Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 16, no. 2, pp. 317–334 (2012)

Visibility Representation of Plane Graphs with Simultaneous Bound for Both Width and Height

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Abstract

The visibility representation (VR for short) is a classical representation of plane graphs. It has various applications and has been extensively studied. A main focus of the study is to minimize the size of the VR. The trivial upper bound is $(n-1)\times(2n-5)$ (height $\times$ width). It is known that there exists a plane graph G with n vertices where any VR of G requires a grid of size at least $\frac{2}{3}n \times (\frac{4}{3}n - 3)$. For upper bounds, it is known that every plane graph has a VR with grid size at most $\frac{2}{3}n \times (2n-5)$, and a VR with grid size at most $(n-1) \times \frac{4}{3}n$. It has been an open problem to find a VR with both height and width simultaneously bounded away from the trivial upper bounds (namely with size at most $c_h n \times c_w n$ with $c_h < 1$ and $c_w < 2$).

In this paper, we provide the first VR construction with this property. We prove that every plane graph of n vertices has a VR with height at most $\frac{23}{24}n + 2\lceil\sqrt{n}\rceil + 10$ and width at most $\frac{23}{12}n$. The area of our VR is larger than the area of some of the previous results. However, bounding one dimension of the VR only requires finding a good st-orientation *or* a good dual s^*t^*-orientation of G. On the other hand, to bound both dimensions of VR simultaneously, one must find a good st-orientation *and* a good dual s^*t^*-orientation at the same time, which is far more challenging. Our VR algorithm is based on an st-orientation of plane graphs with special properties. Since st-orientations are a very useful concept in other applications, this result may be of independent interests.

Submitted:	Reviewed:	Revised:	Accepted:
May 2011	November 2011	February 2012	April 2012
	Final:	Published:	
	April 2012	May 2012	

Article type: Regular paper

Communicated by: A. Symvonis

Research supported in part by NSF Grant CCR-0635104.

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	Plane Graph		4-Connected Plane Graph	
	Width	Height	Width	Height
1	$\leq (2n-5)$ [13, 14]	$\leq (n-1)$ [13, 14]		
2	$\leq \lfloor \frac{3n-6}{2} \rfloor$ [7]			
3	$\leq \lfloor \frac{22n-42}{15} \rfloor$ [10]		$\leq (n-1)$ [8]	
4		$\leq \lfloor \frac{5n}{6} \rfloor$ [18]		
5	$\leq \lfloor \frac{13n-24}{9} \rfloor$ [17]			$\leq \lceil \frac{3n}{4} \rceil$ [16]
6		$\leq \lfloor \frac{4n-1}{5} \rfloor$ [15]		
7		$\leq \frac{2n}{3} + \lfloor 2\sqrt{n} \rfloor$ [6]		
8		$\leq \frac{2n}{3} + 14$ [19]		
9	$\leq \lfloor \frac{4n}{3} \rfloor - 2$ [4]			$\leq \lceil \frac{n}{2} \rceil + 2\lceil \sqrt{\frac{n-2}{2}} \rceil$ [1]
10			$\leq \frac{3}{2}n$ [5]	$\leq \frac{3}{4}n + 2\lceil \sqrt{n} \rceil + 4$ [5]

Table 1: Previous results on the height and the width of VR.

1 Introduction

Drawing plane graphs has emerged as a fast growing research area in recent years (see [3] for a survey). A *visibility representation* (VR for short) is a classical drawing style of plane graphs where the vertices of a graph G are represented by horizontal line segments (called *vertex segment*), and each edge of G is represented by a vertical line segment (called edge segment) touching the vertex segments of its end vertices. Figure 1 shows a VR of a plane graph G. The problem of finding a VR on a compact grid is important not only in algorithmic graph theory, but also in practical applications. A simple linear-time VR algorithm was given by [13, 14] for 2-connected plane graphs. It uses an *st-orientation* of G and the corresponding *st*-orientation of its *st*-dual G^* to construct a VR. Using this approach, the height of the VR is bounded by $(n-1)$ (which is the number of vertices of G minus 1) and the width of the VR is bounded by $(2n-5)$ (which is the number of faces of G minus 1) [13, 14].

As in many other graph drawing problems, one of the main concerns in VR research is to minimize the grid size (i.e., the height and the width) of the representation. For the lower bounds, Zhang and He [18] showed that there exists a plane graph G with n vertices where any VR of G requires a grid of size at least $(\lfloor \frac{2n}{3} \rfloor) \times (\lfloor \frac{4n}{3} \rfloor - 3)$. Some works have been done to reduce the height and width of the VR by carefully constructing special *st*-orientations. Table 1 summarizes related previous results.

Line 1 in Table 1 gives the trivial upper bounds. All other results, except for Line 10, concentrated on one dimension of the VR (either the width or the height). In Table 1, the un-mentioned dimension is bounded by the trivial upper bound, namely, $n-1$ for the height and $2n-5$ for the width. (For Line 8, the original bound given in [19] was Height $\leq 2n/3 + O(1)$. By a more careful calculation, the term $O(1)$ is actually 14.) In [11, 12], heuristic algorithms were developed aiming at reducing the height and the width of VRs at the same time. Line 10 in Table 1 is the only VR construction with simultaneously reduced height and width. However, it only works for 4-connected plane graphs.

In this paper, we prove that every plane graph with n vertices has a VR with height at most $\frac{23}{24}n + 2\lceil\sqrt{n}\rceil + 10$ and width at most $\frac{23}{12}n$.

The present paper is organized as follows. Section 2 introduces preliminaries. Section 3 presents a decomposition lemma for plane graphs. Section 4 presents the construction of VR with the stated height and width. Section 5 concludes the paper.

2 Preliminaries

In this section, we give definitions and preliminary results. Definitions not mentioned here are standard.

All graphs considered in this paper are simple graphs (namely without self-loops and multiple edges). For a graph $G = (V, E)$ and a subset $U \subset V$, $G - U$ denotes the graph obtained from G by deleting the vertices in U (and all edges incident to these vertices). A *planar graph* is a graph $G = (V, E)$ such that the vertices of G can be drawn in the plane and the edges of G can be drawn as non-intersecting curves. Such a drawing is called an *embedding*. The embedding divides the plane into a number of connected regions. Each region is called a *face*. The unbounded face is the *exterior face*. The other faces are *interior faces*. The vertices and edges that are not on the boundary of the exterior face are called *interior vertices* and *edges*, respectively. A *plane graph* is a planar graph with a fixed embedding. A *plane triangulation* is a plane graph where every face is a triangle (including the exterior face). We denote the number of vertices of G by $|G|$. The set of interior vertices of G is denoted by $I(G)$. Thus $|I(G)| = |G| - 3$ for a plane triangulation G.

For a path P, length(P) (or $|P|$) denotes the number of edges in P. For two vertices a, b in P, $P(a, b)$ denotes the sub-path of P from a to b (including a and b).

When discussing VRs, we assume that, without loss of generality, the input graph G is a plane triangulation. (If not, we add *dummy edges* into the faces of G to obtain a triangulation G'. After constructing a VR for G', we can get a VR of G by deleting the vertical line segments for the dummy edges). From now on, G always denotes a plane triangulation.

A *numbering* $\mathcal{O}$ of a set $S = \{a_1, \ldots, a_k\}$ is an one-to-one mapping between S and the set $\{1, 2, \ldots, k\}$. We write $\mathcal{O} = \langle a_{i_1}, a_{i_2}, \ldots, a_{i_k}\rangle$ to indicate $\mathcal{O}(a_{i_1}) = 1$, $\mathcal{O}(a_{i_2}) = 2$, etc. A set S with a numbering written this way is called an *ordered list*. For two elements a_i and a_j, if $\mathcal{O}(a_i) < \mathcal{O}(a_j)$, we write $a_i \prec_{\mathcal{O}} a_j$. Let S_1 and S_2 be two disjoint sets. If $\mathcal{O}_1$ is a numbering of S_1 and $\mathcal{O}_2$ is a numbering of S_2, their concatenation, written as $\mathcal{O} = \langle \mathcal{O}_1, \mathcal{O}_2\rangle$, is the numbering of $S_1 \cup S_2$ defined by:

$$\mathcal{O}(x) = \begin{cases} \mathcal{O}_1(x) & \text{for all } x \in S_1 \\ \mathcal{O}_2(x) + |S_1| & \text{for all } x \in S_2 \end{cases}$$

G is called a *directed graph* (digraph for short) if each edge of G is assigned a direction. An *orientation* of an undirected graph G is a digraph obtained from G by assigning a direction to each edge of G. We use G to denote both the resulting digraph and the underlying undirected graph unless otherwise specified. (Its meaning will be clear from the context.)

Let $G = (V, E)$ be an undirected graph. A numbering $\mathcal{O}$ of V induces an orientation of G as follows: each edge (u, v) of G is directed from u to v if and only if $\mathcal{O}(u) < \mathcal{O}(v)$. The resulting digraph, denoted by $G_{\mathcal{O}}$, is called the *orientation derived from* $\mathcal{O}$ which, obviously, is an acyclic digraph. We use $\text{length}_G(\mathcal{O})$ (or simply $\text{length}(\mathcal{O})$ if G is clear from the context) to denote the length of the longest directed path in $G_{\mathcal{O}}$.

For a 2-connected plane graph G and an exterior edge (s, t), an orientation of G is called an *st-orientation* if the following conditions hold:

- the resulting digraph is acyclic;
- s is the only source and t is the only sink.

Such a digraph is also called an *st-graph.* Lempel et al. [9] showed that for every 2-connected plane graph G and an exterior edge (s, t), there exists an *st*-orientation. The properties of *st*-orientations and *st*-graphs can be found in [2].

Let G be a 2-connected plane graph and (s, t) an exterior edge. An *st-numbering* of G is an one-to-one mapping $\xi : V \to \{1, 2, \dots, n\}$ such that $\xi(s) = 1$, $\xi(t) = n$, and each vertex $v \neq s, t$ has two neighbors u, w with $\xi(u) < \xi(v) < \xi(w)$, where u (w, resp.) is called a *smaller neighbor* (*bigger neighbor*, resp.) of v. Given an *st*-numbering ξ of G, the orientation of G derived from ξ is obviously an *st*-orientation of G. On the other hand, if $G = (V, E)$ has an *st*-orientation $\mathcal{O}$, we can define an one-to-one mapping $\xi : V \to \{1, \dots, n\}$ by topological sort. It is easy to see that ξ is an *st*-numbering and the orientation derived from ξ is $\mathcal{O}$. From now on, we will interchangeably use the term "an *st*-numbering" of G and the term "an *st*-orientation" of G, where each edge of G is directed accordingly.

Definition 1 Let G be a plane graph with an *st*-orientation $\mathcal{O}$, where (s, t) is an exterior edge drawn at the left on the exterior face of G. The *st-dual* G^* of G and the dual orientation $\mathcal{O}^*$ of $\mathcal{O}$ are defined as follows:

- Each face f of G corresponds to a node f^* of G^*. In particular, the unique interior face adjacent to the edge (s, t) corresponds to a node s^* in G^*, the exterior face corresponds to a node t^* in G^*.
- For each edge $e \neq (s, t)$ of G separating a face f_1 on its left and a face f_2 on its right, there is a dual edge e^* in G^* from f_1^* to f_2^*.
- The dual edge of the exterior edge (s, t) is directed from s^* to t^*.

Figure 1 (a) shows an *st*-graph G and its *st*-dual graph G^*. (Circles and solid lines denote the vertices and the edges of G. Squares and dashed lines denote the

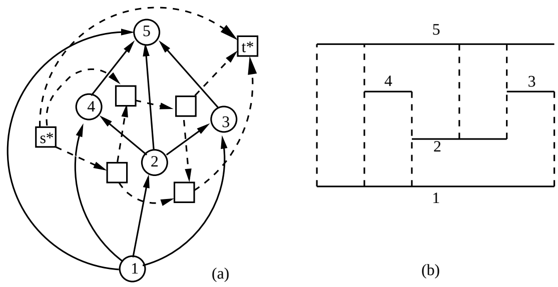

Figure 1: (a) An st-graph G and its st-dual graph G^*; (b) A VR of G.

nodes and the edges of G^*.) It is well known that the st-dual graph G^* defined above is an st-graph with source s^* and sink t^* [2, 13, 14]. The correspondence between an st-orientation $\mathcal{O}$ of G and the dual st-orientation $\mathcal{O}^*$ is an one-to-one correspondence. The following theorem was proved in [13, 14]:

Theorem 1 *Let G be a 2-connected plane graph with an st-orientation $\mathcal{O}$. Let $\mathcal{O}^*$ be the dual st-orientation of G^*. A VR of G can be obtained from $\mathcal{O}$ and $\mathcal{O}^*$ in linear time. The height of the VR is* $\text{length}(\mathcal{O})$. *The width of the VR is* $\text{length}(\mathcal{O}^*)$. *Since G has n vertices and G^* has $2n-4$ nodes, any st-orientation of G leads to a VR with height at most $n-1$ and width at most $2n-5$.*

Figure 1 (b) shows a VR of the graph G shown in Figure 1 (a). The width of the VR is $\text{length}(\mathcal{O}^*) = 5$. The height of the VR is $\text{length}(\mathcal{O}) = 3$.

The following theorems were given in [19, 7, 5], and will be needed later for our VR construction.

Theorem 2 *[19] Every plane triangulation with n vertices has a VR with width at most $2n-5$ and height at most $\frac{2}{3}n+14$, which can be constructed in linear time.*

Theorem 3 *[7] Every plane triangulation with n vertices has a VR with height at most $n-1$ and width at most $\lfloor\frac{3n-6}{2}\rfloor$, which can be constructed in linear time and we can specify s and t arbitrarily on the exterior face.*

Theorem 4 *[5] Every 4-connected plane triangulation with n vertices has a VR with height at most $\frac{3}{4}n+2\lceil\sqrt{n}\rceil+4$ and width at most $\frac{3}{2}n$, which can be constructed in linear time.*

Due to Theorem 1, the results in the above theorems can also be stated in terms of the lengths of the orientations of G. The statement "G has an st-orientation $\mathcal{O}$ such that $\text{length}(\mathcal{O}) \leq x$ and $\text{length}(\mathcal{O}^*) \leq y$" is equivalent to the statement "the VR of G derived from $\mathcal{O}$ has height at most x and width at most y". We will use these two statements interchangeably.

3 A Decomposition Lemma

The basic idea of our VR construction is as follows. First, we divide the input graph G into several subgraphs. Then we use the VR constructions in Theorems 1, 2, 3 and 4 for different subgraphs of G. Some of them have small width and others have small height. The main difficulty of our VR construction is to find a proper balance on the sizes of these subgraphs so that the overall height and width of the VR are both reduced. In this section, we prove a decomposition lemma that is needed by our VR construction to achieve the balance.

Let $G = (V, E)$ be a plane graph. A *triangle* of G is a set of three mutually adjacent vertices. The notation $\triangle = \{a, b, c\}$ denotes a triangle consisting of vertices a, b, c. A triangle divides the plane into an interior region and an exterior region. We say that $\triangle = \{a, b, c\}$ is a *separating triangle* if $G - \{a, b, c\}$ is disconnected. In other words, $\triangle$ is a separating triangle if both its interior and exterior regions contain vertices. The following fact by Whitney is well known,

Fact 1 A plane triangulation G is 4-connected if and only if G has no separating triangles.

Let $\triangle = \{a, b, c\}$ be a separating triangle. Then $G_\triangle$ denotes the subgraph of G induced by $\{a, b, c\} \cup \{v \in V \mid v \text{ is in the interior of } \triangle\}$. We say that $\triangle$ is *maximal* if there is no other separating triangle $\triangle'$ such $G_\triangle \subset G_{\triangle'}$. Two triangles $\triangle_1$ and $\triangle_2$ are *related* if either $G_{\triangle_1} \subseteq G_{\triangle_2}$ or $G_{\triangle_2} \subseteq G_{\triangle_1}$.

Let G_1 and G_2 be two plane triangulations. If G_1 has an internal face f such that the vertex set of f and the vertex set of the outer face of G_2 are identical, we can *embedded* G_2 *into* G_1 by identifying the face f and the exterior face of G_2. The resulting plane triangulation is denoted by $G_1 \oplus_f G_2$ (or simply $G_1 \oplus G_2$).

Definition 2 Let G_1 and G_2 be two plane triangulations such that G_2 can be embedded into G_1 by a common face $f = \{a, b, c\}$. Let $\mathcal{O}_1$ be an *st*-orientation of G_1 and let $\mathcal{O}_2$ be an *st*-orientation of G_2 such that the three edges $\{(a, b), (b, c), (c, a)\}$ are oriented the same way in $\mathcal{O}_1$ and $\mathcal{O}_2$. $\mathcal{O}_{G_1} \oplus \mathcal{O}_{G_2}$ denotes the union of $\mathcal{O}_1$ and $\mathcal{O}_2$, which is an orientation of $G_1 \oplus G_2$.

Lemma 1 *Let G_1, G_2, $\mathcal{O}_1$, and $\mathcal{O}_2$ be as in Definition 2. Then $\mathcal{O}_{G_1} \oplus \mathcal{O}_{G_2}$ is an st-orientation of $G_1 \oplus G_2$.*

Proof: Immediate from the definition. □

Definition 3 The *4-block tree* of a plane triangulation G is a rooted tree T defined as follows:

- If G has no separating triangles (i.e., G is 4-connected), then T consists of a single root r.

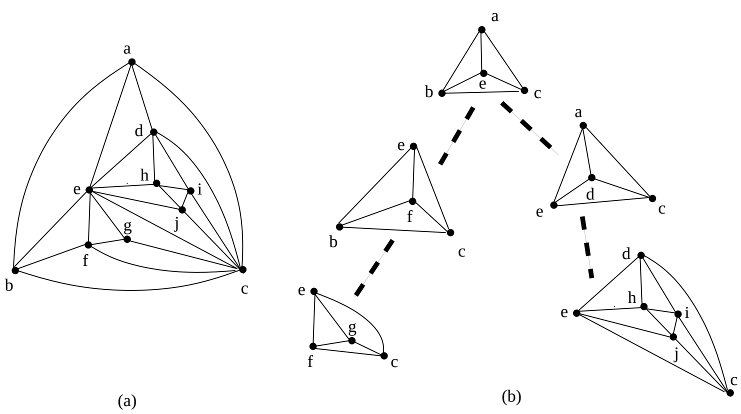

Figure 2: (a) A triangulation G; (b) 4-block components and the 4-block tree T of G.

- If not, let $\triangle_1, \ldots, \triangle_p$ be the maximal separating triangles of G. Let T_i be the 4-block tree of $G_{\triangle_i}$. Then T is the tree with root r and the roots of T_i $(1 \leq i \leq p)$ as the children of r.

From the definition, we have the following properties:

- Each non-root node u of T corresponds to a separating triangle $\triangle_u$ of G.
- For any $u, v \in T$, u and v have ancestor-descendant relation if and only if $\triangle_u$ and $\triangle_v$ are related in G.

For a node u of T, G_u denotes the subgraph $G_{\triangle_u} - (\bigcup_{v \in C(u)} I(G_{\triangle_v}))$ where $C(u)$ is the set of children of u in T. In other words, G_u is obtained from $G_{\triangle_u}$ by deleting all vertices that are in the interior of the maximal separating triangles of $G_{\triangle_u}$. Since G_u has no separating triangles, G_u is 4-connected. Each G_u is called a *4-block component* of G. Figure 2 shows a plane triangulation G, the 4-block components and the 4-block tree of G. For a node $u \in T$, for convenience, we use $|T_u|$ to denote $|G_{\triangle_u}|$.

For example, consider the graph G and its decomposition tree T shown in Figure 2. Let u be the node of T that is the right child of the root of T (consisting of the vertices $\{a, c, d, e\}$.) Then the graph G_u consists of the four vertices $\{a, c, d, e\}$. $G_{\triangle_u}$ is the subgraph of G consisting of the vertices a, e, c and all vertices contained in the interior region of the separating triangle $\triangle = \{a, e, c\}$. We have $|T_u| = 7$.

Lemma 2 *Let G be a triangulation and T be its 4-block tree. Then at least one of the following two conditions holds.*

1. *There exists a node v in T such that $|G_v| \geq \frac{n}{6}$.*

2. *There exists a set of unrelated separating triangles* $\{\triangle_1, \triangle_2, \ldots, \triangle_h\}$ *such that* $|G_{\triangle_i}| \geq 5$ *and* $\frac{n}{4} - 3 \leq \sum_{i=1}^{h} |I(G_{\triangle_i})| \leq \frac{3}{4}n - 3$.

Moreover, the decomposition can be found in linear time.

Proof: Let r be the root of T. Let H be a maximal path in T from r to some node v of T such that for each node $u \in H$, $|T_u| \geq \frac{3n}{4}$ (v can be the root r).

If v is a leaf of T, then $|G_v| \geq \frac{3n}{4} > \frac{n}{6}$. So condition (1) is satisfied.

Now, suppose v is not a leaf. Let $\{v_1, v_2, \ldots, v_p\}$ be the children of v in T. Re-arrange the indices, if necessary, so that $|T_{v_1}| \leq |T_{v_2}| \leq \ldots \leq |T_{v_p}|$. Then, either $\frac{n}{4} \leq |T_{v_p}| < \frac{3n}{4}$; or $|T_{v_i}| < \frac{n}{4}$ for all $v_i \in \{v_1, v_2, \ldots, v_p\}$.

Case A: suppose $|T_{v_i}| < \frac{n}{4}$ for all v_i. Let $i^* \in \{1, \cdots, p\}$ be the index such that $|T_{v_i}| \leq 4$ for all $i \leq i^*$ and $|T_{v_i}| \geq 5$ for all $i > i^*$. There are three sub-cases.

1. $\sum_{i>i^*}(|T_{v_i}| - 3) < \frac{n}{4} - 3$.
   Let $n_1 = |G_v|$. Since G_v is a triangulation with n_1 vertices, G_v has $2n_1 - 5$ internal faces by Euler's formula. Each child v_i of v corresponds to a maximal separating triangle of $G_{\triangle_v}$, and each such separating triangle is one of the interior faces of G_v. Thus, $i^* \leq p \leq 2n_1 - 5$. Since $|I(G_{\triangle_{v_i}})| = 1$ for all $i \leq i^*$, we have

$$\frac{3}{4}n \leq |T_v| = n_1 + \sum_{i \leq i^*} |I(G_{\triangle_{v_i}})| + \sum_{i>i^*} |I(G_{\triangle_{v_i}})| = n_1 + i^* + \sum_{i>i^*} |I(G_{\triangle_{v_i}})|$$
$$\leq n_1 + (2n_1 - 5) + \sum_{i>i^*} |I(G_{\triangle_{v_i}})|$$

   From the assumption $\sum_{i>i^*} |I(G_{\triangle_{v_i}})| < \frac{n}{4} - 3$, we have: $3n_1 - 5 > \frac{3}{4}n - \frac{n}{4} + 3 = \frac{n}{2} + 3$. This implies $|G_v| = n_1 \geq \frac{n}{6} + \frac{8}{3}$. So G_v satisfies condition (1).

2. $\frac{n}{4} - 3 \leq \sum_{i>i^*}(|T_{v_i}| - 3) \leq \frac{3}{4}n - 3$
   This is equivalent to $\frac{n}{4} - 3 \leq \sum_{i>i^*} |I(G_{\triangle_{v_i}})| \leq \frac{3}{4}n - 3$. So the set of unrelated separating triangles $\{\triangle_{v_{i^*+1}}, \triangle_{v_{i^*+2}}, \ldots, \triangle_{v_{i_p}}\}$ satisfies condition (2).

3. $\sum_{i>i^*}(|T_{v_i}| - 3) > \frac{3}{4}n - 3$
   Let i_t be the first index such that $\sum_{i^*<i\leq i_t}(|T_{v_i}| - 3) \geq \frac{n}{4} - 3$. Because $|T_{v_i}| < \frac{n}{4}$ for each i, clearly $\sum_{i^*<i\leq i_t}(|T_{v_i}| - 3) \leq \frac{3}{4}n - 3$. So the set of unrelated separating triangles $\{\triangle_{v_{i^*+1}}, \triangle_{v_{i^*+2}}, \ldots, \triangle_{v_{i_t}}\}$ satisfies condition (2).

Case B: $\frac{n}{4} \leq |T_{v_p}| < \frac{3n}{4}$. If $|T_{v_p}| > 4$, then the separating triangle $\triangle_{v_p}$ satisfies $\frac{n}{4} - 3 \leq |I(G_{\triangle_{v_p}})| \leq \frac{3n}{4} - 3$. So the single separating triangle $\triangle_{v_p}$ satisfies condition (2).

Otherwise, $|T_{v_p}| \leq 4$. This is a special case of Case A (1) (where $i^* = p$). So the claim holds.

For the run time, we first construct the 4-block tree and the 4–block components of G. This can be done in linear time [7]. The sizes of the 4-block components can be easily calculated in linear time. Since the decomposition is solely determined by the sizes of these 4-block components, it can also be done in linear time. □

4 Compact Visibility Representation

In this section, we describe our compact VR construction of a plane triangulation G. In order to keep the VR's height and width small simultaneously, we construct a VR of G by using different VRs for some subgraphs of G. As stated in Theorems 1, 2, 3 and 4, some of these VRs have small height and others have small width. Roughly speaking, we select a set of unrelated separating triangles $\{\triangle_1, \triangle_2, \ldots, \triangle_h\}$ of G. Let G' be the subgraph of G consisting of the vertices that are outside of $\{G_{\triangle_1}, G_{\triangle_2}, \ldots, G_{\triangle_h}\}$. We use a VR of G' with small height. For each $G_{\triangle_i}$, we use a VR with small width. Then, we embed each $G_{\triangle_i}$ into G'.

For convenience, define the function $\mathcal{X}(k) = \lceil \frac{k}{2} - \frac{1}{2} \rceil$ for integers $k \geq 1$. It is easy to verify that

- $\mathcal{X}(k)$ is non-decreasing .
- $\mathcal{X}(k) \geq 1$ and $\mathcal{X}(k) \geq k/3$ for all $k \geq 2$.

Theorem 5 *Let $S = \{\triangle_1, \triangle_2, \ldots, \triangle_h\}$ be a set of unrelated separating triangles of G. Then G has an st-orientation $\mathcal{O}$ such that* $\text{length}(\mathcal{O}) \leq \frac{2n}{3} + \frac{\sum_{i=1}^{h} |I(G_{\triangle_i})|}{3} + 14$ *and* $\text{length}(\mathcal{O}^*) \leq 2n - 5 - \sum_{i=1}^{h} \mathcal{X}(|I(G_{\triangle_i})|)$.

Proof: Define $G_j = G - \bigcup_{i=j+1}^{h} I(G_{\triangle_i})$. (In other words, G_j is obtained from G by deleting all vertices in the interior of the separating triangles $\triangle_i$ for $j+1 \leq i \leq h$.) Note that $G = G_h$.

We will show that G_j $(0 \leq j \leq h)$ has an st-orientation $\mathcal{O}_j$ so that

Claim 1 $\text{length}(\mathcal{O}_j) \leq \frac{2}{3}|G_j| + 14 + \frac{1}{3}\sum_{i=1}^{j} |I(G_{\triangle_i})|$.

Claim 2 $\text{length}(\mathcal{O}_j^*) \leq 2|G_j| - 5 - \sum_{i=1}^{j} \mathcal{X}(|I(G_{\triangle_i})|)$.

Then the theorem follows. We prove these claims by induction.

Base case $j = 0$: From Theorem 2, G_0 has an st-orientation $\mathcal{O}_0$ such that $\text{length}(\mathcal{O}_0) \leq \frac{2}{3}|G_0| + 14$ and $\text{length}(\mathcal{O}_0^*) \leq 2|G_0| - 5$. So the claims hold for the base case.

Induction hypothesis: G_k has an st-orientation $\mathcal{O}_k$ such that

$$\text{length}(\mathcal{O}_k) \leq \frac{2}{3}|G_k| + 14 + \frac{1}{3}\sum_{i=1}^{k} |I(G_{\triangle_i})|$$

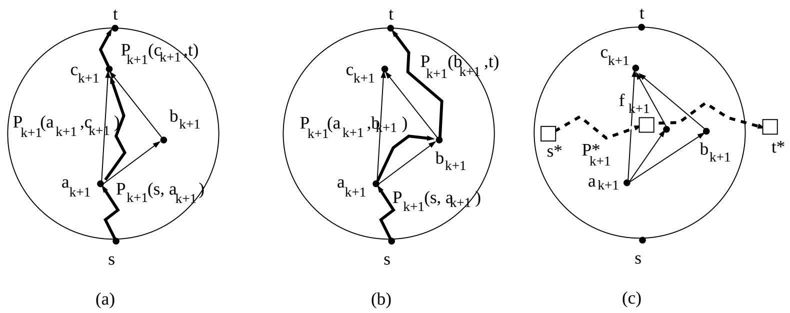

Figure 3: The proof of Theorem 5 (a) Case (ii); (b) Case (iii); (c) Path in the dual graph.

and

$$\text{length}(\mathcal{O}_k^*) \leq 2|G_k| - 5 - \sum_{i=1}^{k} \mathcal{X}(|I(G_{\triangle_i})|).$$

Suppose that $\triangle_{k+1} = \{a_{k+1}, b_{k+1}, c_{k+1}\}$. Without loss of generality, assume that the edges of $\triangle_{k+1}$ are oriented in $\mathcal{O}_k$ as $(a_{k+1} \to b_{k+1}), (b_{k+1} \to c_{k+1}), (a_{k+1} \to c_{k+1})$.

By Theorem 3, $G_{\triangle_{k+1}}$ has an st-orientation $\mathcal{O}_{\triangle_{k+1}}$ from a_{k+1} to c_{k+1} such that $\text{length}(\mathcal{O}_{\triangle_{k+1}}) \leq |G_{\triangle_{k+1}}| - 1$ and $\text{length}(\mathcal{O}^*_{\triangle_{k+1}}) \leq \lfloor \frac{3|G_{\triangle_{k+1}}|-6}{2} \rfloor$.

Let $\mathcal{O}_{k+1} = \mathcal{O}_k \oplus \mathcal{O}_{\triangle_{k+1}}$.

First we show $\text{length}(\mathcal{O}_{k+1}) \leq \frac{2}{3}|G_{k+1}| + 14 + \frac{1}{3}\sum_{i=1}^{k+1} |I(G_{\triangle_i})|$.

Note that $|G_{k+1}| = |G_k| + |I(G_{\triangle_{k+1}})| = |G_k| + |G_{\triangle_{k+1}}| - 3$.

Let P_{k+1} be a longest path in $\mathcal{O}_{k+1}$ from s to t in G_{k+1}; let P_k be a longest path in $\mathcal{O}_k$ from s to t in G_k; and let $P_{\triangle_{k+1}}$ be a longest path in $\mathcal{O}_{\triangle_{k+1}}$ from a_{k+1} to c_{k+1}. There are several cases:

P_{k+1} does not contain any interior edge in $G_{\triangle_{k+1}}$. Then P_{k+1} is a path in G_k. By induction hypothesis,

$$\begin{aligned}\text{length}(\mathcal{O}_{k+1}) = |P_{k+1}| \leq & \quad \tfrac{2}{3}|G_k| + 14 + \tfrac{1}{3}\textstyle\sum_{i=1}^{k} |I(G_{\triangle_i})| \\ < & \quad \tfrac{2}{3}|G_{k+1}| + 14 + \tfrac{1}{3}\textstyle\sum_{i=1}^{k+1} |I(G_{\triangle_i})|.\end{aligned}$$

(ii) P_{k+1} passes through a path in $G_{\triangle_{k+1}}$ from a_{k+1} to c_{k+1} (see Figure 3 (a)). P_{k+1} can be divided into three sub-paths: $P_{k+1}(s, a_{k+1}), P_{k+1}(a_{k+1}, c_{k+1})$, $P_{k+1}(c_{k+1}, t)$. Here $P_{k+1}(s, a_{k+1}), P_{k+1}(c_{k+1}, t)$ are paths in G_k, while $P_{k+1}(a_{k+1}, c_{k+1})$ is a path in $G_{\triangle_{k+1}}$. Since $P_{\triangle_{k+1}}$ is a longest path in $G_{\triangle_{k+1}}$, we have $|P_{k+1}(a_{k+1}, c_{k+1})| \leq |P_{\triangle_{k+1}}|$.

Let P' be the concatenation of $P_{k+1}(s, a_{k+1})$ followed by the edges $(a_{k+1} \to b_{k+1})$ and $(b_{k+1} \to c_{k+1})$; followed by $P_{k+1}(c_{k+1}, t)$. Then P' is a path in

G_k. Thus $|P'| = |P_{k+1}(s, a_{k+1})| + 2 + |P_{k+1}(c_{k+1}, t)| \leq |P_k|$. This implies $|P_{k+1}(s, a_{k+1})| + |P_{k+1}(c_{k+1}, t)| \leq |P_k| - 2$. Hence

$$\begin{aligned}
\text{length}(\mathcal{O}_{k+1}) &= |P_{k+1}| = |P_{k+1}(s, a_{k+1})| + |P_{k+1}(a_{k+1}, c_{k+1})| + |P_{k+1}(c_{k+1}, t)| \\
&\leq |P_k| - 2 + |P_{\triangle_{k+1}}| \\
&\leq \frac{2}{3}|G_k| + 14 + \frac{1}{3}\sum_{i=1}^{k} |I(G_{\triangle_i})| - 2 + |G_{\triangle_{k+1}}| - 1 \\
&= \frac{2}{3}|G_k| + \frac{1}{3}\sum_{i=1}^{k} |I(G_{\triangle_i})| + [|I(G_{\triangle_{k+1}})| + 3] + 14 - 3 \\
&= \frac{2}{3}(|G_k| + |I(G_{\triangle_{k+1}})|) + \frac{1}{3}\sum_{i=1}^{k+1} |I(G_{\triangle_i})| + 14 \\
&= \frac{2}{3}|G_{k+1}| + 14 + \frac{1}{3}\sum_{i=1}^{k+1} |I(G_{\triangle_i})|
\end{aligned}$$

(iii) P_{k+1} passes through a path in $G_{\triangle_{k+1}}$ from a_{k+1} to b_{k+1} (see Figure 3 (b)).

P_{k+1} can be divided into three sub-paths $P_{k+1}(s, a_{k+1}), P_{k+1}(a_{k+1}, b_{k+1})$, $P_{k+1}(b_{k+1}, t)$. Here $P_{k+1}(s, a_{k+1})$ and $P_{k+1}(b_{k+1}, t)$ are paths in G_k, while $P_{k+1}(a_{k+1}, b_{k+1})$ is a path in $G_{\triangle_{k+1}}$. The concatenation of $P_{k+1}(a_{k+1}, b_{k+1})$ and the edge $b_{k+1} \to c_{k+1}$ is a path in $G_{\triangle_{k+1}}$. Hence $|P_{k+1}(a_{k+1}, b_{k+1})| + 1 \leq |P_{\triangle_{k+1}}|$.

The concatenation of $P_{k+1}(s, a_{k+1})$ followed by the edge $a_{k+1} \to b_{k+1}$, followed by $P_{k+1}(b_{k+1}, t)$ is a path in G_k. So we have that $|P_{k+1}(s, a_{k+1})| + 1 + |P_{k+1}(b_{k+1}, t)| \leq |P_k|$. Hence

$$\begin{aligned}
\text{length}(\mathcal{O}_{k+1}) &= |P_{k+1}| = |P_{k+1}(s, a_{k+1})| + |P_{k+1}(a_{k+1}, b_{k+1})| + |P_{k+1}(b_{k+1}, t)| \\
&\leq (|P_k| - 1) + (|P_{\triangle_{k+1}}| - 1) \\
&= |P_k| - 2 + |P_{\triangle_{k+1}}| \\
&\leq \frac{2}{3}|G_{k+1}| + 14 + \frac{1}{3}\sum_{i=1}^{k+1} |I(G_{\triangle_i})|.
\end{aligned}$$

The proof of the last inequality is the same as the proof of case (ii).

(iv) P_{k+1} passes through a path in $G_{\triangle_{k+1}}$ from b_{k+1} to c_{k+1}. The proof is symmetric to case (iii).

Next we prove Claim 2. Let P_k^* be a longest path from s^* to t^* in $\mathcal{O}_k^*$. From the induction hypothesis, we know that $|P_k^*| \leq 2|G_k| - 5 - \sum_{i=1}^{k} \mathcal{X}(|I(G_{\triangle_i})|)$. Let $P_{\triangle_{k+1}}^*$ be a longest path in $G_{\triangle_{k+1}}^*$. By Theorem 3, $|P_{\triangle_{k+1}}^*| \leq \lfloor \frac{3|G_{\triangle_{k+1}}| - 6}{2} \rfloor$.

Let P^*_{k+1} be a longest path from s^* to t^* in $\mathcal{O}^*_{k+1}$. Let f_{k+1} be the face in G_{k+1} that is in the interior of $\triangle_{k+1}$ adjacent to the edge $a_{k+1} \to c_{k+1}$ (see Figure 3 (c).) (In other words, f_{k+1} corresponds to the source node of the dual st-orientation of $G^*_{\triangle_{k+1}}$.) If P^*_{k+1} uses any edge in $G^*_{\triangle_{k+1}}$, it must cross the edge $a_{k+1} \to c_{k+1}$ and enter the face f_{k+1}. There are two cases.

(a) P^*_{k+1} does not go through f_{k+1}. Then P^*_{k+1} is a path in G^*_k and the claim trivially holds:

$$
\begin{aligned}
|P^*_{k+1}| &\leq 2|G_k| - 5 - \sum_{i=1}^{k} \mathcal{X}(|I(G_{\triangle_i})|) \\
&= 2(|G_k| + |I(G_{\triangle_{k+1}})|) - 5 - 2|I(G_{\triangle_{k+1}})| - \sum_{i=1}^{k} \mathcal{X}(|I(G_{\triangle_i})|) \\
&\leq 2|G_{k+1}| - 5 - \sum_{i=1}^{k+1} \mathcal{X}(I(G_{\triangle_i}))
\end{aligned}
$$

(b) P^*_{k+1} passes through f_{k+1}.

$$
\begin{aligned}
\text{length}(\mathcal{O}^*_{k+1}) &= |P^*_k| + |P^*_{\triangle_{k+1}}| - |\{f_{k+1}\}| \\
&\leq 2|G_k| - 5 - \sum_{i=1}^{k} \mathcal{X}(|I(G_{\triangle_i})|) + \left\lfloor \frac{3|G_{\triangle_{k+1}}| - 6}{2} \right\rfloor - 1 \\
&= 2(|G_{k+1}| - |I(G_{\triangle_{k+1}})|) - 5 - \sum_{i=1}^{k} \mathcal{X}(|I(G_{\triangle_i})|) + \\
&\quad + \left\lfloor \frac{3(|I(G_{\triangle_{k+1}})| + 3) - 6}{2} - 1 \right\rfloor \\
&= 2|G_{k+1}| - 5 - \sum_{i=1}^{k} \mathcal{X}(|I(G_{\triangle_i})|) - 2|I(G_{\triangle_{k+1}})| + \\
&\quad + \left\lfloor \frac{3|I(G_{\triangle_{k+1}})|}{2} + \frac{1}{2} \right\rfloor \\
&= 2|G_{k+1}| - 5 - \sum_{i=1}^{k} \mathcal{X}(|I(G_{\triangle_i})|) - \left(\left\lceil \frac{|I(G_{\triangle_{k+1}}|}{2} - \frac{1}{2} \right\rceil \right) \\
&= 2|G_{k+1}| - 5 - \sum_{i=1}^{k+1} \mathcal{X}(|I(G_{\triangle_i})|)
\end{aligned}
$$

This completes the induction. □

Lemma 3 *Let $S = \{\triangle_1, \triangle_2, \ldots, \triangle_h\}$ be a set of unrelated separating triangles of G such that $G' = G - (\bigcup_{i=1}^{h} I(G_{\triangle_i}))$ is a 4-connected graph. Then, G has an* *st-orientation* $\mathcal{O}$ *such that* $\text{length}(\mathcal{O}) \leq \frac{3}{4}n + \frac{\sum_{i=1}^{h}|I(G_{\triangle_i})|}{4} + 2\lceil\sqrt{|G'|}\rceil + 4$ *and* $\text{length}(\mathcal{O}^*) \leq \frac{3}{2}n + \frac{\sum_{i=1}^{h}|I(G_{\triangle_i})|}{2}$.

Proof: The idea of the proof is very similar to the proof of Theorem 5. The only difference is that here the graph G' is assumed to be 4-connected. So we can construct an *st*-orientation of G' in Theorem 4. In contrast, in the proof of Theorem 5, without the 4-connectivity assumption, we use the *st*-orientation in Theorem 2.

Define $G_j = G - \bigcup_{i=j+1}^{h} I(G_{\triangle_i})$. We show, by induction, that G_j has an *st*-orientation $\mathcal{O}_j$ such that

Claim 1: $\text{length}(\mathcal{O}_j) \leq \frac{3}{4}|G_j| + \frac{\sum_{i=1}^{j}|I(G_{\triangle_i})|}{4} + 2\lceil\sqrt{|G'|}\rceil + 4$ and

Claim 2: $\text{length}(\mathcal{O}_j^*) \leq \frac{3}{2}|G_j| + \frac{\sum_{i=1}^{j}|I(G_{\triangle_i})|}{2}$.

Base case $j = 0$: Since $G_0 = G'$ is 4-connected, by Theorem 4, G' has an *st*-orientation $\mathcal{O}'$ such that $\text{length}(\mathcal{O}') \leq \frac{3}{4}|G'| + 2\lceil\sqrt{|G'|}\rceil + 4$ and $\text{length}(\mathcal{O}'^*) \leq \frac{3}{2}|G'|$. The claims are trivially true.

Suppose the claims are true for $j = k$.

Suppose that $\triangle_{k+1} = \{a_{k+1}, b_{k+1}, c_{k+1}\}$. Without loss of generality, assume the edges of $\triangle_{k+1}$ are oriented in $\mathcal{O}_k$ as $(a_{k+1} \to b_{k+1}), (b_{k+1} \to c_{k+1}), (a_{k+1} \to c_{k+1})$.

By Theorem 1, $G_{\triangle_{k+1}}$ has an *st*-orientation $\mathcal{O}_{\triangle_{k+1}}$, with a_{k+1} as the source and c_{k+1} as the sink, such that $\text{length}(\mathcal{O}_{\triangle_{k+1}}) \leq |G_{\triangle_{k+1}}| - 1$ and $\text{length}(\mathcal{O}^*_{\triangle_{k+1}}) \leq 2|G_{\triangle_{k+1}}| - 5$.

We show the orientation $\mathcal{O}_{k+1} = \mathcal{O}_k \oplus \mathcal{O}_{\triangle_{k+1}}$ satisfies the claims.

Both the upper bounds of $\text{length}(\mathcal{O}_j)$ in Theorem 5 and Lemma 3 can be written in the form

$$\text{length}(\mathcal{O}_j) \leq \alpha|G_j| + (1-\alpha)\sum_{i=1}^{j}|I(G_{\triangle_i})| + \beta.$$

Since we use an *st*-orientation with the height at most $\frac{2}{3}|G_0| + 14$ in Theorem 5, α is $\frac{2}{3}$ and β is 14. On the other hand, in the base case of this proof, we use an *st*-orientation with height at most $\frac{3}{4}|G_j| + 2\lceil\sqrt{|G'|}\rceil + 4$. By the same process, the value of α is $\frac{3}{4}$ and β is $2\lceil\sqrt{|G'|}\rceil + 4$ in this case. Hence the proof of Claim 1 is similar to the proof of Claim 1 in Theorem 5. In the following, we prove Claim 2.

By induction hypothesis, G_k has an *st*-orientation $\mathcal{O}_k$ such that $\text{length}(\mathcal{O}_k^*) \leq \frac{3}{2}|G_k| + \frac{\sum_{i=1}^{k}|I(G_{\triangle_i})|}{2}$. Also, we know that $\text{length}(\mathcal{O}^*_{\triangle_{k+1}}) \leq 2|G_{\triangle_{k+1}}| - 5$. As in the proof of Theorem 5, there are two cases for analyzing $\text{length}(\mathcal{O}^*_{k+1})$.

(a) P^*_{k+1} does not pass through f_{k+1}. Then P^*_{k+1} is a path in G^*_k and the claim trivially holds.

(b) P^*_{k+1} passes through f_{k+1}. Then

$$\begin{aligned}\text{length}(\mathcal{O}) &\le \frac{3}{2}|G_k| + \frac{1}{2}\sum_{i=1}^{k}|I(G_{\triangle_i})| + 2|G_{\triangle_{k+1}}| - 5 - 1 \\ &= \frac{3}{2}|G_k| + \frac{1}{2}\sum_{i=1}^{k}|I(G_{\triangle_i})| + 2|I(G_{\triangle_{k+1}})| \\ &= \frac{3}{2}|G_{k+1}| + \frac{1}{2}\sum_{i=1}^{k+1}|I(G_{\triangle_i})|\end{aligned}$$

This completes the induction. □

Theorem 6 *Let G_v be a 4-block component of G associated with a node v of the 4-block tree of G, and let $\triangle_v$ be the separating triangle in G corresponding to node v. Then G has an st-orientation $\mathcal{O}$ such that* $\text{length}(\mathcal{O}) \le \frac{3}{4}n + \frac{1}{4}(n - |G_v|) + 2\lceil\sqrt{|G_v|}\rceil + 5$ *and* $\text{length}(\mathcal{O}^*) \le \frac{3}{2}n + \frac{n-|G_v|}{2}$.

Proof: Let $S = \{\triangle_1, \triangle_2, \ldots, \triangle_h\}$ be the set of maximal separating triangles of $G_{\triangle_v}$. Since G_v is 4-connected, by Lemma 3, $G_{\triangle_v}$ has an *st*-orientation $\mathcal{O}_{\triangle_v}$ such that

$$\begin{aligned}\text{length}(\mathcal{O}_{\triangle_v}) &\le \frac{3}{4}|G_{\triangle_v}| + 2\lceil\sqrt{|G_v|}\rceil + 4 + \frac{\sum_{i=1}^{h}|I(G_{\triangle_i})|}{4} \\ \text{length}(\mathcal{O}^*_{\triangle_v}) &\le \frac{3}{2}|G_{\triangle_v}| + \frac{\sum_{i=1}^{h}|I(G_{\triangle_i})|}{2}.\end{aligned}$$

Let $G_{ext} = G - I(G_{\triangle_v})$. By Theorem 1, G_{ext} has an *st*-orientation such that $\text{length}(\mathcal{O}_{ext}) \le |G_{ext}| - 1$ and $\text{length}(\mathcal{O}^*_{ext}) \le 2|G_{ext}| - 5$. Let $\mathcal{O} = \mathcal{O}_{ext} \oplus \mathcal{O}_{\triangle_v}$. Then

$$\begin{aligned}\text{length}(\mathcal{O}) &\le \text{length}(\mathcal{O}_{ext}) + \text{length}(\mathcal{O}_{\triangle_v}) - 1 \\ &\le (|G_{ext}| - 1) + \frac{3}{4}|G_{\triangle_v}| + 2\lceil\sqrt{|G_v|}\rceil + 4 + \frac{\sum_{i=1}^{h}|I(G_{\triangle_i})|}{4} - 1 \\ &= \frac{3}{4}|G_{ext}| + \frac{1}{4}|G_{ext}| + \frac{3}{4}|G_{\triangle_v}| + 2\lceil\sqrt{|G_v|}\rceil + \frac{\sum_{i=1}^{h}|I(G_{\triangle_i})|}{4} + 2 \\ &= \frac{3}{4}(|G| + 3) + \frac{1}{4}(|V(G_{ext}) \cup (\bigcup_{i=1}^{h} I(G_{\triangle_i}))|) + 2\lceil\sqrt{|G_v|}\rceil + 2 \\ &= \frac{3}{4}(n + 3) + \frac{1}{4}(n - |G_v| + 3) + 2\lceil\sqrt{|G_v|}\rceil + 2 \\ &= \frac{3}{4}n + \frac{1}{4}(n - |G_v|) + 2\lceil\sqrt{|G_v|}\rceil + 5\end{aligned}$$

and

$$\begin{aligned}\text{length}(\mathcal{O}^*) &= \text{length}(\mathcal{O}^*_{ext}) + \text{length}(\mathcal{O}^*_{\triangle_v}) - 1\\ &\leq (2|G_{ext}| - 5) + (\frac{3}{2}|G_{\triangle_v}| + \frac{\sum_{i=1}^h |I(G_{\triangle_i})|}{2}) - 1\\ &= \frac{3}{2}|G_{ext}| + \frac{3}{2}|G_{\triangle_v}| + \frac{1}{2}|G_{ext}| + \frac{\sum_{i=1}^h |I(G_{\triangle_i})|}{2} - 6\\ &= \frac{3}{2}(|G| + 3) + \frac{1}{2}(|I(G_{ext}) \cup (\bigcup_{i=1}^h I(G_{\triangle_i}))| + 3) - 6\\ &= \frac{3}{2}n + \frac{1}{2}(n - |G_v|)\end{aligned}$$

This completes the proof. □

Theorem 7 *Every plane triangulation G of n vertices has a VR with height $\leq \frac{23}{24}n + 2\lceil\sqrt{n}\rceil + 10$ and width $\leq \frac{23}{12}n$. The VR can be constructed in linear time.*

Proof: By Lemma 2, there are two cases.

Case 1: G has a 4-block component with size $n_1 \geq n/6$. By Theorem 6, G has an st-orientation $\mathcal{O}$ such that length$(\mathcal{O}) \leq \frac{3}{4}n + \frac{n-n_1}{4} + 2\lceil\sqrt{n}\rceil + 5$ and length$(\mathcal{O}^*) \leq \frac{3n}{2} + \frac{(n-n_1)}{2}$. Since $n_1 \geq \frac{n}{6}$, we have

$$\begin{aligned}\text{length}(\mathcal{O}) &\leq \frac{23}{24}n + 2\lceil\sqrt{n}\rceil + 5,\\ \text{length}(\mathcal{O}^*) &\leq \frac{23}{12}n.\end{aligned}$$

Case 2: G has a set of unrelated separating triangles $\{\triangle_1, \triangle_2, \dots, \triangle_h\}$ such that

- For all i, $|G_{\triangle_i}| \geq 5$, (which implies $|I(G_{\triangle_i})| \geq 2$).
- $\frac{n}{4} - 3 \leq \sum_{i=1}^h |I(G_{\triangle_i})| \leq \frac{3}{4}n - 3$.

Since $\mathcal{X}(z) \geq z/3$ for all $z \geq 2$, we have

$$\sum_{i=1}^h \mathcal{X}(|I(G_{\triangle_i})|) \geq \frac{\sum_{i=1}^h |I(G_{\triangle_i})|}{3}.$$

By Theorem 5, G has an st-orientation $\mathcal{O}$ such that

$$\begin{aligned}
\text{length}(\mathcal{O}^*) &\leq 2n - 5 - \sum_{i=1}^{h} \mathcal{X}(|I(G_{\triangle_i})|) \\
&\leq 2n - 5 - \frac{\sum_{i=1}^{h} |I(G_{\triangle_i}))}{3} \leq 2n - 5 - \frac{n/4 - 3}{3} < \frac{23}{12}n \\
\text{length}(\mathcal{O}) &\leq \frac{2n}{3} + \frac{\sum_{i=1}^{h} |I(G_{\triangle_i})|}{3} + 14 \\
&\leq \frac{2n}{3} + \frac{3n/4 - 3}{3} + 14 = \frac{11}{12}n + 13 \\
&\leq \frac{23}{24}n + 2\lceil\sqrt{n}\rceil + 10
\end{aligned}$$

The last inequality holds provided $n \geq 3$. In either case, the orientation $\mathcal{O}$ leads to a VR of G with the stated width and height.

To construct the VR, we first find the decomposition in Lemma 2, which can be done in linear time. Then we use the VR constructions in Theorems 1, 2, 3 and 4 for different 4-block components of G. Since all these VRs can be constructed in linear time, our algorithm also takes linear time. □

5 Conclusion

In this paper, we showed that every plane graph of n vertices has a VR with height $\leq \frac{23}{24}n + 2\lceil\sqrt{n}\rceil + 10$ and width $\leq \frac{23}{12}n$. This is the first VR construction for general plane graphs that simultaneously bounds the height and the width away from the trivial upper bound. The gap between the size of our VR and the known lower bound is still large. It would be interesting to find more compact VR constructions.

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