

## Recognition of Unigraphs through Superposition of Graphs

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### Abstract

Unigraphs are graphs uniquely determined by their own degree sequence up to isomorphism. In this paper a structural description for unigraphs is introduced: vertex set is partitioned into three disjoint sets while edge set is divided into two different classes. This characterization allows us to design a new linear time recognition algorithm that works recursively pruning the degree sequence of the graph. The algorithm detects two particular graphs whose superposition generates the given unigraph.

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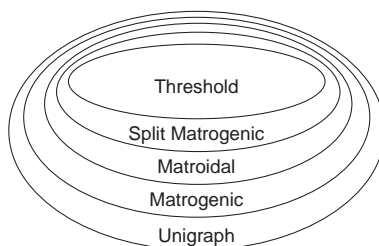


Figure 1: Relationships of inclusion among the subclasses of unigraphs.

## 1 Introduction

In this paper we deal with *unigraphs* [8, 10], i.e. graphs uniquely determined by their own degree sequence up to isomorphism.

Unigraphs are a superclass including *matrogenic graphs*, *matroidal graphs*, *split matrogenic graphs* and *threshold graphs* as shown in Fig. 1.

All these subclasses have been widely studied (e.g. see [1]) and many equivalent definitions have been given. Here we will define these graphs in terms of forbidden induced subgraphs.

A graph  $G = (V, E)$  is:

- *threshold* if and only if it does not contains  $P_4$ , nor chordless  $C_4$  or  $2K_2$  as induced subgraphs [3];
- *split matrogenic* (also called *splitoid*) if and only if it does not contain any of the configurations in Fig. 2 nor any of their complements as induced subgraphs [7];
- *matroidal* if and only if it does not contain any of the configurations in Fig. 3 nor any of their complements as induced subgraph or a chordless  $C_5$  as induced subgraphs [15].
- *matrogenic* if and only if it does not contain any of the configurations in Fig. 3 nor any of their complements as induced subgraph [6];

A property  $\mathcal{P}$  holding for a graph  $G = (V, E)$  is said to be *hereditary* if  $\mathcal{P}$  holds for all the induced subgraphs of  $G$ , too. It has been proved that thresholdness, matroidality and matrogenicity are hereditary properties [11], while unigraficity is not a hereditary property (see Fig. 4).

In [17], unigraphs are characterized following a decomposition theorem stating that any graph and any graphical sequence can be uniquely decomposed into particular components with respect to a decomposition operation (see Section 3). This result, though very interesting from a structural point of view, does not seem to immediately lead to an efficient recognition algorithm; nevertheless, the author, in a private communication, observed that it is somehow possible to

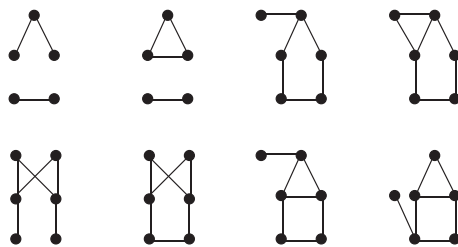


Figure 2: Some forbidden induced subgraphs of a split matrogenic graph.



Figure 3: Some forbidden induced subgraphs of a matrogenic graph.

restrict to unigraphs the algorithm presented in [16] that decomposes arbitrary graphs using results from [17]. To the best of our knowledge, the only published linear time algorithm for recognizing unigraphs is in [9] and works exploiting Ferrer diagrams. On the contrary, it is possible to find linear recognition algorithms for all the subclasses presented in Fig. 1 [3, 4, 6, 11, 12, 13, 14, 18]. In this paper we generalize to unigraphs the pruning algorithm designed for matrogenic graphs in [12] providing a new recognition algorithm for the whole class of unigraphs. It is to notice that the proof of our theorem is not a straightforward generalization of the proof presented in [12], as the latter one is based on the heredity of matrogenicity while this property does not hold for unigraphs.

The algorithm is linear and has a completely different approach with respect to [9], although works on the degree sequence, too. In particular, it partitions the vertex set and the edge set into three and two disjoint sets, respectively, detecting two particular graphs whose superposition generates the given unigraph. This superposition allows us to interpret in a simplified way the unigraph’s structure. Indeed, also the algorithm for the recognition of matrogenic graphs [12] provides a similar superposition that is exploited for solving other problems (e.g. the  $L(2, 1)$ -labeling [2]). It is in the conviction of the authors that the results presented in this paper will be useful for solving such problems, that are NP-hard for general graphs, polynomially solvable for subclasses of unigraphs and still unknown for unigraphs.

This paper is organized as follows: in Section 2 we recall some definitions and useful known results; in Section 3 we resume the decomposition theorem for unigraphs. Section 4 and 5 are the core of the paper and introduce a different characterization for unigraphs and a new recognition algorithm for this class, proving its linearity. Finally in Section 6 we address some conclusions.

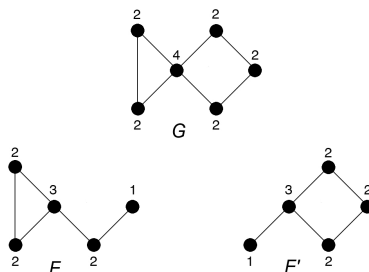


Figure 4: A unigraph whose induced subgraphs are not unigraphs:  $F$  and  $F'$  are both subgraphs of  $G$  and have the same degree sequence even if are not isomorphic.

## 2 Preliminaries

In this section we will recall all definitions and known results that will be useful in the rest of the paper.

We consider only finite, simple, loopless, connected graphs  $G = (V, E)$ , where  $V$  is the vertex set of  $G$  with cardinality  $n$  and  $E$  is the edge set of  $G$  with cardinality  $m$ . Where no confusion arises, we will call  $G = (V, E)$  simply  $G$ .

Let  $DS(G) = \delta_1, \delta_2, \dots, \delta_n$  be the degree sequence of a graph  $G$  sorted by non-increasing values:  $\delta_1 \geq \delta_2 \geq \dots \geq \delta_n \geq 0$ . We call *boxes* the equivalence classes of vertices in  $G$  under equality of degree. In terms of boxes the degree sequence can be compressed as  $d_1^{m_1}, d_2^{m_2}, \dots, d_r^{m_r}, d_1 > d_2 > \dots > d_r \geq 0$ , where  $d_i$  is the degree of the  $m_i$  vertices contained in box  $B_i(G)$ ,  $1 \leq m_i \leq n$ . We call a box *universal (isolated)* if it contains only universal (isolated) vertices, where a vertex  $x \in V$  is called *universal (isolated)* if it is adjacent to all other vertices of  $V$  (no other vertex in  $V$ ); if  $x$  is a universal (isolated) vertex, then its degree is  $d(x) = n - 1$  ( $d(x) = 0$ ).

A graph  $I$  induced by subset  $V_I \subseteq V$  is called *complete* or *clique* if any two distinct vertices in  $V_I$  are adjacent in  $G$ , *stable* or *null* if no two vertices in  $V_I$  are adjacent in  $G$ .

A graph  $G$  is said to be *split* if there is a partition  $V = V_K \cup V_S$  of its vertices such that the induced subgraphs  $K$  and  $S$  are complete and stable, respectively [5].

A set  $M$  of edges is a *perfect matching of dimension  $h$*  of  $X$  onto  $Y$  if and only if  $X$  and  $Y$  are disjoint subsets of vertices with the same cardinality  $h$  and each edge is incident to exactly one vertex  $x \in X$  and to one vertex  $y \in Y$ , and different edges must be incident to different vertices (see Fig. 5.a).

The complement of a perfect matching of dimension  $h$  is called  *$h$ -hyperoctahedron* (see Fig. 5.b).

An *antimatching of dimension  $h$*  of  $X$  onto  $Y$  is a set  $A$  of edges such that  $M(A) = X \times Y - A$  is a perfect matching of dimension  $h$  of  $X$  onto  $Y$  (see

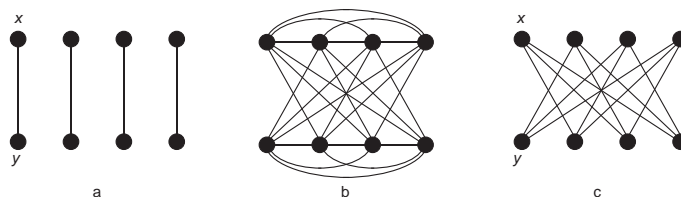


Figure 5: a. a matching of dimension 4; b. a 4-hyperoctahedron; c. an anti-matching of dimension 4.

Fig. 5.c).

Given a graph  $G$ , if its vertex set  $V$  can be partitioned into three disjoint sets  $V_K$ ,  $V_S$ , and  $V_C$  such that every vertex in  $V_C$  is adjacent to every vertex in  $V_K$  and to no vertex in  $V_S$ , then the subgraph induced by  $V_C$  is called *crown*.

### 3 Decomposition of Unigraphs

In this section we state the exhaustive description of the structure of unigraphs based on a decomposition theorem reducing a general graph to its indecomposable components [17].

#### 3.1 Decomposition Theorem

**Definition 1** Given a split graph  $F = (V_K \cup V_S, E(F))$  and a simple graph  $H = (V(H), E(H))$ , their composition is a graph  $G = (V, E) = F \circ H$  defined as follows:

- $V = V_K \cup V_S \cup V(H)$
- $E = E(F) \cup E(H) \cup \{\{a, v\} : a \in V_K, v \in V(H)\}$ .

In other words, the edge set of the complete bipartite graph with parts  $V_K$  and  $V(H)$  is added to the disjoint union  $F \cup H$  (see fig. 6).

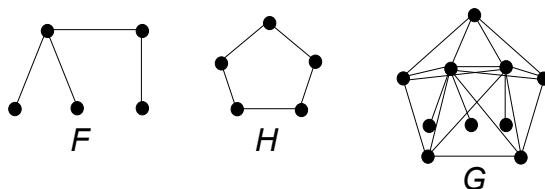


Figure 6: A graph  $G$  as composition of graphs  $F$  and  $H$ .

**Theorem 2** *An  $n$  vertex graph  $G$ , given through its degree sequence  $DS(G) = \delta_1, \delta_2, \dots, \delta_n$  is decomposable as  $F \circ H$ , where  $F$  is a split graph and  $H$  is a simple graph, if and only if there exist nonnegative integers  $p$  and  $q$  such that*

$$0 < p + q < n, \quad \sum_{i=1}^p \delta_i = p(n - q - 1) + \sum_{i=n-q+1}^n \delta_i$$

*and the degree sequences of  $F$  and  $H$  are  $\delta_1, \dots, \delta_p, \delta_{n-q+1}, \dots, \delta_n$  and  $\delta_{p+1}, \dots, \delta_{n-q}$ , respectively. If  $p$  and  $q$  do not exist,  $G$  is said undecomposable.*

Iterating the decomposition proposed by the previous theorem, it is possible to go on until indecomposable components are reached:

**Corollary 3** *(Decomposition theorem) Every graph  $G$  can be decomposed as a composition  $G = F_1 \circ \dots \circ F_c \circ H$  of indecomposable components, where  $F_i, i = 1, \dots, c$  are split graphs and  $H$  is a simple graph.*

### 3.2 Characterization of Unigraphs

Before stating the characterization of unigraphs presented in [17] we highlight that if  $G$  is a unigraph then even its complement  $\overline{G}$  is.

If  $F = (V_K \cup V_S, E)$  is a split graph, its *inverse*  $F^I$  is obtained from  $F$  by deleting the set of edges  $\{\{a_1, a_2\} : a_1, a_2 \in V_K\}$  and adding the set of edges  $\{\{b_1, b_2\} : b_1, b_2 \in V_S\}$ . Observe that  $F^I$  is not uniquely determined by  $F$ , but it holds that if  $F$  is split and indecomposable then  $F^I$  is split and indecomposable, too.

In the following the definitions of some special graphs are recalled from [17]:

$U_2(m, s)$ : it is the disjoint union of perfect matching  $mK_2$  and star  $K_{1,s}$ , for  $m \geq 1, s \geq 2$  (see Fig. 7.a).

$U_3(m)$ : for  $m \geq 1$ , construct this graph fixing a vertex in each component of the disjoint union of the chordless cycle  $C_4$  and  $m$  triangles  $K_3$ , and merging all these vertices in one (see Fig. 7.b).

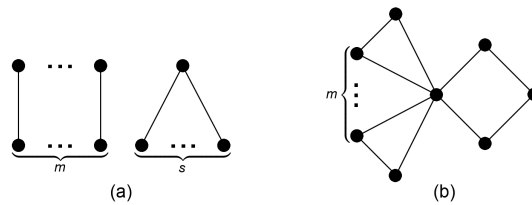


Figure 7: a.  $U_2(m, s)$ ; b.  $U_3(m)$ .

$S_2 = (p_1, q_1; \dots; p_t, q_t)$ : to obtain this graph, add all the edges connecting the centers of  $l$  non isomorphic arbitrary stars  $K_{1,p_i}, i = 1, \dots, t$ , each one occurring  $q_i$  times, where  $p_i, q_i, t \geq 1, q_1 + \dots + q_t = l \geq 2$  (see Fig. 8.a).

$S_3(p, q_1; q_2)$ : take a graph  $S_2(p, q_1; p + 1, q_2)$  where  $p \geq 1$ ,  $q_1 \geq 2$  and  $q_2 \geq 1$ ; add a new vertex  $v$  to the stable part of the graph and add the set of  $q_1$  edges  $\{\{v, w\} : w \in V_K, \deg_{V_S}(w) = p\}$ : the obtained graph is  $S_3$  (see Fig. 8.b).

$S_4(p, q)$ : it is constructed taking a graph  $S_3(p, 2; q)$ ,  $q \geq 1$ , adding a new vertex  $u$  to the clique part and connecting it by the edges with each vertex except  $v$  (see Fig. 8.c).

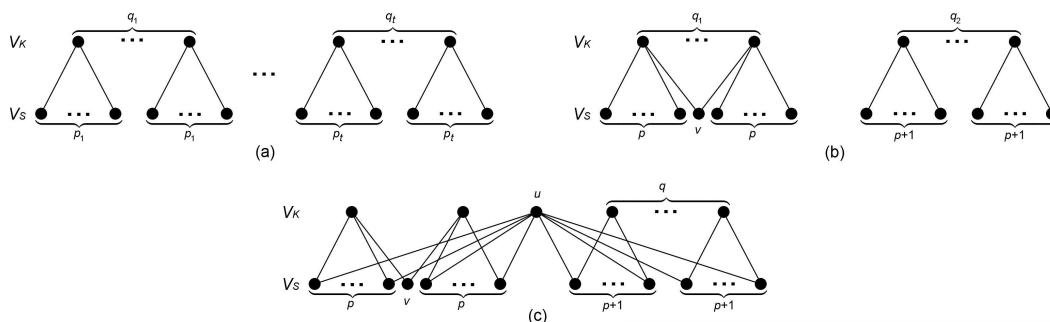


Figure 8: a.  $S_2(p_1, q_1; \dots; p_t, q_t)$ ; b.  $S_3(p, q_1; q_2)$ ; c.  $S_4(p, q)$ . For the sake of clearness, the edges of the clique connecting vertices of  $V_K$  are omitted.

It is easy to see that  $S_2$ ,  $S_3$  and  $S_4$  are split graphs, while  $U_2$  and  $U_3$  are not.

We are now able to state the characterization theorem for unigraphs:

**Theorem 4** [17] *Unigraphs are all graphs of the form  $G_1 \circ \dots \circ G_c \circ G$ , where:*

- $c \geq 0$  if  $G \neq \emptyset$  and  $c \geq 1$  otherwise;
- for each  $i = 1, \dots, c$ , either  $G_i$  or  $\overline{G_i}$  or  $G_i^I$  or  $\overline{G_i^I}$ , is one of the following split unigraphs:

$$K_1, \quad S_2(p_1, q_1; \dots; p_t, q_t), \quad S_3(p, q_1; q_2), \quad S_4(p, q);$$

- if  $G \neq \emptyset$ , either  $G$  or  $\overline{G}$  is one of the following non split unigraphs:

$$C_5, \quad mK_2, m \geq 2, \quad U_2(m, s), \quad U_3(m).$$

## 4 Unigraphs as Superposition of Red and Black Graphs

In this section we present a characterization of unigraphs in terms of superposition of a *red* and a *black* graph. This result generalizes the one holding for matrogenic graphs [12]. It is to notice that the proof of the following theorem is

not a straightforward generalization of the proof presented in [12], as the latter one is based on the heredity of matrogenicity while we know that this property does not hold for unigraphs.

**Theorem 5** *A graph  $G$  is a unigraph if and only if its vertex set can be partitioned into three disjoint sets  $V_K$ ,  $V_S$  and  $V_C$  such that:*

(i)  $V_K \cup V_S$  induces a split unigraph  $F$  in which  $V_K$  is the clique and  $V_S$  is the stable set;

(ii)  $V_C$  induces a crown  $H$  and either  $H$  or  $\overline{H}$  is one of the following graphs:

$$C_5, \quad mK_2, m \geq 2, \quad U_2(m, s), \quad U_3(m);$$

(iii) the edges of  $G$  can be colored red and black so that:

a. the red partial graph is the union of  $H$  and of vertex-disjoint pieces  $P_i, i = 1, \dots, z$ . Each piece  $P_i$  (or  $\overline{P}_i$ , or  $P_i^I$  or  $\overline{P}_i^I$ ) is one of the following graphs:

$$K_1, \quad S_2(p_1, q_1; \dots; p_t, q_t), \quad S_3(p, q_1; q_2), \quad S_4(p, q),$$

considered without the edges in the clique;

b. The linear ordering  $P_1, \dots, P_z$  is such that each vertex in  $V_K$  belonging to  $P_i$  is not linked to any vertex in  $V_S$  belonging to  $P_j$ ,  $j = 1, \dots, i - 1$ , but is linked by a black edge to every vertex in  $V_S$  belonging to  $P_j$ ,  $j = i + 1, \dots, z$ . Furthermore, any edge connecting either two vertices in  $V_K$  or a vertex in  $V_K$  and a vertex in  $V_C$  is black.

Before proving the theorem, we observe that the black graph cited in Theorem 5 is a threshold graph according to one of the equivalent definitions presented in [11]. It is also worthy to be noticed that there is a basic difference between a matching inside the red graph of a split unigraph and a matching constituting the crown of a unigraph: the first one corresponds to a matching whose vertices of one partition are connected in a complete subgraph; the second one corresponds to an  $mK_2$ . An analogous difference holds between an antimatching inside the red graph of a split unigraph and an hyperoctahedron constituting the crown of a unigraph.

**Proof:** Let us prove the 'if' part, first. Items (i) and (ii) and the ordering (iii).b. identify the decomposition in  $F \circ H$  where  $F$  is a split graph and  $H$  is an indecomposable unigraph. Let us now consider graph  $F$ , i.e.  $G - H$ . For item (iii).a  $F$  is the union of vertex-disjoint pieces connected by the black threshold graph; it follows that  $F = F_1 \circ \dots \circ F_c$ , where each  $F_i$  is a piece  $P_i$  plus the black edges in the clique part. This is the crucial point that allows us to bypass the lack of heredity of unigraphicity. So Theorem 4 holds and  $G$  is a unigraph.



Concerning the 'only if' part, observe that the edges added during the composition operation, together with the edges in the cliques of the split components, induce a threshold graph. As  $G$  is a unigraph, items (i) and (ii) derive from this observation and from Theorem 4, while item (iii) is obtained from the following coloring operation: color black all edges coming from the composition operation, all edges induced by  $V_K$  and all edges connecting  $V_K$  and  $V_C$ ; color red all other edges. The elimination of the edges from the complete part of the red pieces in item (iii) is necessary for avoiding that these edges are colored both red and black.  $\square$

**An example.** In Fig. 9.a a unigraph is depicted, and its red and black partial graphs are highlighted in Fig. 9.b and 9.c, respectively. In Fig. 9.b, the pieces  $P_1$ ,  $P_2$  and  $P_3$  are  $S_3(1, 2; 1)$ ,  $K_1$  and  $S_2^I(2, 2)$ , respectively, and the crown is a  $C_5$ . In Fig. 9.c it is highlighted that the black graph is threshold.

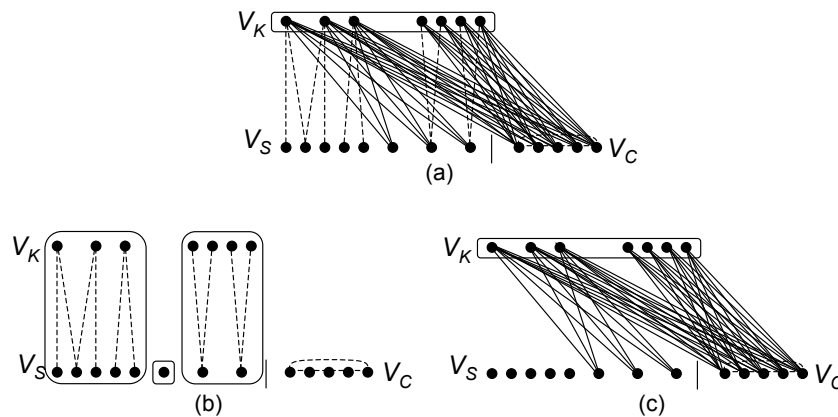


Figure 9: a. A unigraph; b. its red graph; c. its black graph.

Theorem 5 will lead to a consequent new algorithm that is the focus of next two sections.

## 5 A Linear Time Recognition Algorithm for Unigraphs

We present a linear time algorithm that, given in input a graph  $G$  as degree sequence  $d_1^{m_1}, \dots, d_r^{m_r}$ , gives in output the red/black edge coloring if and only if  $G$  is a unigraph. The algorithm exploits Theorem 5 and recognizes the pieces of  $G$  by means of a pruning procedure.

At each step, the algorithm finds an indecomposable split piece  $P_i$  of  $G$  according to part (iii).a. of Theorem 5. To do this, Theorem 2 is exploited, so the first  $p$  and the last  $q$  boxes are checked. The algorithm proceeds on the

pruned graph  $G - P_i$  represented by the sequence  $d_{p+1}^{m_{p+1}}, \dots, d_{r-q}^{m_{r-q}}$  and iterates these steps until either  $G$  is recognized to be a unigraph or some contradiction is highlighted.

In order to detail the algorithm we need to know all the rules that must be respected by any degree sequence  $d_1^{m_1}, \dots, d_r^{m_r}$  corresponding to graph  $F \circ H$ . For the sake of clearness, we prefer to present the algorithm first, postponing the list of these conditions to the successive subsection.

## 5.1 The Recognition Algorithm

Algorithm **Pruning-Unigraphs** uses  $imax$  and  $imin$  as indices of the first and last boxes of the current degree sequence. Three fundamental steps are highlighted: in the first one the crown is recognized if it exists, while the second and the third ones are related to split components: step 2 considers piece  $K_1$  and step 3 deals with pieces  $S_2, S_3, S_4$ . Observe that, in view of item (iii).a of Theorem 5, each time a piece  $S_i, i = 2, 3, 4$ , is considered, 4 conditions must be checked, i.e. COND.  $S_i$ , COND.  $\overline{S}_i$ , COND.  $S_i^I$  and COND.  $\overline{S}_i^I$ , while if the piece is  $K_1$  the conditions are simply COND.  $U$  (universal box) and COND.  $I$  (isolated box).

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ALGORITHM **Pruning-Unigraphs**

INPUT: a graph  $G$  by means of its degree sequence  $d_1^{m_1}, \dots, d_r^{m_r}$   
 OUTPUT: a red/black edge coloring if  $G$  is an unigraph, "failure" otherwise.

$imax \leftarrow 1; imin \leftarrow r; n \leftarrow \sum_{j=imax}^{imin} m_j;$

REPEAT

**Step 1** (non split indecomposable component, i.e. crown)

IF  $imax = imin$  AND (COND.  $K_2$  OR COND.  $\overline{K}_2$  OR COND.  $C_5$ )

THEN color by red all edges of the crown;

$n \leftarrow 0;$

ELSE

IF  $imax = imin - 1$  AND (COND.  $U_2$  OR COND.  $U_3$  OR COND.  $\overline{U}_2$  OR COND.  $\overline{U}_3$ )

THEN color by red all edges of the crown;

$n \leftarrow 0;$

ELSE

**Step 2** (universal or isolated box)

IF COND.  $U$

THEN FOR  $i = imax + 1$  TO  $imin$  DO

$d_i \leftarrow d_i - m_{imax};$

$imax \leftarrow imax + 1;$

$n \leftarrow n - m_{imax};$

ELSE

IF COND.  $I$

THEN  $imin \leftarrow imin - 1;$

$n \leftarrow n - m_{imin};$

ELSE

**Step 3** (split indecomposable components)

IF COND.  $S_3$  OR COND.  $\overline{S}_3$  OR COND.  $\overline{S}_4$  OR COND.  $S_4^I$

THEN color by red all the edges of the split component but the edges of its clique;

FOR  $i = imax + 1$  TO  $imin - 2$  DO

$d_i \leftarrow d_i - m_{imax};$

$imax \leftarrow imax + 1;$

```

    imin ← imin - 2;
    n ← n - mimax - mimin - mimin-1;
ELSE
IF COND.  $\overline{S_3}$  OR COND.  $S_3^I$  OR COND.  $S_4$  OR COND.  $\overline{S_4^I}$ 
THEN color by red all the edges of the split component but the edges of its clique;
    FOR i = imax + 2 TO imin - 1 DO
        di ← di - mimax - mimax+1;
        imax ← imax + 2;
        imin ← imin - 1;
        n ← n - mimax - mimax+1 - mimin;
ELSE
IF mimax ≤ mimin AND there exists an x ≥ 1 s.t. COND. S2 OR COND.  $\overline{S_2^I}$ 
THEN color by red all the edges of the split component but the edges of its clique;
    FOR i = imax + x TO imin - 1 DO
        di ← di - mimax - ... - mimax+x-1;
        imax ← imax + x;
        imin ← imin - 1;
        n ← n - mimax - ... - mimax+x-1 - mimin;
ELSE
IF mimax > mimin AND there exists an x ≥ 1 s.t. COND.  $\overline{S_2}$  OR COND. S2I
THEN color by red all the edges of the split component but the edges of its clique;
    FOR i = imax + 1 TO imin - x DO
        di ← di - mimax;
        imax ← imax + 1;
        imin ← imin - x;
        n ← n - mimax - mimin-x+1 - ... - mimin;
ELSE
Step 4 (G is not an unigraph)
    RETURN “failure”;
UNTIL n = 0;
color by black all the uncolored edges of G;
RETURN the red/black edge coloring of G.

```

**An example.** Although the details of the conditions will be presented in the next subsection, we prefer to give here an example of the execution of the algorithm, in order to help the reader to a complete comprehension of it.

Let  $16^3, 12^4, 9^5, 5^2, 3^1, 2^1, 1^4$  be the degree sequence of an input graph  $G$ . On this sequence,  $\text{COND. } S_3 = \text{true}$ , hence  $B_1 \cup B_6 \cup B_7$  induce  $S_3(1, 2; 1)$  (see Fig. 10).

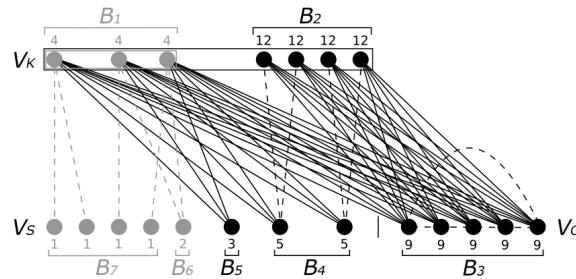


Figure 10: The graph in input, where component  $S_3(1, 2; 1)$  is highlighted.

After the pruning operation, the new degree sequence is  $9^4, 6^5, 2^2, 0^1$  and represents the graph in Fig. 11.

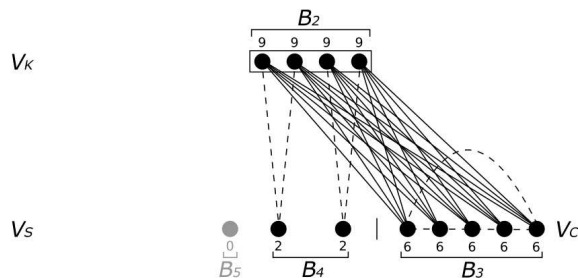


Figure 11: The graph in input after pruning component  $S_3(1, 2; 1)$ .

For this degree sequence,  $\text{COND. } I = \text{true}$ , and therefore  $B_5$  induces  $K_1 \in V_S$  (see Fig. 11). After the removal of this box (see Fig. 12), the pruned degree sequence is  $9^4, 6^5, 2^2$  and on it  $\text{COND. } \overline{S_2} = \text{true}$ .

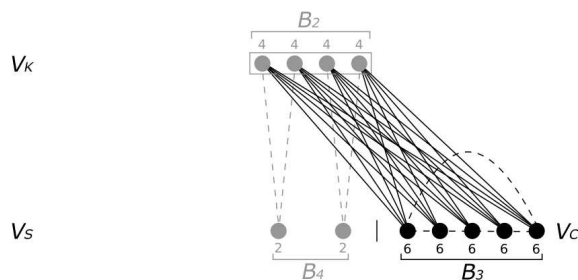


Figure 12: The graph in Figure 11 after pruning component  $K_1 \in V_S$ .

We deduce that  $B_2 \cup B_4$  induces  $\overline{S_2(2, 2)}$ , as shown in Fig. 12. The algorithm prunes the sequence, producing graph in Fig. 13, corresponding to sequence  $2^5$ , that verifies  $\text{COND. } C_5$ . It follows that  $B_3$  induces  $C_5$ .

Also this sequence is pruned and the reduced graph is empty, hence the algorithm successfully terminates recognizing that  $G$  is a unigraph and returning a red/black edge coloring of it.

We observe that, if Step 1 of the algorithm never occurs, the recognized unigraph is a split unigraph, while if only a sequence of Step 2 occurs, then the recognized unigraph is a threshold graph.

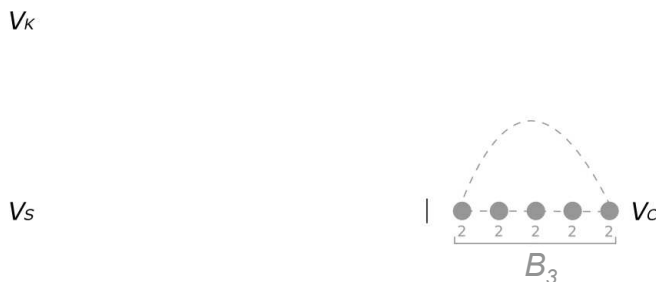


Figure 13: The graph in Figure 12 after pruning component  $\overline{S_2(2,2)}$ .

### 5.2 List of Conditions for Recognizing Indecomposable Parts

In this subsection we list which conditions the degree sequence  $d_{imax}^{m_{imax}}, \dots, d_{imin}^{m_{imin}}$  must satisfy to guarantee that its first  $p$  and last  $q$  boxes identify one of the indecomposable pieces. Remind that  $n = \sum_{i=imax}^{imin} m_i$ .

We will call each condition with the name of the graph that it identifies, although in the algorithm we add the word COND. to each of them.

Indecomposable non-split graphs: These graphs, if they exist, identify the crown (cf. item (ii) of Theorem 5) and hence are the last pieces to be recognized and colored by our algorithm. Therefore, it must be  $imin = imax$  when  $H$  is either a matching, or an hyperoctahedron, or  $C_5$ , and  $imin = imax + 1$  when  $H$  is either  $U_2(m, s)$ , or  $U_3(m)$ , or their complements.

It follows the list of the conditions specifying the different possibilities.

- $H = mK_2, m \geq 2$   
 In view of the definition of matching, it must hold:  

$$d_{imax} = 1, m_{imax} \text{ is even and } m_{imax} \geq 4 \tag{K_2}$$
 and in such a case  $m = \frac{m_{imax}}{2}$ .
- $H = \overline{mK_2}, m \geq 3$   
 Easily, it must be:  

$$d_{imax} = m_{imax} - 2, m_{imax} \text{ is even and } m_{imax} \geq 6 \tag{\overline{K_2}}$$
 and in such a a case  $m = \frac{m_{imax}}{2}$ . Observe that we have excluded  $m = 2$  as, in this case,  $H = \overline{2K_2} = 2K_2$ .
- $H = C_5 (= \overline{C_5})$   
 In this case, we have:  

$$d_{imax} = 2 \text{ and } m_{imax} = 5 \tag{C_5}$$
- $H = U_2(m, s)$   
 From the definition of  $U_2$ , it follows:

$$m_{imax} = 1, d_{imin} = 1, 2 \leq d_{imax} \leq n - 3 \text{ and } n \geq 5 \quad (U_2)$$

and in this case  $m = \frac{m_{imin} - d_{imax}}{2}$  and  $s = d_{imax}$ .

- $H = U_3(m)$

From the definition of  $U_3$  we deduce:

$$m_{imax} = 1, d_{imin} = 2, d_{imax} = n - 2, n \geq 6 \text{ and } n \text{ is even} \quad (U_3)$$

and in this case  $m = \frac{m_{imin} - 3}{2}$ .

- $H = \overline{U_2(m, s)}$

$$m_{imin} = 1, 2 \leq d_{imin} \leq n - 3, d_{imin} \text{ is even, } d_{imax} = n - 2 \text{ and } n \geq 5 \quad (\overline{U_2})$$

and in this case  $m = \frac{d_{imin}}{2}$  and  $s = m_{imax} - d_{imin}$ .

- $H = \overline{U_3(m)}$

$$m_{imin} = 1, d_{imin} = 1, d_{imax} = n - 3, m_{imax} \text{ is odd, } n \text{ is even and } n \geq 6 \quad (\overline{U_3})$$

and in this case  $m = \frac{m_{imax} - 3}{2}$ .

Indecomposable split graphs: In order to identify the indecomposable split pieces, the algorithm considers separately the case in which  $P_i$  is  $K_1$ , since in this case only one box – either the first one or the last one – is involved.

- $P_i = K_1$  in  $K$

In this case the first box is universal:

$$d_{imax} = n - 1 \quad (U)$$

- $P_i = K_1$  in  $S$

In this case the last box is isolated and the condition is:

$$d_{imin} = 0 \quad (I)$$

In all the other cases of item (iii).a of Theorem 5, we have to check a larger number of boxes in order to identify the corresponding indecomposable piece  $P_i$ . For this reason, the conditions we will show in the following are given by means of tables and may appear more complicate, although the way to compute them follows the same analysis of the structure of the piece as before.

Moreover, in order not to overburden the exposition, when we speak about one of these graphs (i.e.  $S_2, S_3$  and  $S_4$ ) we refer also to their inverses, their complements and the inverses of their complements, even if not explicitly underlined.

$P_i$	$d_1$	$d_2$	$d_3$
$S_3$	$p + q_1 + q_2$	$q_1$	1
$\overline{S_3}$	$p(q_1 + q_2) + q_1 + 2q_2 - 1$	$p(q_1 + q_2) + 2q_2$	$p(q_1 + q_2) + q_2 - p$
$S_3^I$	$p(q_1 + q_2) + q_1 + q_2$	$p(q_1 + q_2) + q_2 + 1$	$p + 1$
$\overline{S_3^I}$	$p(q_1 + q_2) + q_1 + 2q_2 - p - 1$	$q_1 + q_2 - 1$	$q_2$
$S_4$	$2p + 2q + pq + 2$	$p + q + 3$	2
$\overline{S_4}$	$2p + 2q + pq + 1$	$p + q + pq$	1
$S_4^I$	$2p + q + pq + 2$	$2p + q + pq$	$p + 1$
$\overline{S_4^I}$	$p + 2q + pq + 2$	$q + 3$	$q + 1$

Table 1: Values of  $d_i$ ,  $i = 1, 2, 3$  in the degree sequence of graphs  $S_3$  and  $S_4$ .

$P_i$	$m_1$	$m_2$	$m_3$
$S_3$	$q_1 + q_2$	1	$p(q_1 + q_2) + q_2$
$\overline{S_3}$	$p(q_1 + q_2) + q_2$	1	$q_1 + q_2$
$S_3^I$	1	$p(q_1 + q_2) + q_2$	$q_1 + q_2$
$\overline{S_3^I}$	$q_1 + q_2$	$p(q_1 + q_2) + q_2$	1
$S_4$	1	$q + 2$	$2p + q + pq + 1$
$\overline{S_4}$	$2p + q + pq + 1$	$q + 2$	1
$S_4^I$	$2p + q + pq + 1$	1	$q + 2$
$\overline{S_4^I}$	$q + 2$	1	$2p + q + pq + 1$

Table 2: Values of  $m_i$ ,  $i = 1, 2, 3$  in the degree sequence of graphs  $S_3$  and  $S_4$ .

- $P_i = S_3(p, q_1; q_2)$  or  $P_i = S_4(p, q)$

In both these cases three boxes are involved. In Tables 1 and 2 values of  $d_i$  and  $m_i$  for these cases are reported. According to the structure of  $S_3$  and  $S_4$  they can be either the first one and the last two or the first two and the last one.

In the first case, the indices 1, 2 and 3 of  $d$  in the tables correspond to  $imax$ ,  $imin-1$  and  $imin$ , respectively, while in the second case, indices 1, 2 and 3 correspond to  $imax$ ,  $imax+1$  and  $imin$ , respectively. Consequently, the following conditions  $\alpha$  and  $\beta$  must hold in the first and second case, respectively:

$$d_{imin}m_{imin} + d_{imin-1}m_{imin-1} = m_{imax}(d_{imax} - n + m_{imin} + m_{imin-1} + 1) \tag{\alpha}$$

$$d_{imin}m_{imin} = m_{imax}(d_{imax} - n + m_{imin} + 1) + m_{imax+1}(d_{imax+1} - n + m_{imin} + 1) \tag{\beta}$$

Tables 3 and 4 report the values of the parameters of  $S_3$  and  $S_4$ , respectively.

$P_i$	$p$	$q_1$	$q_2$
$S_3$	$\frac{m_{imin} - m_{imax} + d_{imin-1}}{m_{imax}}$	$d_{imin-1}$	$m_{imax} - d_{imin-1}$
$\overline{S_3}$	$m_{imax} - d_{imin}$	$m_{imin}(m_{imax} - d_{imin} + 1) - m_{imax}$	$m_{imax} - m_{imin}(m_{imax} - d_{imin})$
$S_3^I$	$d_{imin} - 1$	$m_{imin}d_{imin} - m_{imax+1}$	$m_{imax+1} - m_{imin}(d_{imin} - 1)$
$\overline{S_3^I}$	$\frac{m_{imin-1} - d_{imin}}{m_{imax}}$	$m_{imax} - d_{imin}$	$d_{imin}$

Table 3: Parameters of  $S_3$ .

$P_i$	$p$	$q$
$S_4$	$\frac{m_{imin} - m_{imax+1} + 1}{m_{imax+1}}$	$m_{imax+1} - 2$
$\overline{S_4}$	$m_{imax} - d_{imin-1} - 1$	$m_{imin-1} - 2$
$S_4^I$	$d_{imin} - 1$	$m_{imin} - 2$
$\overline{S_4^I}$	$\frac{m_{imin} - d_{imin}}{d_{imin} + 1}$	$d_{imin} - 1$

Table 4: Parameters of  $S_4$ .

Summarizing, we derive a set of conditions, each one univocally identifying a different graph  $P_i$  of type either  $S_3$  or  $S_4$ . All these sets are listed in Table 5. So, for example, when in the algorithm COND.  $S_3$  must be checked, we require to verify that condition  $\alpha$  is true, and  $m_{imin-1} = 1$ ,  $d_{imin} = 1$ ,  $p \geq 1$ ,  $q_1 \geq 2$  and  $q_2 \geq 1$  are also all true.

$P_i$	Conditions					
$S_3$	$\alpha$	$m_{imin-1} = 1$	$d_{imin} = 1$	$p \geq 1$	$q_1 \geq 2$	$q_2 \geq 1$
$\overline{S_3}$	$\alpha$	$m_{imin-1} = 1$	$d_{imin} > 1$	$p \geq 1$	$q \geq 1$	-
$S_4$	$\alpha$	$m_{imin-1} > 1$	$d_{imin-1} \geq m_{imin-1}$	$p \geq 1$	$q \geq 1$	-
$\overline{S_4}$	$\alpha$	$m_{imin-1} > 1$	$d_{imin-1} < m_{imin-1}$	$p \geq 1$	$q_1 \geq 2$	$q_2 \geq 1$
$S_3^I$	$\beta$	$m_{imax} = 1$	$m_{imax+1} > m_{imin}$	$p \geq 1$	$q_1 \geq 2$	$q_2 \geq 1$
$\overline{S_3^I}$	$\beta$	$m_{imax} = 1$	$m_{imax+1} < m_{imin}$	$p \geq 1$	$q \geq 1$	-
$S_4^I$	$\beta$	$m_{imax} > 1$	$d_{imin} < m_{imin}$	$p \geq 1$	$q \geq 1$	-
$\overline{S_4^I}$	$\beta$	$m_{imax} > 1$	$d_{imin} \geq m_{imin}$	$p \geq 1$	$q_1 \geq 2$	$q_2 \geq 1$

Table 5: Set of conditions for the recognition of components  $S_3$  and  $S_4$ .

- $P_i = S_2(p_1, q_1; \dots; p_t, q_t)$

In this case  $x+1$  boxes are involved, where  $x$  is a not null integer variable. Again, these boxes can be either the first one and the last  $x$  or the first  $x$  and the last one. More precisely, when  $P_i$  is either  $\overline{S_2}$  or  $S_2^I$ , boxes with indices  $imax$ ,  $imin - x + 1, \dots, imin$  are involved and we have that  $m_{imax} \geq m_{imin}$ . When  $P_i$  is either  $S_2$  or  $\overline{S_2^I}$ , boxes with indices  $imax$ ,



...,  $imax + x - 1$ ,  $imin$  are involved and  $m_{imax} \leq m_{imin}$ . Observe that when  $m_{imax} = m_{imin}$  then  $x = 1$  and the two cases collapse as  $S_2 = S_2^I$  and  $\overline{S_2} = \overline{S_2^I}$ .

Analyzing the degree sequences of graph  $S_2$ , we derive the values of  $p_i$  and  $q_i$ ,  $i = 1, \dots, x$ , summarized in Table 6. Furthermore, we find a set of conditions, each one univocally identifying a different graph  $P_i$  of type  $S_2$ . All these sets are listed in Table 7.

$P_i$	$p_i$	$q_i$
$S_2$	$d_i - n + m_{imin} + 1$	$m_i$
$\overline{S_2^I}$	$n - 1 - d_{x-i+1}$	$m_{x-i+1}$
$S_2^I$	$m_{imax} - d_{imin-i+1}$	$m_{imin-i+1}$
$\overline{S_2^I}$	$d_{imin-x+i}$	$m_{imin-x+i}$

Table 6: Parameters of  $S_2$ . Index  $i$  varies between 1 and  $x$ .

$P_i$	Conditions	
$S_2$	$m_{imax} \leq m_{imin}$	$m_{imin} = \sum_{i=1}^x m_i(d_i - n + m_{imin} + 1)$
	$d_{imin} = 1$	$p_i, q_i \geq 1$
$\overline{S_2}$	$m_{imax} \geq m_{imin}$	$m_{imax} = \sum_{i=imin-x+1}^{imin} m_i(m_{imax} - d_i)$
	$d_{imax} = n - 2$	$p_i, q_i \geq 1$
$S_2^I$	$m_{imax} \geq m_{imin}$	$m_{imin} = \sum_{i=1}^x m_i(n - 1 - d_i)$
	$d_{imax} = n - \sum_{i=imin-x+1}^{imin} m_i$	$p_i, q_i \geq 1$
$\overline{S_2^I}$	$m_{imax} \leq m_{imin}$	$m_{imin} = \sum_{i=1}^x m_i(n - 1 - d_i)$
	$d_{imin} = \sum_{i=1}^x m_i - 1$	$p_i, q_i \geq 1$

Table 7: Set of conditions for the recognition of components  $S_2$ .

Let us now explain how to determine  $x$ . Without loss of generality, let us assume that  $m_{imax} \leq m_{imin}$ : the other case can be treated analogously. Let us start from the set of conditions that individuate graph  $S_2$  (table 7). At the beginning we check whether all the conditions in the set are verified with  $x = 1$ . If it is so,  $S_2$  has been found; otherwise, if the sum is smaller than  $m_{imin}$  but all the other conditions hold, we increment  $x$  and check the set of conditions again. This loop stops when the sum becomes greater than or equal to  $m_{imin}$ . If the equality holds,  $S_2$  has been found, otherwise, we deduce that the component cannot be an  $S_2$  and we move to check next set of conditions (i.e. that one individuating graph  $\overline{S_2^I}$ ). If also this graph cannot be individuated, then we conclude that the current component is not an  $S_2$ .

### 5.3 Correctness and complexity

We conclude this section showing that algorithm **Pruning-Unigraphs** correctly recognizes if  $G$  is a unigraph, and red/black edge color it. Moreover, we show that the algorithm runs in linear time.

**Theorem 6** *Algorithm Pruning-Unigraphs produces a red/black edge coloring of  $G$  if and only if  $G$  is a unigraph.*

**Proof:** Conditions of steps 1, 2, 3 and 4 of the algorithm are mutually exclusive, so at each iteration only one step may occur.

Checking the first  $p$  and the last  $q$  boxes of the degree sequence it is possible to recognize each piece  $P_i$  of the split component  $F$  (cf. item (i) of Theorem 5), if it exists. Item (iii).a. of Theorem 5 identifies all the components of  $F$  for a unigraph and the conditions listed in Subsection 5.2 univocally determine each component and indicate the values of  $p$  and  $q$  for each component in terms of boxes.

Item (iii).b. of Theorem 5 guarantees that the pruning operation can be iteratively applied. Indeed, the algorithm, at each step, red-colors and eliminates a complete piece  $P_i$  and all the edges connecting  $P_i$  to  $P_j$ , for  $j = i + 1, \dots, z$  that will be black-colored at the end; hence, if the original graph  $G$  is a unigraph, the reduced graph  $G - P_i$  is a unigraph, too as for it Theorem 5 holds. Finally, the crown, if it exists, is specified by item (ii) of Theorem 5 and by the conditions in Subsection 5.2, and is red-colored. The correctness of the algorithm follows.  $\square$

**Theorem 7** *Algorithm Pruning-Unigraphs runs in  $O(n)$  time.*

**Proof:** Each indecomposable component  $P_i$  involves a certain number of boxes  $r_i, 1 \leq r_i \leq r$  and  $\sum r_i = r$ , where the sum is performed over all the found indecomposable components. Observe that all the indecomposable components, except  $S_2$  (and its complement, its inverse and the inverse of its complement) involve a constant number of boxes (either 1 or 2 or 3). From the other hand, determining  $x$ , and hence recognizing  $S_2$  as explained at the end of Subsection 5.2, takes time  $\Theta(x)$ . Since the recognition of component  $S_2$  is executed as last possibility, it follows that the recognition of each indecomposable component  $P_i$ , involving  $r_i$  boxes, always takes  $\Theta(r_i)$  time. Also the update of the degree sequence can be run in the same time provided that a clever implementation is performed. Indeed, in Algorithm **Pruning-Unigraphs**, it is not necessary to update at each step the degree sequence as we do in order to highlight the pruning technique: instead, it is enough to keep an integer variable *prune* and to check conditions on  $(d_i - \textit{prune})$  instead of  $d_i$  in order to guarantee that no additional time is used to prune the sequence. Hence the algorithm recognizes if  $G$  is a unigraph in time  $\Theta(\sum r_i) = \Theta(r) = O(n)$ .  $\square$

## 6 Conclusions

In this paper we have presented a linear time algorithm for recognizing unigraphs exploiting a new characterization for these graphs that generalizes a known characterization for matrogenic graphs. It partitions the vertex set and the edge set into three and two disjoint sets, respectively, detecting two particular graphs whose superposition generates the given unigraph.

As we have already observed, the proof of the known characterization for matrogenic graphs is based on the hereditary property, that holds for matrogenic graphs but not for unigraphs; so, we have weakened our requirements proving that only some special subgraphs of a unigraph are still unigraphs. This is the picklock of the proof of our characterization.

We are convinced that the results presented in this paper can be helpful to solve some of those problems that are NP-hard in general, polynomially solved for subclasses of unigraphs and still unknown for unigraphs. This will be a future direction of our work.

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