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Degree-constrained edge partitioning in graphs arising from discrete tomography

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Abstract

Starting from the basic problem of reconstructing a 2-dimensional image given by its projections on two axes, one associates a model of edge coloring in a complete bipartite graph. The complexity of the case with $k = 3$ colors is open. Variations and special cases are considered for the case $k = 3$ colors where the graph corresponding to the union of some color classes (for instance colors 1 and 2) has a given structure (tree, vertex-disjoint chains, 2-factor, etc.). We also study special cases corresponding to the search of 2 edge-disjoint chains or cycles going through specified vertices. A variation where the graph is oriented is also presented.

In addition we explore similar problems for the case where the underlying graph is a complete graph (instead of a complete bipartite graph).

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1 Introduction

Our aim is to explore some problems that arise from a graph-theoretical formulation of the basic image reconstruction problem in discrete tomography (see [11, 12]). These models are also frequently used for problems of timetabling and scheduling (see [6] for instance). Additional applications are presented in [11, 12].

Assume we are given an $m \times n$ array $A = (a_{ij})$ where each entry contains one of the colors $1, \dots, k$. This matrix models an image of $m \times n$ pixels: for each $s \in \{1, \dots, k\}$, we define h_i^s (resp. v_j^s) as the number of pixels with color s for each row $i, i \in \{1, \dots, m\}$ (resp. each column $j, j \in \{1, \dots, n\}$).

The basic image reconstruction problem consists in assigning a color in $\{1, \dots, k\}$ to each entry of A so that in each row i (resp. column j) there are exactly h_i^s (resp. v_j^s) entries with color s. Formally, we have:

BASIC RECONSTRUCTION PROBLEM

Input: $(h_1^s, \dots, h_m^s), (v_1^s, \dots, v_n^s)$ for each $s \in \{1, \dots, k\}$

Output: an array $A = (a_{ij})$ such that in each row i (resp. column j) there are exactly h_i^s (resp. v_j^s) entries with color s, $\forall i \in \{1, \dots, m\}, \forall j \in \{1, \dots, n\}, \forall s \in \{1, \dots, k\}$.

Clearly the values h_i^s and v_j^s must satisfy some (necessary) conditions:

$$\sum_{s=1}^{k} h_i^s = n \qquad \forall i \in \{1, \dots, m\} \tag{1}$$

$$\sum_{s=1}^{k} v_j^s = m \qquad \forall j \in \{1, \dots, n\} \tag{2}$$

$$\sum_{i=1}^{m} h_i^s = \sum_{j=1}^{n} v_j^s \quad \forall s \in \{1, \dots, k\} \tag{3}$$

It is known that for $k = 2$, one can efficiently decide whether there exists or not an image corresponding to values h_i^s, v_j^s satisfying (1)-(3). Indeed in [15], Ryser gives, for the case $k = 2$, necessary and sufficient conditions to be verified by the values h_i^s and v_j^s for a solution to exist. Furthermore these conditions can be checked in polynomial time. They can be formulated as follows for each color s: we rearrange the values h_i^s in non increasing order and call them $r_1 \geq r_2 \geq ... \geq r_m$. We do the same for the values v_j^s and call them $s_1 \geq s_2 \geq ... \geq s_n$. Then the Ryser conditions are $\sum_{i=1}^{m} \min\{k, r_i\} \geq \sum_{j=1}^{k} s_j$ for each $k \in \{1, ..., n\}$ with equality for $k = n$.

For $k = 4$, the decision problem associated with the basic reconstruction problem was shown to be $NP-$complete [4], while for $k = 3$ its status is to our knowledge still open.

In this paper we essentially consider the case where we have $k = 3$ colors unless stated otherwise. Some tractable cases have been discussed in [5] and [6] in particular. [11, 12] contain also some related results for special cases of the general problem.

The graph-theoretical model we will associate with this problem is the following (see [2] for all graph theoretical terms not defined here). We have a complete bipartite graph $K_{X,Y}$ with $|X| = m$, $|Y| = n$. Each row i of A corresponds to vertex i; X is the set of these vertices. Similarly each column j of A corresponds to vertex j and Y is the set of these vertices. Each entry a_{ij} of A corresponds to the color of the edge $[i, j]$, $i \in X$ and $j \in Y$.

Given the values h_i^s and v_j^s, the reconstruction problem consists in partitioning the edge set $E(K_{X,Y})$ into k subsets $E^1, \dots, E^k$ (E^s is the subset of edges which will be given color s) such that for each $s \in \{1, \dots, k\}$

$$h_i^s \quad \textit{is the number of edges of } E^s \textit{ adjacent to vertex } i, \forall i \in X \tag{4}$$

$$v_j^s \quad \textit{is the number of edges of } E^s \textit{ adjacent to vertex } j, \forall j \in Y \tag{5}$$

For the rest of the paper, when working in complete bipartite graphs, we assume that conditions (1)-(3) are verified as well as the Ryser conditions for each color $s \in \{1, ..., k\}$.

For all these problems we also examine the corresponding problem in the case where instead of having an underlying graph which is bipartite (as was $K_{X,Y}$) we have now a complete graph K_X on $|X| = m$ vertices. So we are given for each vertex i in K_X and for each color $s \in \{1, ..., k\}$ a non negative integer h_i^s. Our problem then consists in finding a partition of the edge set $E(K_X)$ into k subsets $E^1, ..., E^k$ such that for each vertex $i \in \{1, \dots, m\}$ and for each color $s \in \{1, \dots, k\}$:

$$h_i^s \quad \textit{is the number of edges of } E^s \textit{ adjacent to vertex } i \textit{ in } X \tag{6}$$

Clearly for a solution to exist the following conditions must hold:

$$\sum_{s=1}^{k} h_i^s = m - 1 \quad \forall i \in \{1, \dots, m\} \tag{7}$$

$$\sum_{i \in X} h_i^s \text{ is even} \quad \forall s \in \{1, \dots, k\} \tag{8}$$

For the rest of the paper, when working in complete graphs, we assume that conditions (7)-(8) are satisfied. Moreover, for $k = 2$, one can efficiently decide whether there exists or not an edge partition satisfying the values h_i^1. Indeed in [7], Erdös and Gallai give necessary and sufficient conditions to be verified by the values h_i^1. Furthermore these conditions can be checked in polynomial time.

Finally, we will also study the case where, instead of a complete bipartite graph, the underlying graph is a digraph. The paper is organized as follows.

In Section 2, we give complexity results for the case where the graph is a complete graph G_X.

In general no requirement is imposed on the structure of the graphs generated by E^s or by $E^{st} \equiv E^s \bigcup E^t$ besides satisfying (4)-(5). Here we shall first examine some variations where the union of some subsets E^s has to satisfy some additional constraints. We will focus on these subsets and we will not care about the other subsets corresponding to the remaining colors. We shall show that for these special cases of the problem with $k = 3$ colors solutions can be obtained in polynomial time. These special cases correspond in discrete tomography to situations where one has some additional information on the occurrences of some colors or of some combinations of two or more colors.

Let us observe that from constraints (1)-(2) we see that there are indeed $k-1$ independent colors, the last one, say color k, will be the ground color (the number of its occurrences in each row and in each column is determined by the occurrences of the $k-1$ other colors). Since we assume $k = 3$, we will have to determine disjoint sets E^1, E^2 and $E^3 = E - (E^1 \cup E^2)$ will be automatically determined and it will satisfy (4) and (5).

In Section 3, we give sufficient conditions based on the maximum degree in E^{12} for a solution to exist for the case of non oriented complete bipartite graphs and complete graphs (see [18] for additional results on the sequences of degrees of a graph). This exhibits a solvable case of the basic image reconstruction problem with $k = 3$ colors.

In Section 4 we examine some variations of the problem with $k = 3$ colors where E^{12} has to satisfy some additional constraints. We give necessary and sufficient conditions for the cases where the edges of E^{12} form a tree or a collection of vertex-disjoint chains. These conditions can be checked in polynomial time.

Finally, in Section 5, we will consider the problem corresponding to special values (0, 1 or 2) of h_i^s (and v_j^s), i.e., the search of two edge-disjoint chains or cycles going through specified vertices in complete bipartite graphs or complete graphs (see [10, 13] for additional results on the sequences of degrees of a graph forming two (or three) edge-disjoint forests).

2 Complexity results for complete graphs

Here, we study the complexity status of the degree-constrained edge partitioning problem. We will show the NP-completeness of the problem in case of complete graphs.

Although this has no immediate connection with discrete tomography as before, this result is given for its interest in a graph theoretical context.

Let us consider a complete graph K_X on a set X of m vertices.

As already mentioned in the introduction we now want to find a partition $E^1, \dots, E^k$ of the edge set $E(K_X)$ such that in E^s there are exactly h_i^s edges adjacent to vertex i for each $i \in \{1, \dots, m\}$ and each $s \in \{1, \dots, k\}$ (condition (6)). E^s is usually called a *b-factor*: we recall that in a graph where to each vertex v is associated a nonnegative integer $b(v)$ a b-factor is a subset E^* of edges such that each vertex v is adjacent to exactly $b(v)$ edges of E^*; conditions of existence are given in [16]. E^s can be constructed in polynomial time by using a b-matching algorithm [16]. We shall assume that these conditions hold for each E^s (otherwise our problem has no solution). These can be formulated as follows for each color s: for each partition T, U, W of X, the subgraph induced by T has at most $\sum_{i \in U} h_i^s - \sum_{i \in W} h_i^s + 2|E[W,W]| + |E[T,W]|$ components K with $\sum_{i \in K} h_i^s + |E[K,W]|$ odd, where $E[A,B] = \{[a,b] \in E(G) | a \in A, b \in B\}$.

Furthermore we assume that every vertex in $K_{X,Y}$ is adjacent to at least one edge of $E^{1,\dots,k-1}$(otherwise we simply delete the vertices not adjacent to any edge of $E^{1,\dots,k-1}$ and consider the remaining graph). This assumption can be stated as follows:

$$V(E^1) \cup V(E^2) \cup \ldots \cup V(E^{k-1}) = V(G) \tag{9}$$

The following proposition sheds some light on the relative complexity of the decision versions of the edge k-partitioning problems in a complete graph and in a complete bipartite graph.

Proposition 1 *The degree-constrained edge k-partitioning problem in a complete graph is at least as difficult as in a complete bipartite graph.*

Proof: We are given a degree-constrained edge k-partitioning problem P defined by a complete bipartite graph $K_{X,Y}$ and values h_i^s $(i \in X)$, v_j^s $(j \in Y)$ for each $s \in \{1, \dots, k\}$. We recall that conditions (1)-(3) do hold. We construct a complete graph $G' = K_{X \cup Y}$ on $X \cup Y$ by introducing in $K_{X,Y}$ a clique on X and a clique on Y. Let $m = |X|$ and $n = |Y|$. For each $i \in X$ we set $h_i^{'1} = h_i^1 + m - 1$, $h_i^{'s} = h_i^s$ for each $s \in \{2, ..., k\}$ and for each $j \in Y$ we set $v_j^{'2} = v_j^2 + n - 1$, $v_j^{'s} = v_j^s$ for each $s \in \{1, \dots, k\} \setminus \{2\}$. This defines a problem P' on G'. Clearly if P has a solution S, we can derive a solution S' to P' by keeping the colors of the edges $[x_i, y_j]$ of G', by giving color 1 to all edges $[x_r, x_t] \in X \times X$ and color 2 to all edges $[y_u, y_v] \in Y \times Y$. Conversely assume that P' has a solution S' in G'. Then all edges with both ends in X (resp. in Y) have color 1 (resp. color 2): suppose an edge $[x_i, x_j]$ has some color $c \neq 1$; then x_i and x_j are adjacent to $m-2$ edges of color 1 (instead of $m-1$) with both ends in X; so the number of edges of color 1 going out of X will be at least $\sum_{i \in X} h_i^1 + 2 > \sum_{j \in Y} v_j^1 = \sum_{j \in Y} v_j^{'1}$ which is at least as large as the number of edges of color 1 which may have one or two ends in Y. This is impossible. For color 2, the same holds (interchanging the roles of X and Y). Then by keeping the colors of all edges $[x, y]$ of $K_{X \cup Y}$, we get a solution for P in $K_{X,Y}$. □

From Proposition 1 and from the NP-completeness of the image reconstruction problem for $k = 4$ [4], we obtain the following:

Corollary 2 *For any fixed $k \geq 4$, the degree-constrained edge k-partitioning problem in a complete graph is NP-complete.*

3 Sufficient conditions for a solution to exist when $k = 3$

In this section, we shall give a sufficient condition for a partition E^1, E^2, E^3 satisfying (4)-(5) (or (6)) to exist.

First we study the case of a complete bipartite graph $K_{X,Y}$ (i.e. the case of image reconstruction problem) The condition involves the largest degree p in $E^{12} = E^1 \bigcup E^2$. We shall assume that $p \geq 2$ in this section (since the case $p = 1$ is trivial).

Proposition 3 *In a complete bipartite graph $K_{X,Y}$ let $p = \max_{i \in X, j \in Y}\{h_i^1 + h_i^2, v_j^1 + v_j^2\} \geq 2$. There exists a partition E^1, E^2, E^3 of E satisfying (4)-(5) if $|E^{12}| \geq 2p(p-2) + 3$.*

Proof: By [15], we know how to construct separately E^1 and E^2. If there are no cycles of length 2, then we are done since E^1 and E^2 are disjoint and the remaining (uncolored) edges will necessarily belong to E^3.

Otherwise we have at least one cycle of length 2, consisting of the parallel edges $[x, y]_1, [x, y]_2$, where $x \in X$, $y \in Y$, $[x, y]_1 \in E^1$ and $[x, y]_2 \in E^2$. If we can find an edge $[z, t] \in E^{12}$ $(z \in X, t \in Y)$ of color $s \in \{1, 2\}$ such that $[x, t], [z, y] \notin E^{12}$, then by replacing $[x, y]_s$ and $[z, t]$ by $[x, t]$ and $[z, y]$ which get color s, conditions (4)-(5) are still satisfied and we have at least one less cycle of length 2. By repeating this procedure, we get 2 disjoint edge sets E^1, E^2 satisfying (4)-(5) and thus a solution of our problem.

Let us now show in which case we can always find an edge $[z, t] \in E^{12}$ such that $[x, t], [z, y] \notin E^{12}$. Such an edge will be called a *good edge*. Notice that x and y are considered as linked by two edges. Clearly all edges having as endvertices x or y are not good edges. We have at most $2(p-1)$ such edges. Furthermore, all edges $[u, v] \in E^{12}$ such that $[x, v]$ or $[u, y]$ belongs to E^{12} are not good edges; there are at most $2(p-2)(p-1)$ such edges. Clearly every other edge in E^{12} not belonging to these two sets will be a good edge. Thus if we require that E^{12} contains at least $2(p-1) + 2(p-2)(p-1) + 1 = 2p(p-2) + 3$ edges, there will always be a good edge and hence all cycles of length 2 can be replaced by two disjoint edges. □

We will now deal with the analogous case where the graph is a complete graph K_X.

Proposition 4 *In a complete graph K_X, let $p = \max_{i \in X}\{h_i^1 + h_i^2\}$, $p \geq 2$. There exists a partition E^1, E^2, E^3 of the edge set E satisfying (6) if $|E^{12}| \geq p^2 - 2p + 3$.*

Proof: We proceed in a similar way as in the proof of Proposition 3, i.e., we first construct E^1 and E^2 independently (using the construction given in [16]). If they are disjoint, then we are done. Otherwise there is at least one cycle of length 2 consisting of the parallel edges $[x, y]_1, [x, y]_2$, where $[x, y]_1 \in E^1$ and $[x, y]_2 \in E^2$.

If we find a (good) edge $[z, t] \in E^{12}$ such that $[x, z], [y, t] \notin E^{12}$ or $[x, t], [y, z] \notin E^{12}$, then we can replace $[z, t]$ and one edge of the cycle by one of these pairs of edges and condition (6) will still be satisfied. Repeating this procedure will necessarily lead to a solution.

Now we will show a sufficient condition for a good edge $[z, t]$ to exist. Clearly all edges incident to x or y in E^{12} are not good edges. We have at most $2(p-1)$ such edges. Let q denote the number of vertices which are neighbors of both x and y in E^{12}. Then all edges incident to these vertices are not good edges either. We have at most $q(p-2)$ of them different from edges $[x, y]_1, [x, y]_2$. Finally each edge $[u, v]$ joining two neighbors of x (resp. y) which are not neighbors of y (resp. x) is not a good edge. We have at most $(p-2-q)(p-3-q)$ such edges. It is easy to see that every other edge will be a good edge. Thus if we require that $|E^{12}| \geq 2(p-1)+q(p-2)+(p-2-q)(p-3-q)+1$, then we can always find a good edge and hence replace the cycle of length 2. If we consider the extreme cases where $q = 0$ and $q = p-2$, we find that $|E^{12}| \geq \max(p^2-2p+3, p^2-3p+5)$ and thus $|E^{12}| \geq p^2 - 2p + 3$ since $p \geq 2$. □

4 The case where E^{12} is acyclic

4.1 Trees in complete bipartite graphs

The first problem which we consider can be formulated as follows: given a complete bipartite graph $K_{X,Y}$ and the values $h_i^1, h_i^2, v_j^1, v_j^2$ (for each $i \in \{1, \ldots, m\}$, for each $j \in \{1, \ldots, n\}$) find two disjoint subsets E^1, E^2 of edges of $E(K_{X,Y})$ such that (4) - (5) hold for each $s \in \{1, 2\}$ and in addition E^{12} forms a tree. Observe that $h_i^1 + h_i^2$ (resp. $v_j^1 + v_j^2$) will be the degree of vertex i in X (resp. vertex j in Y) in the tree E^{12}.

To avoid dealing with trivial cases, we shall assume that our problem is not degenerate so that each one of colors 1, 2 occurs on at least one edge.

We shall first state two lemmas that will be repeatedly used to construct the required subset E^{12} of edges by reducing the number of connected components. These two lemmas hold for both complete bipartite graphs and complete graphs.

Recoloring Lemma *Let C_1, C_2 be two connected components of E^{12} satisfying (4)-(5) and such that C_2 contains at least one cycle.*

Assume one can find an edge $[x_1, y_1]$ in C_1 and an edge $[x_2, y_2]$ belonging to some cycle C of C_2 such that $[x_1, y_1]$ and $[x_2, y_2]$ have the same color (both are in E^1 or both are in E^2).

Then by replacing $[x_1, y_1]$, $[x_2, y_2]$ by $[x_1, y_2]$, $[x_2, y_1]$ and by giving them the same color as the removed edges we get a single connected component C' which *still satisfies (4)-(5).*

Proof: Since $[x_1, y_2]$, $[x_2, y_1]$ get the same color as $[x_1, y_1]$, $[x_2, y_2]$, (4)-(5) are still satisfied. One verifies that x_2 and y_2 are still connected in $C - [x_2, y_2]$; furthermore in C' x_1 and y_1 are connected by a chain consisting of edge $[x_1, y_2]$ followed by $C - [x_2, y_2]$ and by edge $[x_2, y_1]$. It follows that there is at least a chain between any two vertices of C', and so C' is connected. Notice furthermore that C' is still bipartite if C_1 and C_2 were bipartite. □

Recycling Lemma *Assume we have a connected component C of E^{12} containing some cycle $\overline{C}$ and let e be an edge of C not contained in any cycle. If there is a chain $\tilde{C}$ in C containing e and starting with some edge $[x_2, y_2]$ in $\overline{C}$ and ending with an edge $[x_1, y_1] \neq e$ in $C - \overline{C}$ with the same color as $[x_2, y_2]$, then one may replace edges $[x_1, y_1], [x_2, y_2]$ by $[x_1, y_2], [x_2, y_1]$ so that (4)-(5) still hold and e is on a cycle.*

Proof: Notice that $[x_1, y_2], [x_2, y_1]$ are not in E^{12} (otherwise e would be in a cycle). Replacing $[x_1, y_1], [x_2, y_2]$ by $[x_1, y_2], [x_2, y_1]$ and assigning them the same color as $[x_1, y_1], [x_2, y_2]$ gives another connected component where (4)-(5) still hold. It can be checked that there is a cycle C' (possibly of length 2) containing e which goes either through x_1, y_2 or through x_2, y_1. □

Remark 4.1 *We shall use later an oriented version of these lemmas; the translation to the new case will be immediate.* □

Proposition 5 *In $K_{X,Y}$ there exist two disjoint subsets E^1, E^2 of edges such that (4)-(5) hold and E^{12} is a tree if and only if:*

(a) $\sum_{i \in X} (h_i^1 + h_i^2) = \sum_{j \in Y} (v_j^1 + v_j^2) = (m + n - 1)$

(b) $\sum_{i \in X} h_i^s = \sum_{j \in Y} v_j^s \leq (m_s + n_s - 1)$ *for each $s \in \{1, 2\}$, where m_s (resp. n_s) is the number of vertices i in X (resp. j in Y) with $h_i^s > 0$ (resp. $v_j^s > 0$) for each $s \in \{1, 2\}$.*

Proof:

- $\Rightarrow$ If E^{12} is a tree it does satisfy (a) and E^1, E^2 cannot contain any cycle, so they are forests and (b) is verified.
- $\Leftarrow$ From [15] we know how to construct E^1 and E^2, since the Ryser conditions are satisfied. Notice that some edges may appear in both E^1 and E^2, creating cycles of length 2 (in this case the graph is not simple). But these will be removed later in the process.

  If E^{12} is connected, it is a tree from (a) and (9) and we are done.

Otherwise, E^{12} consists of $p \geq 2$ connected components $C_1, \ldots, C_p$. From (a) there is at least one such component, say C_1, which is a tree and at least one that contains cycles. By (9), C_1 contains at least one edge.

As long as we can find two edges $[x_1, y_1], [x_2, y_2]$ of the same color (1 or 2) in two connected components and such that in addition $[x_1, y_1]$ is in some cycle, we can reduce these components to a single component by the Recoloring Lemma. This will not create any new cycle of length 2 since the new edges join distinct vertices of two different connected components.

When we cannot find such pairs of edges anymore, either we are done or we are necessarily in the following situation: all connected components that are trees are monochromatic and all have the same color, say 1. Furthermore there is exactly one additional connected component C that contains cycles (otherwise we could have used the Recoloring Lemma); all edges belonging to cycles in C have color 2.

Notice that in C there must be at least one edge of color 1, otherwise (b) would be violated for color 2. From (b) we know that C must also contain an edge $[x_2, y_2]$ of color 2 which is not incident to any cycle of C. It is linked to some vertex x^* of a cycle $\overline{C}$ by a chain Q containing at least one edge e of color 1. Now take some edge $[x_1, y_1]$ of $\overline{C}$. Applying the Recycling Lemma, we replace $[x_1, y_1]$ and $[x_2, y_2]$ by $[x_1, y_2]$ and $[x_2, y_1]$; it gives a connected component where edge e (of color 1) now belongs to some cycle (which may possibly be of length 2). Since there is at least one connected component which is a tree and where all edges have color 1, we can apply the Recoloring Lemma.

□

4.2 Trees in complete graphs

Here, we give two statements analogous to Proposition 5 in case of complete graphs and complete symmetric oriented graphs.

Proposition 6 *In a complete graph K_X there exist disjoint subsets E^1, E^2 of edges such that (6) holds for each vertex i and for each $s \in \{1, 2\}$ and E^{12} is a tree, if and only if*

(a) $\sum_{i \in X}(h_i^1 + h_i^2) = 2\ (|X| - 1)$

(b) $\sum_{i \in X} h_i^s \leq 2\ (m_s - 1)$ *for each $s \in \{1, 2\}$ where m_s is the number of vertices i with $h_i^s > 0$.*

The proof follows the same lines as the proof of Proposition 5 (except that we do not have to take care of the bipartite character of E^{12} when connecting different components).

In order to further generalize the previous formulations of these variations on the basic image reconstruction problem we could consider that the underlying graph G is now oriented with arcs (x, y) instead of edges $[x, y]$. We shall assume that when two vertices x, y are linked in G, there may be several arcs (x, y). This will be needed for constructing an initial solution.

Let us consider here the case where $G = \overrightarrow{K}_X$ is a complete symmetric oriented graph on a set X of m vertices.

For each vertex i in X we are given $2k$ integers h_i^{+s}, h_i^{-s} for each $s \in \{1, \ldots, k\}$. We have to find a partition $\overrightarrow{E}^1, \ldots, \overrightarrow{E}^k$ of the arc set $\overrightarrow{E}(G)$ such that for each color s we have:

$$h_i^{+s} \text{ is the number of arcs of } \overrightarrow{E}^s \text{ leaving vertex } i, \forall i \in X \tag{10}$$

$$h_i^{-s} \text{ is the number of arcs of } \overrightarrow{E}^s \text{ entering vertex } i, \forall i \in X \tag{11}$$

Clearly for a solution to exist we must have:

$$\sum_{s=1}^{k} h_i^{+s} = d_G^+(i) \quad \text{(out degree of } i \text{ in } G), \forall i \in X \tag{12}$$

$$\sum_{s=1}^{k} h_i^{-s} = d_G^-(i) \quad \text{(in degree of } i \text{ in } G), \forall i \in X \tag{13}$$

$$\sum_{i \in X} h_i^{+s} = \sum_{i \in X} h_i^{-s} \qquad \forall s \in \{1, \ldots, k\} \tag{14}$$

We assume that (12)-(14) are verified. As before we shall assume that for each color s the values h_i^{+s}, h_i^{-s} are such that there exists a subset $\overrightarrow{E}^s$ satisfying (10) and (11). Necessary and sufficient conditions for the existence of such a subset $\overrightarrow{E}^s$ are given in [2] (chapter 6); to construct such a subset $\overrightarrow{E}^s$, we have to find a b-factor in a bipartite graph $G = (X, X', U)$ obtained by introducing for every vertex $i \in X$ a vertex $i' \in X'$ and linking every $i \in X$ to every $j' \in X'$ (with $i \neq j$) by an arc (i, j'). We set $b(i) = h_i^{+s}$ for each $i \in X$ and $b(i') = h_i^{-s}$ for every $i' \in X'$. Finding a b-factor can be done in polynomial time with network flow techniques (see [16]).

As in the previous sections, we shall consider here the case of $k = 3$ colors. We assume w.l.o.g. that there is no vertex with $h_i^{+1} = h_i^{-1} = h_i^{+2} = h_i^{-2} = 0$.

Proposition 7 *Let $\overrightarrow{K}_X$ be an oriented complete symmetric graph with values h_i^{+s}, h_i^{-s} given for each vertex i in X and for each $s \in \{1, 2\}$.*

There exist disjoint subsets $\overrightarrow{E}^1, \overrightarrow{E}^2$ of the arc set $\overrightarrow{E}(\overrightarrow{K}_X)$ satisfying (10), (11) and such that $\overrightarrow{E}^{12} = \overrightarrow{E}^1 \bigcup \overrightarrow{E}^2$ is a tree if and only if

(a) $\sum_{i \in X} (h_i^{+1} + h_i^{+2}) = \sum_{i \in X} (h_i^{-1} + h_i^{-2}) = |X| - 1$

(b) $\sum_{i \in X} h_i^{+s} = \sum_{i \in X} h_i^{-s} \leq m_s - 1$ *for each* $s \in \{1, 2\}$

where m_s is the number of vertices i in X with $h_i^{+s} + h_i^{-s} > 0$ (i.e. vertices adjacent to at least one arc of color s).

Proof: Condition (a) is necessary for $\overrightarrow{E}^{12}$ to be a tree. Furthermore if there is a solution, then $\overrightarrow{E}^1$ and $\overrightarrow{E}^2$ have to be forests, so (b) must hold.

Let us now show that the conditions are sufficient. By our assumptions one can find subsets $\overrightarrow{E}^1, \overrightarrow{E}^2$ of $\overrightarrow{E}(G)$ satisfying (10) and (11). Notice that $\overrightarrow{E}^1$ and $\overrightarrow{E}^2$ may use parallel arcs $(x, y)_1$ $(x, y)_2, \ldots$, or $(y, x)_1$, $(y, x)_2, \ldots$, between pairs of vertices $x \in X$, $y \in X$. But since $\overrightarrow{E}^{12}$ has to be a tree, these parallel arcs will have to be removed during the process.

Consider $\overrightarrow{E}^{12} = \overrightarrow{E}^1 \bigcup \overrightarrow{E}^2$; if it generates a connected graph, it is a tree from (a) and we are done.

Otherwise $\overrightarrow{E}^{12}$ generates several connected components; at least one of them is a tree (from (a)). Now let us suppose that we can find two connected components C_1, C_2 such that C_1 contains at least one cycle C. Take an arc (x_1, y_1) in C and assume there is in C_2 an arc (x_2, y_2) of the same color.

Replacing (x_1, y_1) and (x_2, y_2) by (x_1, y_2) and (x_2, y_1) gives a single connected component and the conditions (10) and (11) are still satisfied. No new pair of parallel arcs is created and the graph generated by $\overrightarrow{E}^{12}$ has one less connected component. We can then apply systematically the Recoloring Lemma and the Recycling Lemma as in the bipartite case.

This can be repeated until we get a tree for $\overrightarrow{E}^{12}$. □

4.3 The case of vertex disjoint chains

First, we study the case of a complete bipartite graph $K_{X,Y}$. We are now given values h_i^s, v_j^s which satisfy

$$1 \le h_i^1 + h_i^2 \le 2 \quad \text{for each } i \text{ in } X \tag{15}$$
$$1 \le v_j^1 + v_j^2 \le 2 \quad \text{for each } j \text{ in } Y \tag{16}$$

Here E^{12} will have to consist of a collection of elementary open chains having their endvertices at vertices r (resp. t) with $h_r^1 + h_r^2 = 1$ (resp. $v_t^1 + v_t^2 = 1$). These will be called *odd vertices.* Clearly we must have an even positive number of odd vertices for the existence of a solution.

Notice that we exclude cycles in a solution, i.e., we have to show that we only have open chains.

Proposition 8 *In a complete bipartite graph $K_{X,Y}$ there exist subsets E^1, E^2 of edges of $E(K_{X,Y})$ satisfying (4)-(5) and such that E^{12} is a collection of vertex-disjoint open chains covering the vertices of $K_{X,Y}$ if and only if (15)-(16) hold and:*

- *(a) For each color s, there is at least one vertex which has to be adjacent to exactly one edge of color s.*
- *(b) There exists a vertex $i \in X$ with $h_i^1 + h_i^2 = 1$ or a vertex $j \in Y$ with $v_j^1 + v_j^2 = 1$.*

Proof: Clearly (15)-(16) are necessary.

It follows from (b) and from (3) that the number of odd vertices is even and positive.

If (a) does not hold, there is one color s such that every vertex is adjacent to two edges or to no edge of color s. Clearly the edges of color s cannot be on a chain of E^{12}.

To show that the conditions are sufficient, we start from a set E^{12} satisfying (15)-(16); E^1 and E^2 can be constructed separately since the Ryser conditions are assumed to hold. As before E^{12} may contain cycles of length two. If it contains no connected component which is a cycle we are done. Otherwise consider a cycle C; since there is at least one odd vertex (from (b)) there is a chain C' in E^{12}; if we can find a pair of edges $[x_1, y_1]$ in C', $[x_2, y_2]$ in C of the same color, we use the Recoloring Lemma. When we cannot use this lemma anymore we are in the situation where we have monochromatic cycles (all of the same color, say 1) and monochromatic chains (all of color 2) between odd vertices. But this is not possible: from (a) for color 1, there must be a vertex adjacent to exactly one edge of color 1. Hence we do not have this case and we can construct a set E^{12} satisfying (15)-(16) and consisting of open chains. □

Now, we consider the case of a complete graph K_X where E^{12} is a collection of chains between odd vertices (i.e. $h_i^1 + h_i^2 = 1$). If we have exactly two odd vertices then the problem amounts to finding a subset E^{12} (satisfying the degree requirements) which is a Hamiltonian chain with fixed end vertices.

We may as well consider the case where a Hamiltonian cycle has to be constructed while taking the condition (6) into account.

Proposition 9 *Given values h_i^1, h_i^2 , satisfying $h_i^1 + h_i^2 = 2$ for each vertex i of a complete graph K_X, there are disjoint subsets E^1, E^2 of the edge set $E(K_X)$ such that (6) holds for each vertex i and for each $s \in \{1, 2\}$ and in addition E^{12} is a Hamiltonian cycle, if and only if there exists at least one vertex with $h_i^1 = h_i^2 = 1$.*

Proof: If the condition does not hold, no connected solution can be found. The sufficiency is shown by the Recoloring Lemma: the only case where it cannot be applied is when E^{12} consists of two disjoint elementary cycles which are monochromatic (one with color 1, the other one with color 2), but this is impossible from the condition. □

5 Cases where each one of E^1, E^2 is structured

We shall now examine additional cases where $k = 3$ and the subsets E^1, E^2 have a given structure.

5.1 E^1, E^2 are Hamiltonian chains

We study the situation where both E^1 and E^2 are Hamiltonian chains in $K_{X,Y}$. For this graph to have Hamiltonian chains we shall assume $|X| = |Y|$ (notice that we could have $|X| \leq |Y| \leq |X|+1$ but for simplicity we will limit our study to the case where $|X| = |Y|$) and each chain has an endvertex in X and the other one in Y. Let $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_n\}$ and let $a \in X, b \in Y$ be the endvertices of the Hamiltonian chain forming E^1; this means that we have

$$h^1(x_i) = \begin{cases} 1 & \text{if } x_i = a, \\ 2 & \text{if } x_i \neq a. \end{cases}$$

and

$$v^1(y_i) = \begin{cases} 1 & \text{if } y_i = b, \\ 2 & \text{if } y_i \neq b. \end{cases}$$

Let $u \in X, v \in Y$ be the endvertices of the Hamiltonian chain E^2; we will have similarly

$$h^2(x_i) = \begin{cases} 1 & \text{if } x_i = u, \\ 2 & \text{if } x_i \neq u. \end{cases}$$

and

$$v^2(y_i) = \begin{cases} 1 & \text{if } y_i = v, \\ 2 & \text{if } y_i \neq v. \end{cases}$$

In terms of tomography, it would mean that each one of the colors $1, 2$ occurs twice in some fixed rows, once in two specific rows and it does not occur in the other rows and moreover one can define for each color in $\{1, 2\}$ a sequence of moves from one entry containing this color to a next one in the same row or the same column, and this can be done for this color in such a way that we visit exactly once each row and each column containing this color.

Proposition 10 *In $K_{X,Y}$ (with $|X| = |Y| = n$) there exist two disjoint Hamiltonian chains E^1 (with arbitrary endvertices a, b) and E^2 (with arbitrary endvertices u, v) if and only if $n \geq 5$.*

Proof: If $n \leq 4$ one cannot find two disjoint Hamiltonian chains with $a = u$ and $b = v$. We assume now that $n \geq 5$. Let $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$ be a numbering of the vertices in X and in Y such that $x_1 = a,\ y_1 = b$. We construct E^1 by taking edges $[x_i, y_{i+1}], [x_{i+1}, y_i]$ for each $i \in \{1, ..., n-1\}$ and $[x_n, y_n]$. We have several cases to consider for E^2.

a) $u = a, v = b$
   We construct E^2 as follows (see Figure 1): we build the sequence of indices of vertices which will be visited by E^2 by taking first the odd indices in increasing order followed by the even indices in increasing order but where we just interchange 4 with the largest even index; the sequence $\overrightarrow{C}$ obtained in this way is then completed by the same sequence $\overleftarrow{C}$ in

reverse order; then we assign labels x and y alternately to all terms of the sequence $\overrightarrow{C} \oplus \overleftarrow{C}$; we get thus $x_1, y_3, ..., x_3, y_1$. This gives clearly a Hamiltonian chain E^2 with endvertices x_1, y_1 which is disjoint from E^1 since E^2 contains neither edges $[x_i, y_j]$ with $|i - j| = 1$ nor $[x_n, y_n]$.

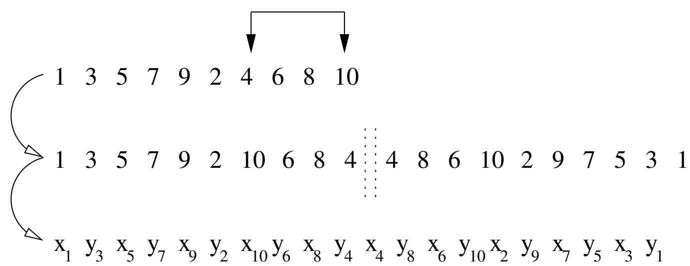

Figure 1: Construction of E^2 in the case of $n = 10$.

b) $u = a, v \neq b$
Notice that v is a fixed arbitrary vertex of Y with $v \neq y_1$. We can w.l.o.g. assume that from the beginning our vertices have been numbered in such a way that $v = y_3$. Now replacing $[x_1, y_3]$ by $[x_1, y_1]$ in the E^2 constructed in a) gives a new Hamiltonian chain E^2 disjoint from E^1 and with endvertices u, v.

c) $u \neq a, v \neq b$
Again u and v are fixed arbitrary vertices of X and Y respectively. We can w.l.o.g. assume that from the beginning our vertices have been numbered in such a way that $u = x_4, v = y_4$. We obtain E^2 by replacing $[x_4, y_4]$ by $[x_1, y_1]$ in the E^2 constructed in a). Clearly E^2 is a Hamiltonian chain disjoint from E^1.

□

Remark 5.1 *One can easily verify that Proposition 10 can be extended to the case of a complete graph on m vertices K_X with $m \geq 4$ if $a \neq u$ and $b \neq v$, $m \geq 5$ if $a = u$ and $b \neq v$, $m \geq 6$ if $a = u$ and $b = v$.*

5.2 E^1 and E^2 are cycles

Consider now the case where for each color $s \in \{1, 2\}$ we have $h_i^s, v_j^s \in \{0, 2\}$ for all $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$. $V(E^s)$ will be the set of vertices with $h_i^s = 2$ or $v_j^s = 2$ for each $s \in \{1, 2\}$. The problem consists then in finding two edge-disjoint cycles E^1, E^2 in $K_{X,Y}$ through specified vertex sets $V(E^1)$ and $V(E^2)$. W.l.o.g. we can assume that (9) holds: $V(G) = V(E^1) \cup V(E^2)$.

The reconstruction problem where both E^1 and E^2 are collections of vertex-disjoint cycles in $K_{X,Y}$ was studied in [5] under the name $RPB(m, n, p = 2)$ (see also [3, 17]). It was shown that a solution can be constructed if and only if one does not have one of the four following pathological cases:

a) $\sum_{i\in X} h_i^1 = 4 = \sum_{i\in X} h_i^2$, $|X| \leq 3$, $|Y| \leq 3$

b) $\sum_{i\in X} h_i^1 = 4$, $\sum_{i\in X} h_i^2 = 6$, $|X| = 3$, $|Y| \leq 4$

c) $\sum_{i\in X} h_i^1 = 6 = \sum_{i\in X} h_i^2$, $|X| = 3$, $|Y| \leq 5$

d) $\sum_{i\in X} h_i^1 = 6$, $\sum_{i\in X} h_i^2 = 8$, $|X| = 4 = |Y|$

For more results on edge-disjoint cycles in graphs, we refer the reader to [14] where the case of Hamiltonian cycles is considered.

Goddyn and Stacho give in [9] the following theorem concerning general graphs:

Theorem 11 *Let $G = (V, E)$ be a finite undirected simple graph of order m, let $W \subseteq V$, $|W| \geq 3$, and let k be a positive integer. Suppose that $G[W]$ is $2k$-connected, and that*

$$\max(d_G(u), d_G(v)) \geq \frac{m}{2} + 2(k-1) \tag{17}$$

for every $u, v \in W$ such that $dist_{G[W]}(u, v) = 2$. Then G contains k pairwise edge-disjoint cycles $E^1, ..., E^k$ such that $W \subseteq V(E^i)$, for each $i \in \{1, \ldots, k\}$.

(Here, $d_G(v)$ is the degree of v in G, $G[W]$ is the subgraph induced by W, and $dist_G(u, v)$ is the distance from u to v in G.)

Notice that for $k \geq 2$ condition (17) cannot be verified in the case of bipartite graphs. For complete bipartite graphs we give the following proposition. W.l.o.g. we assume that $\sum_i h_i^1 \leq \sum_i h_i^2$.

Proposition 12 *Let $K_{X,Y}$ be a complete bipartite graph. There exist two edge-disjoint cycles E^1, E^2 through specified vertex sets $V(E^1)$ and $V(E^2)$ satisfying (4)-(5) if and only if we are not in one of the four pathological cases and we do not have the forbidden configuration F given in Figure 2.*

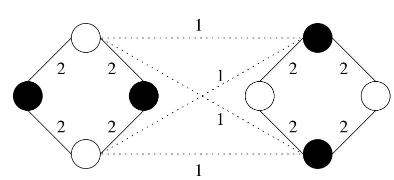

Figure 2: Configuration F. The black vertices belong to X and the white ones to Y.

Proof: First, if $\sum_i h_i^2 \geq 10$, then, whenever (1.1)-(1.3) are satisfied, there always exist two edge-disjoint cycles E^1 and E^2 satisfying (4)-(5). Indeed, remove a cycle of length $\sum_i h_i^1$ (the cycle E^1 through $V(E^1)$ corresponding to edges of color 1) from $K_{X,Y}$. Then, if we consider any p by p subgraph of the resulting graph (and so, in particular, the one induced by the $\sum_i h_i^2 = 2p$ vertices adjacent to edges of color 2), it has a minimum degree of $p-2$. Since $\sum_i h_i^2 \geq 10$, we have $p \geq 5$ and so $p-2 \geq \frac{p+1}{2}$. Thus, by [1] (Chapter 7, Section 3), there exists a Hamiltonian cycle in this (sub)graph which will correspond to E^2. This implies that we can obtain two disjoint cycles (one for color 1 and one for color 2) respecting the given projections.

Second, if $\sum_i h_i^1 = \sum_i h_i^2 = 8$, then, as previously, there always exist two edge-disjoint cycles E^1 and E^2 satisfying (4)-(5) whenever (1.1)-(1.3) are satisfied. Take any cycle E^1 on $V(E^1)$. Say $E^1 = \{[x_1, y_1], [y_1, x_2], ..., [x_4, y_4], [y_4, x_1]\}$. Consider the cycle $C = \{[x'_1, y'_1], [y'_1, x'_2], ..., [x'_4, y'_4], [y'_4, x'_1]\}$, where $x'_i = x_i$ if $h^2(x_i) = 2$ and $x'_i = z_i$ if $h^2(x_i) = 0$, where $z_i \neq x_j$, for each $j \in \{1,2,3,4\}$, is some vertex in X with $h^2(z_i) = 2$ as well as $y'_i = y_i$ if $v^2(y_i) = 2$ and $y'_i = t_i$ if $v^2(y_i) = 0$, where $t_i \neq y_j$, for each $j \in \{1,2,3,4\}$, is some vertex in Y with $v^2(t_i) = 2$. We construct E^2 by linking each vertex v in C to the two vertices in C which are at distance three of v.

Finally, let us deal with the case where $\sum_i h_i^1 \leq 6$ and $\sum_i h_i^2 \leq 8$. First, it is easy to check in Figure 2 (where $\sum_i h_i^1 = 4$ and $\sum_i h_i^2 = 8$) that F is the unique configuration satisfying these degree requirements h_i^1 and h_i^2 for each $i \in \{1, \ldots, m\}$, and that, in F, E^2 consists of two vertex-disjoint cycles. If $\sum_i h_i^1 \leq 6$ and $\sum_i h_i^2 \leq 6$, then, if we are not in one of the four pathological cases, there exists a solution and this solution necessarily consists of two edge-disjoint cycles (since there can be no cycle of length three or less). The last case to consider is when $\sum_i h_i^2 = 8$ and $\sum_i h_i^1 \leq 6$. Assume that we are not in one of the four pathological cases (and thus there exist two disjoint subsets of edges E'^1 and E'^2 satisfying (4)-(5), such that E'^1 is a cycle) and that the subgraph induced by $V(E'^1) \cup V(E'^2)$ is not the one in Figure 2. If E'^2 consists of one C_8, we are done (here C_k denotes as usual a chordless cycle on k vertices). Otherwise (E'^2 consists of two C_4), if there exists in the solution a C_4 with edges of color $2, 3, 2, 3$ (recall that we have $k = 3$) then color 2 and color 3 can be interchanged in the C_4: this provides an equivalent solution in which the edges of color 2 form a C_8. If such a cycle does not exist, then anyway there exists at least one edge of color 3 between the vertices of the two C_4 of color 2 (since otherwise there are 8 edges of color 1 between these vertices, and this contradicts $\sum_i h_i^1 \leq 6$). This implies that there exist four edges of color 1 between these vertices (since otherwise there is a C_4 with edges of color 2,3,2,3), which form a C_4. Therefore, this C_4 is *the* cycle of color 1: hence, we obtain the graph in Figure 2, which is a contradiction. □

Consider now the case where E^1 and E^2 are required to be two edge-disjoint cycles in a complete graph K_X with $|X| = m$, i.e. $h_i^s = 0$ or $h_i^s = 2$ for each $s \in \{1,2\}$ and each $i \in X$. We shall assume that $V(E^1) \cup V(E^2) = X$ and $|V(E^1)|, |V(E^2)| \geq 3$. We have the following:

Proposition 13 *In K_X with $|X| = m$, one can find two edge-disjoint cycles E^1 on $V(E^1)$ and E^2 on $V(E^2)$ satisfying (6) if and only if we are not in the following cases:*

(a) $m \leq 4$,

(b) $3 \leq |V(E^1) \cap V(E^2)| \leq 4$ *and* $m = 5$.

Proof: The conditions are clearly necessary. If we are in case (a) or (b), one can easily check that E^1 and E^2 cannot be disjoint, by enumerating all possible configurations (case (a) or (b) and different lengths of the cycles, namely 3, 4, or 5).

Consider now the case where $m \geq 5$.

From Theorem 11 we deduce that the proposition is true for $|V(E^1) \cap V(E^2)| \geq 5$. Indeed, in a complete graph there are no two vertices at distance 2 and $V(E^1) \cap V(E^2)$ is 4-connected if $|V(E^1) \cap V(E^2)| \geq 5$. Thus, the conditions of Theorem 11 are verified and the graph contains 2 edge-disjoint cycles through $V(E^1)$ and $V(E^2)$.

If $|V(E^1) \cap V(E^2)| < 5$, $V(E^1) \cap V(E^2)$ is not 4-connected and the conditions do not hold anymore. Let us now study the different cases.

If $|V(E^1) \cap V(E^2)| \leq 1$, clearly E^1 and E^2 will be edge-disjoint.

If $V(E^1) \cap V(E^2) = \{a, b\}$, then w.l.o.g. $|V(E^1) \setminus (V(E^1) \cap V(E^2))| \geq 2$, say $V(E^1) \setminus (V(E^1) \cap V(E^2)) = \{c, d, ..., z\}$. We construct a cycle $E^1 = a, c, b, d, ..., z, a$ and for E^2 a cycle consisting of the union of $[a, b]$ and a chain between a and b in $V(E^2) \setminus (V(E^2) \cap V(E^1))$. These subsets E^1, E^2 will be disjoint.

If $|V(E^1) \cap V(E^2)| = 3$ or 4 and $m \geq 6$, it is easy to obtain two edge-disjoint cycles. We give the construction for $|V(E^1) \cap V(E^2)| = 4$ (the case $|V(E^1) \cap V(E^2)| = 3$ can be treated similarly). Let $V(E^1) \cap V(E^2) = \{a, b, c, d\}$. Then $|V(E^1) - V(E^2)| + |V(E^2) - V(E^1)| \geq 2$. If $e \in V(E^1) - V(E^2)$ and $f \in V(E^2) - V(E^1)$, we can take for E^1 the cycle beginning with vertices $a, b, c, d, e, ...$ followed by the remaining vertices of $V(E^1) - V(E^2)$. For E^2 we can take the cycle beginning with vertices $c, a, d, b, f, ...$ followed by the remaining vertices of $V(E^2) - V(E^1)$. If f does not exist, i.e., $V(E^2) \subseteq V(E^1)$ then we have $V(E^1) - V(E^2) = \{e, g, ...\}$. In this case we take for E^1 the cycle beginning with vertices $a, b, e, c, d, g, ...$ followed by the remaining vertices of $V(E^1) - V(E^2)$. For E^2, we take the cycle (a, d, b, c, a). Clearly E^1, E^2 will be disjoint. □

We have considered here the problem of constructing in a complete (bipartite) graph two disjoint subsets of edges satisfying some requirements on their degrees at every vertex. Since the given values h_i^s, v_j^s determine the cardinalities $|E^1|, |E^2|$, we have in fact to find if there exist two disjoint subsets of edges with given cardinalities which satisfy some additional requirements (on their degrees). This problem is related to the following NP-complete problem (see [8]): given a bipartite graph G and two integers $p > q > 0$, does G contain two edge-disjoint matchings M_p, M_q with $|M_p| = p, |M_q| = q$? If the values h_i^s, v_j^s are 0 or 1, then E^1, E^2 are matchings of a fixed size. But we know which vertices belong to E^1 or E^2 and the graph G is a complete bipartite graph. This situation has been studied in [5] under the name $RPU(m, n, p)$.

6 Conclusion

We have investigated some graph theoretical problems related to the image reconstruction problem in discrete tomography. We have exhibited a solvable case of the basic image reconstruction problem with $k = 3$ colors. The complexity of the related problem in a complete graph has already been settled for a fixed $k \geq 4$.

We imposed the structure of the graph formed by the union of two colors. Here having a tree allowed us to find solutions whenever they exist. The choice was adequate since it eliminated the cycles that were introduced by the parallel edges or arcs needed in the model (the presence of parallel edges in E^{12} would have meant that the corresponding entries of the array A received several colors). In fact we have imposed constraints on the cardinalities of E^1, E^2 and/or E^{12} and it is worth observing that if we introduce some requirements on $|E^1|$ (for instance $|E^1| \leq f(m, n)$ where f is a linear function of the size of the array $A = (a_{ij})$), this additional piece of information does not simplify the problem in the following sense: one may transform any reconstruction problem P with $k = 3$ colors into a larger problem P' with $k = 3$ colors where the first color class will satisfy a requirement of the form $|E^1| \leq f(m', n')$ and P' will have a solution if and only if P has one. This can be seen easily by embedding array A in a corner of a larger $m' \times n'$ array A' where we impose color 3 to all entries of A' outside of A.

It would be interesting to examine other structures for the graph associated to the union of several colors.

Finally we mention the case where each one of E^1, E^2 is a spanning tree; this problem seems to be of interest. As far as we know problems consisting of packing some special types of graphs, like trees, have not been explored intensively when there are requirements on the degrees of the vertices.

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References

[1] A. Asratian, T. Denley, and R. Häggkvist. *Bipartite graphs and their applications.* Cambridge Tracks in Mathematics 131. Cambridge University Press, 1998.

[2] C. Berge. *Graphes.* Gauthier-Villars, 1983.

[3] B. Bollobás and G. Brightwell. Cycles through specified vertices. *Combinatorica*, 13:147–155, 1993.

[4] M. Chrobak and C. Dürr. Reconstructing polyatomic structures from x-rays: Np-completeness proof for three atoms. *Theoretical Computer Science*, 259(1):81–98, 2001.

[5] M.-C. Costa, D. de Werra, and C. Picouleau. Using graphs for some discrete tomography problems. *Discrete Applied Mathematics*, 154:35–46, 2006.

[6] M.-C. Costa, D. de Werra, C. Picouleau, and B. Ries. On the use of graphs in discrete tomography. *4OR*, 6(2):101–123, 2006.

[7] P. Erdös and T. Gallai. Graphs with prescribed degree of vertices. *Mat. Lapok*, 11:35–46, 1960.

[8] H. Gabow, T. Nishizeki, O. Kariv, D. Leven, and O. Tereda. Algorithms for edge-coloring graphs. University of Colorado, 1983.

[9] L. Goddyn and L. Stacho. Edge disjoint cycles through specified vertices. *Journal of Graph Theory*, 50:246–260, 2005.

[10] M. Koren. Edge disjoint realizations of two forest-realizable sequences. *SIAM Journal of Appl. Math.*, 30:625–629, 1976.

[11] A. Kuba and G. T. Hermann. *Discrete Tomography: Foundations, Algorithms and Applications.* Birkhauser, 1999.

[12] A. Kuba and G. T. Hermann. *Advances in Discrete Tomography and its Applications.* Birkhauser, 2007.

[13] S. Kundu. Disjoint representation of three tree realizable sequences. i. *SIAM Journal of Appl. Math.*, 28:290–302, 1975.

[14] G. Li. Edge disjoint hamilton cycles in graphs. *Journal of Graph Theory*, 35:8–20, 2000.

[15] H. Ryser. Combinatorial properties of matrices of zeros and ones. *Canad. J. Math.*, 9:371–377, 1957.

[16] A. Schrijver. *Combinatorial Optimization, Polyhedra and Efficiency.* Springer, 2003.

[17] F. Zeffirelli. Les cycles franciscains de giotto, 1967. private communication, 14mn.

[18] I. Zverovich and V. E. Zverovich. Contributions to the theory of graphical sequences. *Discrete Mathematics*, 105:293–303, 1992.