = 2, 3-1 = 2, 2 \\
 U_3 &= \text{reuse}(U_2) = \text{reuse}_3(2, 2) = 3, 3, 4-2, 3, 3, 4-2 = 3, 3, 2, 3, 3, 2 \\
 U_4 &= \text{reuse}(U_3) = \text{reuse}_4(3, 3, 2, 3, 3, 2) \\
 &= 4, 4, 4, 5-3, 4, 4, 4, 5-3, 4, 4, 4, 5-2, 4, 4, 4, 5-3, 4, 4, 4, 5-3, 4, 4, 4, 5-2 \\
 &= 4, 4, 4, 2, 4, 4, 4, 2, 4, 4, 4, 3, 4, 4, 4, 2, 4, 4, 4, 2, 4, 4, 4, 3.
 \end{aligned}$$

Corbett proved that these ‘canonical’ reuse sequences are Hamilton sequences for rotator graphs.

Theorem 1 ([3]) *The reuse sequence U_n is a Hamilton sequence for the rotator graph \mathcal{G}_n .*

For example, we can verify Theorem 1 for $n = 4$ by creating the associated list of strings.

$$\begin{aligned} & \text{path}(4321, U_4) && (3) \\ &= \text{path}(4321, 4, 4, 4, 2, 4, 4, 4, 2, 4, 4, 4, 3, 4, 4, 4, 2, 4, 4, 4, 2, 4, 4, 4, 3) \\ &= 4321 \sigma_4 3214 \sigma_4 2143 \sigma_4 1432 \sigma_2 4132 \sigma_4 1324 \sigma_4 3241 \sigma_4 2413 \sigma_2 \\ & \quad 4213 \sigma_4 2134 \sigma_4 1342 \sigma_4 3421 \sigma_3 4231 \sigma_4 2314 \sigma_4 3142 \sigma_4 1423 \sigma_2 \\ & \quad 4123 \sigma_4 1234 \sigma_4 2341 \sigma_4 3412 \sigma_2 4312 \sigma_4 3124 \sigma_4 1243 \sigma_4 2431 \sigma_3 \end{aligned}$$

Notice that every string in Π_4 appears once and that the list is cyclic. Thus, U_4 is a Hamilton sequence for \mathcal{G}_4 . Also notice that U_4 contains at least one copy of each symbol in $\{2, 3, 4\}$. More generally, a simple inductive argument proves that U_n contains at least one copy of every symbol in $\{2, 3, \dots, n\}$, and thus U_n is not the Hamilton sequence of any (strictly) restricted rotator graph.

To understand how Theorem 1 works in general, we present one remark and one lemma. To illustrate the remark, observe that every fourth string in (3) is followed by its three ‘full rotations’. For example, 4231 is followed by $4231 \sigma_4 = 2314$, $4231 \sigma_4^2 = 3142$, and $4231 \sigma_4^3 = 1423$. In general, this immediately follows from the fact that $\text{reuse}_n(r)$ begins with $n - 1$ copies of n .

Remark 6 *For any sequence S , if $\mathbf{b} \in \text{nodes}(\mathbf{a}, \text{reuse}(S))$, then for all $1 \leq i \leq n - 1$ it is also true that $\mathbf{b} \sigma_n^i \in \text{nodes}(\mathbf{a}, \text{reuse}(S))$. In other words, reused sequences create lists of strings that are closed under full rotations.*

To describe the lemma, let $\varsigma_i = (n \ n-1 \ \dots \ n-i+1)$ denote the *suffix-rotation* operation. Consider the following cycle of strings

$$4321 \varsigma_3 4132 \varsigma_3 4213 \varsigma_2 4231 \varsigma_3 4123 \varsigma_3 4312 \varsigma_2 . \tag{4}$$

Notice that (4) contains the strings in Π_4 starting with 4 in the same order they appear in (3). Furthermore, the sequence of suffix-rotations between these strings is 3, 3, 2, 3, 3, 2 and this sequence is precisely U_3 . Thus, $U_4 = \text{reuse}(U_3)$ has converted the prefix-rotations of U_3 into suffix-rotations. Lemma 1 states this result formally.

Lemma 1 *If \mathbf{a} is a string of length n , and $r \in \{2, 3, \dots, n - 1\}$, then*

$$\text{node}(\mathbf{a}, \text{reuse}_n(r)) = \mathbf{a} \varsigma_r .$$

Proof: The proof is obtained from the following derivation

$$\text{node}(\mathbf{a}, \text{reuse}_n(r)) = \mathbf{a} \sigma_n^{n-1} \sigma_{n-r+1} = \mathbf{a} \circ_n \sigma_{n-r+1} = \mathbf{a} \varsigma_r$$

Ruskey and Williams proved that these ‘canonical’ recycle sequences are Hamilton sequences for restricted rotator graphs using σ_{n-1} and σ_n .

Theorem 3 ([18]) *The canonical recycle sequence C_n is a Hamilton sequence for the restricted rotator graph $\mathcal{R}_n(R)$ with $R = \{n - 1, n\}$.*

For example, we can verify Theorem 3 for $n = 4$ by creating the associated list of strings.

$$\begin{aligned} & \text{path}(4321, U_4) && (5) \\ &= \text{path}(4321, 4, 4, 3, 3, 4, 4, 3, 3, 4, 4, 3, 4, 4, 4, 3, 3, 4, 4, 3, 3, 4, 4, 3, 4, 4, 3, 4) \\ &= 4321 \sigma_4 3214 \sigma_4 2143 \sigma_3 1423 \sigma_3 4213 \sigma_4 2134 \sigma_4 1342 \sigma_3 3412 \sigma_3 \\ & \quad 4132 \sigma_4 1324 \sigma_4 3241 \sigma_3 2431 \sigma_4 4312 \sigma_4 3124 \sigma_4 1243 \sigma_3 2413 \sigma_3 \\ & \quad 4123 \sigma_4 1234 \sigma_4 2341 \sigma_3 3421 \sigma_3 4231 \sigma_4 2314 \sigma_4 3142 \sigma_3 1432 \sigma_4 \end{aligned}$$

Notice that every string in Π_4 appears once and the that list is cyclic. Thus, C_4 is a Hamilton sequence for $\mathcal{R}_4(R)$ where $R = \{3, 4\}$.

To shed some light on how Theorem 3 works in general, we present one lemma. Let $\sigma'_i = (2\ 3 \cdots i+1)$ denote a modified prefix-rotation that begins at the second symbol instead of the first symbol. Consider the following cycle of strings

$$4321 \sigma'_3 4213 \sigma'_3 4132 \sigma'_2 4312 \sigma'_3 4123 \sigma'_3 4231 \sigma'_2 . \tag{6}$$

Notice that (6) contains the strings in Π_4 starting with 4 in the same order they appear in (5). Furthermore, the sequence of modified prefix-rotations between these strings is 3, 3, 2, 3, 3, 2 and this sequence is precisely C_3 . Thus, $C_4 = \text{recycle}(C_3)$ has converted the prefix-rotations of C_3 into modified prefix-rotations. Lemma 2 states this result formally; it was proven in a slightly different form as the second identity of Lemma 2 in [9].

Lemma 2 ([9]) *If \mathbf{a} is a string of length n , and $r \in \{2, 3, \dots, n - 1\}$, then*

$$\text{node}(\mathbf{a}, \text{recycle}_n(r)) = \mathbf{a} \sigma'_r.$$

Proof: The proof is obtained from the following derivation

$$\text{node}(\mathbf{a}, \text{recycle}_n(r)) = \mathbf{a} \sigma_n \sigma_n \sigma_{n-1}^{r-1} \sigma_n^{n-r-1} = \mathbf{a} \sigma_n^2 \sigma_{n-1}^{r-1} \sigma_n^{n-r-1} = \mathbf{a} \sigma'_r$$

where the last equality is known as the “recycling identity” in [20]. □

While Lemma 2 gives a concise expression for every n th string in lists of the form $\text{path}(n\ n-1 \cdots 1, \text{recycle}(S))$, the intermediate strings are more difficult to describe. In other words, there is no simple recycling analogue to Remark 6 on reusing. For this reason, additional results on recycling have been handled on a case-by-case basis. For example, the preliminary version of this article proved that Corbett’s canonical reuse sequences are recyclable.

Theorem 4 ([20]) *If $S = U_{n-1}$ is the canonical reuse sequence, then $\text{recycle}(S)$ is a Hamilton sequence for the restricted rotator graph $\mathcal{R}_n(R)$ with $R = \{n-1, n\}$.*

Theorem 4 was proven in [20] to answer a conjecture in [9] involving short-hand universal cycles of permutations. It also provided the basis for the main algorithm presented in [20]. In this version of the article we simplify the presentation by instead basing our main algorithm on Theorem 3. We return to the question of recycling Hamilton sequences in the open problems of Section 7.1.

4.3 Rewinding and Incomplete Rotator Graphs

The reuse and recycle operations are similar in the sense that they transform each individual value into n values. On the other hand, the rewind operation repeats an entire sequence m times, but each time it leaves off one or two symbols, and inserts n in appropriate positions to make up for loss.

To define the operation, suppose $S = r_1, r_2, \dots$ is a sequence in which the largest symbol $n - 1$ appears in the last two positions. For convenience, we let T denote the remaining sequence in S . That is, $S = T, n - 1, n - 1$ where $\max(S) = n - 1$. The *rewind operation* creates a sequence according to the following subsequences along with a finishing ‘tail’:

- The first subsequence is $T, n - 1, n$.
- The second through $(m - 1)$ st subsequences are T, n .
- The last subsequence is $T, n - 1, n$.
- The tail is n, n, \dots, n ($m - 2$ copies).

Observe that each subsequence omits the last one or two symbols of S , and the tail inserts copies of n to ensure the resulting sequence has length $m \cdot |S|$. We visualize this operation below.

$$\begin{array}{c}
 \xrightarrow{(S=r_1, r_2, \dots)} \boxed{\text{rewind}(S)} \xrightarrow{\begin{array}{c} T, n-1, n, \overbrace{T, n, \dots, T, n}^{m-2 \text{ copies}}, T, n-1, n, \overbrace{n, \dots, n}^{m-2 \text{ copies}} \\ (n = \max(S) + 1) \quad (m = n) \end{array}} \\
 \xrightarrow{(S=r_1, r_2, T)}
 \end{array}$$

Again, the value of n is either explicitly given, or is implicitly set to the largest value in S plus one. Similarly, the value of m is either explicitly given, or is implicitly set to n . In other words, by default the rewind operation results in a sequence of length $n \cdot |S|$. By convention, when both ‘optional’ parameters are explicitly provided, n is first and m is second, and when only one ‘optional’ parameter is provided it is m . Thus, $\text{rewind}_{j,k}(S)$ explicitly sets $m = j$ and $n = k$, while $\text{rewind}_j(S)$ has $m = j$ and implicitly $n = \max(S) + 1$. To describe repeated applications of the operation, let $W_2 = 2, 2$ and $W_k = \text{rewind}(W_{k-1})$, where n and m are implicit. Observe that in this one-parameter *rewind sequence* W_n has length $n!$. For example, the ‘canonical’ one-parameter rewind sequence for $n = 3$ is $W_3 = 2, 3, 3, 2, 3, 3$ and appears below for $n = 4$ with its associated

subsequences underlined

$$\begin{aligned} W_4 &= \text{rewind}(W_3) \\ &= \text{rewind}_{4,4}(2, 3, 3, 2, 3, 3) \\ &= \underline{2, 3, 3, 2, 3, 4}, \underline{2, 3, 3, 2, 4}, \underline{2, 3, 3, 2, 4}, \underline{2, 3, 3, 2, 3, 4}, 4, 4. \end{aligned}$$

More generally, let $W_{k,j} = \text{rewind}_j(W_{k-1})$ for $j \in \{2, 3, \dots, n\}$. Observe that in this two-parameter *rewind sequence* $W_{n,m}$ has length $m \cdot (n - 1)!$. For example, the ‘canonical’ two-parameter rewind sequences appear below for $n = 4$ and $m = 3, 2$

$$\begin{aligned} W_{4,3} &= \text{rewind}_3(W_3) \\ &= \text{rewind}_{4,3}(2, 3, 3, 2, 3, 3) \\ &= \underline{2, 3, 3, 2, 3, 4}, \underline{2, 3, 3, 2, 4}, \underline{2, 3, 3, 2, 3, 4}, 4 \\ W_{4,2} &= \text{rewind}_2(W_3) \\ &= \text{rewind}_{4,2}(2, 3, 3, 2, 3, 3) \\ &= \underline{2, 3, 3, 2, 3, 4}, \underline{2, 3, 3, 2, 3, 4}. \end{aligned}$$

Ponnuswamy and Chaudhary proved that this sequence operation creates Hamilton cycles in incomplete rotator graphs.

Theorem 5 ([15]) *Repeated rewinding gives the Hamilton sequence $W_{n,m}$ for the incomplete rotator graph $\mathcal{M}_n(m)$.*

To understand how Theorem 5 works, let us consider the associated list of strings for $n = m = 4$, where the result of applying each successive subsequence is shown on an individual row, and the crossed out strings will be explained later.

$$\begin{aligned} \text{path}(4123, W_4) &= \text{path}(4321, 2, 3, 3, 2, 3, 4, 2, 3, 3, 2, 4, 2, 3, 3, 2, 4, 2, 3, 3, 2, 3, 4, 4, 4) \\ &= 4321 \sigma_2 \ 3421 \sigma_3 \ 4231 \sigma_3 \ 2341 \sigma_2 \ 3241 \sigma_3 \ 2431 \sigma_4 \\ &\quad 4312 \sigma_2 \ 3412 \sigma_3 \ 4132 \sigma_3 \ 1342 \sigma_2 \ 3142 \sigma_4 \ ~~1432~~ \\ &\quad 1423 \sigma_2 \ 4123 \sigma_3 \ 1243 \sigma_3 \ 2413 \sigma_2 \ 4213 \sigma_4 \ ~~2143~~ \\ &\quad 2134 \sigma_2 \ 1234 \sigma_3 \ 2314 \sigma_3 \ 3124 \sigma_2 \ 1324 \sigma_3 \ 3214 \sigma_4 \\ &\quad 2143 \sigma_4 \ 1432 \sigma_4 \end{aligned} \tag{7}$$

Notice that the first row of (7) starts with a string ending in 1 and then contains every other string ending in 1. Similarly, the fourth row starts with a string ending in 4 and then contains every other string ending in 4. On the other hand, the second and third rows start with strings ending in 2 and 3, respectively, and then contain all but one of the other strings ending in 2 and 3, respectively. The fact that these rows contain all strings in $\Pi_{4,1}$ and $\Pi_{4,4}$, and all but one string in $\Pi_{4,2}$ and $\Pi_{4,3}$, follows from the initial string of the respective rows and the following remark.

Remark 7 *If $T, n-1, n-1$ is a Hamilton sequence for \mathcal{G}_{n-1} and $\mathbf{a} \in \Pi_{n,x}$, then*

1. $\text{nodes}(\mathbf{a}, T, n-1) = \Pi_{n,x}$, and
2. $\text{nodes}(\mathbf{a}, T) = \Pi_{n,x} - \{\mathbf{a} \circ_{n-1}\}$.

Observe that the second point of Remark 7 explicitly states the string that is ‘lost’ when applying the intermediate subsequence. For example, the second row of (7) begins with 4312, and $4312 \circ_3 = 1432$ is lost. Similarly, the third row begins with 1423, and $1423 \circ_3 = 2143$ is ‘lost’. In both cases, these lost strings are crossed out in (7).

While Remark 7 describes the set of strings obtained from each subsequence, Remark 8 explicitly states the last string obtained by applying each subsequence. In this remark we let $\tau_i = (i-1\ i)$ be permutation that *swaps* the i th and $(i-1)$ st of a string. Observe that if \mathbf{a} is a string of length n , then $\mathbf{a} \circ_{n-1} \sigma_n = \mathbf{a} \tau_n$.

Remark 8 *If $T, n-1, n-1$ is a Hamilton sequence for \mathcal{G}_{n-1} and $\mathbf{a} \in \Pi_{n,x}$, then*

1. $\text{node}(\mathbf{a}, T, n-1, n) = \mathbf{a} \circ_{n-1} \sigma_n = \mathbf{a} \tau_n$, and
2. $\text{node}(\mathbf{a}, T, n) = \mathbf{a} \circ_{n-1} \circ_{n-1} \sigma_n = \mathbf{a} \circ_{n-1} \tau_n$.

To see Remark 8 in action, note that the first strings in each row of (7) are

$$4321 \tau_4 \quad 4312 \circ_3 \tau_4 \quad 1423 \circ_3 \tau_4 \quad 2134 \tau_4 \quad 2143,$$

where successive strings follow from the first, second, second, and first points.

Besides Remarks 7 and 8 there are three ‘tricks’ to Ponnuswamy and Chaudhary’s order. First, each successive subsequence is applied to a string whose last symbol is one larger than for the previous subsequence. For example, the first four rows in (7) begin with strings ending in 1, 2, 3, and 4, respectively. Second, the strings that are ‘lost’ while applying the rotations in the intermediate subsequence are ‘found’ when applying the rotation in the tail. For example, 1432 and 2143 appear in the last row of (7). Third, the order can be finished ‘early’. For example, if σ_3, σ_4 is applied at the end of the second row in (7) instead of σ_4 , then the cyclic order finishes with 1432, thereby avoiding all strings ending in 3 or 4. Similarly, if $\sigma_3, \sigma_4, \sigma_4$ is applied at the end of the third row in (7) instead of σ_4 , then the cyclic order finishes with 2143, 1432, thereby avoiding all strings ending in 4.

Using this outline we generalize Theorem 5 to any Hamilton sequence ending with two copies of its largest symbol. Furthermore, if the Hamilton sequence is for a restricted rotator graph on strings of length $n-1$, then rewinding only adds σ_n to the set of rotations used.

Theorem 6 *Hamilton sequences for restricted rotator graphs can be rewound into Hamilton sequences for incomplete rotator graphs. More specifically, if S is a Hamilton sequence for $\mathcal{R}_{n-1}(R)$ whose last two symbols are $n-1$, then $\text{rewind}_m(S)$ is a Hamilton sequence for $\mathcal{B}_n(R \cup \{n\}, m)$ for any $m \in \{2, 3, \dots, n\}$.*

Proof: Following our earlier conventions, let $S = T, n - 1, n - 1$. Since S only contains symbols in R , it follows that $\text{rewind}_m(S)$ only contains symbols in $R \cup \{n\}$ by the definition of rewind . Also, $|\text{rewind}_m(S)| = m|S| = m \cdot (n - 1)!$. Thus, we need to prove that $\text{nodes}(n \ n-1 \ \cdots \ 1, \text{rewind}_m(S))$ contains every string in $\Pi_{n,x}$ for each x satisfying $1 \leq x \leq m$, and that the resulting path is a Hamilton cycle by showing $\text{node}(n \ n-1 \ \cdots \ 1, \text{rewind}_m(S)) = n \ n-1 \ \cdots \ 1$.

Towards our goal, let us focus on the last string obtained by applying each successive subsequence. Let $\mathbf{a}_i = \text{node}(\mathbf{a}, R_1, R_2, \dots, R_{i-1})$ for all $1 \leq i \leq m+1$ and $\mathbf{a} = n \ n-1 \ \cdots \ 1$, where $R_1 = T, n - 1, n$ and $R_j = T, n$ for $2 \leq j \leq m - 1$ and $R_m = T, n - 1, n$ are the subsequences. By Remark 8 we have

$$\begin{aligned} \mathbf{a}_1 &= n \ n-1 \ \cdots \ 1 \\ \mathbf{a}_2 &= \mathbf{a}_1 \ \tau_n = n \ n-1 \ \cdots \ 3 \ 1 \ 2 \\ \mathbf{a}_3 &= \mathbf{a}_2 \ \tau_{n-1} \ \tau_n = 1 \ n \ n-1 \ \cdots \ 4 \ 2 \ 3 \\ \mathbf{a}_4 &= \mathbf{a}_3 \ \tau_{n-1} \ \tau_n = 2 \ 1 \ n \ n-1 \ \cdots \ 5 \ 3 \ 4 \\ &\vdots \\ \mathbf{a}_m &= \mathbf{a}_{m-1} \ \tau_{n-1} \ \tau_n = m-2 \ m-3 \ \cdots \ 1 \ n \ n-1 \ \cdots \ m+1 \ m-1 \ m \\ \mathbf{a}_{m+1} &= \mathbf{a}_m \ \tau_n = m-2 \ m-3 \ \cdots \ 1 \ n \ n-1 \ \cdots \ m-1. \end{aligned}$$

Observe that \mathbf{a}_x ends with i for $1 \leq x \leq m$. Therefore, by Remark 7, each of our desired strings are in $\text{nodes}(n \ n-1 \ \cdots \ 1, \text{rewind}_m(S))$ except possibly for the ‘lost’ strings. The second point of Remark 7 allows us to give an expression for the ‘lost’ strings. We provide this expression below and then compare to the strings obtained by applying the $m - 2$ copies of n in the tail of $\text{rewind}_m(S)$ as rotations to \mathbf{a}_{m+1}

$$\begin{aligned} &\{\mathbf{a}_2 \ \tau_{n-1}, \mathbf{a}_3 \ \tau_{n-1}, \dots, \mathbf{a}_{m-1} \ \tau_{n-1}\} \\ &= \{1 \ n \ n-1 \ \cdots \ 2, 2 \ 1 \ n \ n-1 \ \cdots \ 3, \dots, m-1 \ m-2 \ \cdots \ 1 \ m\} \\ &\subseteq \text{nodes}(m-2 \ m-3 \ \cdots \ 1 \ n \ n-1 \ \cdots \ m-1, n, n, \dots, n) \\ &= \text{nodes}(\mathbf{a}_{m+1}, n, n, \dots, n). \end{aligned}$$

Therefore, all of the ‘lost’ strings are ‘found’. Furthermore, $\text{node}(\mathbf{a}_{m+1}, n, n, \dots, n) = n \ n-1 \ \cdots \ 1$, so the tail finishes at the starting node to complete the cycle. \square

5 New Sequences

In this section we combine our results from Section 4 in several ways. As in Section 4 we visualize sequence building by using small diagrams. In particular, we let $\boxed{\text{recycle}^k}$ denote k successive applications of the recycle operation. Thus,

$$\xrightarrow{1} \boxed{\text{recycle}^k} \xrightarrow{C_k} = \xrightarrow{1} \boxed{\text{recycle}} \rightarrow \boxed{\text{recycle}} \rightarrow \dots \rightarrow \boxed{\text{recycle}} \xrightarrow{C_k}.$$

In each of our results we use the above sequence as our “base sequence”. This results in a presentation that is slightly simpler than our preliminary article which used Theorem 4 in this capacity.

Now we present our three main results. Theorem 7 proves that the following sequence gives a Hamilton sequence using only the rotations $\sigma_2, \sigma_3,$ and σ_n .

$$\xrightarrow{1} \boxed{\text{recycle}^{n-1}} \xrightarrow{C_{n-1}} \boxed{\text{reuse}} \rightarrow \text{Hamilton sequence for } \mathcal{R}_n(R) \text{ with } R=\{2,3,n\}.$$

Theorem 7 *The sequence reuse(S) for S = C_{n-1} is a Hamilton sequence for the restricted rotator graph $\mathcal{R}_n(R)$ with $R = \{2, 3, n\}$.*

Proof: By Theorem 3, $S = C_{n-1}$ is a Hamilton sequence for the restricted rotator graph $\mathcal{R}_{n-1}(X)$ with $X = \{n - 2, n - 1\}$. By Theorem 2, reuse(S) is a Hamilton sequence for the restricted rotator graph $\mathcal{R}_n(R)$ with

$$R = \{n - (n - 2) + 1, n - (n - 1) + 1, n\} = \{2, 3, n\}.$$

□

Before providing our next two results, we must address a minor issue between reusing and recycling on one side, and rewinding on the other side. Recall that Theorem 6 requires a sequence to end in two copies of the largest symbol before it can be properly rewound. Unfortunately, this precondition is not always met after recycling a sequence, nor is it ever met after reusing a sequence. To bridge this incompatibility, we take advantage of Remark 2 from Section 1. Recall that every shift of a Hamilton sequence in a restricted rotator graph is also a Hamilton sequence. Furthermore, it is easy to observe that both reusing and recycling result in sequences that begin with two copies of the largest symbol. Thus, we can shift these sequences twice so that they meet the necessary condition.

Remark 9 *If S is a sequence, then reuse(S) and recycle(S) begin with two copies of the largest symbol max(S)+1. Thus, shift²(reuse(S)) and shift²(recycle(S)) end with two copies of the largest symbol max(S) + 1.*

Visually, we denote this double-shift by $\boxed{\text{shift}^2}$. Theorem 8 proves that the following sequence gives a Hamilton sequence for any incomplete rotator graph using only the rotations $\sigma_{n-2}, \sigma_{n-1},$ and σ_n .

$$\xrightarrow{1} \boxed{\text{recycle}^{n-1}} \xrightarrow{C_{n-1}} \boxed{\text{shift}^2} \xrightarrow[\text{m}]{\text{shift}^2(C_{n-1})} \boxed{\text{rewind}_m} \rightarrow \text{Hamilton sequence for } \mathcal{B}_n(R,m) \text{ with } R=\{n-2,n-1,n\} \text{ and } m \in \{2,3,\dots,n\}.$$

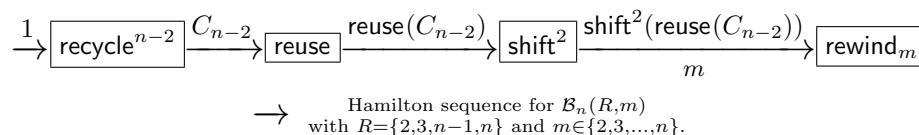
Theorem 8 *The sequence rewind_m(S) for S = shift²(C_{n-1}) is a Hamilton sequence for the restricted and incomplete rotator graph $\mathcal{B}_n(R, m)$ with $R = \{n - 2, n - 1, n\}$ and any $m \in \{2, 3, \dots, n\}$.*

Proof: By Theorem 3, C_{n-1} is a Hamilton sequence for the restricted rotator graph $\mathcal{R}_{n-1}(X)$ with $X = \{n-2, n-1\}$. By Remarks 2 and 9, $S = \text{shift}^2(C_{n-1})$ is also Hamilton sequence for this restricted rotator graph, and moreover S ends with two copies of $n-1$. By Theorem 6, $\text{rewind}_m(S)$ is a Hamilton sequence for the restricted and incomplete rotator graph $\mathcal{B}_n(R, m)$ with

$$R = \{n-2, n-1\} \cup \{n\} = \{n-2, n-1, n\}$$

and any $m \in \{2, 3, \dots, n\}$. □

Theorem 9 proves that the following sequence gives a Hamilton sequence for any incomplete rotator graph using only the rotations $\sigma_2, \sigma_3, \sigma_{n-1}$ and σ_n .



Theorem 9 *The sequence $\text{rewind}_m(S)$ for $S = \text{shift}^2(\text{reuse}(C_{n-2}))$ is a Hamilton sequence for the restricted and incomplete rotator graph $\mathcal{B}_n(R, m)$ with $R = \{2, 3, n-1, n\}$ and any $m \in \{2, 3, \dots, n\}$.*

Proof: By Theorem 3, C_{n-2} is a Hamilton sequence for the restricted rotator graph $\mathcal{R}_{n-2}(X)$ with $X = \{n-3, n-2\}$. By Theorem 2, $\text{reuse}(C_{n-2})$ is a Hamilton sequence for the restricted rotator graph $\mathcal{R}_n(Y)$ with

$$Y = \{n-1 - (n-3) + 1, n-1 - (n-2) + 1, n-1\} = \{2, 3, n-1\}.$$

By Remarks 2 and 9, $S = \text{shift}^2(\text{reuse}(C_{n-2}))$ is also Hamilton sequence for this restricted rotator graph, and moreover S ends with two copies of $n-1$. By Theorem 6, $\text{rewind}_m(S)$ is a Hamilton sequence for the restricted and incomplete rotator graph $\mathcal{B}_n(R, m)$ with

$$R = \{2, 3, n-1\} \cup \{n\} = \{2, 3, n-1, n\}$$

and any $m \in \{2, 3, \dots, n\}$. □

6 Algorithms

In this section we describe how each of the sequences in Section 5 can be generated by a loopless algorithm. In this context, a *loopless algorithm* creates each successive symbol in a sequence in worst-case $O(1)$ -time, where the hidden constant is independent of n and m . As is customary, the algorithms call a visit routine each time the next symbol in the sequence is created. To clarify an important point, a loopless algorithm can take more than $O(1)$ -time to initialize itself and visit the first symbol in the sequence.

As a starting point, observe that Theorems 7, 8, and 9 all involve canonical recycle sequences. A loopless algorithm for generating these sequences was

presented in [18], where 0 and 1 were used to represent n and $n - 1$, respectively. Besides this minor difference, we also use the opportunity to slightly simplify the algorithm and correct a problem with its last iteration. The result is `RecycleSequence` found in Algorithm 1. In the algorithm, the *Iverson bracket* on line 6 converts a boolean value into an integer as follows: $\llbracket \text{true} \rrbracket = 1$ and $\llbracket \text{false} \rrbracket = 0$. The purpose of line 15 is explained later in this section.

Algorithm 1 `RecycleSequence` generates the recycle sequence C_n , which is a Hamilton sequence for the restricted rotator graph $\mathcal{R}_n(R)$ with $R = \{n - 1, n\}$.

Procedure `RecycleSequence`(n)

```

1:  $a_1 \cdots a_n \leftarrow 0 \cdots 0$ 
2:  $f_1 \cdots f_n \leftarrow 1 \cdots n$ 
3: while  $f_1 < n$ 
4:    $j \leftarrow f_1$ 
5:    $f_1 \leftarrow 1$ 
6:   visit( $n - (j + \llbracket a_j \leq 1 \rrbracket \bmod 2)$ )
7:    $a_j \leftarrow a_j + 1$ 
8:   if  $a_j = n - j$ 
9:      $a_j \leftarrow 0$ 
10:     $f_j \leftarrow f_{j+1}$ 
11:     $f_{j+1} \leftarrow j + 1$ 
12:   end
13: end
14: visit( $n - (n \bmod 2)$ )
15:  $f_1 \leftarrow 1$ 

```

Theorem 10 ([18]) `RecycleSequence`(n) generates each successive symbol in the canonical recycle sequence C_n in worst-case $O(1)$ -time.

To adapt Theorem 10 for the sequences in Section 5, we must reuse and rewind the recycle sequence. To reuse a sequence we can replace the `visit` calls in its associated algorithm with calls to the following `reuse` procedure.

Procedure `reuse`(n, r)

```

1: for  $i = 1$  to  $n - 1$ 
2:   visit( $n$ )
3: end
4: visit( $n - r + 1$ )

```

Let `RecycleSequence'`(n) be the result of replacing each `visit`(x) by `reuse`($n+1, x$) on lines 6 and 14 of `RecycleSequence`(n). Thus, a call to `RecycleSequence'`($n-1$) generates `reuse`(C_{n-1}), which is the sequence from Theorem 7.

To rewind a sequence we can call its associated algorithm m times. More precisely, if $S = T, n-1, n-1$ is a sequence where $n-1 = \max(S)$, and `A`($n-1$) is an algorithm that generates T , then the following procedure will generate `rewindm`(S).

Procedure `rewind`(n, m, A)

```

1: A( $n - 1$ )
2: visit( $n - 1$ )
3: visit( $n$ )
4: for  $i = 1$  to  $m - 2$ 
5:   A( $n - 1$ )
6:   visit( $n$ )
7: end
8: A( $n - 1$ )
9: visit( $n - 1$ )
10: visit( $n$ )
11: for  $i = 1$  to  $m - 2$ 
12:   visit( $n$ )
13: end

```

The above procedure is loopless, so long as `A`($n-1$) is loopless and initially calls `visit` in $O(1)$ -time. At first glance, this is not true of `RecycleSequence`(n) since its initialization on lines 1 and 2 takes $O(n)$ -time. However, this initialization does not need to be repeated on successive calls to `RecycleSequence`(n) since the two arrays are reset to their initial values after line 15 is executed. This fact is stated in Remark 10, and can be understood by referring to Algorithm M and Algorithm H for generating multi-radix numbers in Knuth [11].

Remark 10 *When `RecycleSequence`(n) terminates, its array variables are set to $a_1 \cdots a_n \leftarrow 0 \cdots 0$ and $f_1 \cdots f_n \leftarrow 1 \cdots n$.*

The desired portion of $\text{shift}^2(C_{n-1})$ is generated by `RecycleSequence`($n - 1$), so long as we ‘skip’ the first two visited values. Let `RecycleSequence''`(n) denote this necessary modification of `RecycleSequence`(n), with the understanding that successive calls also skip over lines 1 and 2. The result is that a call to `rewind`(n, m, A) for $A = \text{RecycleSequence}''$ will generate $\text{rewind}_m(\text{shift}^2(C_{n-1}))$, which is the sequence from Theorem 8. Similarly, these ideas can be used to generate the sequence from Theorem 9. Collectively, these results imply the following theorem.

Theorem 11 *The sequences in Theorems 7, 8, and 9 can be generated by a loopless algorithm.*

Theorem 11 can be strengthened in the sense that multiple symbols can be visited at the same time. To clarify this point, consider an algorithm that maintains an array of length n that holds the next n symbols in the sequence of length $n!$ it is generating. In this situation, the algorithm would visit $(n - 1)!$ subsequences of n symbols, instead of $n!$ individual symbols. The goal in this situation is to update the entire array in worst-case $O(1)$ -time. This can goal can only be accomplished if successive subsequences differ in a constant number of positions. For an example of how this situation applies to the sequences in this article, notice that each subsequence of n symbols in $\text{reuse}(C_{n-1})$ has the

following form

$$n, n, \overbrace{n-1, \dots, n-1}^{n-2 \text{ copies}} \quad \text{or} \quad n, n, \overbrace{n-1, \dots, n-1}^{n-3 \text{ copies}}, n.$$

Therefore, successive subsequences of length n differ in at most one position, and so the associated array of length n can be updated worst-case $O(1)$ -time. See [20] for further details on this type of optimization.

7 Final Remarks

In this article we have considered the reuse operation by Corbett, the recycle operation by Ruskey and Williams, and the rewind operation by Ponnuswamy and Chaudhary. We have shown that these operations can be combined to create Hamilton sequences for restricted and incomplete rotator graphs using a limited number of prefix-rotations σ_i for ‘small’ and ‘large’ values of i . In two cases our set of rotations is ‘optimal’ in the sense of providing minimal sets of operations that ensure the existence of Hamilton cycles. Furthermore, we have provided loopless algorithms for generating these Hamilton sequences.

7.1 The Recycling Question

While the reuse and rewind operations have been made fairly transparent by the results of Section 4, the recycling operation remains somewhat mysterious. In general, we are interested in the following open problem.

Question 1 *If S is a Hamilton sequence for the rotator graph \mathcal{G}_{n-1} , then under what conditions on S is $\text{recycle}(S)$ a Hamilton sequence for the restricted rotator graph $\mathcal{R}_n(\{n-1, n\})$?*

We would like to determine necessary and sufficient conditions on S for answering this question. As mentioned in Section 4.2, the answer to Question 1 is ‘yes’ for the canonical reuse U_{n-1} and recycle sequence C_{n-1} . We conjecture this is also true for the canonical rewind sequence W_{n-1} .

Conjecture 1 *If $S = W_{n-1}$ is the canonical rewind sequence, then $\text{recycle}(S)$ is a Hamilton sequence for the restricted rotator graph $\mathcal{R}_n(R)$ with $R = \{n-1, n\}$.*

Question 1 also has an affirmative answer for any Hamilton sequence using the two ‘largest’ rotations. This result was proven as Theorem 13 in the context of shorthand universal cycles for permutations in [9].

Theorem 12 ([9]) *If S is a Hamilton sequence for the restricted rotator graph $\mathcal{R}_{n-1}(\{n-2, n-1\})$, then $\text{recycle}(S)$ is a Hamilton sequence for the restricted rotator graph $\mathcal{R}_n(\{n-1, n\})$.*

On the other hand, we now illustrate that the answer to Question 1 is not always ‘yes’. Consider the following Hamilton sequence for \mathcal{G}_4 and Hamilton cycle

$$\begin{aligned}
 S &= 3, 3, 2, 3, 4, 2, 3, 4, 2, 3, 3, 4, 4, 2, 3, 3, 2, 3, 4, 4, 4, 3, 4, 4 \\
 \text{path}(4321, S) &= 4321 \sigma_3 \underline{3241} \sigma_3 \underline{2431} \sigma_2 \underline{4231} \sigma_3 2341 \sigma_4 3412 \sigma_2 4312 \sigma_3 3142 \sigma_4 \\
 &\quad 1423 \sigma_2 4123 \sigma_3 1243 \sigma_3 2413 \sigma_4 4132 \sigma_4 1324 \sigma_2 3124 \sigma_3 1234 \sigma_3 \\
 &\quad 2314 \sigma_2 3214 \sigma_3 2134 \sigma_4 1342 \sigma_4 \underline{3421} \sigma_4 \underline{4213} \sigma_3 2143 \sigma_4 1432 \sigma_4 .
 \end{aligned}$$

By Lemma 2, the underlined edges imply that $\text{path}(54321, \text{recycle}(S))$ contains the following two subpaths

$$\begin{aligned}
 \text{path}(52431, \text{recycle}(2)) &= 52431 \sigma_5 24315 \sigma_5 43152 \sigma_4 31542 \sigma_5 \underline{15423} \sigma_5 54231 \\
 \text{path}(53421, \text{recycle}(4)) &= 53421 \sigma_5 34215 \sigma_5 42153 \sigma_4 21543 \sigma_4 \underline{15423} \sigma_4 54213.
 \end{aligned}$$

The two underlined copies of node 15423 imply that $\text{recycle}(S)$ is not a Hamilton sequence, and thus the answer to Question 1 is ‘no’ for this choice of S . The reader can also verify that the answer to Question 1 is ‘no’ for $\text{reuse}(S)$ and $\text{rewind}(S)$. In other words, Theorem 4 and Conjecture 1 suggest there is something special about the canonical reuse and rewind sequences, as opposed to individual applications of the reuse and rewind operation.

7.2 Additional Open Problems

1. Proposition 1 proves that the restricted incomplete rotator graph $\mathcal{B}_n(R, m)$ does not have a Hamilton cycle for $R = \{n-1, n\}$ when $m \leq n-2$. On the other hand, $\mathcal{B}_n(R, n) = \mathcal{R}_n(R)$, so there is a Hamilton cycle when $m = n$ [18]. This leaves the remaining case of $m = n - 1$ open.
2. The existence of Hamilton paths and Hamilton cycles for the restricted rotator graph $\mathcal{R}_n(R)$ for $R = \{3, n\}$ and $R = \{2, n\}$ are open subject to Remarks 4 and 5, respectively. These questions are also open for restricted incomplete rotator graphs when $R = \{3, n-1, n\}$ and $R = \{2, n-1, n\}$.
3. A third generalization of rotator graphs features nodes that are the permutations of a multiset instead of a set. Williams proved that all “multiset rotator graphs” have Hamilton cycles [22], and this is also true when the rotations are restricted to $R = \{n-1, n\}$ [23]. On the other hand, Cheng proved that $R = \{2, n\}$ is not sufficient, even when the multiset contains only two distinct symbols and when σ_n is allowed [1]. The Hamiltonicity of incomplete multiset rotator graphs has not been explored.
4. This article notes that $R = \{2, 3, n\}$ is a minimal set of rotations required for a Hamilton cycle in the restricted rotator graph $\mathcal{R}_n(R)$ and provides an efficient algorithm for generating such a Hamilton cycle. Can an efficient algorithm be found for every minimal set of rotations?

5. There are a number of factors involved when choosing the ‘best’ topology for a given type of computer network. For interconnection networks, a partial list of relevant factors includes connectivity, diameter, symmetry, degree, bisection width, narrowness, expansion increments and the existence of Hamilton cycles, and proposed topologies include meshes, rings, hypercubes, shuffle exchange networks, butterfly networks, pancake graphs, star graphs and rotator graphs (see Siegel [19] and Hsu and Lin [10] for textbooks in this area). Further study of restricted incomplete rotator graphs will reveal if they offer any advantages in this area.

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