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Constrained Simultaneous and Near-Simultaneous Embeddings

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Abstract

A *geometric simultaneous embedding* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with a bijective mapping of their vertex sets $\gamma : V_1 \rightarrow V_2$ is a pair of planar straight-line drawings Γ_1 of G_1 and Γ_2 of G_2, such that each vertex $v_2 = \gamma(v_1)$, with $v_1 \in V_1$ and $v_2 \in V_2$, is mapped in Γ_2 to the same point where v_1 is mapped in Γ_1.

In this paper we examine several constrained versions and a relaxed version of the geometric simultaneous embedding problem. We show that assuming that the input graphs do not share common edges does not yield larger classes of graphs that can be simultaneously embedded. Further, if a prescribed combinatorial embedding for each input graph must be preserved, then we can answer some of the problems that are still open in the standard geometric simultaneous embedding setting. Finally, we present some results on the near-simultaneous embedding problem, in which vertices are not forced to be placed exactly at the same, but just at "nearby" points in different drawings.

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1 Introduction

Graph drawing techniques are commonly used to visualize relationships between objects, where the objects are the vertices of the graph and the relationships are captured by the edges in the graph. The simultaneous embedding problem arises when visualizing two or more relationships defined on the same set of objects. If the graphs corresponding to these relationships are planar, the aim of simultaneous embedding is to find point locations in the plane for the vertices of the graphs, so that each of the graphs can be realized on the same point-set without edge crossings. To ensure good readability of the drawings, it is preferable that the edges are drawn as straight-line segments. This problem is known as *geometric simultaneous embedding.* It has been shown that only a few classes of graphs, such as paths, cycles, and caterpillars admit pairwise geometric simultaneous embeddings.

Nevertheless, simultaneous embedding techniques have been used to obtain new result in geometric graph thickness [6] and have been helpful in visualizing evolving and dynamic graphs [7]. Many intriguing simultaneous embedding problems remain open. For example, it is not yet fully understood which graph classes always admit a geometric simultaneous embedding. There are also no practical algorithms guaranteeing high quality layouts for evolving and dynamic graphs. With this in mind, we consider three further variants of the geometric simultaneous embedding problem: one in which we do not allow the input graphs to share any edges, another in which the input graphs must realize given combinatorial embeddings, i.e. given cyclic orderings of the edges incident to each vertex, and yet another in which we allow corresponding vertices to be placed not exactly at the same point, but just at nearby points.

1.1 Related Work

Brass *et al.* [3], Erten and Kobourov [8], and Geyer *et al.* [12] respectively showed examples of three paths, of a planar graph and a path, and of two trees that do not admit geometric simultaneous embeddings. Recently, Estrella-Balderrama *et al.* [9] proved that determining whether two planar graphs admit a geometric simultaneous embedding is an *NP*-hard problem. On the positive side, efficient algorithms for geometric simultaneous embedding of pairs of paths, pairs of cycles and pairs of caterpillars have been developed [3].

As geometric simultaneous embedding turns out to be very restrictive, it is natural to relax some of the constraints of the problem. Not insisting on straight-line edges led to positive results such as a linear-time algorithm by Erten and Kobourov for embedding any pair of planar graphs with at most three bends per edge, or any pair of trees with at most two bends per edge [8]. The results of Pach and Wenger [15], and of Badent *et al.* [1] concerning graph embeddings with vertices at fixed locations directly imply that any number of planar graphs can be embedded simultaneously using $O(n)$ bends per edge. In such results it is allowed for an edge connecting a pair of vertices to be represented by different Jordan curves in different drawings, something not possible when edges are straight-line segments. As this can be detrimental to the readability of the drawings, several papers considered a slightly more constrained version of this problem, namely, *simultaneous embedding with fixed edges*. In this version of the problem bends are allowed, provided that an edge connecting the same pair of vertices is drawn in exactly the same way in all drawings. Di Giacomo and Liotta [5] showed that any outerplanar graph can be simultaneously embedded with fixed edges with a path or with a cycle using at most one bend per edge. Frati [11] showed that a planar graph and a tree can also be simultaneously embedded with fixed edges, however the algorithm described in [11] constructs drawings in which the edges have a large number of bends.

1.2 Our Results

In this paper we consider several variants of the simultaneous embedding problem. In Section 3, we consider the variants in which the input graphs are assumed to not share any edge, called *geometric simultaneous embedding with no common edges* and the one in which the input graphs must realize given combinatorial embeddings, called *geometric simultaneous embedding with fixed embedding*.

Most of the proofs about the non-existence of simultaneous embeddings exploit the presence of common edges between the graphs that have to be drawn, since such edges create a barrier that cannot be traversed by any other edge of the graphs. Hence, it is natural to ask whether larger classes of graphs admit geometric simultaneous embedding if no edges are shared by the input graphs. Quite surprisingly, we show that common edges are not the only geometric obstruction for simultaneous embeddability, in fact we show that there exist a planar graph and a path that do not share edges and that do not allow for a geometric simultaneous embedding, generalizing the result in [8], where it is shown that a planar graph and a path that do share edges may not have a geometric simultaneous embedding. We further conjecture that an analogous negative result holds for two trees not sharing edges. We remark that, if we allow edges to have bends, then any number of planar graphs not sharing edges admit a simultaneous embedding [1, 15].

The second problem we consider is that of simultaneously drawing graphs with a fixed planar embedding. Drawing graphs with fixed embeddings has been studied before but not in the context of simultaneous embedding of multiple graphs. Specifically, it is well-known that every planar embedding can be realized with straight-line segments [10, 16, 18] and that testing whether a cyclic ordering of the edges incident to each vertex corresponds to a planar embedding can be done in linear time [14]. Other well-known results on drawing graphs with fixed embeddings concern upward drawings [2] and orthogonal drawings [17]. Besides, understanding the geometric simultaneous embedding with fixed embedding problem could be helpful, in our opinion, for solving some open questions on the geometric simultaneous embedding problem. While all negative results known for geometric simultaneous embedding remain valid in the fixed embedding setting, in this more restrictive setting we are able to re-

	Geometric	*Disj.*	*Fixed Emb.*	*Disj. Fixed Emb.*
path + path	YES [3]	YES [3]	YES [3]	YES [3]
star + path	YES [3]	YES [3]	YES *Th. 4*	YES *Th. 4*
double-star + path	YES [3]	YES [3]	?	YES *Th. 5*
caterpillar + path	YES [3]	YES [3]	?	?
caterpillar + caterpillar	YES [3]	YES [3]	NO *Th. 3*	NO *Th. 3*
3 *paths*	NO [3]	?	NO [3]	?
tree + path	?	?	?	?
tree + caterpillar	?	?	NO *Th. 3*	NO *Th. 3*
outerplanar + path	?	?	NO *Th. 2*	NO *Th. 2*
tree + tree	NO [12]	?	NO [12]	NO *Th. 3*
outerplanar + outerplanar	NO [3]	?	NO [3]	NO *Th. 3*
planar + path	NO [8]	NO *Th. 1*	NO [8]	NO *Th. 1*

Table 1: Known results and our contribution on geometric simultaneous embedding (Geometric), geometric simultaneous embedding with no common edges (Disj.), geometric simultaneous embedding with fixed embedding (Fixed Emb.), geometric simultaneous embedding with fixed embedding and no common edges (Disj. Fixed Emb.).

solve several problems that are still open in more general settings. We show that there exist an outerplanar graph and a path that have no geometric simultaneous embedding with fixed embedding (the same problem is not yet solved if the embedding of the outerplanar graph is not fixed). Moreover, we show that some classes of graphs that have geometric simultaneous embeddings do not admit one with fixed combinatorial embeddings. In particular, we prove such a negative result for caterpillar-caterpillar pairs. Motivated by understanding which classes of caterpillars always admit a geometric simultaneous embedding with fixed embedding, we show that a star and a path always admit a geometric simultaneous embedding with fixed embedding and that a double-star and a path always admit a geometric simultaneous embedding with fixed embedding if they do not share edges.

In the quest for more practical settings where we can still guarantee some theoretical properties of the resulting embeddings, in Section 4 we study a variant of geometric simultaneous embedding which we call *geometric near-simultaneous embedding*. In a geometric near-simultaneous embedding edges are drawn as straight-line segments but vertices representing the same entity in different input graphs can be placed not exactly at the same point but at points that are just nearby each other. We show that even this version is restrictive, namely, assuming that the vertices are placed on an integer grid, we prove that there exist pairs of n-vertex planar graphs in which vertices that represent the same entity in different graphs must be placed in points that are at distance $\Omega(n)$ each other. Further, we discuss some interesting research directions for geometric near-simultaneous embedding, among which proving bounds for the near-simultaneous embedding of sequences of *similar* planar graphs.

2 Preliminaries

Here we summarize some of the basic terminology used in this paper; further graph drawing definitions can be found in the surveys by Di Battista *et al.* [4] and by Kaufmann and Wagner [13].

A *straight-line drawing* of a graph is a mapping of each vertex to a unique point in the plane and of each edge to a segment between the endpoints of the edge. A *grid* drawing is one in which every vertex is placed at a point with integer coordinates in the plane. A *planar drawing* is one in which no two edges intersect. A *planar graph* is a graph that admits a planar drawing. It is a well-known result [10] that every planar graph admits a planar straight-line drawing. A planar drawing of a graph determines a cyclic ordering of the edges incident to each vertex. Two drawings of the same graph are *equivalent* if they determine the same cyclic ordering around each vertex. A *combinatorial embedding*, or *planar embedding*, is an equivalence class of planar drawings. A planar drawing partitions the plane into topologically connected regions, called *faces*. The unbounded face is the *external face*. An embedding of a graph G completely determines the faces in any drawing of G in which the cyclic order of the edges incident to each vertex is the same specified by the embedding, even though the embedding does not determine which face is the *external face*. A combinatorial embedding together with a choice for the external face is called *plane embedding*. A graph is *triconnected* if for every pair of distinct vertices there exist three vertex-disjoint paths connecting them. A triconnected graph has a unique embedding, up to a reversal of the cyclic lists of the edges incident to each node [19].

An *outerplanar graph* is a graph that admits an *outerplanar drawing*, that is, a planar drawing in which all the vertices are incident to the same face. The embedding of the outerplanar graph in an outerplanar drawing is called an *outerplanar embedding*. *Trees* are connected acyclic graphs and they are a subclass of the outerplanar graphs. The *degree* of a vertex is the number of its neighbors. A *leaf* is a vertex of a tree with degree 1. A *path* is a tree in which every vertex, other than the leaves, has degree 2. A *caterpillar* is a tree in which the removal of all the leaves and their incident edges yields a path. A *star* (*double-star*) is a caterpillar with only one vertex (two vertices), called *center* (*centers*), of degree greater than one.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two n-vertex planar graphs with a bijective mapping $\gamma : V_1 \rightarrow V_2$ between their vertices. A *geometric simultaneous embedding* is a pair of straight-line drawings Γ_1 and Γ_2 of G_1 and of G_2 such that: (i) each of Γ_1 and Γ_2 is planar, and (ii) each vertex $v_2 = \gamma(v_1)$, with $v_1 \in V_1$ and $v_2 \in V_2$, is mapped in Γ_2 to the same point where v_1 is mapped in Γ_1.

3 Simultaneous Embedding with Disjoint Edges and with Fixed Embedding

In this section we consider the problem of constructing geometric simultaneous embeddings of graphs that do not share common edges and of graphs that have fixed combinatorial embeddings, deriving some negative and some positive results, described in Section 3.1 and in Section 3.2, respectively.

3.1 Negative Results

First, we deal with planar graphs and paths not sharing edges. The strategy for proving the existence of a planar graph and a path that do not share common edges and that do not admit any geometric simultaneous embedding is as follows. First, we show a planar graph G^* and a path P^* that do not share common edges and that do not admit any geometric simultaneous embedding if the plane embedding $\mathcal{E}^*$ of G^* is fixed. Second, we show a planar graph G and a path P such that in every planar drawing of G there is a subgraph G_i^* of G isomorphic to G^* which has plane embedding $\mathcal{E}^*$ and such that the subgraph of P induced by the vertices of G_i^* is isomorphic to P^*. A similar strategy is used later in order to prove the existence of an outerplanar graph and a path that do not allow for a geometric simultaneous embedding with fixed embedding.

Let G^* be the triconnected planar graph on nine vertices $v_1, v_2, \ldots, v_9$ shown in Fig. 1(a). Since G^* is triconnected, it has the same faces in each of its planar embeddings. Let F^* denote the triangular face $\Delta v_1 v_3 v_9$ and let P^* be the path $(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)$.

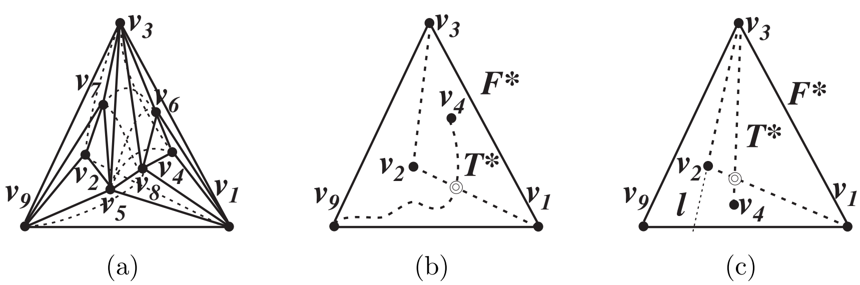


Figure 1: (a) Triconnected planar graph G^* drawn with solid edges and path P^* drawn with dashed edges; (b) Embedding vertex v_4 inside T^* creates a crossing between the subpath of P^* connecting v_1 and v_3 and the subpath of P^* connecting v_4 and v_9; (c) Embedding vertex v_4 outside T^* creates a crossing between edges (v_1, v_2) and (v_3, v_4) of P^*.

Lemma 1 *There does not exist a geometric simultaneous embedding of G^* and P^* in which the external face of G^* is F^*.*

Proof: Note that all vertices of G^*, other than v_1, v_3 and v_9, are contained inside F^* as F^* is the external face of G^*. Consider the triangle T^* formed by the edges (v_1, v_2), (v_2, v_3) of P^*, and by the edge (v_1, v_3) of G^*. Since vertex v_9 is incident to F^*, it must lie outside T^*. Let l be the line passing through v_2 and v_3; l separates the plane in two open half-planes, one containing v_9, called the *exterior part* of l, and one not containing v_9, called the *interior part* of l. We show that every placement of v_4 leads to a crossing in the drawing of the path if the planarity of the drawing of G^* is preserved. If v_4 is placed inside T^* then the subpath of P^* composed of the edges (v_1, v_2) and (v_2, v_3) crosses the subpath of P^* connecting v_4, that lies inside T^*, and v_9, that lies outside T^*; see Fig. 1(b). Suppose v_4 is placed outside T^*. Since vertex v_4 lies inside triangle $\Delta v_1 v_3 v_5$ and vertex v_2 lies inside triangle $\Delta v_3 v_5 v_9$, the clockwise order of the edges around v_3, i.e. the clockwise order of the edges $(v_3, v_1), (v_3, v_5), (v_3, v_9)$ of G^* and of the edges $(v_3, v_4), (v_3, v_2)$ of P^*, is $(v_3, v_1), (v_3, v_4), (v_3, v_5), (v_3, v_2), (v_3, v_9)$. Therefore v_4 is in the *interior part* of l and hence edge (v_1, v_2) crosses edge (v_3, v_4) in P^*; see Fig. 1(c). □

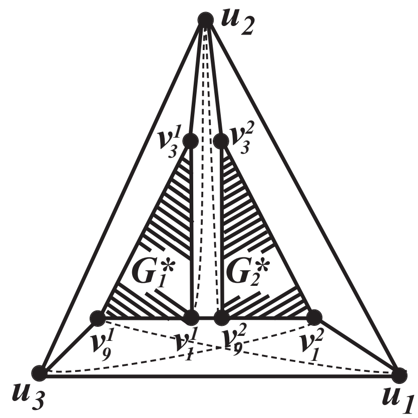


Figure 2: Triconnected planar graph G drawn with solid edges and path P drawn with dashed edges.

Theorem 1 *There exist a planar graph G, a path P, and a mapping between their vertices such that: (i) G and P do not share edges, and (ii) G and P have no geometric simultaneous embedding.*

Proof: We will construct graph G and path P out of two copies of G^* and P^* described above. In particular, let G_1^* and G_2^* be two copies of the planar graph G^*. G_1^* and G_2^* have nine vertices each, and we denote by v_i^j the vertex of G_j^* that corresponds to the vertex v_i in G^*, where $j = 1, 2$ and $i = 1, \ldots, 9$.

Let G be the graph composed of G_1^* and G_2^* together with three additional vertices u_1, u_2, and u_3 and eight additional edges (u_1, u_2), (u_1, u_3), (u_2, u_3), (u_1, v_1^2), (u_2, v_3^1), (u_2, v_3^2), (u_3, v_9^1), and (v_1^1, v_9^2); see Fig. 2. Since G is planar and triconnected by construction, it has the same faces in each of its planar drawings.

Let P be the path $(u_1, v_9^1, v_8^1, v_7^1, v_6^1, v_5^1, v_4^1, v_3^1, v_2^1, v_1^1, u_2, v_9^2, v_8^2, v_7^2, v_6^2, v_5^2, v_4^2, v_3^2, v_2^2, v_1^2, u_3)$. It is easy to verify that G and P do not share edges. Note that the subpaths of P induced by the vertices of G_1^* and by the vertices of G_2^* play the same role that path P^* plays for graph G^* in Lemma 1.

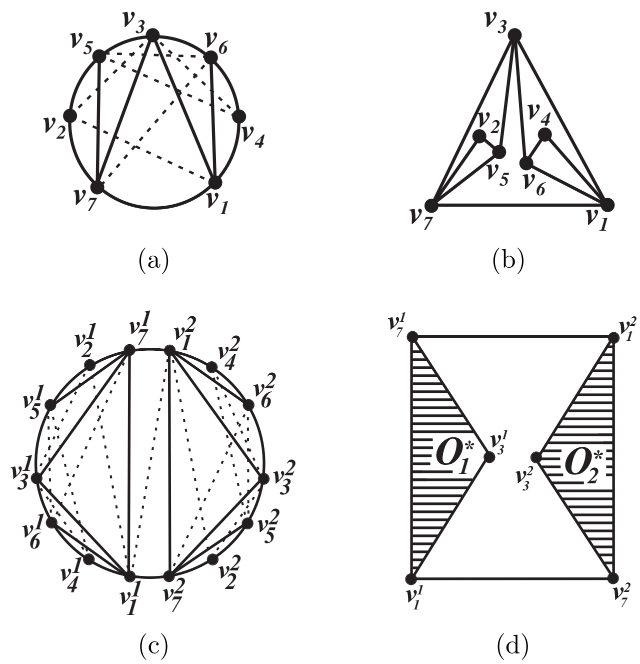


Figure 3: (a) Outerplanar graph O^*, drawn with solid edges, and path P^*, drawn with dashed edges. (b) Embedding $\mathcal{E}^*$ of O^*. (c) Outerplanar graph O, drawn with solid edges, and path P, drawn with dashed edges. (d) Embedding $\mathcal{E}$ of O.

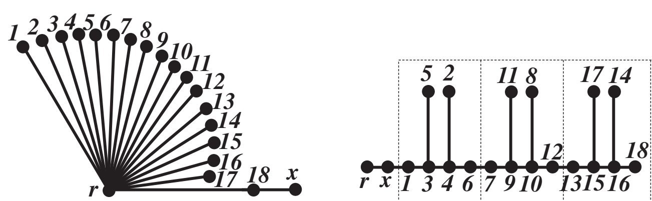
We now show that every plane drawing Γ of G determines a non-planar drawing of P. Let F_1^* and F_2^* denote cycles (v_1^1, v_3^1, v_9^1) and (v_1^2, v_3^2, v_9^2), respectively. Consider the plane embedding $\mathcal{E}_G$ of G obtained by choosing $\Delta u_1 u_2 u_3$ as external face; see Fig. 2. Choosing any face external to F_1^* in $\mathcal{E}_G$ as external face of Γ leaves G_1^* embedded with external face F_1^*. Choosing any face external to F_2^* in $\mathcal{E}_G$ as external face of Γ leaves G_2^* embedded with external face F_2^*. Since each face of G is external to either F_1^* or F_2^* in $\mathcal{E}_G$, we can apply Lemma 1 and conclude that there does not exist a simultaneous embedding of G and P.□

Next, we show that there exist an outerplanar graph and a path that do not share edges and that admit no simultaneous embedding in which the outerplanar graph has a fixed combinatorial embedding. Let O^* be the outerplanar graph shown in Fig. 3(a) and let $\mathcal{E}^*$ be the plane embedding of O^* shown in Fig. 3(b). Further, let P^* be the dashed path in Fig. 3(a). Note that O^* and P^* do not share edges. Using techniques similar to the ones in the proof of Lemma 1, it is possible to show that there exists no geometric simultaneous embedding of O^* and P^* in which the plane embedding of O^* is $\mathcal{E}^*$.

Out of two copies O_1^* and O_2^* of O^*, with plane embeddings $\mathcal{E}_1^*$ and $\mathcal{E}_2^*$ corresponding to the plane embedding $\mathcal{E}^*$ of O^*, we construct an outerplanar graph O (see Fig. 3(c)) with a fixed combinatorial embedding $\mathcal{E}$ (see Fig. 3(d)). Further, we construct a path P such that O and P do not share edges and that the subpaths of P induced by the vertices of O_1^* (of O_2^*) play for O_1^* (resp. for O_2^*) the same role that path P^* plays for graph O^*. By using arguments similar to the ones in the proof of Theorem 1, it is possible to show that every plane drawing of O with combinatorial embedding $\mathcal{E}$ determines a non-planar drawing of P. In fact, in every plane drawing of O with combinatorial embedding $\mathcal{E}$, either O_1^* has plane embedding $\mathcal{E}_1^*$ or O_2^* has plane embedding $\mathcal{E}_2^*$, hence implying that the subpath of P induced by the vertices of O_1^* or the subpath of P induced by the vertices of O_2^* is self-intersecting. Thus, we have the following result:

Theorem 2 *There exist an outerplanar graph O, a combinatorial embedding $\mathcal{E}$ of O, a path P, and a mapping between their vertices such that: (i) O and P do not share edges, and (ii) O and P have no geometric simultaneous embedding in which the combinatorial embedding of O is $\mathcal{E}$.*

Finally, we deal with pairs of caterpillars with fixed embeddings. Recall that in the standard setting of geometric simultaneous embedding we can always simultaneously embed pairs of caterpillars. This is no longer true if we insist on preserving the given combinatorial embedding of the caterpillars:



(a) (b)

Figure 4: Caterpillars C_1 (a) and C_2 (b) with fixed embeddings.

Theorem 3 *There exist two caterpillars C_1 and C_2 with fixed embeddings $\mathcal{E}_1$ and $\mathcal{E}_2$, and a mapping γ between their vertices such that:*

1. *C_1 and C_2 do not share edges, and*

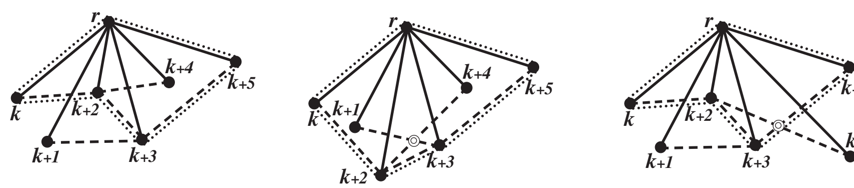
2. *C_1 and C_2 do not admit a geometric simultaneous embedding with combinatorial embeddings $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively.*

Proof:

Let C_1 and C_2 be the two caterpillars with embeddings $\mathcal{E}_1$ and $\mathcal{E}_2$ shown Fig. 4(a)-(b). Let $\gamma(x) = x$ be the bijective mapping between their vertices and note that C_1 and C_2 do not share edges. We show that there does not exist a geometric simultaneous embedding of C_1 and C_2 in which C_1 and C_2 respect the given embeddings $\mathcal{E}_1$ and $\mathcal{E}_2$.

Construct any straight-line drawing Γ_1 of C_1. The placement of the vertices of C_1 in Γ_1 also defines a drawing Γ_2 of C_2, in which the combinatorial embedding of C_2 is supposed to be $\mathcal{E}_2$. The combinatorial embedding $\mathcal{E}_1$ of C_1 forces the vertices $1, 2, \dots, 18$ to appear in this order around r in Γ_1. Consider the subtrees of C_1 induced by the vertices $r, 1, 2, \dots, 6$, by the vertices $r, 7, 8, \dots, 12$, and by the vertices $r, 13, 14, \dots, 18$. Since such subtrees appear consecutively around r, then at least one of them is drawn in a wedge rooted at r with angle less than π. Let T be such a subtree and let $k, k+1, \dots, k+5$ be the vertices of T, with $k = 1, 7$ or 13. Without loss of generality, let r be the uppermost point of this wedge. It follows that vertices $k, k+1, \dots, k+5$ have y-coordinate less than the one of r.

Denote by Q the polygon composed of the edges (r, k) and $(r, k+5)$ of C_1 and of the edges $(k, k+2)$, $(k+2, k+3)$, and $(k+3, k+5)$ of C_2. Note that vertices $k+1$ and $k+4$ are either both inside or both outside Q. In fact, placing one of them inside and the other outside Q is not consistent with the embedding constraints of $\mathcal{E}_2$; see Fig. 5(a).



(a) (b) (c)

Figure 5: (a) A placement of the vertices of T not respecting the embedding constraints of $\mathcal{E}_2$. The polygon Q is drawn with dotted segments, the edges of C_1 (of C_2) are drawn as solid (dashed) segments; (b) Placing vertices $k+1$ and $k+4$ inside Q leads to an intersection between edges $(k+1, k+3)$ and $(k+2, k+4)$ of T; (c) Placing vertices $k+1$ and $k+4$ outside Q leads to an intersection between edge $(k+2, k+4)$ and one out of the edges $(k+1, k+3)$ or $(k+3, k+5)$ of T.

If both vertices $k+1$ and $k+4$ are placed inside Q, then the embedding constraints of $\mathcal{E}_1$ and $\mathcal{E}_2$ and the fact that r is the vertex with greatest y-coordinate among the vertices of T imply that edge $(k+2, k+4)$ crosses edge $(r, k+3)$ and that edge $(k+1, k+3)$ crosses edge $(r, k+2)$. It follows that there is an intersection between edges $(k+2, k+4)$ and $(k+1, k+3)$, both belonging to T; see Fig. 5(b). Similarly, if both vertices $k+1$ and $k+4$ are placed outside Q, then, by the embedding constraints of $\mathcal{E}_1$ and $\mathcal{E}_2$, vertex $k+2$ is placed inside the polygon formed by the edges $(r, k+1)$, $(r, k+5)$ of C_1 and by the edges

$(k+1, k+3)$, $(k+3, k+5)$ of C_2, otherwise edge $(k+1, k+3)$ would cross edge $(k, k+2)$. Hence, edge $(k+2, k+4)$ crosses such a polygon either in edge $(k+1, k+3)$ or in edge $(k+3, k+5)$ and this concludes the proof; see Fig. 5(c). □

3.2 Positive Results

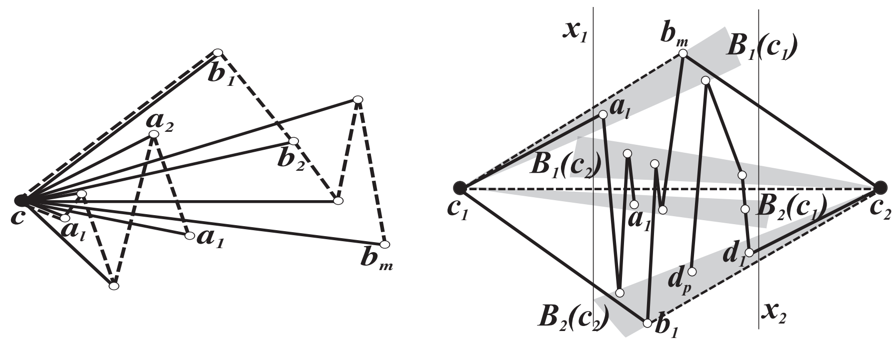
First, we show that a star and a path always admit a simultaneous embedding, even if the star has a fixed combinatorial embedding. Let P be an n-vertex path and let S be an n-vertex star with fixed combinatorial embedding $\mathcal{E}$ and *center* c. Notice that every cyclic ordering of the edges incident to each vertex of a tree provides a planar embedding of the tree. Note also that S and P share at least one and at most two edges. Let $P = (a_1, a_2, \dots, a_l, c, b_1, b_2, \dots, b_m)$, where one among the sequences $(a_1, a_2, \dots, a_l)$ and $(b_1, b_2, \dots, b_m)$ could be empty. We show that P and S have a simultaneous embedding in which the combinatorial embedding of S is $\mathcal{E}$. Draw S with c as the leftmost point and all of the edges in an order around c consistent with $\mathcal{E}$ and so that edge (c, b_1), if it exists, is the *uppermost* edge of S. It is easy to ensure that:

- the x-coordinate of a vertex b_i is greater than the x-coordinate of a vertex a_j, with $1 \leq i \leq m$ and $1 \leq j \leq l$,
- that the x-coordinate of a vertex b_i is greater than the x-coordinate of a vertex b_j, with $1 \leq j < i \leq m$, and
- that the x-coordinate of a vertex a_i is greater than the x-coordinate of a vertex a_j, with $1 \leq i < j \leq l$.

The resulting drawing of S is clearly planar; see Fig. 6(a). Further, path P is not self-intersecting as it is realized by two x-monotone curves that lie on disjoint x-intervals and that are joined by an edge that is higher than every other edge of P. This yields the following result:

Theorem 4 *Any n-vertex star and any n-vertex path admit a geometric simultaneous embedding in which the star has a prescribed combinatorial embedding.*

Next, we show that a double-star and a path not sharing edges have a simultaneous embedding even if the combinatorial embedding of the double-star is fixed in advance. Let P be an n-vertex path and let D be an n-vertex double-star with combinatorial embedding $\mathcal{E}$ and with *centers* c_1 and c_2. Suppose that D and P do not share edges. Let $P = (a_1, a_2, \dots, a_l, c_1, b_1, b_2, \dots, b_m, c_2, d_1, d_2, \dots, d_p)$. Note that the sequences $(a_1, a_2, \dots, a_l)$ and $(d_1, d_2, \dots, d_p)$ could be empty. In fact, sequence $(a_1, a_2, \dots, a_l)$ is empty if one of the end-vertices of P is mapped to c_1. Analogously, sequence $(d_1, d_2, \dots, d_p)$ is empty if one of the end-vertices of P is mapped to c_2. On the other hand, sequence $(b_1, b_2, \dots, b_m)$ has at least two elements, i.e. $m \geq 2$. In fact, if the sequence is empty then P and D share edge (c_1, c_2); if $m = 1$ then P and D share either edge (c_1, b_1) or edge (b_1, c_2), depending on whether b_1 is a neighbor of c_1 or of c_2 in D, respectively. Observe



(a) (b)

Figure 6: (a) Simultaneous embedding of a star and a path; (b) Simultaneous embedding of a double-star and a path not sharing edges.

also that b_1 is a neighbor of c_2 and b_m is a neighbor of c_1 in D, otherwise D and P would share edge (b_1, c_1) or edge (b_m, c_2), respectively; see Fig. 6(b). The edges incident to c_1 (incident to c_2), except for (c_1, c_2), are grouped into two *bundles* $B_1(c_1)$ and $B_2(c_1)$ (resp. $B_1(c_2)$ and $B_2(c_2)$). $B_1(c_1)$ is made up of the edges starting from (c_1, b_m) until, but not including, (c_1, c_2) in the clockwise order of the edges incident to c_1. $B_2(c_1)$ is made up of the edges starting from (c_1, c_2) until, but not including, (c_1, b_m) in the clockwise order of the edges incident to c_1. The other two bundles $B_1(c_2)$ and $B_2(c_2)$ are defined analogously. P is divided into three subpaths: a subpath $P_1 = (c_1, a_l, a_{l-1}, \dots, a_2, a_1)$, a subpath $P_2 = (c_1, b_1, b_2, \dots, b_m, c_2)$, and a subpath $P_3 = (c_2, d_1, d_2, \dots, d_p)$.

Draw (c_1, c_2) as a horizontal line segment, with c_1 on the left. $B_1(c_1)$ and $B_2(c_1)$ ($B_1(c_2)$ and $B_2(c_2)$) are drawn inside *wedges* centered at c_1 (resp. centered at c_2) and directed rightward (resp. directed leftward), with $B_1(c_1)$ above (c_1, c_2) and $B_2(c_1)$ below (c_1, c_2) (resp. with $B_1(c_2)$ above (c_2, c_1) and $B_2(c_2)$ below (c_2, c_1)). Such wedges are disjoint and have the further property that there exists an interval $[x_1, x_2]$ of the x-axis that is common to all the wedges, where $[x_1, x_2]$ is a sub-interval of the x-extension of the edge (c_1, c_2). Draw each edge inside the wedge of its bundle, respecting $\mathcal{E}$ and so that the following rules are observed:

- the x-coordinate of a vertex b_i is greater than the x-coordinate of a vertex a_j, with $1 \leq i \leq m$ and $1 \leq j \leq l$;
- the x-coordinate of a vertex d_i is greater than the x-coordinate of a vertex b_j, with $1 \leq i \leq p$ and $1 \leq j \leq m$;
- the x-coordinate of a vertex a_i is greater than the x-coordinate of a vertex a_j, with $1 \leq i < j \leq l$;
- the x-coordinate of a vertex b_i is greater than the x-coordinate of a vertex b_j, with $1 \leq j < i \leq m$; and

- the x-coordinate of a vertex d_i is greater than the x-coordinate of a vertex d_j, with $1 \leq i < j \leq p$.

Note that each vertex has an x-coordinate in the open interval (x_1, x_2). Further, edge (c_1, b_m) (edge (c_2, b_1)) of D is drawn so high (resp. so low) that edge (c_2, b_m) (resp (c_1, b_1)) of P does not create crossings with the other edges of the path. The absence of crossings in the drawing of D follows from the fact that its edges are drawn inside disjoint regions of the plane. The absence of crossings in the drawing of P follows from: (i) the absence of crossings in the drawings of its subpaths, which in turn follows from the strictly increasing or decreasing x-coordinates of its vertices; and (ii) from the fact that the subpaths occupy disjoint regions, except for edges (c_1, b_1) and (c_2, b_m) which do not create crossings, as already discussed. Thus, we have the following result:

Theorem 5 *Any n-vertex double-star and any n-vertex path not sharing edges admit a geometric simultaneous embedding in which the double-star has a prescribed combinatorial embedding.*

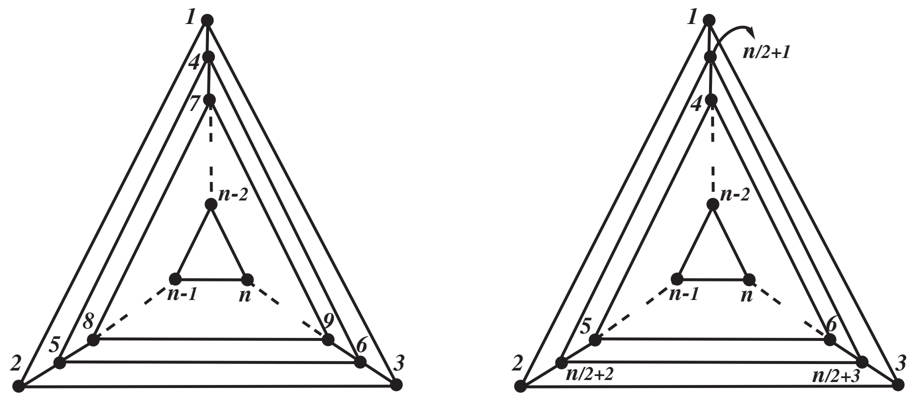
4 Near-Simultaneous Embedding

In this section we study the variation of geometric simultaneous embedding in which vertices that represent the same entity in different graphs are allowed to be placed at different points. The relaxation of the constraint that forces vertices to be placed exactly at the same point should allow us to near-simultaneously embed larger classes of graphs. However, in order to preserve the viewer's "mental map" corresponding vertices should be placed as close as possible. This turns out to be impossible for general planar graphs, as Theorem 6 below shows.

Define the *displacement* of a vertex v between two drawings Γ_1 and Γ_2 as the distance between the location of v in Γ_1 and the location of v in Γ_2. We show that there exist two n-vertex planar graphs G_1 and G_2 with a bijection γ between their vertices such that for any two planar straight-line grid drawings Γ_1 and Γ_2 of G_1 and G_2, respectively, there exists a vertex v that has a displacement $\Omega(n)$ between Γ_1 and Γ_2.

A *nested triangles graph* G is a triconnected planar graph that admits a plane embedding $\mathcal{E}$ where a sequence $c_1, c_2, \ldots, c_{n/3}$ of vertex-disjoint 3-cycles can be found such that c_i contains c_{i+1} in its interior, for any $1 \leq i < n/3$. Denote by $F(G)$ the external face of a nested triangles graph G in such a plane embedding $\mathcal{E}$.

Let G_1 and G_2 be two nested triangles graphs, each on n vertices; see Fig. 7. Suppose the mapping γ between the vertices in $V(G_1)$ and the vertices in $V(G_2)$ is the one shown in Fig. 7. Formally, the mapping can be defined by the following procedure: embed G_1 and G_2 with external faces $F(G_1)$ and $F(G_2)$, respectively. Starting from G_1, for $i = 1, \ldots, n/3$, remove from the current graph the three vertices of the external face and label them $3i, 3i-1$, and $3i-2$. Then, starting from G_2, for $i = 1, \ldots, n/3$, remove from the current graph the three vertices of the external face and label them as follows: If i is odd the three vertices get



(a) (b)

Figure 7: (a) Nested triangles graph G_1; (b) Nested triangles graph G_2.

labels $\frac{3(i+1)}{2}$, $\frac{3(i+1)}{2} - 1$, and $\frac{3(i+1)}{2} - 2$; if i is even the three vertices get labels $\frac{n+3i}{2}$, $\frac{n+3i}{2} - 1$, and $\frac{n+3i}{2} - 2$. Considering a near-simultaneous embedding of G_1 and G_2 leads to the following:

Theorem 6 *There exist two planar graphs G_1 and G_2 such that, in any pair of of planar straight-line grid drawings Γ_1 of G_1 and Γ_2 of G_2, a vertex representing the same entity in G_1 and G_2 has displacement $\Omega(n)$ between Γ_1 and Γ_2.*

Proof: Let G be a nested triangles graph that admits a plane embedding $\mathcal{E}$ where a sequence $c_1, c_2, \ldots, c_t$ of vertex-disjoint 3-cycles can be found such that c_i contains c_{i+1} in its interior, for any $1 \leq i < t$. A nested triangles graph is triconnected, hence the only degree of freedom for obtaining a plane embedding of such a graph is given by the choice of its external face. Choosing any external face f for G leads to a plane embedding in which two sequences of nested 3-cycles $T_1(G)$ and $T_2(G)$ can be found, one with t_1 and the other with t_2 3-cycles, with $0 \leq t_1, t_2 \leq t$ and with $t_1 + t_2 = t$. In fact, choosing the face delimited by c_1 as external face for G leads to a plane embedding in which c_i contains c_{i+1} in its interior, for $1 \leq i < t$. Analogously, choosing the face delimited by c_t as external face for G leads to a plane embedding in which c_{i+1} contains c_i in its interior, for $1 \leq i < t$. Finally, choosing any face that is between two cycles c_k and c_{k+1} in $\mathcal{E}$ as external face for G leads to a plane embedding in which c_i contains c_{i+1} in its interior, for $k+1 \leq i < t$, and c_{i+1} contains c_i in its interior, for $1 \leq i < k$. In any plane embedding of G with external face f, the two sequences of nested 3-cycles $T_1(G)$ and $T_2(G)$ cover disjoint portions of the plane. It is easy to see that there exist indices i and j, with $i, j \in \{1, 2\}$, such that $T_i(G_1)$ and $T_j(G_2)$ share a linear number of vertices.

In a grid drawing of a nested triangles graph in which the external face is chosen to be a triangular face, if two vertices v_1 and v_2 belong to two different

3-cycles that are separated by t 3-cycles in the nested structure, then the x-coordinate or the y-coordinate of the two vertices differs by at least t units. Consider the sub-drawing of Γ_2 corresponding to the subgraph T^* of $T_j(G_2)$ made of $k = an+b$ most deeply nested 3-cycles of $T_j(G_2)$, with a and b constants. Note that such a subgraph has a fixed plane embedding with outer face O. Choose a and b so that the vertices incident to O belong also to $T_i(G_1)$. Now consider the three most deeply nested 3-cycles c_1^*, c_2^*, and c_3^* of T^*, such that c_1^* is nested inside c_2^* that is nested inside c_3^*. We now have two cases to consider:

- If there is a vertex v of c_1^*, of c_2^*, or of c_3^* that does not belong to $T_i(G_1)$, then it will be embedded outside T^* in Γ_1. Since T^* is made of $\Omega(n)$ nested 3-cycles, v is mapped in Γ_2 into a point at distance $\Omega(n)$ from the point where v is mapped in Γ_1.

- Otherwise every vertex of c_1^*, c_2^*, and c_3^* belongs to $T_i(G_1)$. Note that the labels of the vertices of c_1^* (of c_2^*) differ from the labels of the vertices of c_2^* (resp. of c_3^*) by at least $\frac{n}{2} - 5$ units and that in $T_i(G_1)$ there are $\Omega(n)$ 3-cycles separating c_1^* and c_2^*. This implies that either the position of the vertices of c_1^*, or the position of the vertices of c_2^*, or the position of the vertices of c_3^* in Γ_1 and in Γ_2 is at distance $\Omega(n)$. □

The lower bound in Theorem 6 concerning the distance between two consecutive placements of a vertex in two different drawings is easily matched by an upper bound obtained by independently drawing each planar graph in $O(n) \times O(n)$ area: Each vertex is displaced by at most the length of the diagonal of the drawing's bounding box. Clearly, such a diagonal has length $O(n)$.

The above result shows that we cannot hope to guarantee near-simultaneous embeddings for arbitrary pairs of planar graphs. However, near-simultaneous embeddings of graphs can still lead to interesting theoretical results as well as practical layout algorithms and heuristics. We next present several observations and simple results:

- *Different optimization criteria.* Earlier we defined the goal of near-simultaneous embedding as the minimization of the maximum displacement of one of the vertices of the graphs. However, several other goals could be pursued. It is not difficult to modify the proof of Theorem 6 in order to prove that a *linear* number of vertices has linear displacement among consecutive drawings, hence it does not seem useful to consider global optimization criteria such as the sum of the displacement of the vertices (that would be quadratic, in the worst case). Nevertheless, it might be useful to find other optimization functions that capture mental map preservation and for which good theoretical bounds can be proved.

- *Restriction to sub-classes of planar graphs.* It would be worthwhile to address the near-simultaneous embedding problem for certain subclasses of planar graphs. In fact, the lower bound of Theorem 6 strongly relies on the nested structures of cycles in the considered graphs. For several subclasses of planar graphs, e.g. for trees and outerplanar graphs, no such a nesting can be forced, hence it is may be possible to find better theoretical bounds on the maximum displacement of the vertices among consecutive drawings.

- *Restriction to similar graphs.* In practice, consecutive graphs arising from dynamic processes are *similar*. Intuitively, the concept of similarity has to take into account the topology of the graph with respect to the labeling of the vertices. The notion of similarity could be formalized in several different ways. For example, if any two vertices are at a graph-theoretical distance d in a certain graph of a sequence, in the next graph the two vertices should be at distance at least $d-k$ and at most $d+k$, for some positive constant k. Drawing a sequence of consecutive paths by simply placing each vertex v of a path P at point $(\delta(v), 0)$, where $\delta(v)$ is the position of the vertex in the path, guarantees that, if the paths in the sequence are similar, the displacement of each vertex is at most k. Concerning trees, an analogous definition of similarity guarantees that near-simultaneous embeddings can be found. Two rooted ordered trees T_1 and T_2 are similar if each vertex v that has depth d in T_1 has depth at least $d-k$ and at most $d+k$ in T_2 and if a tree traversal (e.g., pre-order, in-order, post-order or breadth-first-search) has vertex v in position $\delta(v)$ in T_1 and in position in the range $(\delta(v)-k, \delta(v)+k)$ in T_2, for some positive constant k. Then, we can draw each tree by placing each vertex v at point $(\delta(v), d_T(v))$, where $d_T(v)$ is the depth of v in the rooted tree T, guaranteeing planar drawings with constant displacement.

5 Conclusions and Future Work

In this paper we have considered several variations of the simultaneous graph embedding problem. In particular, we studied the case in which no edges are shared by the input graphs, the case in which the input graphs have fixed embeddings, and the case in which vertices are allowed to be placed at nearby points in different drawings.

In the case of geometric simultaneous embedding without common edges, we provided a negative result that indicates that assuming the input graphs to not share edges does not yield to much larger classes of graphs always admitting a geometric simultaneous embedding. Further, we believe that there exist two trees not sharing common edges that have no geometric simultaneous embedding. This would extend the result in [12] where an example of two trees sharing edges and not admitting any simultaneous embedding is shown. We now describe two trees that do not share edges and that we conjecture to have no simultaneous embedding. Consider the two isomorphic rooted trees $T_1(h,k)$ and $T_2(h,k)$ shown in Fig. 8 and whose topology is described below. Let γ be the mapping between their vertices also shown in Fig. 8 and defined below. Notice that $T_1(h,k)$ and $T_2(h,k)$ do not have any edges in common.

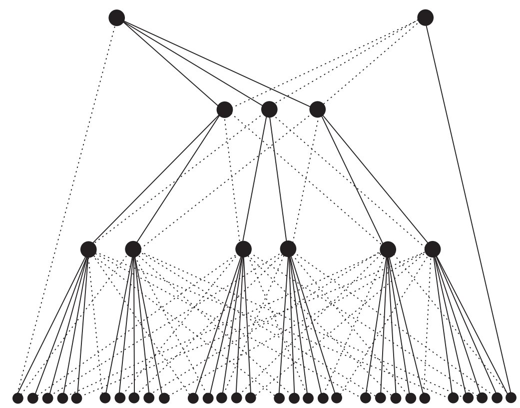


Figure 8: Trees $T_1(3,3)$ and $T_2(3,3)$ with the mapping γ between their vertices. $T_1(3,3)$ has solid edges and $T_2(3,3)$ has dashed edges.

- the root of $T_1(h,k)$ (of $T_2(h,k)$) has k children;
- each vertex of $T_1(h,k)$ (of $T_2(h,k)$) at distance i from the root, with $1 \leq i < h$, has a number of children equal to the number of vertices at distance i from the root in $T_1(h,k)$ (resp. in $T_2(h,k)$) minus one;
- exactly one vertex of $T_1(h,k)$ (of $T_2(h,k)$) at distance h from the root has one child;
- each child of the root of $T_1(h,k)$ is mapped to a distinct child of the root of $T_2(h,k)$;
- for each pair of vertices v_1 of $T_1(h,k)$ and v_2 of $T_2(h,k)$, $v_2 \neq \gamma(v_1)$, that are at distance i from the root of their own tree, there exists a child of v_1 that is mapped to a child of v_2;
- the only vertex of $T_1(h,k)$ (of $T_2(h,k)$) that is at distance $h+1$ from the root is mapped to the root of $T_2(h,k)$ (resp. to the root of $T_1(h,k)$).

Conjecture 1 *For sufficiently large h and k, $T_1(h,k)$ and $T_2(h,k)$ do not admit a geometric simultaneous embedding with mapping γ between their vertices.*

For the problem of drawing graphs simultaneously with fixed embedding, we provided more negative results than in the usual setting for geometric simultaneous embedding, while providing only two positive results strengthening analogous ones already known for geometric simultaneous embedding. We believe that studying the problem of constructing simultaneous embeddings of a tree and a path in which the tree has a fixed combinatorial embedding could be useful for the same problem in the non-fixed embedding setting.

The time complexity of testing whether two graphs not sharing edges or having fixed embeddings admit a geometric simultaneous embedding is not yet known, even though it was recently shown that testing whether two graphs admit a geometric simultaneous embedding is *NP*-hard [9].

Concerning the near-simultaneous embedding problem, we provided two n-vertex planar graphs and a bijective mapping of their vertices such that, in any pair of planar straight-line grid drawings of the graphs, there exist corresponding vertices that must have linear displacement. In Section 4 we discussed several research directions for near-simultaneous embedding and several related problems. We would like to emphasize here a problem which seems to us particularly intriguing. Let G_1 and G_2 be two n-vertex graphs and let γ be a bijective mapping among the vertices of G_1 and G_2. Suppose that, if two vertices u and v of G_1 are at graph-theoretic distance d in G_1, then vertices $\gamma(u)$ and $\gamma(v)$ are at graph-theoretic distance at least $d - k_1$ and at most $d + k_1$ in G_2, for some constant k_1. Does a pair of straight-line grid drawings of G_1 and G_2 exist such that, for every vertex u of G_1, the displacement of u from $\gamma(u)$ is at most k_2, for some constant k_2 depending only on k_1?

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