Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 12, no. 1, pp. 97–112 (2008)

Drawing Bipartite Graphs on Two Parallel Convex Curves

Emilio Di Giacomo *Luca Grilli* *Giuseppe Liotta*

Dipartimento di Ingegneria Elettronica e dell'Informazione
Università degli Studi di Perugia
http://gdv.diei.unipg.it
{digiacomo,luca.grilli,liotta}@diei.unipg.it

Abstract

Let G be a bipartite graph, and let λ_e, λ_i be two parallel convex curves; we study the question about whether G admits a planar straight-line drawing such that the vertices of one partite set of G lie on λ_e and the vertices of the other partite set lie on λ_i. A characterization is presented that gives rise to linear time testing algorithm. We also describe a drawing algorithm that runs in linear time if the curves are two concentric circles and the real RAM model of computation is adopted.

Article Type	Communicated by	Submitted	Revised
Regular paper	M. Kaufmann and D. Wagner	December 2006	September 2007

Research partially supported by MIUR under Project "ALGO-NEXT": Algorithms for the Next Generation Internet and Web: Methodologies, Design and Applications.

1 Introduction

Common requirements for drawing a bipartite graph are that the bipartition is highlighted in the visualization by representing the vertices on two distinct layers, the edges have as few bends as possible, and the number of edge crossings is minimized. A bipartite graph is a *biplanar graph* if it has a straight-line crossing-free drawing where the vertices of one partite set are on a horizontal layer and the vertices of the other partite set are on a separate parallel horizontal layer [6]. Biplanar graphs have been independently characterized in [5, 8, 11]. Also, the problem of computing straight-line drawings of bipartite graphs with the vertices on two horizontal layers and minimum number of crossings has been well studied; see, e.g. [2, 3, 7, 9, 10] for some basic references on this and related topics.

This paper studies planar drawings of bipartite graphs where vertices are constrained to be on two parallel convex curves, which generalizes the case of horizontal layers. Let G be a bipartite graph, and let λ_e, λ_i be two parallel convex curves; we want to answer the question about whether G admits a planar straight line drawing such that the vertices of one partite set of G lie on λ_e and the vertices of the other partite set lie on λ_i.

Our interest in this question is in part motivated by the observation that the class of bipartite graphs that admit a planar straight line drawing on two horizontal lines is quite restricted and that one may hopefully enlarge this class by allowing some curvature on the two layers. Indeed, there is already some evidence in the literature that if the vertices in a drawing are not constrained to be collinear but instead can lie on curves, the family of representable graphs for specific drawing conventions can increase significantly; see, e.g. [4] for drawings of planar graphs with at most one bend per edge and vertices constrained to be on a given curve.

The problem addressed in this paper is also related to the study of *radial planarity testing* initiated by Bachmaier, Brandenburg and Forster [1]. In [1] the input is a k-partite graph G and k-concentric circles; the question is whether G has a crossing-free drawing where the vertices of the same partite set are points of the same radial level (circle) and the edges are simple Jordan curves in the outward direction. Here, we study radial planarity testing for bipartite graphs with the additional constraint that the edges are straight-line segments (indeed, two concentric circles are a special case of two parallel convex curves).

Our contribution is as follows. The family of bipartite graphs which admit a planar straight-line drawing with the vertices constrained to be on two parallel convex curves and with no two vertices of the same partite set on different curves is characterized. The characterization gives rise to a linear time testing algorithm. The proof of sufficiency uses a linear time (real RAM) drawing algorithm in the case of two concentric circles.

2 Preliminaries

A graph $G = (V, E)$ is *bipartite* if there exists a partition $V = V_0 \cup V_1$ of the vertices of G such that $E \subseteq V_0 \times V_1$. The two sets V_0 and V_1 are called *partite sets* of G. A bipartite graph with a given planar embedding is *maximal* if every internal face of G consists of four edges.

A simple curve λ in the Euclidean plane is a *closed* curve if it partitions the plane into two topologically connected regions; λ is an *open* curve otherwise. Curve λ is *convex* if any straight line intersects λ in at most two points. Note that a circle is a special case of closed convex curve.

Let p, q be two distinct points of λ. If λ is an open curve we say that p *precedes* q on λ if p is encountered before q when traversing λ in the clockwise direction. If λ is a closed curve, let p and q be two distinct points of λ such that the portion of λ traversed when going from p to q in the clockwise direction is shorter than the portion of λ traversed when going from q to p; we say that p *precedes* q and that q *follows* p on λ.

Two convex curves are *parallel* if every normal to one curve is also a normal to the other curve and the distance between the points where the normals intersect the two curves is a constant. In the rest of this paper we denote with λ_e, λ_i two parallel convex curves such that the curvature of λ_e is less than the curvature of λ_i; λ_e is the *external curve*, λ_i is the *internal curve* (in the special case of two concentric circles, λ_e is the circle with larger radius). Curves λ_e, λ_i are *paired* if there exist two points $p \in \lambda_i$ and $q \in \lambda_e$ such that the straight-line segment $\overline{pq}$ intersects λ_i twice. A straight-line segment with the property of $\overline{pq}$ is said to *cross* curve λ_i. Observe that two concentric circles are paired. Two curves will be called *non-paired* if they are parallel, convex, but are not paired. Figure 1 shows an example of two paired and two non-paired curves, respectively. In particular, if the curves are non-paired and ℓ is the straight line through a point p of λ_i, see Figure 1(b), two cases are possible: either ℓ intersects λ_e in a point q without crossing λ_i or ℓ crosses λ_i in a point x without intersecting λ_e.

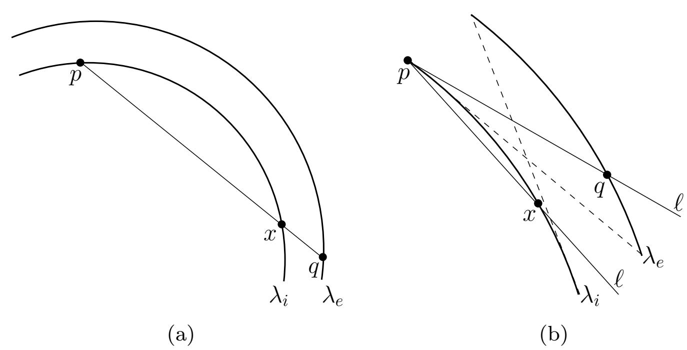


Figure 1: (a) Two paired curves. (b) Two non-paired curves.

Let λ_e, λ_i be two parallel convex curves. A bipartite graph G is *curve bipla-*

nar on λ_e, λ_i if it admits a *curve biplanar drawing*, i.e. a planar straight-line drawing such that all vertices of a partite set of G are represented as points on λ_e and the vertices of the other partite set are represented as points on λ_i. A graph is a *caterpillar* if deleting all vertices of degree one produces a (possibly empty) path. A *leaf* is a vertex with degree 1. A 2-*claw* is the graph consisting of one degree-3 vertex (called the *centre*) which is adjacent to three degree-2 vertices, each of which is adjacent to the centre and to one leaf. Figure 2 depicts an example of a caterpillar and a 2-claw. Note that a graph G is a forest of caterpillars if and only if G is acyclic and contains no 2-claw, see [5, 8, 11]. As the next theorem shows, if λ_e, λ_i are non-paired, the family of curve biplanar graphs coincides with the family of biplanar graphs characterized in [5, 8, 11].

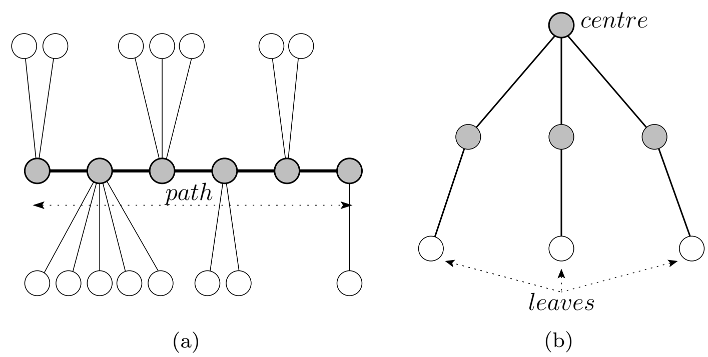


Figure 2: (a) A caterpillar. (b) A 2-claw.

Theorem 1 *A bipartite graph admits a curve biplanar drawing on two non-paired curves if and only if it is a forest of caterpillars.*

Proof: The proof is an easy adaptation of the arguments in [5, 8, 11]. Initially we prove that if a graph G admits a biplanar drawing on two non-paired curves, then it is a forest of caterpillars.

We prove first that G is acyclic. Since the curves are non-paired and the edges are drawn as straight-line segments, every edge is contained in the annulus between the two curves. Let $v_0, u_0, v_1, u_1, \dots, v_{h-1}, u_{h-1}$ be a path such that $v_j \in V_0$ and $u_j \in V_1$ $(0 \le j \le h-1)$. In order to avoid crossings in the drawing of this path, each vertex v_j must precede vertex v_{j+1} on one of the two curves, say λ_e, and each vertex u_j must precede vertex u_{j+1} $(j = 0, 1, \dots, h-2)$ on the other curve, i.e. λ_i. If one wants to close a cycle by adding edge (v_0, u_{h-1}), then this edge would cross all edges of the path, because these edges are completely contained in the annulus between the two curves and the two curves are non-paired. It follows that G is acyclic.

We prove now that G does not contain a 2-claw and hence it is a forest of caterpillars. Assume, by contradiction, that G contains a 2-claw. The centre v of the 2-claw is drawn on one curve, say λ_e, and it is adjacent to the three degree-2 vertices that are on λ_i. Let u_0, u_1 and u_2 be these degree-2 vertices and assume that u_1 follows u_0 and precedes u_2 on λ_i. The leaf v_1 adjacent to u_1 either precedes v on λ_e, in which case there is a crossing between (v, u_0) and (u_1, v_1), or it follows v on λ_e, in which case there is a crossing between (v, u_2) and (u_1, v_1). It follows that G cannot contain a 2-claw.

Let G be a forest of caterpillars and let λ_e, λ_i be two non-paired curves. We prove the sufficiency in the case that G is one caterpillar. If G is not connected, every connected component can be drawn independently. Let $\Pi = v_0, u_0, v_1, u_1, \ldots, v_{h-1}, u_{h-1}$ be the path obtained by removing all leafs from G. Draw each vertex v_j on one of the two curves, say λ_e, and each vertex u_j on the other curve, i.e. λ_i $(j = 0, \ldots, h-1)$, so that v_j precedes v_{j+1} on λ_e and u_j precedes u_{j+1} on λ_i $(j = 0, \ldots, h-2)$. For each vertex v_j, draw the leafs adjacent to v_j as points of λ_i which follow u_{j-1} and precede u_j. For each vertex u_j, draw the leafs adjacent to u_j as points of λ_e which follow v_j and precede v_{j+1}. □

Motivated by Theorem 1, we will investigate the family of bipartite graphs that admit a curve biplanar drawing on two paired curves. We first show how to draw a specific family of graphs, namely *bipartite fans*, and then present a complete characterization of curve biplanar graphs on two paired curves.

3 How to Draw a Bipartite Fan

Let G be a biconnected bipartite graph with a given planar embedding. G is a *bipartite fan* if it has a vertex u, called *apex*, that is shared by all its faces (including the external one). The edges incident on u are the *radial edges* of the fan. Let $u, v_0, v_1, \ldots, v_{n-2}$ be the vertices of a fan G in the counterclockwise order they have on the external face. Edges (u, v_0) and (u, v_{n-2}) are called *first edge* and *last edge* of the fan, respectively. Any three vertices $v_{2j}, v_{2j+1}, v_{2j+2}$ $(0 \leq j \leq \frac{n-4}{2})$ form a *fan triplet* of G. Notice that v_{2j+1} belongs to the same partite set as u. See Figure 3(a) for an illustration of a bipartite fan.

We show how to compute a curve biplanar drawing of a bipartite fan on two paired curves such that the drawing is contained in a suitable region of the plane called a *wedge* and defined as follows. Let λ_e, λ_i be two paired curves, let p, q, r be three points such that:

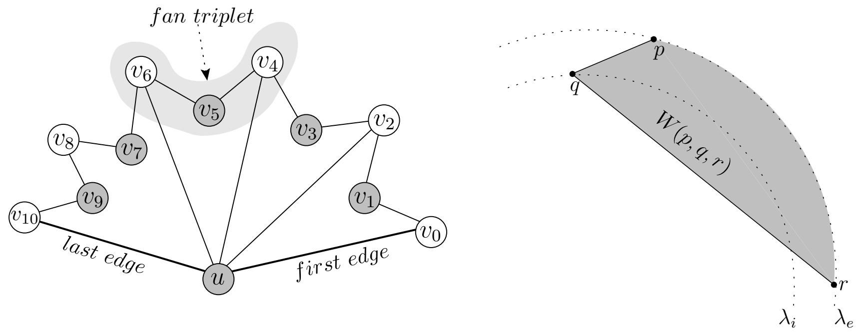
(i) $p, r \in \lambda_e$ and p precedes r on λ_e;

(ii) $q \in \lambda_i$;

(iii) segment $\overline{pq}$ does not cross curve λ_i;

(iv) segment $\overline{qr}$ crosses λ_i.

Let λ_{pr} be the portion of λ_e consisting of all points $x \in \lambda_e$ such that x follows p and precedes r. The closed bounded region delimited by $\overline{pq}$, $\overline{qr}$ and λ_{pr} is a wedge of λ_e, λ_i and is denoted as $W(p, q, r)$ (see Figure 3(b) for an example).



(a) (b)

Figure 3: (a) A bipartite fan. (b) A wedge $W(p, q, r)$.

Lemma 1 *Let G be a bipartite fan with n vertices and apex u. Let λ_e, λ_i be two paired curves and let $W(p, q, r)$ be a wedge of λ_e, λ_i. Fan G admits a curve biplanar drawing on λ_e, λ_i contained inside $W(p, q, r)$ such that:*

(i) the first and the last edge of G are represented by segments $\overline{qr}$ and $\overline{pq}$, respectively;

(ii) for every fan triplet $v_{2j}, v_{2j+1}, v_{2j+2}$ of G ($0 \le j \le \frac{n-4}{2}$), the three points representing the triplet define a wedge of λ_e, λ_i.

Moreover, if λ_e and λ_i are circles the drawing can be computed in $O(n)$ time.

Proof: We assume that G is maximal, i.e. that every internal face consists of four edges and four vertices; if not, we can split each internal face f having more than four edges by connecting u to all vertices of f that are not adjacent to u and do not belong to the same partite set of u (it is immediate to see that the resulting augmented graph is still a bipartite fan). Let $u, v_0, v_1, \ldots, v_{n-2}$ be the vertices of fan G in the counterclockwise order they have on its external face, (see Figure 3(a)), in what follows we denote as r_i points where vertices v_i ($0 \le i \le n-2$) are placed and we adopt the following notation for the faces of G: f_0 is the external face; f_{j+1} with $0 \le j \le \frac{n-4}{2}$ is the internal face containing the apex u and the fan triplet $v_{2j}, v_{2j+1}, v_{2j+2}$.

First, we prove that a bipartite fan can be drawn inside a wedge $W(p, q, r)$ in such a way that (see Figure 4 for an example):

- condition (ii) is satisfied;
- the apex u is drawn on the point q;
- the first edge of the fan is represented by the segment $\overline{qr}$;
- points p, q, r_{n-2} define a wedge $W(p, q, r_{n-2})$ entirely contained in $W(p, q, r)$.

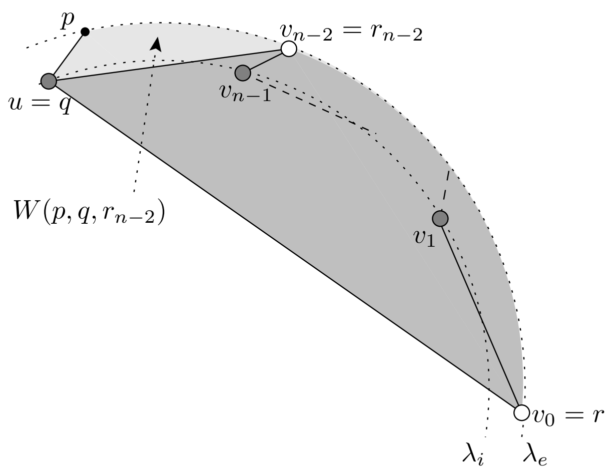


Figure 4: Wedge $W(p, q, r_{n-2})$ obtained after the application of the recursive drawing technique.

The proof is by induction on the number n_{fi} of the internal faces of G.

Base Case: $n_{fi} = 1$. G is a 4-cycle and it is sufficient to show how its internal face f_1 must be drawn. The apex u is drawn at point q and the vertex v_0 is drawn at point r_0 coincident with r, as depicted in Figure 5(a). Thus, the first edge of G is represented by the segment $\overline{qr_0}$. Let x_1 be the point where segment $\overline{qr_0}$ crosses λ_i. Since λ_i is convex, there exists a straight line through r_0 and tangent to λ_i at a point t_1 such that t_1 follows q and precedes x_1 on λ_i; since $\overline{pq}$ does not cross λ_i, then t_1 is inside wedge $W(p, q, r)$. Choose a point q_1 that follows q and precedes t_1 on λ_i and let ℓ_1 be the straight line through q and q_1. Draw vertex v_2 at point $r_2 = \ell_1 \cap \lambda_e$ (see Figure 5(a)). Let t'_1 be the point on λ_i that follows q_1 and where the tangent to λ_i through r_2 intersects λ_i; note that t'_1 may not exist. Draw vertex v_1 at any point r_1 of λ_i that follows q_1 and precedes t_1 and t'_1 when it exists.

Segment $\overline{r_1 r_0}$ crosses λ_i because r_1 precedes t_1 on λ_i and λ_i is convex. Let t''_1 be the point of λ_i that precedes q_1 and where the tangent to λ_i through r_2 intersects λ_i; t''_1 follows q and precedes r_1; since t'_1 follows r_1 the segment $\overline{r_2 r_1}$ does not cross λ_i.

Therefore points r_0, r_1, r_2, which represent the vertices of the fan triplet v_0, v_1, v_2, define a wedge $W(r_2, r_1, r_0)$. Also, segment $\overline{qr_2}$ crosses λ_i, and therefore points p, q, and r_2 define a wedge $W(p, q, r_2)$, as shown in Figure 5(b).

Inductive Case: $n_{fi} = k+1$. Suppose that the internal faces $f_1, f_2, \ldots, f_k$ have already been drawn, and let $v_{2k-2}, v_{2k-1}, v_{2k}$ be the fan triplet of f_k. By the inductive hypothesis vertex v_{2k} is drawn at point r_{2k}, the segment $\overline{qr_{2k}}$ crosses λ_i and points p, q, r_{2k} define a wedge $W(p, q, r_{2k})$ (see Figure 6(a) for an example).

In order to draw face f_{k+1}, we only need to draw vertices v_{2k+1}, v_{2k+2} be-

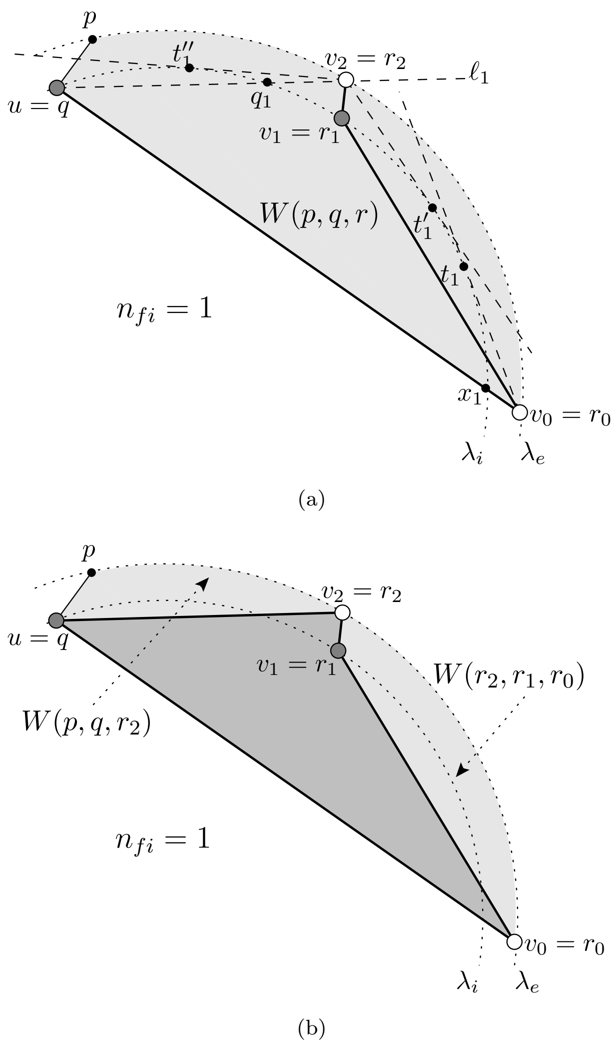
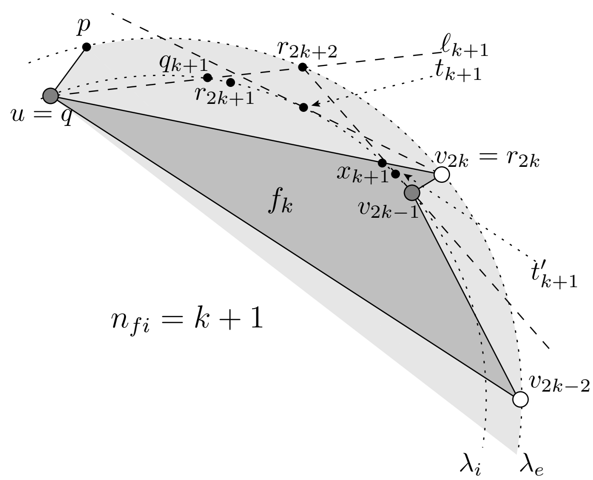
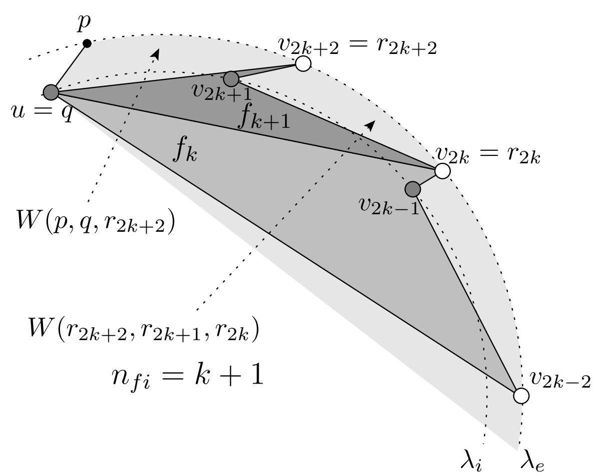


Figure 5: Illustration of the first step of the proof of Lemma 1.



(a)



(b)

Figure 6: Illustration of the inductive step of the proof of Lemma 1.

cause the apex u and vertex v_{2k} have already been drawn. It can be done by applying the same technique that we used to draw v_1 and v_2 inside wedge $W(p, q, r)$ in the base case of the inductive proof (see also Figure 6(b)).

To complete the proof, we need to show that the drawing of the fan can be modified in such a way that condition (i) be satisfied. Namely, although the first edge is represented by the segment $\overline{qr}$ as required, the last edge of the fan is not represented by the segment $\overline{pq}$. However, this can be accomplished by changing the drawing of the last internal face $f_{\frac{n}{2}-1}$ as follows (see Figure 7 for an illustration).

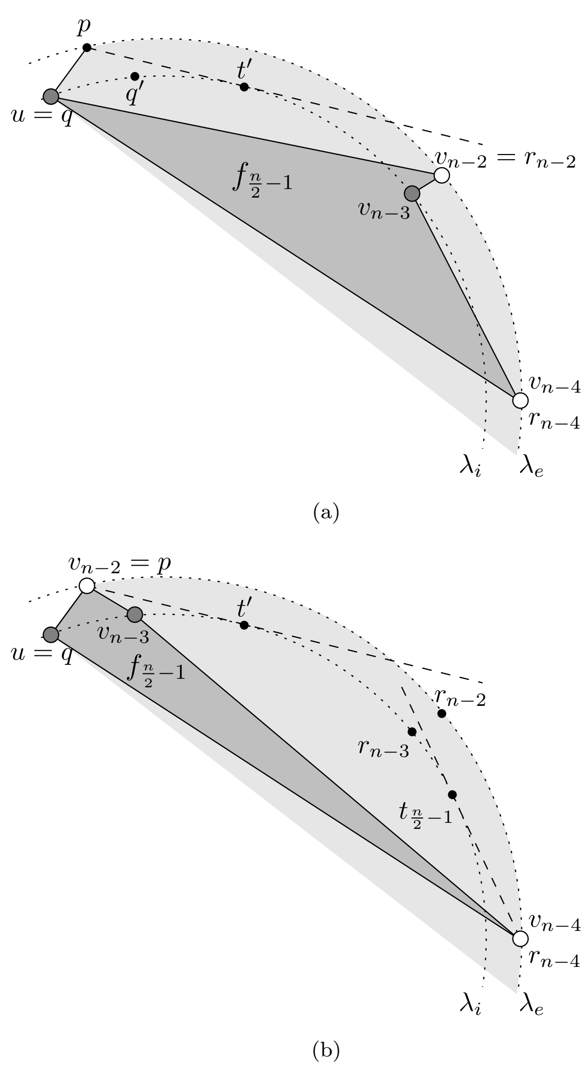


Figure 7: Redrawing technique of the last internal face.

Let t' be the point on λ_i that follows q and where the tangent to λ_i through p intersects λ_i. Move vertex v_{n-2} to point p and move vertex v_{n-3} to any point q' that follows q and precedes both t' and r_{n-3}. This choice ensures that segment $\overline{pq'}$ does not cross λ_i. Instead, since q' precedes r_{n-3} it precedes $t_{\frac{n}{2}-1}$, thus segment $\overline{q'r_{n-4}}$ crosses λ_i. Hence, condition (i) is satisfied and condition (ii) is preserved.

Note that, moving vertices v_{n-2}, v_{n-3} as explained above does not introduce crossings with other elements of the drawing, because vertices v_{n-2} and v_{n-3} have been moved inside the wedge $W(p, q, r_{n-4})$, while all the other vertices (and edges) are drawn in the plane region delimited by the segment $\overline{qr_{n-4}}$, the segment $\overline{qr}$ and the portion of λ_e from point r_{n-4} to r.

Concerning the time complexity, it is immediate to see that the augmentation that makes G maximal can be performed by visiting each vertex at most once. Also, computing the intersection and the tangent to the circle requires $O(1)$ time in the real RAM model of computation (note that this may not be the case for other types of curves).

Finally, the drawing algorithm computes the coordinates for the four vertices of each face of a fan by solving at most four quadratic equations. □

4 Curve Biplanar Graphs

We start with a sufficient condition whose proof uses the following definition. Let $G = (V, E)$ be a connected graph. A subset of vertices $S \subset V$ is a *cut-set* if the removal of S disconnects G. Let $G_0, \ldots, G_{k-1}$ be the connected components of $G - S$ (possibly isolated vertices). The *S-components* of G are the subgraphs of G induced by sets $V(G_j) \cup S$ $(0 \leq j \leq k-1)$, where $V(G_j)$ denotes the set of vertices of G.

Lemma 2 *Let G be a biconnected bipartite graph with a given planar embedding such that all vertices in one partite set belong to the external face. Then G is curve biplanar on two paired curves.*

Proof: We describe now how to compute an embedding preserving curve biplanar drawing of G. To this aim, we decompose it into subgraphs that are bipartite fans and draw each fan by using the technique described in Lemma 1.

Let V_0 and V_1 be the two partite sets of G and assume that all vertices of V_0 belong to the external face of G in the given embedding. Since G is bipartite, there exists a vertex $u \in V_1$ such that u belongs to the external face of G. Let F_u be the subgraph of G induced by all vertices that share an internal face with u (F_u exists because G is biconnected). Note that F_u is a bipartite fan, which we call *the fan of u*. Let λ_e and λ_i be two paired curves, let $W(p, q, r)$ be an arbitrarily chosen wedge of λ_e, λ_i (such a wedge always exists because λ_e and λ_i are paired). By Lemma 1, F_u can be drawn inside $W(p, q, r)$ so that its first and last edge are represented by segments $\overline{qr}$ and $\overline{pq}$, respectively.

Let $u, v_0, v_1, \ldots, v_{n-2}$ be the vertices of F_u in the counterclockwise order they have on the external face of F_u. Since $u \in V_1$, vertices v_{2j} $(j = 0, 1, \ldots, \frac{n-2}{2})$

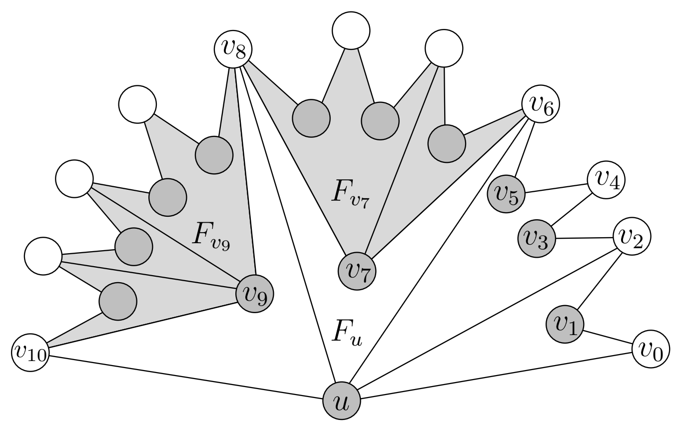


Figure 8: Illustration of the decomposition technique described in Lemma 2. White vertices belong all to the external face. Fan F_u has 12 vertices: $u, v_0, v_1, \ldots, v_{10}$; the fan triplets v_6, v_7, v_8 and v_8, v_9, v_{10} are cut sets for the whole graph because vertices v_7 and v_9 do not belong to the external face.

belong to V_0 and are on the external face of G. This implies that every fan triplet $\tau = \{v_{2j}, v_{2j+1}, v_{2j+2}\}$ is a cut-set for G unless v_{2j+1} is on the external face. Indeed, since G is biconnected, the boundary of its external face is a cycle C, and if $v_{2j+1} \in \tau$ is not on the external face, then the vertices of τ induce a path that splits C in two paths C' and C''; since no edge of G can connect a vertex of C' to a vertex of C'' (otherwise G would not be planar), then τ is a cut-set for G (see Figure 8 for an illustration).

The τ-component of G that does not contain u is a planar bipartite graph that satisfies the condition expressed by the statement. Also, by Lemma 1, points $r_{2j}, r_{2j+1}, r_{2j+2}$ representing the vertices of τ in the drawing of F_u define a wedge $W(r_{2j}, r_{2j+1}, r_{2j+2})$. Therefore, the τ-component of G that does not contain u can be recursively drawn inside $W(r_{2j}, r_{2j+1}, r_{2j+2})$. □

The previous sufficient condition can be extended also to non-biconnected graphs by using the augmentation technique described in the following lemma.

Lemma 3 *Let G be a non-biconnected bipartite graph and let V_0 be one of the partite sets of G. If G has a planar embedding with all vertices of V_0 on the external face, then G can be augmented with dummy vertices and edges so that the resulting graph G' is biconnected, bipartite, has a partite set V_0' such that $V_0 \subseteq V_0'$, and has a planar embedding with all vertices of V_0' on the external face.*

Proof: Consider the embedding of G such that all vertices of V_0 belong to the external face and let v be a cut-vertex of G. If $v \in V_0$, then it is on the external face. If $v \in V_1$, all its adjacent vertices are on the external face. It follows that v belongs to the external face also in this case.

Suppose that v belongs to h biconnected components. In the cyclic ordering of the edges incident on v there are h consecutive pairs that belong to different biconnected components. Consider one such pair (v, u) and (v, w). We augment G by adding a path u, x, w for $h - 1$ of these pairs, where x is a dummy vertex. Since G is bipartite, then v belongs to one partite set and u, w belong to the other one. To maintain the bipartiteness x is assigned to the same partite set containing v. All dummy vertices are added on the external face and each vertex that was on the external face before adding the paths remains on the external face. By repeating this augmentation until there are no more cut-vertices, we obtain a biconnected bipartite graph G' that satisfies the statement. □

Theorem 2 *A bipartite graph G is curve biplanar on two paired curves if and only if it admits a planar embedding such that all vertices in one partite set belong to the external face. Also, if G is curve biplanar on two concentric circles, a curve biplanar drawing of G can be computed in $O(n)$ time in the real RAM model of computation, where n is the number of vertices of G.*

Proof: We first prove the sufficiency. Let V_0 be a partite set of G such that all vertices of V_0 belong to the external face of a planar embedding of G. If G is biconnected, the sufficiency follows from Lemma 2. Otherwise, as shown by Lemma 3, G can be augmented by adding dummy vertices and edges such that the augmented graph G' is biconnected and bipartite, one of its partite sets is V'_0 with $V_0 \subseteq V'_0$, and has a planar embedding with the vertices of V'_0 on the external face. It follows that G' has a biplanar drawing on two paired curves by Lemma 2 and hence G is curve biplanar on two paired curves.

We now prove the necessity. Let Γ be a curve biplanar drawing of a graph G on two paired curves λ_e and λ_i. All vertices drawn as points of λ_e are on the external face of Γ because the curves are convex and the drawing is straight-line. Since all vertices on the same curve are in the same partite set, G admits a planar embedding such that all vertices in one partite set belong to the external face.

Time complexity. The augmentation technique can be performed in $O(n)$ time by visiting the vertices on the external face. The fan F_u of u can be computed in time proportional to the number n_u of vertices in F_u by visiting all faces containing u. As stated in Lemma 3, drawing each fan F_u requires $O(n_u)$ time when the curves are two concentric circles and the real RAM model of computation is adopted. Moreover, since $\sum_{F_u} O(n_u) = O(n)$ it follows that the overall time complexity is $O(n)$. □

Theorem 3 *Let G be a bipartite planar graph with n vertices. The curve biplanarity of G on two paired curves can be tested in $O(n)$ time.*

Proof: A curve biplanarity test can be performed by executing at most two planarity tests with the additional constraint that all vertices in one of the partite sets of G belong to the external face. □

We conclude this section by describing how the result of Theorem 2 is related to radial planarity. A k-partite graph $G = (V_0, \ldots, V_{k-1}, E)$ is *radial k-level* *planar* if it admits a planar drawing on k concentric circles $C_0, C_1, \dots, C_{k-1}$, with the vertices of partition V_i drawn on circle C_i ($0 \le i \le k-1$) and the edges drawn as strictly monotone curves from inner to outer circles. Radial planarity has been studied by Bachmaier et al. [1] who present a linear time algorithm for radial planarity testing and embedding. Based on the fact that two concentric circles are two paired curves and that straight-line segments with the end-vertices on two different circles are a special case of strictly monotone curves from inner to outer circles, the following results immediately follows from Theorems 2 and 3.

Corollary 1 *A bipartite graph G is radial 2-level planar with straight-line edges if and only if it admits an embedding such that all vertices in one partite set belong to the external face. Also, there exists an $O(n)$-time algorithm that tests whether a bipartite graph G is radial 2-level planar with straight-line edges.*

5 Open Problems

In this paper we have studied the curve biplanar drawability problem of bipartite graphs. It may be considered a generalization of the well known biplanar drawability problem, when parallel layers are allowed to be curves.

We have shown that, if the curves are non-paired, the families of graph that admits a curve biplanar drawing is the forest of caterpillars, e.g. the same family that characterizes the biplanar graphs [5, 8, 11]. Instead, if the curves are paired, we have proved that a bipartite graph is curve biplanar if and only if it admits a planar embedding with all vertices of one partite set on the external face. We have provided also a drawing algorithm that runs in linear time if the curves are two concentric circles and the real RAM model of computation is adopted. Moreover, we have proven that the curve biplanarity can be tested in linear time.

We conclude by listing some open problems that in our opinion should be investigated.

- Extend the study to k-partite graphs and k parallel curves with $k > 2$. In particular, it would be interesting to study radial planarity testing with straight-line edges and more than two concentric circles.
- Study the complexity of the following problem: Let G be a planar bipartite graph and let c be a positive integer. Does G have a curve biplanar subgraph (not necessarily induced) with at least c edges?
- Study the complexity of the edge crossing minimization problem for straight-line drawings of bipartite graphs on two parallel convex curves.
- Investigate what happens when non-convex curves are considered: does the class of curve biplanar (bipartite) graphs change? If yes, how does it depend on the profile of the curves?

- The presented algorithm may require very large resolution, expressed as the ratio between the longest edge and the shortest edge. What is a lower bound for the resolution? How does it depend on the distance between the two curves?

Acknowledgments

The authors wish to thank Walter Didimo for useful discussions and the anonymous referees for their suggestions.

References

[1] C. Bachmaier, F. J. Brandenburg, and M. Forster. Radial level planarity testing and embedding in linear time. *J. Graph Algorithms Appl.*, 9(1):53–97, 2005.

[2] T. C. Biedl. Drawing planar partitions I: LL-drawings and LH-drawings. In *SCG '98: Proceedings of the fourteenth annual symposium on Computational geometry*, pages 287–296, New York, NY, USA, 1998. ACM Press.

[3] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing*. Prentice Hall, Upper Saddle River, NJ, 1999.

[4] E. Di Giacomo, W. Didimo, G. Liotta, and S. K. Wismath. Curve-constrained drawings of planar graphs. *Comput. Geom.: Theory and Appl.*, 30:1–23, 2005.

[5] P. Eades, B. D. McKay, and N. C. Wormald. On an edge crossing problem. In *Proc 9th Austr. Comp. Sci. conf.*, pages 327–334. Australian National University, 1986.

[6] P. Eades and S. H. Whitesides. Drawing graphs in two layers. *Theoretical Computer Science*, 131(2):361–374, 1994.

[7] U. Fößmeier and M. Kaufmann. Nice drawings for planar bipartite graphs. In G. C. Bongiovanni, D. P. Bovet, and G. Di Battista, editors, *Algorithms and Complexity, Third Italian Conference, CIAC '97*, volume 1203 of *Lecture Notes in Computer Science*, pages 122–134. Springer-Verlag, 1997.

[8] F. Harary and A. Schwenk. A new crossing number for bipartite graphs. *Utilitas Mathematica*, 1:203–209, 1972.

[9] M. Kaufmann and D. Wagner, editors. *Drawing Graphs*, volume 2025 of *Lecture Notes in Computer Science*. Springer-Verlag, 2001.

[10] M. Suderman. Proper and planar drawings of graphs on three layers. In *Graph Drawing (Proc. GD'05)*, volume 3843 of *Lecture Notes in Computer Science*, pages 434–445. Springer, 2005.

[11] N. Tomii, Y. Kambayashi, and S. Yajima. On planarization algorithms of 2-level graphs. Technical Report EC77-38, Inst. of Elect. and Comm. Eng. Japan, 1977.