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On a Class of Planar Graphs with Straight-Line Grid Drawings on Linear Area

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Abstract

A straight-line grid drawing of a planar graph G is a drawing of G on an integer grid such that each vertex is drawn as a grid point and each edge is drawn as a straight-line segment without edge crossings. It is well known that a planar graph of n vertices admits a straight-line grid drawing on a grid of area $O(n^2)$. A lower bound of $\Omega(n^2)$ on the area-requirement for straight-line grid drawings of certain planar graphs are also known. In this paper, we introduce a fairly large class of planar graphs which admits a straight-line grid drawing on a grid of area $O(n)$. We give a linear-time algorithm to find such a drawing. Our new class of planar graphs, which we call "doughnut graphs," is a subclass of 5-connected planar graphs. We show several interesting properties of "doughnut graphs" in this paper. One can easily observe that any spanning subgraph of a "doughnut graph" also admits a straight-line grid drawing with linear area. But the recognition of a spanning subgraph of a "doughnut graph" seems to be a non-trivial problem, since the recognition of a spanning subgraph of a given graph is an NP-complete problem in general. We establish a necessary and sufficient condition for a 4-connected planar graph G to be a spanning subgraph of a "doughnut graph." We also give a linear-time algorithm to augment a 4-connected planar graph G to a "doughnut graph" if G satisfies the necessary and sufficient condition.

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1 Introduction

Recently automatic aesthetic drawings of graphs have created intense interest due to their broad applications in computer networks, VLSI layout, information visualization etc., and as a consequence a number of drawing styles have come out [4, 11, 14, 16]. A classical and widely studied drawing style is the "straight-line drawing" of a planar graph. A *straight-line drawing* of a planar graph G is a drawing of G such that each vertex is drawn as a point and each edge is drawn as a straight-line segment without edge crossings. A *straight-line grid drawing* of a planar graph G is a straight-line drawing of G on an integer grid such that each vertex is drawn as a grid point as shown in Figure 1(b).

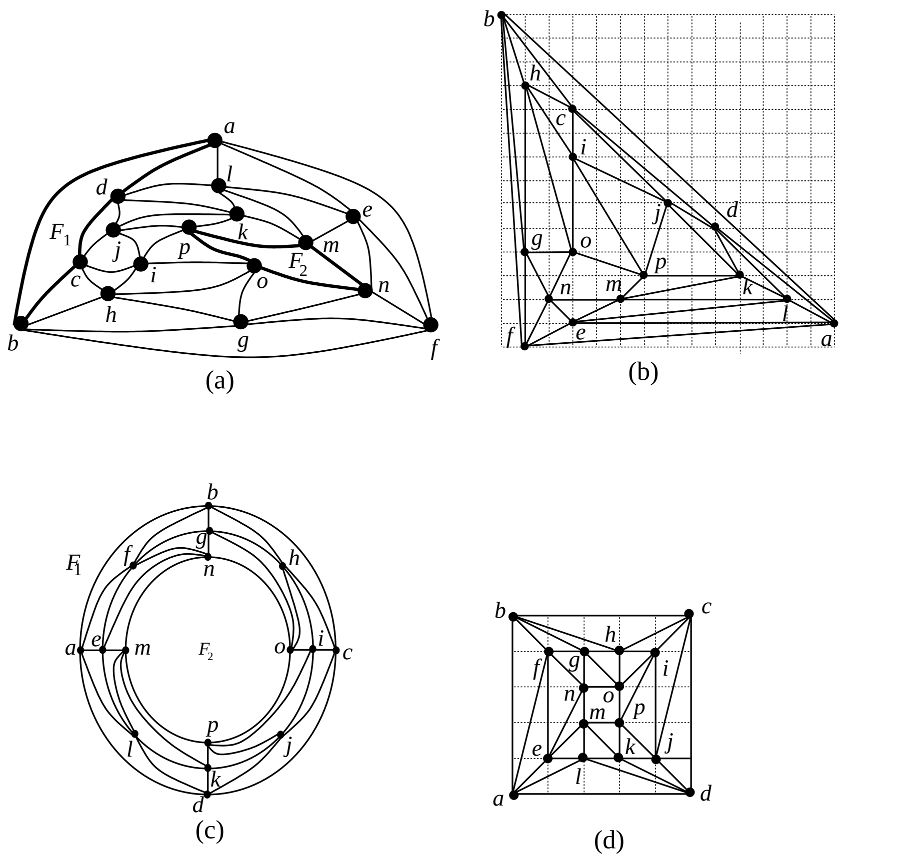

Figure 1: (a) A planar graph G, (b) a straight-line grid drawing of G with area $O(n^2)$, (c) a doughnut embedding of G and (d) a straight-line grid drawing of G with area $O(n)$.

Wagner [19], Fary [6] and Stein [18] independently proved that every planar graph G has a straight-line drawing. Their proofs immediately yield polynomial-time algorithms to find a straight-line drawing of a given plane graph. However, the area of a rectangle enclosing a drawing on an integer grid obtained by these algorithms is not bounded by any polynomial of the number n of vertices in G. In fact, to obtain a drawing of area bounded by a polynomial remained as an open problem for long time. In 1990, de Fraysseix *et al.* [3] and Schnyder [17] showed by two different methods that every planar graph of $n \geq 3$ vertices has a straight-line drawing on an integer grid of size $(2n-4) \times (n-2)$ and $(n-2) \times (n-2)$, respectively. Figure 1(b) illustrates a straight-line grid drawing of the graph G in Figure 1(a) with area $O(n^2)$. A natural question arises: what is the minimum size of a grid required for a straight-line drawing? de Fraysseix *et al.* showed that, for each $n \geq 3$, there exists a plane graph of n vertices, for example nested triangles, which needs a grid size of at least $\lfloor 2(n-1)/3 \rfloor \times \lfloor 2(n-1)/3 \rfloor$ for any grid drawing [2, 3]. Recently Frati and Patrignani showed that $n^2/9 + \Omega(n)$ area is necessary for any planar straight-line drawing of a nested triangles graph [7]. (Note that a plane graph is a planar graph with a given embedding.) It has been conjectured that every plane graph of n vertices has a grid drawing on a $\lceil 2n/3 \rceil \times \lceil 2n/3 \rceil$ grid, but it is still an open problem. For some restricted classes of graphs, more compact straight-line grid drawings are known. For example, a 4-connected plane graph G having at least four vertices on the outer face has a straight-line grid drawing with area $(\lceil n/2 \rceil - 1) \times (\lfloor n/2 \rfloor)$ [15]. Garg and Rusu showed that an n-node binary tree has a planar straight-line grid drawing with area $O(n)$ [9]. Although trees admit straight-line grid drawings with linear area, it is generally thought that triangulations may require a grid of quadratic size. Hence finding nontrivial classes of planar graphs of n vertices richer than trees that admit straight-line grid drawings with area $o(n^2)$ is posted as an open problem in [1]. Garg and Rusu showed that an outerplanar graph with n vertices and maximum degree d has a planar straight-line drawing with area $O(dn^{1.48})$ [10]. Recently Di Battista and Frati showed that a "balanced" outerplanar graph of n vertices has a straight-line grid drawing with area $O(n)$ and a general outerplanar graph of n vertices has a straight-line grid drawing with area $O(n^{1.48})$ [5].

In this paper, we introduce a new class of planar graphs which has a straight-line grid drawing on a grid of area $O(n)$. We give a linear-time algorithm to find such a drawing. Our new class of planar graphs is a subclass of 5-connected planar graphs, and we call the class "doughnut graphs" since a graph in this class has a doughnut-like embedding as illustrated in Figure 1(c). In an embedding of a "doughnut graph" of n vertices, there are two vertex-disjoint faces each having exactly $n/4$ vertices and each of all the other faces has exactly three vertices. Figure 1(a) illustrates a "doughnut graph" of 16 vertices where each of the two faces F_1 and F_2 contains four vertices and each of all other faces contains exactly three vertices. Figure 1(c) illustrates a doughnut-like embedding of G where F_1 is embedded as the outer face and F_2 is embedded as an inner face. A straight-line grid drawing of G with area $O(n)$ is illustrated in Figure 1(d). The outerplanarity of a "doughnut graph" is 3. Thus "doughnut graphs" introduce a subclass of 3-outerplanar graphs that admits straight-line grid drawing with linear area. One can easily observe that any spanning subgraph of a "doughnut graph" also admits a straight-line grid drawing with linear area. But the recognition of a spanning subgraph of a given graph is an NP-complete problem in general. We establish a necessary and sufficient condition for a 4-connected planar graph to be a spanning subgraph of a "doughnut graph." We also provide a linear-time algorithm to augment a 4-connected graph G to a "doughnut graph" if G satisfies the necessary and sufficient condition. This gives us a new class of graphs which is a subclass of 4-connected planar graphs that admits straight-line grid drawings with linear area.

The remainder of the paper is organized as follows. In Section 2, we give some definitions. Section 3 provides some properties of the class of "doughnut graphs." Section 4 deals with straight-line grid drawings of "doughnut graphs." Section 5 provides the characterization for a 4-connected planar graph to be a spanning subgraph of a "doughnut graph." Finally Section 6 concludes the paper. Early versions of this paper have been presented at [12] and [13].

2 Preliminaries

In this section we give some definitions.

Let $G = (V, E)$ be a connected simple graph with vertex set V and edge set E. Throughout the paper, we denote by n the number of vertices in G, that is, $n = |V|$, and denote by m the number of edges in G, that is, m=$|E|$. An edge joining vertices u and v is denoted by (u, v). The degree of a vertex v, denoted by $d(v)$, is the number of edges incident to v in G. G is called *r-regular* if every vertex of G has degree r. We call a vertex v a *neighbor* of a vertex u in G if G has an edge (u, v). The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph K_1. G is called *k-connected* if $\kappa(G) \geq k$. We call a vertex of G a cut-vertex of G if its removal results in a disconnected or single-vertex graph. For $W \subseteq V$, we denote by $G - W$ the graph obtained from G by deleting all vertices in W and all edges incident to them. A *cut-set* of G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component or $G - S$ is a single vertex graph. A *path* in G is an ordered list of distinct vertices $v_1, v_2, ..., v_q \in V$ such that $(v_{i-1}, v_i) \in E$ for all $2 \leq i \leq q$. Vertices v_1 and v_q are end-vertices of the path $v_1, v_2, ..., v_q$. Two paths are *vertex-disjoint* if they do not share any common vertex except their end vertices. The *length* of a path is the number of edges on the path. We call a path P an *even path* if the number of edges on P is even. We call a path P an *odd path* if the number of edges on P is odd.

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A *plane graph* is a planar graph with a fixed embedding. A plane graph G divides the plane into connected regions called $faces$. A bounded region is called an *inner face* and the unbounded region is called the *outer face*. For a face F in G we denote by $V(F)$ the set of vertices of G on the boundary of face F. Two faces F_1 and F_2 are *vertex-disjoint* if $V(F_1) \bigcap V(F_2) = \emptyset$. Let F be a face in a plane graph G with $n \geq 3$. If the boundary of F has exactly three vertices then we call F a *triangulated* face. One can divide a face F of p ($p \geq 3$) vertices into $p-2$ triangulated faces by adding $p-3$ extra edges. The operation above is called *triangulating a face*. If every face of a graph is triangulated, then the graph is called a *triangulated plane graph*. We can obtain a triangulated plane graph G' from a non-triangulated plane graph G by triangulating all faces of G.

A *maximal planar* graph is one to which no edge can be added without losing planarity. Thus the boundary of every face of G is a triangle in any embedding of a maximal planar graph G with $n \geq 3$, and hence an embedding of a maximal planar graph is often called a triangulated plane graph. It can be derived from Euler's formula for planar graphs that if G is a maximal planar graph with n vertices and m edges then $m = 3n - 6$, for more details see [16]. We call a face a *quadrangle face* if the face has exactly four vertices.

For any 3-connected planar graph the following fact holds.

Fact 1 *Let G be a 3-connected planar graph and let Γ and Γ' be any two planar embeddings of G. Then any facial cycle of Γ is a facial cycle of Γ' and vice versa.*

Let G be a 5-connected planar graph, let Γ be any planar embedding of G and let p be an integer such that $p \geq 4$. We call G a *p-doughnut* graph if the following conditions (d_1) and (d_2) hold:

(d_1) Γ has two vertex-disjoint faces each of which has exactly p vertices, and all the other faces of Γ has exactly three vertices; and

(d_2) G has the minimum number of vertices satisfying condition (d_1).

In general, we call a p-doughnut graph for $p \geq 4$ a *doughnut graph*. Since a doughnut graph is a 5-connected planar graph, Fact 1 implies that the decomposition of a doughnut graph into its facial cycles is unique. Throughout the paper we often mention faces of a doughnut graph G without mentioning its planar embedding where the description of the faces is valid for any planar embedding of G.

3 Properties of Doughnut Graphs

In this section we will show some properties of a *p-doughnut* graph. We have the following lemma on the number of vertices of a graph satisfying condition (d_1).

Lemma 2 *Let G be a 5-connected planar graph, let Γ be any planar embedding of G, and let p be an integer such that $p \geq 4$. Assume that Γ has two vertex-disjoint faces each of which has exactly p vertices, and all the other faces of Γ has exactly three vertices. Then G has at least $4p$ vertices.*

Proof: Let F_1 and F_2 be the two faces of Γ each of which contains exactly p vertices. Let x be the number of vertices in G which are neither on F_1 nor on F_2. Then G has $x + 2p$ vertices.

We calculate the number of edges in G as follows. Faces F_1 and F_2 of Γ are not triangulated since $p \geq 4$. If we triangulate F_1 and F_2 of Γ then the resulting graph G' is a maximal planar graph. Using Euler's formula, G' has exactly $3(x+2p)-6 = 3x+6p-6$ edges. To triangulate each of F_1 and F_2, we need to add $p-3$ edges and hence the number of edges in G is exactly

$$(3x+6p-6)-2(p-3) \quad = \quad 3x+4p. \tag{1}$$

Since G is 5-connected, using the degree-sum formula, we get $2(3x+4p) \geq 5(x+2p)$. This relation implies

$$x \quad \geq \quad 2p. \tag{2}$$

Therefore G has at least $4p$ vertices. $\mathcal{Q.E.D.}$

Lemma 2 implies that a p-doughnut graph has $4p$ or more vertices. We now show that $4p$ vertices are sufficient to construct a p-doughnut graph as in the following lemma.

Lemma 3 *For an integer p, $p \geq 4$, one can construct a p-doughnut graph G with $4p$ vertices.*

To prove Lemma 3 we first construct a planar embedding Γ of G with $4p$ vertices by the construction **Construct-Doughnut** given below and then show that G is a p-doughnut graph.

Construct-Doughnut. Let C_1, C_2, C_3 be three vertex-disjoint cycles such that C_1 contains p vertices, C_2 contains $2p$ vertices and C_3 contains p vertices. Let $x_1, x_2, ..., x_p$ be the vertices on C_1, $y_1, y_2, ..., y_p$ be the vertices on C_3, and $z_1, z_2, .., z_{2p}$ be the vertices on C_2. Let R_1, R_2 and R_3 be three concentric circles on a plane with radius r_1, r_2 and r_3, respectively, such that $r_1 > r_2 > r_3$. We embed C_1, C_2 and C_3 on R_1, R_2 and R_3 respectively, as follows. We put the vertices $x_1, x_2 ... x_p$ of C_1 on R_1 in clockwise order such that x_1 is put on the leftmost position among the vertices $x_1, x_2, ..., x_p$. Similarly, we put vertices $z_1, z_2, ..., z_{2p}$ of C_2 on R_2 and $y_1, y_2, ..., y_p$ of C_3 on R_3. We add edges between the vertices on C_1 and C_2, and between the vertices on C_2 and C_3 as follows. We have two cases to consider.

Case 1: k is even in z_k.

In this case, we add two edges $(z_k, x_{k/2})$, (z_k, x_i) between C_2 and C_1, and one edge (z_k, y_i) between C_2 and C_3 where $i = 1$ if $k = 2p$, and $i = k/2+1$ otherwise.

Case 2: k is odd in z_k.

In this case, we add two edges $(z_k, y_{\lceil k/2 \rceil})$, (z_k, y_i) between C_2 and C_3, and one edge $(z_k, x_{\lceil k/2 \rceil})$ between C_1 and C_2 where $i = 1$ if $k = 2p-1$, and $i = \lceil k/2 \rceil + 1$ otherwise.

We thus constructed a planar embedding Γ of G. Figure 2 illustrates the construction above for the case of $p = 4$. □

We have the following lemma on the construction **Construct-Doughnut**.

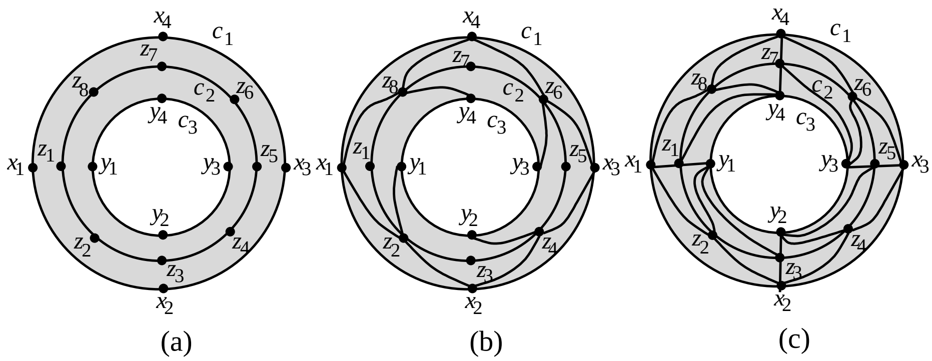

Figure 2: Illustration for the construction of a planar embedding Γ of a p-doughnut graph G for $p = 4$; (a) embedding of the three cycles C_1, C_2 and C_3 on three concentric circles, (b) addition of edges for the case where k is even in z_k and (c) Γ.

Lemma 4 *Let Γ be the plane graph of $4p$ vertices obtained by the construction* **Construct-Doughnut**. *Then Γ has exactly two vertex-disjoint faces F_1 and F_2 each of which has exactly p vertices, and the rest of the faces are triangulated.*

Proof: The construction of Γ implies that cycle C_1 is the boundary of the outer face F_1 of Γ and cycle C_3 is the boundary of an inner face F_2. Each of F_1 and F_2 has exactly p vertices. Clearly the two faces F_1 and F_2 of Γ are vertex-disjoint. Thus it is remained to show that the rest of the faces of Γ are triangulated. The rest of the faces can be divided into two groups; (i) faces having vertices on both the cycles C_1 and C_2, and (ii) faces having the vertices on both the cycles C_2 and C_3.

We only prove that each face in group (i) is triangulated, since the proof for group (ii) is similar.

From our construction each vertex z_i with even i has exactly two neighbors on C_1, and the two neighbors of z_i on C_1 are consecutive. Hence we get a triangulated face for each z_i with even i which contains z_i and the two neighbors of z_i on C_1.

We now show that the remaining faces in group (i) are triangulated. Clearly each of the remaining faces in group (i) must contain a vertex z_i with odd i since a vertex on a face in group (i) is either on C_1 or on C_2 and a vertex on C_2 has at most two neighbors on C_1. Let z_i, z_{i+1} and z_{i+2} be three consecutive vertices on C_2 with even i. Then z_i and z_{i+2} has a common neighbor x on C_1. One can observe from our construction that x is also the only neighbor of z_{i+1} on C_1. Then exactly two faces in group (i) contain z_{i+1} and the two faces are triangulated. This implies that for each z_i on C_2 with odd i there are exactly two faces in group (i) which contain z_i, and the two faces are triangulated.

$\mathcal{Q.E.D.}$

We are ready to prove Lemma 3.

Proof of Lemma 3

We construct a planar embedding Γ of a graph G with $4p$ vertices by the construction **Construct-Doughnut** and show that G is a p-doughnut graph. To prove this claim we need to prove that G satisfies the following properties (a)–(c):

(a) the graph G is a 5-connected planar graph;

(b) any planar embedding Γ' of G has exactly two vertex-disjoint faces each of which has exactly p vertices, and all the other faces are triangulated; and

(c) G has the minimum number of vertices satisfying (a) and (b).

(a) G is a planar graph since it has a planar embedding Γ as illustrated in Figure 2(c). To prove that G is 5-connected, we show that the size of any cut-set of G is 5 or more. We first show that G is 5-regular. From the construction, one can easily see that each of the vertex of C_2 has exactly three neighbors in $V(C_1) \cup V(C_3)$. Hence the degree of each vertex of C_2 is exactly 5. We only prove that the degree of each vertex of C_1 is exactly 5 since the proof is similar for the vertices of C_3. Each even index vertex v of C_2 has two neighbors on C_1 and the two neighbors of v are consecutive on C_1 by construction. Each vertex u of C_1 has at most two even index neighbors on C_2, since C_2 has p even index vertices, C_1 has p vertices, and Γ is a planar embedding. Assume that a vertex u of C_1 has two even index neighbors y_i and y_{i+2} on C_2. Since Γ is a planar embedding y_{i+1} can have only one neighbor on C_1 which is u. Thus a vertex u on C_1 has at most three neighbors on C_2. Since there are exactly $3p$ edges each of which has one end point on C_1 and the other on C_2, and a vertex on C_1 has at most three neighbors on C_2, each vertex of C_1 has exactly three neighbors on C_2. Hence the degree of a vertex on C_1 is 5. Therefore G is 5-regular. We next show that the size of any cut-set of G is 5 or more. Assume for a contradiction that G has a cut-set of less than five vertices. In such a case, G would have a vertex of degree less than five, a contradiction. (Note that G is 5-regular, the vertices of G lie on three vertex disjoint cycles C_1, C_2 and C_3, none of the vertices of C_1 has a neighbor on C_3, each of the faces of G is triangulated except faces F_1 and F_2.)

(b) By Lemma 4, G has a planar embedding Γ such that Γ has exactly two vertex-disjoint faces F_1 and F_2 each of which has exactly p vertices, and the rest of the faces are triangulated. Since G is 5-connected, Fact 1 implies that any planar embedding Γ' of G has exactly two vertex-disjoint faces each of which has exactly p vertices, and all the other faces are triangulated.

(c) We have constructed the graph G with $4p$ vertices and proofs for (a) and (b) imply that G satisfies properties (a) and (b). G is a 5-connected planar graph and hence satisfies condition (d_1) of the definition of a p-doughnut graph. By Lemma 2, $4p$ is the minimum number of vertices of such a graph. $\mathcal{Q.E.D.}$

Condition (d_2) of the definition of a p-doughnut graph and Lemmas 2 and 3 imply that a p-doughnut graph G has exactly $4p$ vertices. Then the value of x in Eq. (2) is $2p$ in G. By Eq. (1), G has exactly $3x + 4p = 10p$ edges. Since G is 5-connected, every vertex has degree 5 or more. Then the degree-sum formula implies that every vertex of G has degree exactly 5. Thus the following theorem holds.

Theorem 1 *Let G be a p-doughnut graph. Then G is 5-regular and has exactly $4p$ vertices.*

For a cycle C in a plane graph G, we denote by $G(C)$ the plane subgraph of G inside C excluding C. Let C_1, C_2 and C_3 be three vertex-disjoint cycles in a planar graph G such that $V(C_1) \cup V(C_2) \cup V(C_3){=}V(G)$. Then we call a planar embedding Γ of G a *doughnut embedding* of G if C_1 is the outer face and C_3 is an inner face of Γ, $G(C_1)$ contains C_2 and $G(C_2)$ contains C_3. We call C_1 the *outer cycle*, C_2 the *middle cycle* and C_3 the *inner cycle* of Γ. We next show that a p-doughnut graph has a doughnut embedding. To prove the claim we need the following lemmas.

Lemma 5 *Let G be a p-doughnut graph. Let F_1 and F_2 be the two faces of G each of which contains exactly p vertices. Then $G - \{V(F_1) \cup V(F_2)\}$ is connected and contains a cycle.*

Proof: Since G is 5-connected, $G' = G - \{V(F_1) \cup V(F_2)\}$ is connected; otherwise, G would have a cut-set of 4 vertices - two of them are on F_1 and the other two are on F_2, a contradiction. Clearly G' has exactly $2p$ vertices. Since G is 5-regular and has exactly $4p$ vertices by Theorem 1, one can observe following the degree-sum formula that G' contains at least $2p$ edges; if there is no edge between a vertex of F_1 and a vertex of F_2 in G then G' contains exactly $2p$ edges, otherwise G' contains more than $2p$ edges. Since G' is connected, has $2p$ vertices and has at least $2p$ edges, G' must have a cycle. $\mathcal{Q.E.D.}$

Lemma 6 *Let G be a p-doughnut graph. Let F_1 and F_2 be the two faces of G each of which contains exactly p vertices. Let Γ be a planar embedding of G such that F_1 is embedded as the outer face. Let C be a cycle in $G - \{V(F_1) \cup V(F_2)\}$. Then $G(C)$ in Γ contains F_2.*

Proof: Assume that $G(C)$ does not contain F_2 in Γ. Since F_1 is embedded as the outer face of Γ, F_2 will be an inner face of Γ as illustrated in Figure 3. Then there would be edge crossings in Γ among the edges from the vertices on C to the vertices on F_1 and F_2 as illustrated in Figure 3, a contradiction to the assumption that Γ is a planar embedding of G. (Note that G is 5-connected, 5-regular and has $10p$ edges.) Therefore $G(C)$ contains F_2. $\mathcal{Q.E.D.}$

Lemma 7 *Let G be a p-doughnut graph. Let F_1 and F_2 be the two faces of G each of which contains exactly p vertices. Then the following (a) - (c) hold.*

(a) There is no edge between a vertex of F_1 and a vertex of F_2.

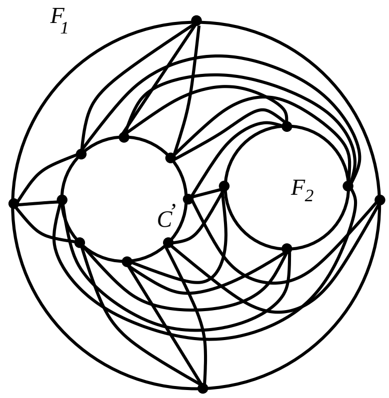

Figure 3: An embedding Γ' of G where F_1 is embedded as the outer face and $G(C')$ does not contain F_2.

(b) $G - \{V(F_1) \cup V(F_2)\}$ *contains exactly* $2p$ *edges and has exactly one cycle.*

(c) All the vertices of $G - \{V(F_1) \cup V(F_2)\}$ *are contained in a single cycle.*

Proof: *(a)* Let Γ be a planar embedding of G such that F_1 is embedded as the outer face. Let C be a cycle in $G - \{V(F_1) \cup V(F_2)\}$. Since $G(C)$ contains F_2 by Lemma 6, there is no edge between a vertex of F_1 and a vertex of F_2; otherwise, an edge between a vertex of F_1 and a vertex of F_2 would cross an edge on C, a contradiction to the assumption that Γ is a planar embedding of G.

(b) Since there is no edge between a vertex of F_1 and a vertex of F_2 by Lemma 7(a), $G - \{V(F_1) \cup V(F_2)\}$ contains exactly $2p$ edges as mentioned in the proof of Lemma 5. Since $G - \{V(F_1) \cup V(F_2)\}$ is connected, contains exactly $2p$ vertices and has exactly $2p$ edges, $G - \{V(F_1) \cup V(F_2)\}$ contains exactly one cycle.

(c) Let Γ be a planar embedding of G such that F_1 is embedded as the outer face. Let C be a cycle in $G - \{V(F_1) \cup V(F_2)\}$. By Lemma 7(b), C is the only cycle in $G - \{V(F_1) \cup V(F_2)\}$. Assume that the cycle C does not contain all the vertices of $G - \{V(F_1) \cup V(F_2)\}$. Then there is at least a vertex in $G - \{V(F_1) \cup V(F_2)\}$ whose degree is one in $G - \{V(F_1) \cup V(F_2)\}$. Let v be a vertex of degree one in $G - \{V(F_1) \cup V(F_2)\}$. We may assume that the vertex v is outside of the cycle C in Γ since the proof is similar if v is inside of the cycle C. Then the four neighbors of v must be on F_1, since $G - \{V(F_1) \cup V(F_2)\}$ contains exactly one cycle by Lemma 7(b), the vertex v is outside of the cycle C, and $G(C)$ contains F_2 by Lemma 6. Then either G would not be 5-regular or Γ would not be a planar embedding of G, a contradiction. Hence all the vertices of $G - \{V(F_1) \cup V(F_2)\}$ are contained in cycle C. $\mathcal{Q.E.D.}$

We now prove the following theorem.

Theorem 2 *A* p*-doughnut graph always has a doughnut embedding.*

Proof: Let F_1 and F_2 be two faces of G each of which contains exactly p vertices. Let Γ be a planar embedding of G such that F_1 is embedded as the outer face. By Lemma 7(c), all the vertices of $G - \{V(F_1) \cup V(F_2)\}$ are contained in a single cycle C. By Lemma 6, $G(C)$ contains F_2. Then in Γ, $G(F_1)$ contains C and $G(C)$ contains F_2, and hence Γ is a doughnut embedding. $\mathcal{Q.E.D.}$

A 1-*outerplanar* graph is an embedded planar graph where all vertices are on the outer face. It is also called 1-*outerplane* graph. An embedded graph is a k-*outerplane* $(k > 1)$ if the embedded graph obtained by removing all vertices of the outer face is a $(k-1)$-outerplane graph. A graph is k-*outerplanar* if it admits a k-outerplanar embedding. A planar graph G has *outerplanarity* k $(k > 0)$ if it is k-outerplanar and it is not j-outerplanar for $0 < j < k$.

In the rest of this section, we show that the outerplanarity of a p-doughnut graph G is 3. Since none of the faces of G contains all vertices of G, G does not admit 1-outerplanar embedding. We thus need to show that G does not admit a 2-outerplanar embedding. We have the following fact.

Fact 8 *A graph G having outerplanarity 2 has a cut-set of four or less vertices.*

Proof: Deletion of all vertices on the outer face from a 2-outerplane graph leaves a 1-outerplane graph. Since all vertices of a 1-outerplane graph are on the outer face, a 1-outerplane graph has a cut-set of at most two vertices. Then one can observe that a graph G having outerplanarity 2 has a cut-set of four or less vertices. $\mathcal{Q.E.D.}$

Since G is 5-connected graph, G has no cut-set of four or less vertices. Hence by Fact 8 the graph G has outerplanarity greater than 2. Thus the following lemma holds.

Lemma 9 *Let G be a p-doughnut graph for $p \geq 4$. Then G is neither a 1-outerplanar graph nor a 2-outerplanar graph.*

We now prove the following theorem.

Theorem 3 *The outerplanarity of a p-doughnut graph G is 3.*

Proof: A doughnut embedding of G immediately implies that G has a 3-outerplanar embedding. By Lemma 9, G is neither a 1-outerplanar graph nor a 2-outerplanar graph. Therefore the outerplanarity of a p-doughnut graph is 3. $\mathcal{Q.E.D.}$

4 Drawings of Doughnut Graphs

In this section we give a linear-time algorithm for finding a straight-line grid drawing of a doughnut graph on a grid of linear area.

Let G be a p-doughnut graph. Then G has a doughnut embedding by Theorem 2. Let Γ be a doughnut embedding of G as illustrated in Figure 4(a). Let C_1, C_2 and C_3 be the outer cycle, the middle cycle and the inner cycle of Γ, respectively. We have the following facts.

Fact 10 *Let G be a p-doughnut graph and let Γ be a doughnut embedding of G. Let C_1, C_2 and C_3 be the outer cycle, the middle cycle and the inner cycle of Γ, respectively. For any two consecutive vertices z_i, z_{i+1} on C_2, one of z_i, z_{i+1} has exactly one neighbor on C_1 and the other has exactly two neighbors on C_1.*

Fact 11 *Let G be a p-doughnut graph and let Γ be a doughnut embedding of G. Let C_1, C_2 and C_3 be the outer cycle, the middle cycle and the inner cycle of Γ, respectively. Let z_i be a vertex of C_2, then either the following (a) or (b) holds.*

(a) z_i has exactly one neighbor on C_1 and exactly two neighbors on C_3.

(b) z_i has exactly one neighbor on C_3 and exactly two neighbors on C_1.

Before describing our algorithm we need some definitions. Let z_i be a vertex of C_2 such that z_i has two neighbors on C_1. Let x and x' be the two neighbors of z_i on C_1 such that x' is the counter clockwise next vertex to x on C_1. We call x the *left neighbor* of z_i on C_1 and x' the *right neighbor* of z_i on C_1. Similarly we define the left neighbor and the right neighbor of z_i on C_3 if a vertex z_i on C_2 has two neighbors on C_3. We are now ready to describe our algorithm.

We embed C_1, C_2 and C_3 on three nested rectangles R_1, R_2 and R_3, respectively on a grid as illustrated in Figure 4(b). We draw rectangle R_1 on grid with four corners on grid point (0, 0), $(p+1, 0)$, $(p+1, 5)$ and (0, 5). Similarly the four corners of R_2 are (1, 1), $(p, 1)$, $(p, 4)$, (1, 4) and the four corners of R_3 are (2, 2), $(p-1, 2)$, $(p-1, 3)$, (2, 3).

We first embed C_2 on R_2 as follows. Let z_1, z_2, ..., z_{2p} be the vertices on C_2 in counter clockwise order such that z_1 has exactly one neighbor on C_1. We put z_1 on (1, 1), z_p on $(p, 1)$, z_{p+1} on $(p, 4)$ and z_{2p} on (1, 4). We put the other vertices of C_2 on grid points of R_2 preserving the relative positions of vertices of C_2.

We next put vertices of C_1 on R_1 as follows. Let x_1 be the neighbor of z_1 on C_1 and let x_1, x_2, ..., x_p be the vertices of C_1 in counter clockwise order. We put x_1 on (0, 0) and x_p on (0, 5). Since z_1 has exactly one neighbor on C_1, by Fact 10, z_{2p} has exactly two neighbors on C_1. Since z_1 and z_{2p} are on a triangulated face of G having vertices on both C_1 and C_2, x_1 is a neighbor of z_{2p}. One can easily observe that x_p is the other neighbor of z_{2p} on C_1. Clearly the edges (x_1, z_1), (x_1, z_{2p}), (x_p, z_{2p}) can be drawn as straight-line segments without edge crossings as illustrated in Figure 4(b). We next put neighbors of z_p and z_{p+1}. Let x_i be the neighbor of z_p on C_1 if z_p has exactly one neighbor on C_1, otherwise let x_i be the left neighbor of z_p on C_1. We put x_i on $(p+1, 0)$ and x_{i+1} on $(p+1, 5)$. In case of z_p has exactly one neighbor on C_1, by Fact 10, z_{p+1} has two neighbors on C_1, and x_i and x_{i+1} are the two neighbors of z_{p+1} on C_1. Clearly the edges (z_p, x_i), (z_{p+1}, x_i) and (z_{p+1}, x_{i+1}) can be drawn as straight-line segments without edge crossings, as illustrated in Figure 4(b). In case of z_p has exactly two neighbors x_i and x_{i+1} on C_1, the edges between neighbors of z_p and z_{p+1} on C_1 can be drawn without edge crossings as illustrated in Figure 5. We put the other vertices of C_1 on grid points of R_1 arbitrarily preserving their relative positions on C_1.

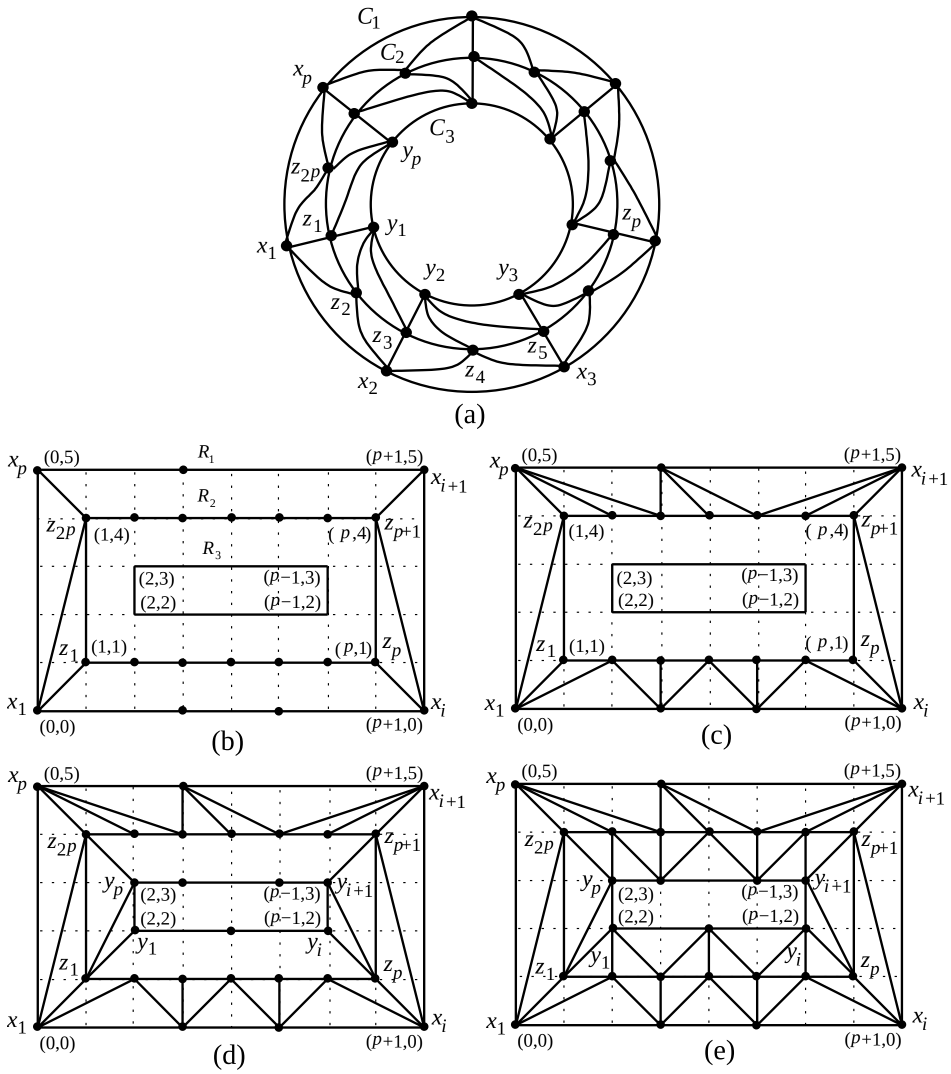

Figure 4: (a) A doughnut embedding of a p-doughnut graph of G, (b) edges between four corner vertices of R_1 and R_2 are drawn as straight-line segments, (c) edges between vertices on R_1 and R_2 are drawn, (d) edges between four corner vertices of R_2 and R_3 are drawn as straight-line segments, and (e) a straight-line grid drawing of G.

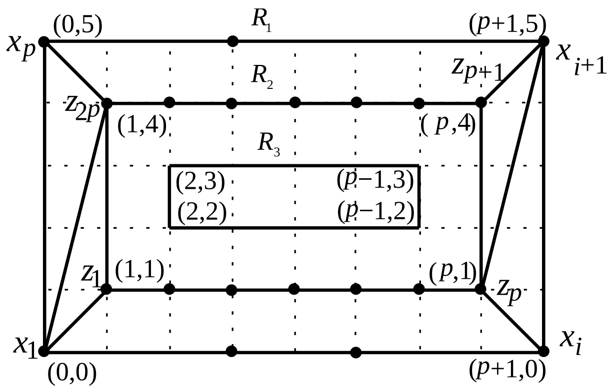

Figure 5: Illustration for the case where z_p has two neighbors on C_1.

One can observe that all the edges of G connecting vertices in $\{z_2, z_3, ..., z_{p-1}\}$ to vertices in $\{x_2, x_3, .., x_{i-1}\}$, and connecting vertices in $\{z_{p+2}, z_{p+2}, ..., z_{2p-1}\}$ to vertices in $\{x_{i+2}, x_{i+3}, .., x_{p-1}\}$ can be drawn as straight-line segments without edge crossings. See Figure 4(c).

We finally put the vertices of C_3 on R_3 as follows. Since z_1 has exactly one neighbor on C_1, by Fact 11(a), z_1 has exactly two neighbors on C_3. Then, by Fact 11(b), z_{2p} has exactly one neighbor on C_3. Let $y_1, y_2, ..., y_p$ be the vertices on C_3 in counter clockwise order such that y_1 is the right neighbor of z_1. Then y_p is the left neighbor of z_1. We put y_1 on (2, 2) and y_p on (2, 3). Clearly the edges (y_1, z_1), (y_p, z_{2p}), (y_p, z_1) can be drawn as straight-line segments without edge crossings, as illustrated in Figure 4(d). We next put neighbors of z_p and z_{p+1} on C_3 as we have put the neighbors of z_p and z_{p+1} on C_1 at the other two corners of R_3 in a counter clockwise order as illustrated in Figure 4(d). We put the other vertices of C_3 on grid points of R_3 arbitrarily preserving their relative positions on C_3. It is not difficult to show that edges from the vertices on C_2 to the vertices on C_3 can be drawn as straight-line segments without edge crossings. Figure 4(e) illustrates the complete straight-line grid drawing of a p-doughnut graph.

The area requirement of the straight-line grid drawing of a p-doughnut graph G is equal to the area of rectangle R_1 and the area of R_1 is $= (p + 1) \times 5 = (n/4 + 1) \times 5 = O(n)$, where n is the number of vertices in G. Thus we have a straight-line grid drawing of a p-doughnut graph on a grid of linear area. Clearly the algorithm takes linear time. Thus the following theorem holds.

Theorem 4 *A doughnut graph G of n vertices has a straight-line grid drawing on a grid of area $O(n)$. Furthermore, the drawing of G can be found in linear time.*

5 Spanning Subgraphs of Doughnut Graphs

In Section 4, we have seen that a doughnut graph admits a straight-line grid drawing with linear area. One can easily observe that a spanning subgraph of a doughnut graph also admits a straight-line grid drawing with linear area. Figure 6(b) illustrates a straight-line grid drawing with linear area of a graph G' in Figure 6(a) where G' is a spanning subgraph of a doughnut graph G in Figure 1(a). Using a transformation from the "subgraph isomorphism" problem [8], one can easily prove that the recognition of a spanning subgraph of a given graph is an NP-complete problem in general. Hence the recognition of a spanning subgraph of a doughnut graph seems to be a non-trivial problem. We thus restrict ourselves only to 4-connected planar graphs. In this section, we give a necessary and sufficient condition for a 4-connected planar graph to be a spanning subgraph of a doughnut graph as in the following theorem.

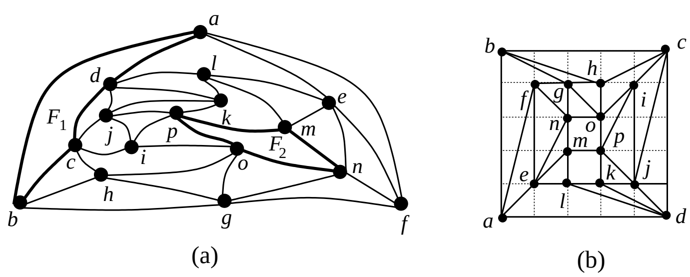

Figure 6: (a) A spanning subgraph G' of G in Figure 1(a), and (b) a straight-line grid drawing of G' with area $O(n)$.

Theorem 5 *Let G be a 4-connected planar graph with $4p$ vertices where $p > 4$ and let $\Delta(G) \leq 5$. Let Γ be a planar embedding of G. Assume that Γ has exactly two vertex disjoint faces F_1 and F_2 each of which has exactly p vertices. Then G is a spanning subgraph of a p-doughnut graph if and only if the following conditions $(a) - (e)$ hold.*

(a) G has no edge (x, y) such that $x \in V(F_1)$ and $y \in V(F_2)$.

(b) Every face f of Γ has at least one vertex $v \in \{V(F_1) \cup V(F_2)\}$.

(c) For any vertex $x \notin \{V(F_1) \cup V(F_2)\}$, the total number of neighbors of x on faces F_1 and F_2 are at most three.

(d) Every face f of Γ except the faces F_1 and F_2 has either three or four vertices.

(e) For any x-y path P such that $V(P) \cap \{V(F_1) \cup V(F_2)\} = \emptyset$ and x has exactly two neighbors on face $F_1(F_2)$. Then the following conditions hold.

(i) If P is even, then the vertex y has at most two neighbors on face $F_1(F_2)$ and at most one neighbor on face $F_2(F_1)$.

(ii) If P is odd, then the vertex y has at most one neighbor on face $F_1(F_2)$ and at most two neighbors on face $F_2(F_1)$.

Fact 1 implies that the decomposition of a 4-connected planar graph G into its facial cycles is unique. Throughout the section we thus often mention faces of G without mentioning its planar embedding where the description of the faces is valid for any planar embedding of G, since $\kappa(G) \geq 4$ for every graph G considered in this section.

Before proving the necessity of Theorem 5, we have the following fact.

Fact 12 *Let G be a 4-connected planar graph with $4p$ vertices where $p > 4$ and let $\Delta(G) \leq 5$. Assume that G has exactly two vertex disjoint faces F_1 and F_2 each of which has exactly p vertices. If G is a spanning subgraph of a doughnut graph then G can be augmented to a 5-connected 5-regular graph G' through triangulation of all the non-triangulated faces of G except the faces F_1 and F_2.*

One can easily observe that the following fact holds from the construction **Construct-Doughnut** given in Section 3.

Fact 13 *Let G be a doughnut graph, and let P be any x-y path such that $V(P) \cap \{V(F_1) \cup V(F_2)\} = \emptyset$ and x has exactly two neighbors on face $F_1(F_2)$. Then the following conditions (i) and (ii) hold.*

(i) If P is even, then the vertex y has exactly two neighbors on face $F_1(F_2)$ and exactly one neighbor on face $F_2(F_1)$.

(ii) If P is odd, then the vertex y has exactly one neighbor on face $F_1(F_2)$ and exactly two neighbors on face $F_2(F_1)$.

We are ready to prove the necessity of Theorem 5.

Proof for the Necessity of Theorem 5

Assume that G is a spanning subgraph of a p-doughnut graph. Then by Theorem 1 G has $4p$ vertices. Clearly $\Delta(G) \leq 5$ and G satisfies the conditions (a), (b) and (c), otherwise G would not be a spanning subgraph of a doughnut graph. The necessity of condition (e) is obvious by Fact 13. Hence it is sufficient to prove the necessity of condition (d) only.

(d) G does not have any face of two or less vertices since G is a 4-connected planar graph. Then every face of G has three or more vertices. We now show that G has no face of more than four vertices. Assume for a contradiction that G has a face f of q vertices such that $q > 4$. Then f can be triangulated by adding $q - 3$ extra edges. These extra edges increase the degrees of $q - 2$ vertices, and the sum of the degrees will be increased by $2(q - 3)$. Using the pigeonhole principle, one can easily observe that there is a vertex among the $q(> 4)$ vertices whose degree will be raised by at least 2 after a triangulation of f. Then G' would have a vertex of degree six or more where G' is a graph obtained after triangulation of f. Hence we cannot augment G to a 5-regular graph through triangulation of all the non-triangulated faces of G other than the faces F_1 and F_2. Therefore G cannot be a spanning subgraph of a doughnut graph by Fact 12, a contradiction. Hence each face f of G except F_1 and F_2 has either three or four vertices. $\mathcal{Q.E.D.}$

In the remaining of this section we give a constructive proof for the sufficiency of Theorem 5. Assume that G satisfies the conditions in Theorem 5. We have the following lemma.

Lemma 14 *Let G be a 4-connected planar graph satisfying the conditions in Theorem 5. Assume that all the faces of G except F_1 and F_2 are triangulated. Then G is a doughnut graph.*

Proof: To prove the claim, we have to prove that (i) G is 5-connected, (ii) G has two vertex disjoint faces each of which has exactly p, $p > 4$, vertices, and all the other faces of G has exactly three vertices, and (iii) G has the minimum number of vertices satisfying the properties (i) and (ii).

(i) We first prove that G is a 5-regular graph. Every face of G except F_1 and F_2 is a triangle. Furthermore each of F_1 and F_2 has exactly p, $p > 4$, vertices. Then G has $3(4p) - 6 - 2(p-3) = 10p$ edges. Since none of the vertices of G has degree more than five and G has exactly $10p$ edges, each vertex of G has degree exactly five. We next prove that the vertices of G lie on three vertex-disjoint cycles C_1, C_2 and C_3 such that cycles C_1, C_2, C_3 contain exactly p, $2p$ and p vertices, respectively. We take an embedding Γ of G such that F_1 is embedded as the outer face and F_2 is embedded as an inner face. We take the contour of face F_1 as cycle C_1 and contour of face F_2 as cycle C_3. Then each of C_1 and C_2 contains exactly p, $p > 4$, vertices. Since G satisfies conditions (a), (b) and (c) in Theorem 5 and all the faces of G except F_1 and F_2 are triangulated, the rest $2p$ vertices of G form a cycle in Γ. We take this cycle as C_2. $G(C_2)$ contains C_3 since G satisfies condition (b) in Theorem 5. Clearly C_1, C_2 and C_3 are vertex-disjoint and cycles C_1, C_2, C_3 contain exactly p, $2p$ and p vertices, respectively. We finally prove that G is 5-connected. Assume for a contradiction that G has a cut-set of less than five vertices. In such a case G would have a vertex of degree less than five, a contradiction.

(ii) The proof of this part is obvious since G has two vertex disjoint faces each of which has exactly p vertices and all the other faces of G has exactly three vertices.

(iii) The number of vertices of G is $4p$. Using Lemma 2, we can easily prove that the minimum number of vertices required to construct a graph G that satisfies the properties (i) and (ii) is $4p$.

$\mathcal{Q.E.D.}$

We thus assume that G has a non-triangulated face f except faces F_1 and F_2. By condition (d) in Theorem 5, f is a quadrangle face. It is sufficient to show that we can augment the graph G to a doughnut graph by triangulating each of the quadrangle faces of G. However, we cannot augment G to a doughnut graph by triangulating each quadrangle face arbitrarily. For example, the graph G in Figure 7(a) satisfies all the conditions in Theorem 5 and it has exactly one quadrangle face $f_1(a, b, c, d)$. If we triangulate f_1 by adding an edge (a, c) as illustrated in Figure 7(b), the resulting graph G' would not be a doughnut graph since a doughnut graph does not have an edge (a, c) such that $a \in V(F_1)$ and $c \in V(F_2)$. But if we triangulate f_1 by adding an edge (b, d) as illustrated in Figure 7(c), the resulting graph G' is a doughnut graph. Hence every triangulation of a quadrangle face is not always valid to augment G to a doughnut graph. We call a triangulation of a quadrangle face f of G a *valid triangulation* if the resulting graph G' obtained after the triangulation of f does not contradict any condition in Theorem 5. We call a vertex v on the contour of a quadrangle face f a *good vertex* if v is one of the end vertex of an edge which is added for a valid triangulation of f.

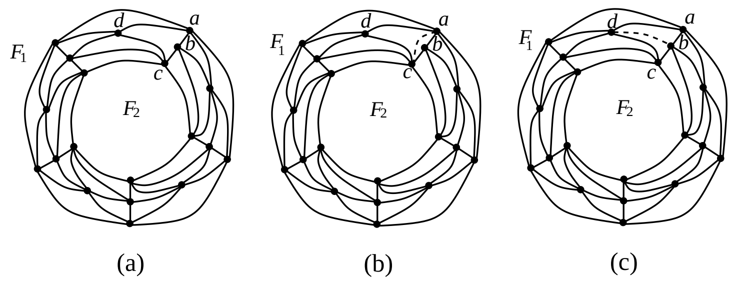

Figure 7: (a) $f_1(a, b, c, d)$ is a quadrangle face, (b) the triangulation of f_1 by adding the edge (a, c) and (c) the triangulation of f_1 by adding the edge (b, d).

We call a quadrangle face f of G an α-*face* if f contains at least one vertex from each of the faces F_1 and F_2. Otherwise, we call a quadrangle face f of G a β-*face*. In Figure 8, $f_1(a, b, c, d)$ is an α-face whereas $f_2(p, q, r, s)$ is a β-face.

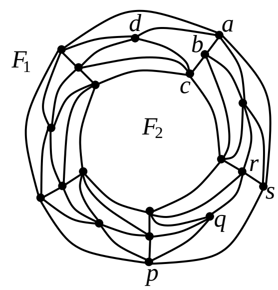

Figure 8: $f_1(a, b, c, d)$ is an α-face and $f_2(p, q, r, s)$ is a β-face.

In a valid triangulation of an α-face f of G no edge is added between any two vertices x, $y \in V(f)$ such that $x \in V(F_1)$ and $y \in V(F_2)$. Hence the following fact holds on an α-face f.

Fact 15 *Let G be a 4-connected planar graph satisfying the conditions in Theorem 5. Let f be an α-face in G. Then f admits a unique valid triangulation and the triangulation is obtained by adding an edge between two vertices those are not on F_1 and F_2.*

Faces $f_1(a,b,c,d)$ and $f_2(p,q,r,s)$ in Figure 9(a) are two α-faces and Figure 9(b) illustrates the valid triangulations of f_1 and f_2. Vertices b and d of f_1 and vertices q and s of f_2 are good vertices.

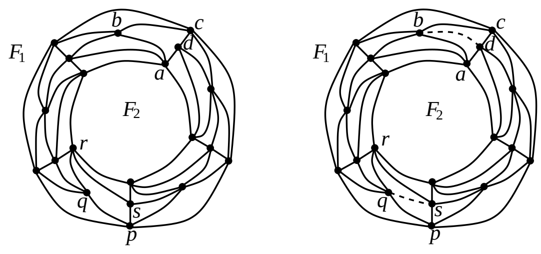

Figure 9: (a) $f_1(a,b,c,d)$ and $f_2(p,q,r,s)$ are two α-faces, and (b) valid triangulations of f_1 and f_2.

We call a β-face a β_1*-face* if the face contains exactly one vertex either from F_1 or from F_2. Otherwise we call a β-face a β_2*-face.* In Figure 10, $f_1(a,b,c,d)$ is a β_1-face whereas $f_2(p,q,r,s)$ is a β_2-face. We call a vertex v on the contour of a β_1-face f a *middle* vertex of f if the vertex is in the middle position among the three consecutive vertices other than the vertex on F_1 or F_2. In Figure 10, vertex c of f_1 and vertex r of f_2 are the middle vertices of f_1 and f_2, respectively.

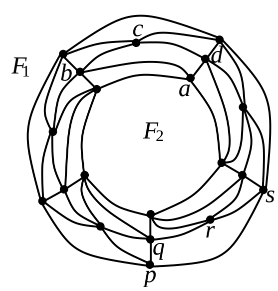

Figure 10: $f_1(a,b,c,d)$ is a β_1-face and $f_2(p,q,r,s)$ is a β_2-face.

In a valid triangulation of a β_1-face f of G no edge is added between any two vertices $x,y \in V(f)$ such that $x,y \notin V(F_1) \cup V(F_2)$. Hence the following fact holds on a β_1-face f.

Fact 16 *Let G be a 4-connected planar graph satisfying the conditions in Theorem 5. Let f be a β_1-face of G. Then f admits a unique valid triangulation and the triangulation is obtained by adding an edge between the vertex on F_1 or F_2 and the middle vertex.*

Faces $f_1(a,b,c,d)$ and $f_2(p,q,r,s)$ in Figure 11(a) are two β_1-faces and Figure 11(b) illustrates the valid triangulations of f_1 and f_2. Vertices a and c of f_1 and vertices p and r of f_2 are good vertices.

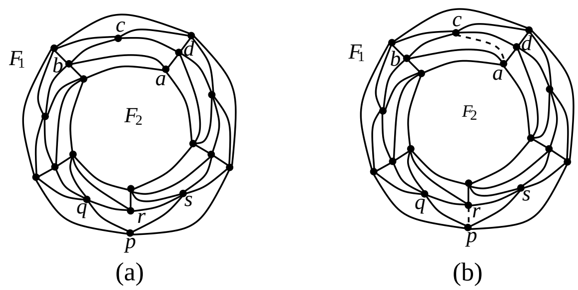

Figure 11: (a) $f_1(a, b, c, d)$ and $f_2(p, q, r, s)$ are two β_1-faces, and (b) valid triangulations of f_1 and f_2.

In a valid triangulation of a β_2-face f of G no edge is added between any two vertices $x, y \in V(f)$ where $x \in V(F_1)(V(F_2))$, $y \notin \{V(F_1) \cup V(F_1)\}$ and G has either (i) an even q-y path P such that q has exactly two neighbors on $F_2(F_1)$ and $V(P) \cap \{V(F_1) \cup V(F_2)\} = \emptyset$, or (ii) an odd q-y path P such that q has exactly two neighbors on $F_1(F_2)$ and $V(P) \cap \{V(F_1) \cup V(F_2)\} = \emptyset$. Hence the following fact holds on a β_2-face f.

Fact 17 *Let G be a 4-connected planar graph satisfying the conditions in Theorem 5. Let f be a β_2-face of G. Then f admits a unique valid triangulation and the triangulation is obtained by adding an edge between a vertex on face F_1 or F_2 and a vertex $z \notin V(F_1) \cup V(F_2)$.*

Face $f_1(a, b, c, d)$ in the graph in Figure 12(a) is a β_2-face and the graph has an even u-d path P such that u has exactly two neighbors g and h on F_2, and $V(P) \cap \{V(F_1) \cup V(F_2)\} = \emptyset$. Figure 12(c) illustrates the valid triangulation of f_1. Vertices a and c are the good vertices of f_1. Face $f_2(l, m, n, o)$ in the graph in Figure 12(b) is a β_2-face and the graph has an odd v-o path P such that v has exactly two neighbors s and t on F_1, and $V(P) \cap \{V(F_1) \cup V(F_2)\} = \emptyset$. Figure 12(d) illustrates the valid triangulation of f_2. Vertices l and n are the good vertices of f_2.

Before giving a proof for the sufficiency of Theorem 5 we need to prove the following Lemmas 18 and 19.

Lemma 18 *Let G be a 4-connected planar graph satisfying the conditions in Theorem 5. Then any quadrangle face f of G admits a unique valid triangulation such that after triangulation $d(v) \leq 5$ holds for any vertex v in the resulting graph.*

Proof: By Facts 15, 16 and 17, f admits a unique valid triangulation. Since a valid triangulation increases the degree of a good vertex by one, it is sufficient to show that each good vertex of f has degree less than five in G. Assume for a contradiction that a good vertex v has degree more than four in G. Then one can observe that G would violate a condition in Theorem 5. $\mathcal{Q.E.D.}$

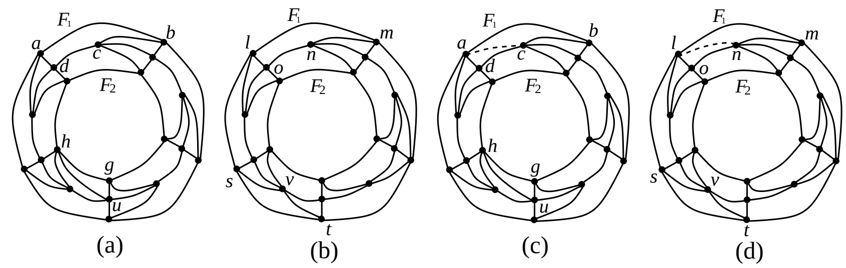

Figure 12: Illustration for valid triangulation of β_2-face; (a) a β_2 face $f_1(a, b, c, d)$ in a graph satisfying condition (i), (b) a β_2 face $f_2(l, m, n, o)$ in a graph satisfying condition(ii), (c) the valid triangulation of f_1 and (d) the valid triangulation of f_2.

Lemma 19 *Let G be a 4-connected planar graph satisfying the conditions in Theorem 5. Also assume that G has quadrangle faces. Then no two quadrangle faces f_1 and f_2 have a common vertex which is a good vertex for both the faces f_1 and f_2.*

Proof: Assume that u is a common vertex between two quadrangle faces f_1 and f_2. If u is neither a good vertex of f_1 nor a good vertex of f_2, then we have done. We thus assume that u is a good vertex of f_1 or f_2. Without loss of generality, we assume that u is a good vertex of f_1. Then u is not a good vertex of f_2, otherwise u would not be a common vertex of f_1 and f_2, a contradiction. $\mathcal{Q.E.D.}$

Proof for the Sufficiency of Theorem 5

Assume that the graph G satisfies all the conditions in Theorem 5. If all the faces of G except F_1 and F_2 are triangulated, then G is a doughnut graph by Lemma 14. Otherwise, we triangulate each quadrangle face of G, using its valid triangulation. Let G' be the resulting graph. Lemmas 18 and 19 imply that $d(v) \leq 5$ for each vertex v in G'. Then the graph G' satisfies the conditions in Theorem 5, since G satisfies the conditions in Theorem 5, G' is obtained from G using valid triangulations of quadrangle faces and $d(v) \leq 5$ for each vertex v in G'. Hence G' is a doughnut graph by Lemma 14. Therefore G is a spanning subgraph of a doughnut graph. $\mathcal{Q.E.D.}$

We now have the following lemma.

Lemma 20 *Let G be a 4-connected planar graph satisfying the conditions in Theorem 5. Then G can be augmented to a doughnut graph in linear time.*

Proof: We first embed G such that F_1 is embedded as the outer face and F_2 is embedded as an inner face. We then triangulate each of the quadrangle faces of G using its valid triangulation if G has quadrangle faces. Let G' be the resulting graph. As shown in the sufficiency proof of Theorem 5, G' is a doughnut graph. One can easily find all quadrangle faces of G and perform their valid triangulations in linear time, hence G' can be obtained in linear time. $\mathcal{Q.E.D.}$

In Theorem 5 we have given a necessary and sufficient condition for a 4-connected planar graph to be a spanning subgraph of a doughnut graph. As described in the proof of Lemma 20, we have provided a linear-time algorithm to augment a 4-connected planar graph G to a doughnut graph if G satisfies the conditions in Theorem 5. We have thus identified a subclass of 4-connected planar graphs that admits straight-line grid drawings with linear area as stated in the following theorem.

Theorem 6 *Let G be a 4-connected planar graph satisfying the conditions in Theorem 5. Then G admits a straight-line grid drawing on a grid of area $O(n)$. Furthermore, the drawing of G can be found in linear time.*

Proof: Using the method described in the proof of Lemma 20, we augment G to a doughnut graph G' by adding dummy edges (if required) in linear time. By Theorem 4, G' admits a straight-line grid drawing on a grid of area $O(n)$. We finally obtain a drawing of G from the drawing of G' by deleting the dummy edges (if any) from the drawing of G'. By Lemma 20, G can be augmented to a doughnut graph in linear time and by Theorem 4, a straight-line grid drawing of a doughnut graph can be found in linear time. Moreover, the dummy edges can also be deleted from the drawing of a doughnut graph in linear time. Hence the drawing of G can be found in linear time. $\mathcal{Q.E.D.}$

6 Conclusion

In this paper we introduced a new class of planar graphs, called doughnut graphs, which is a subclass of 5-connected planar graphs. A graph in this class has a straight-line grid drawing on a grid of linear area, and the drawing can be found in linear time. We showed that the outerplanarity of a doughnut graph is 3. Thus we identified a subclass of 3-outerplanar graphs that admits straight-line grid drawing with linear area. One can easily observe that any spanning subgraph of a doughnut graph also admits straight-line grid drawing with linear area. However, the recognition of a spanning subgraph of a doughnut graph seems to be a non-trivial problem. We established a necessary and sufficient condition for a 4-connected planar graph G to be a spanning subgraph of a doughnut graph. We also gave a linear-time algorithm to augment a 4-connected planar graph G to a doughnut graph if G satisfies the necessary and sufficient condition. By introducing the necessary and sufficient condition, in fact, we have identified a subclass of 4-connected planar graphs that admits straight-line grid drawings with linear area. Finding other nontrivial classes of planar graphs that admit straight-line grid drawings on grids of linear area is also left as an open problem.

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