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Building Blocks of Upward Planar Digraphs

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Abstract

The *upward planarity testing* problem consists of testing if a digraph admits a drawing Γ such that all edges in Γ are monotonically increasing in the vertical direction and no edges in Γ cross. In this paper we reduce the problem of testing a digraph for upward planarity to the problem of testing if its blocks admit upward planar drawings with certain properties. We also show how to test if a block of a digraph admits an upward planar drawing with the aforementioned properties.

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1 Introduction

A digraph is said to be *upward planar* if it admits a drawing that is both upward (i.e. all edges are monotonically increasing in the vertical direction) and planar (i.e. no edges cross). Fig. 1(a) shows a planar, but not upward, drawing of a digraph, Fig. 1(b) shows an upward, but not planar, drawing of a digraph, and Fig. 1(c) is an example of an upward planar drawing of a digraph. The

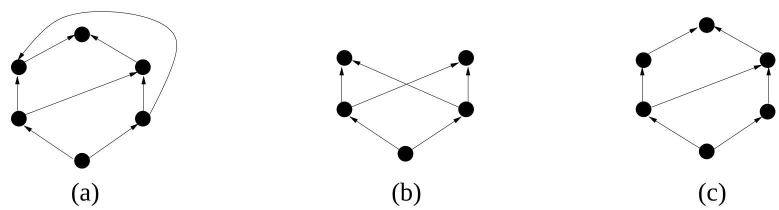


Figure 1: A planar drawing (a); an upward drawing (b); an upward planar drawing (c).

upward planarity of digraphs is a classical topic in the area of *graph drawing*. The field of graph drawing is concerned with the construction of geometric representations of graphs and is surveyed by Di Battista, Eades, Tamassia and Tollis [6]. Graph drawing algorithms have uses in many areas including in the design of layouts of printed circuit boards (PCB) and very large scale integration (VLSI) and for visualising information. Hierarchical network structures such as isa-hierarchies, PERT networks, family trees, call graphs and organisation charts are naturally modeled by acyclic digraphs. For the purposes of visualisation, acyclic digraphs are usually represented by upward drawings so as to reveal the intrinsic (partial) order of their nodes. Planar drawings are also desirable because it has been empirically shown that the presence of edge-crossings in a drawing of a graph make it more difficult for a person to absorb the information being modeled by the graph [21]. Thus, drawings which combine both these properties (i.e. upward planar drawings), if possible, are an effective means of visualising acyclic digraphs.

Whitney [24] has shown that a graph is planar if and only if its biconnected components are planar. Although similar to the upward planarity testing problem, the planarity testing problem is in many ways an easier problem. For instance planarity testing can be performed in linear time [16] while upward planarity testing is an *NP*-complete problem [11]. Thus, it is not surprising that the statement: "a digraph is upward planar if and only if its biconnected components are upward planar", is not in general true. For example, consider the digraph H of Fig. 2: H is not upward planar even though its biconnected components are upward planar. In this paper we convert the problem of testing if a digraph is upward planar to the problem of testing if its biconnected components have upward planar drawings with certain properties. We also describe how to test if a biconnected digraph admits upward planar drawings with these properties. This decomposition strategy is a key component of a previously described parameterised upward planarity testing algorithm [15].

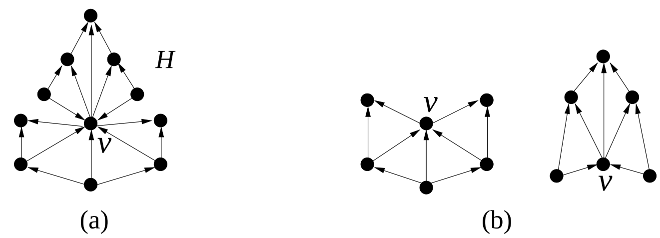


Figure 2: H is not upward planar (a); but its blocks are upward planar (b).

This paper has the following structure. We begin by giving a synopsis of previous work on the upward planarity testing problem in Section 2 followed by preliminary definitions in Section 3. In Section 4 we present results (Theorems 4–7) which show that G admits certain classes of upward planar drawing if and only if the components of G with respect to any vertex v of G have certain properties. A *component of G with respect to v* is formed from a connected component G' of $G \setminus v$ by adding to G' the vertex v and all edges between v and G'. Using the results presented in Section 4 we describe a recursive upward planarity testing algorithm in Section 5. Following this we present two worked examples of this algorithm in Section 6 and conclude the paper in Section 7 by discussing our results and suggesting open problems.
Note: The results presented in Section 4 are a more extensive treatment of a paper presented by the authors at the 12th International Symposium on Graph Drawing [13]. Theorem 4 is one of the main results of Section 4 and is concerned with the conditions under which the union of an arbitrary number of upward planar digraphs sharing exactly one vertex and no edges is upward planar. Following completion of that research work we learned that Chan [5] had independently discovered a result similar to Theorem 4.

2 Review of Upward Planarity Testing Literature

An *st*-digraph is defined as an acyclic digraph with exactly one source, exactly one sink, and with an edge from the source to the sink. Kelly [18] and Di Battista and Tamassia [9] have both shown that a digraph is upward planar if and only if it is a spanning subgraph of a planar *st*-digraph. Di Battista and Tamassia [9] give an $\mathcal{O}(n' \log n')$-time algorithm for constructing an upward planar straight-line drawing of a planar st-digraph with n' nodes. Di Battista, Tamassia and Tollis [10] have revealed a family $\mathcal{G}_n$ of upward planar digraphs such that $\mathcal{G}_n$ has $2n + 2$ nodes and any upward planar straight-line drawing of $\mathcal{G}_n$ has $\Omega(2^n)$

area. Garg and Tamassia [11] have shown that upward planarity testing is *NP*-complete for general digraphs. Consequently, research efforts have focused on developing efficient algorithms for special classes of digraphs and parameterised algorithms.

Di Battista, Liu, and Rival [8] have shown that a bipartite digraph is upward planar if and only if it is planar. Thus a bipartite digraph can be tested for upward planarity in linear time using any linear-time planarity testing algorithm (e.g. [16]). Papakostas [19] has developed a quadratic-time algorithm that tests if an outerplanar digraph is upward planar. Two efficient algorithms are known for testing the upward planarity of an n-vertex single-source digraph. Hutton and Lubiw [17] give an algorithm with $\mathcal{O}(n^2)$ running time that utilises Thomassen's [23] characterisation of upward planar embedded single source digraphs. Bertolazzi, Di Battista, Mannino and Tamassia [3] have improved upon this algorithm by developing a linear-time upward planarity testing algorithm for single-source digraphs. Bertolazzi, Di Battista, Liotta and Mannino [2] have presented a characterisation of upward planar embedded digraphs that yields a quadratic-time algorithm for testing the upward planarity of embedded digraphs. Bertolazzi, Di Battista and Didimo [1] have developed a branch and bound algorithm for testing the upward planarity of biconnected digraphs. Given a digraph $G = (V, E)$ with n vertices, c cutvertices and t triconnected components, Chan [5] has given a fixed parameter tractable algorithm for testing if G is upward planar that runs in $\mathcal{O}(t! \cdot 8^t \cdot n^3 + 2^{3 \cdot 2^c} \cdot t^{3 \cdot 2^c} \cdot t! \cdot 8^t \cdot n)$ time. Healy and Lynch [15] improved on Chan's algorithm with an $\mathcal{O}(2^t \cdot t! \cdot n^2)$-time upward planarity testing algorithm and have also presented an algorithm that can test the upward planarity of $G = (V, E)$ in $\mathcal{O}(n^2 + k^4(2k + 1)!)$ time, where $k = |E| - |V|$. Di Battista and Liotta [7] have developed a linear-time verification algorithm, that is optimal both in terms of efficiency and in terms of degree, for checking the upward planarity of drawings of a very significant family of digraphs. Members of this family include rooted trees, planar st-digraphs and upward planar single-source digraphs and most algorithms for constructing upward planar drawings receive such digraphs as input.

3 Preliminaries

In this section we introduce terms and definitions which are used in later sections. We assume a familiarity with basic graph theoretical notions such as graph, digraph, subgraph, edge, and vertex and refer readers unacquainted with such terms to one of the many introductory texts on the subject (e.g. [12]).

3.1 Standard Terminology

We believe that the terminology presented in this subsection will be familiar to most readers but include it as it is necessary for the understanding of subsequent sections. Let G be a digraph. We denote the *vertex set* of G by $V(G)$ and the *edge set* of G by $E(G)$. The *union* of digraphs G and F, denoted $G \cup F$, is the digraph with vertex set $V(G \cup F) = V(G) \cup V(F)$ and edge set $E(G \cup F) = E(G) \cup E(F)$. Let $e = \langle u, v \rangle$ be an edge in G. We refer to the first vertex in e as the *tail* of e, denoted $tail(e)$, and to the last vertex in e as the *head* of e, denoted $head(e)$. A vertex of degree one is referred to as a *leaf node*. We shall use the acronym *wrt* to represent *with respect to*. Given a real number y we use $abs(y)$ to represent the absolute value of y.

A *walk* w_1 in G is an alternating sequence $w_1 = \langle v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n \rangle$ of vertices and edges, such that e_i equals either $\langle v_i, v_{i-1} \rangle$ or $\langle v_{i-1}, v_i \rangle$ for $i = 1, \ldots, n$. A *diwalk* w_2 in G is an alternating sequence $w = \langle v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n \rangle$ of vertices and edges, such that $e_i = \langle v_{i-1}, v_i \rangle$ for $i = 1, \ldots, n$. A *closed walk* in G is a walk in which the first vertex and the last vertex are identical. A *path* p_1 in G is a walk in which no vertex is repeated, except possibly the first and last vertex. A *dipath* p_2 in G is a diwalk in which no vertex is repeated, except possibly the first and last vertex. A *cycle* is a path in which the first vertex and last vertex are identical. A *dicycle* is a dipath in which the first vertex and last vertex are identical. A digraph is said to be *acyclic* if it contains no dicycle. A *cutset* of G is defined as a minimal set of vertices whose removal increases the number of connected components of G. A cutset of size one is called a *cutvertex*. A *block* of G is a maximal connected subgraph B of G such that no vertex of B is a cutvertex of B.

A *planar drawing* of G is a drawing of G in which no two distinct edges intersect. A digraph is said to be *planar* if it admits a planar drawing. An *upward drawing* of G is a drawing of G in which all edges are monotonically increasing in the vertical direction. An *upward planar drawing* of G is a drawing of G which is both planar and upward. A digraph is said to be *upward planar* if it admits an upward planar drawing. A *straight-line drawing* of G is a drawing of G in which each edge is drawn as a straight line segment. An *upward planar straight-line (UPSL) drawing* of G is a drawing of G which is both upward planar and straight-line.

An *embedded digraph* G_ϕ is an equivalence class of planar drawings of digraph G with the same clockwise orderings, ϕ, of the edges incident upon each node. Such a choice ϕ for a clockwise ordering of the edges incident on each node is called an *embedding* of G. Observe that two drawings of G with the same embedding have the same set of faces, though they may have different external faces. A *strongly embedded digraph* G_φ is an equivalence class of planar drawings of G with the same clockwise orderings of the edges incident upon each node and the same external face. Such a choice, φ, for a clockwise ordering of the edges incident on each node and of an external face is called a *strong embedding* of G. We denote the set of faces in G_φ by $faces(G_\varphi)$. If v is a vertex in G_φ we use $\varphi(v)$ to denote the clockwise ordering of the edges incident on v in G_φ. If e is an edge incident on v we use $\varphi^e(v)$ to denote the linear sequence of edges with first edge e that is consistent with $\varphi(v)$. A strongly embedded digraph is said to be upward planar if it contains an upward planar drawing. A node v of an embedded or strongly embedded digraph is *bimodal* if the outgoing (or incoming) edges incident on v appear consecutively around v. If all the nodes of an embedded (resp., strongly embedded) digraph are bimodal then it is said to be *bimodal* and such an embedding (resp., strong embedding) is said to be a *bimodal embedding* (resp., *bimodal strong embedding*) of G. A digraph is said to be *bimodal planar* if it allows a bimodal embedding.

A *bipartite flow network* $\mathcal{N}$ is a bipartite digraph in which

(i) each source (sink) has a nonnegative integer called its supply (demand) associated with it;

(ii) each edge $\langle u, v\rangle$ has a positive integer c_{uv} called its capacity associated with it.

A *flow* fl in $\mathcal{N}$ is an assignment from the set of nonnegative integers to the edges of $\mathcal{N}$ that satisfies the following two properties:

(i) for each edge $\langle u, v\rangle$ of $\mathcal{N}$, $fl\,(\langle u, v\rangle) \leq c_{uv}$;

(ii) for each source s (sink t) of $\mathcal{N}$, the sum of the values assigned to the edges incident on s (t) by fl is not more than the supply (demand) of s (t).

The *value* of a flow fl in $\mathcal{N}$ is equal to the sum of the values assigned to the edges of $\mathcal{N}$ by fl. Let P be a path in $\mathcal{N}$. An edge e in P is said to be a *forward edge* if *tail*(e) appears before *head*(e) in P and a *backward edge* otherwise. The path P is said to be a *flow augmenting path* if it starts at a source, ends at a sink, each forward edge in P has a flow which is less than its capacity and each backward edge has a positive flow.

3.2 Specialised and Novel Terminology

In this subsection we present more specialised and novel terminology.

Angles. The *angles* of a strongly embedded digraph G_φ are ordered triples $\langle a, v, b\rangle$, where a and b are edges and v is a node incident with both a and b, such that either a directly precedes b in $\varphi(v)$ or v is a node of degree 1. An angle $\langle a, v, b\rangle$ of G_φ is said to be *centred at* v and v is said to be the *centre of* $\langle a, v, b\rangle$. An angle $\langle a, v, b\rangle$ is said to be an S*-angle* (resp., T*-angle*) if both a and b *leave* (resp., *enter*) v and an I*-angle* if one of the edges a, b leaves v and the other enters v. The angles of G_φ are mapped to geometric angles in an UPSL drawing Γ of G_φ. Let $\langle a, v, b\rangle$ be an angle of Γ. If $a \neq b$ the size of the corresponding geometric angle of $\langle a, v, b\rangle$ in Γ equals the number of radians one has to rotate a in the clockwise direction around v in order to reach b. If $a = b$ the size of the corresponding geometric angle of $\langle a, v, b\rangle$ is 2π. An angle of Γ is said to be *large* (resp., *small*) if its corresponding geometric angle is greater (resp., smaller) than π. If v is a node we use $L_a(v)$ to represent the number of (*large* S*-angles* + *large* T*-angles*) centred at v. We are interested in five types of angles, each of which we represent by a two letter code. We denote large S-angles and T-angles by `LS` and `LT` respectively, small S-angles and T-angles by `SS` and `ST` respectively, and I-angles by `II`.

Facial Boundary. Let f be a face of G_ϕ. We use the term *facial boundary* of f to refer to the closed walk w defined by traversing the boundary of f such that w contains a subsequence $\langle x, y, z\rangle$, where x and z are edges and y is a node if and only if $\langle x, y, z\rangle$ is an angle in G_ϕ. A face f is said to *contain* an edge, angle, or node w if w is a subsequence of the facial boundary of f and w is said to be *contained by* f. The ideas introduced in this paragraph are illustrated in Fig. 3; the *facial boundary* of f is $\langle v_2, e_1, v_1, e_8, v_8, e_7, v_6, e_6, v_7, e_6, v_6, e_5, v_5, e_4, v_3, e_3, v_4, e_3, v_3, e_2\rangle$, the *S-angles* f contains are $\langle e_6, v_7, e_6\rangle$, $\langle e_4, v_3, e_3\rangle$ and $\langle e_2, v_2, e_1\rangle$, the *T-angles* f contains are $\langle e_7, v_6, e_6\rangle$, $\langle e_6, v_6, e_5\rangle$ and $\langle e_3, v_4, e_3\rangle$, and the *I-angles* f contains are $\langle e_5, v_5, e_4\rangle$, $\langle e_3, v_3, e_2\rangle$, $\langle e_1, v_1, e_8\rangle$, and $\langle e_8, v_8, e_7\rangle$. We use $S_{angs}(f)$ (resp., $L_a(f)$) to represent the number of S-angles (resp., *(large S-angles + large T-angles)*) contained by f.

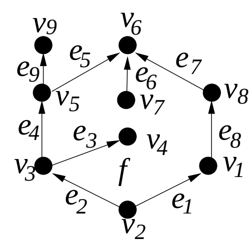


Figure 3: Facial boundaries of G_ϕ.

Cut Components. Let G be a connected digraph with a vertex v. A *component of G wrt v* is formed from a connected component H of $G \setminus v$ by adding to H the vertex v and all edges between v and H. Let C be a component of G wrt v. We use $G \setminus C$ to denote the digraph derived from G by deleting all the nodes in $C \setminus v$ from G. C is said to be a *source-component*, *sink-component*, or *internal-component* of G wrt v if v is a source node, sink node or internal node respectively in C. Fig. 4 shows the components of a digraph G wrt the vertex v. We use $S(v)$, $T(v)$, and $I(v)$ to refer to the subdigraph of G consisting of the union of all source-components, sink-components, and internal-components of G wrt v, respectively.

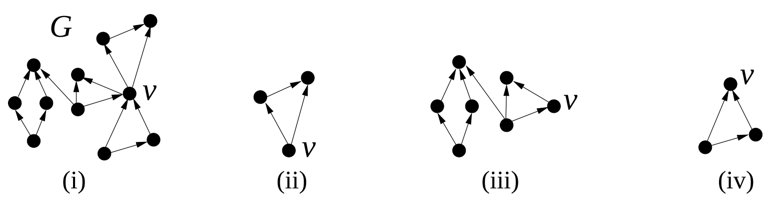


Figure 4: A digraph G with a cutvertex v (i); a source-component of G wrt v (ii); an internal-component of G wrt v (iii); a sink-component of G wrt v (iv).

Adding Two Strongly Embedded Digraphs. Definition 1 attempts to formalise the intuitive operation of adding two strongly embedded digraphs by identifying a vertex from one with a vertex from the other (see Fig. 5).

Definition 1 *Let A_α and B_β be strongly embedded digraphs such that $V(A) \cap V(B) = \{u\}$, the external face h_a of A_α contains the angle $\langle a_1, u, a_2\rangle$ and the face f_b of B_β contains the angle $\langle b_1, u, b_2\rangle$. The result of adding A_α and B_β by inserting $\langle a_1, u, a_2\rangle$ within $\langle b_1, u, b_2\rangle$ is the strongly embedded digraph G_φ, defined as follows:*

1. $G = A \cup B$,

2.

$$\varphi(v) = \begin{cases} \alpha(v) & \text{for all } v \in V(A) \setminus u, \\ \beta(v) & \text{for all } v \in V(B) \setminus u, \\ \langle \alpha^{a_2}(v), \beta^{b_2}(v) \rangle & v = u. \end{cases}$$

3. The external face of G_φ *is the face whose facial boundary contains the facial boundary of the external face of* B_β *as a (not necessarily proper) subsequence.*

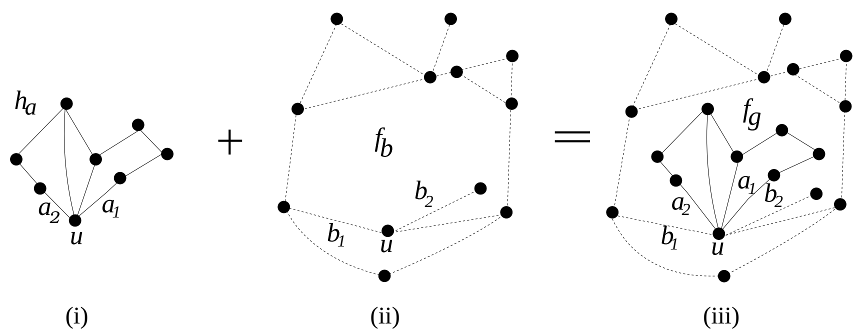


Figure 5: A_α (i); B_β (ii); G_φ (iii).

The facial boundary of f_g is the concatenation of the facial boundaries of h_a and f_b. That is, the facial boundary of f_g equals $\langle u, a_2, ..., a_1, u, b_2, ..., b_1 \rangle$, where the list of nodes and edges from a_2 to a_1 concatenated with u is the facial boundary of h_a and the list of nodes and edges from b_2 to b_1 concatenated with u is the facial boundary of f_b. Property 1 follows. We refer to the faces h_a, f_b, and f_g as the *merge-faces* of A_α, B_β, and G_φ respectively.

Property 1 *The face* f_g *contains all the angles contained by* h_a *except* $\langle a_1, u, a_2 \rangle$, *all the angles contained by* f_b *except* $\langle b_1, u, b_2 \rangle$, *and the "new" angles* $\langle b_1, u, a_2 \rangle$ *and* $\langle a_1, u, b_2 \rangle$.

There is an obvious one-to-one correspondence ρ between the set of faces $faces(A_\alpha) \cup faces(B_\beta) \setminus \{f_b, h_a\}$ and the set of faces $faces(G_\varphi) \setminus f_g$ such that if f is a face in $faces(A_\alpha) \cup faces(B_\beta) \setminus \{f_b, h_a\}$ then $\rho(f)$ is the unique face in $faces(G_\varphi) \setminus f_g$ whose facial boundary is identical with the facial boundary of f. Also if $\rho(f) = f'$ then we say $\rho^{-1}(f') = f$.

3.3 Known Properties of Upward Planar Digraphs

We now present previously published properties of upward planar digraphs that are used in the proofs of many of our results. Theorem 1 is independently due to Di Battista and Tamassia [9], as well as Kelly [18].

Theorem 1 (Di Battista and Tamassia; Kelly) *For any digraph G the following statements are equivalent.*

1. *G is upward planar.*
2. *G admits an UPSL drawing.*
3. *G is the spanning subgraph of a planar st-digraph.*

The remainder of this subsection is concerned with Bertolazzi et al.'s [2] work on upward planar embedded digraphs. Lemma 1 gives a formula representing the number of large angles that a face of an UPSL drawing contains as well as showing that the number of large angles with centre v depends only on whether v is an internal vertex or not.

Lemma 1 (Bertolazzi et al. [2]) *The following consistency properties hold for any UPSL drawing Γ of a digraph G:*

$$L_a(f) = \begin{cases} S_{angs}(f) + 1 & \text{if } f \text{ is the external face of } \Gamma, \\ S_{angs}(f) - 1 & \text{if } f \text{ is an internal face of } \Gamma. \end{cases}$$

$$L_a(v) = \begin{cases} 0 & \text{if } v \text{ is an internal node,} \\ 1 & \text{if } v \text{ is a source node or a sink node.} \end{cases}$$

Let G_φ be a strongly embedded digraph. Consider an assignment $\mathcal{M}$ that maps each source or sink v of G_φ to a face $\mathcal{M}(v)$ of G_φ which contains v. Such an assignment $\mathcal{M}$ is said to be *consistent* if the number of nodes assigned to the external face h of G_φ equals $S_{angs}(h) + 1$ and the number of nodes assigned to each internal face f of G_φ equals $S_{angs}(f) - 1$. For each face z in G_φ we use $\mathcal{M}^{-1}(z)$ to denote the set of nodes assigned to z by $\mathcal{M}$. Theorem 2 is a characterisation of upward planar embedded digraphs [2].

Theorem 2 (Bertolazzi et al. [2]) *A (strongly) embedded digraph is upward planar if and only if it is acyclic, bimodal and admits a consistent assignment of sources and sinks to its faces.*

Corollary 1 *A (strongly) embedded digraph is upward planar if and only if it admits an UPSL drawing.*

Although Corollary 1 is not explicitly stated by Bertolazzi et al. [2], they do describe an algorithm for constructing an UPSL drawing of a (strongly) embedded digraph that is acyclic, bimodal and admits a consistent assignment of sinks and sources to its faces. Following from Theorem 1 and Corollary 1 in the remainder of this paper we restrict our attention to UPSL drawings. Theorem 2 yields a quadratic-time algorithm for testing the upward planarity of a (strongly) embedded digraph.

Theorem 3 (Bertolazzi et al. [2]) *A (strongly) embedded digraph G_ϕ, with n nodes can be tested for upward planarity in $\mathcal{O}(n^2)$ time.*

4 The Characterisation

In this section we present necessary and sufficient conditions for when the union of a set of digraphs $G_i, i = 1, ..., l$ with exactly one common vertex u (i.e. $V(G_a) \cap V(G_b) = \{u\}$ if $1 \leq a, b \leq l$ and $a \neq b$) admits an UPSL drawing, an UPSL drawing whose external face contains a prescribed vertex and an UPSL drawing whose external face contains a given class of angle with a prescribed centre. The main results of this section are Theorems 4– 7.

4.1 Two Digraphs - Sufficient Conditions

In this subsection we define sufficient conditions for when the union of two upward planar digraphs with exactly one common vertex is also an upward planar digraph. Throughout this subsection let A and B be upward planar digraphs such that $V(A) \cap V(B) = \{u\}$. Let w_a be a vertex in A such that $w_a \neq u$ and let w_b be a vertex in B such that $w_b \neq u$. Let $G = A \cup B$. We need Property 2 in the proof of Lemmas 2 - 7.

Property 2 *If G is the union of two distinct upward planar digraphs A and B with exactly one common vertex u then G is acyclic.*

Lemmas 2 - 7 are all concerned with sufficient conditions for merging two upward planar strongly embedded digraphs that share exactly one vertex into an upward planar strongly embedded digraph with certain properties. The proofs of all five lemmas display a high amount of overlap and thus we only present the proof of Lemma 2 here. The interested reader can obtain the proofs of Lemmas 3-7 here [14]. In the following let `X` denote an arbitrary element of the set {`LS`, `LT`, `SS`, `ST`, `II`}. Lemma 2 is concerned with the case when u is a source in both A and B or a sink in both A and B and is illustrated by Fig. 6.

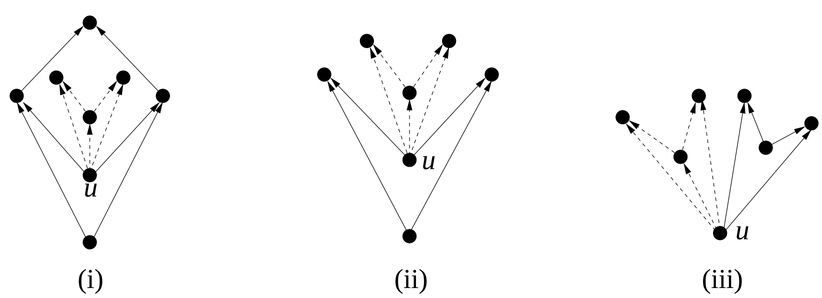


Figure 6: An illustration of Lemma 2, in the case when u is a source in both A and B. Dashed (Solid) edges are used to represent A (B): Statement 2.1 (i); Statement 2.2 (ii); Statement 2.3 (iii).

Lemma 2 *If u is a source (sink) in both A and B, and A has an UPSL drawing whose external face contains u then the following three statements hold.*

2.1 G is upward planar.

2.2 G admits an UPSL drawing whose external face contains u if B admits an UPSL drawing whose external face contains u.

2.3 G admits an UPSL drawing whose external face contains a large angle with centre u if A and B both admit UPSL drawings whose external face contains a large angle with centre u.

Proof: Assume that u is a source in both A and B (the case when u is a sink in both A and B follows by symmetry) and that A admits an UPSL drawing $\mathcal{A}$ whose external face h_a contains u. It follows that h_a contains an angle with centre u which we shall call $\langle a_1, u, a_2 \rangle$. Also there exists a strong embedding α of A such that A_α is bimodal, admits a consistent assignment $\mathcal{M}_a$ of the sinks and sources of A_α to its faces, and $\mathcal{A} \in A_\alpha$ (from Theorem 2). Let $\mathcal{M}_a(u) = f_a$.

Assume that B is upward planar. It follows that B admits an upward planar strong embedding β. Let $\mathcal{M}_b$ denote a consistent assignment of the sinks and sources of B_β to the faces of B_β and let $\mathcal{M}_b(u) = f_b$. Let $\mathcal{B}$ denote an UPSL drawing of B_β that corresponds to $\mathcal{M}_b$ and let $\langle b_1, u, b_2 \rangle$ be the large angle with centre u in $\mathcal{B}$ (observe that f_b contains the angle $\langle b_1, u, b_2 \rangle$).

Let G_φ be the strongly embedded digraph that results from adding A_α and B_β by inserting $\langle a_1, u, a_2 \rangle$ within $\langle b_1, u, b_2 \rangle$. We will now show that G_φ is acyclic, bimodal and has a consistent assignment of sinks and sources to its faces (i.e. that G_φ satisfies the three sufficient conditions for upward planarity specified by Theorem 2).

Acyclic. That G is acyclic follows directly from Property 2.

Bimodal. It follows from Definition 1 that $\varphi(v) = \alpha(v)$ for all $v \in V(A) \setminus u$ and that $\varphi(v) = \beta(v)$ for all $v \in V(B) \setminus u$. But A_α and B_β are both bimodal. Therefore all nodes in $V(G) \setminus u$ are bimodal. But u is also bimodal (because u is a source). It follows that G_φ is bimodal.

Consistent Assignment. The set of faces of G_φ consists of all the faces of A_α except h_a, all the faces of B_β except f_b, and the "new" face f_g. Consider the assignment $\mathcal{M}$ of the sinks and sources of G to the faces of G_φ which is defined as follows:

P1. For each source or sink $v \in V(G) \cap V(A)$,

$$\mathcal{M}(v) = \begin{cases} \rho(\mathcal{M}_a(v)) & \text{if } \mathcal{M}_a(v) \neq h_a, \\ f_g & \text{if } \mathcal{M}_a(v) = h_a. \end{cases}$$

P2. For each source or sink $v \in V(G) \cap V(B) \setminus \{u\}$,

$$\mathcal{M}(v) = \begin{cases} \rho(\mathcal{M}_b(v)) & \text{if } \mathcal{M}_b(v) \neq f_b, \\ f_g & \text{if } \mathcal{M}_b(v) = f_b. \end{cases}$$

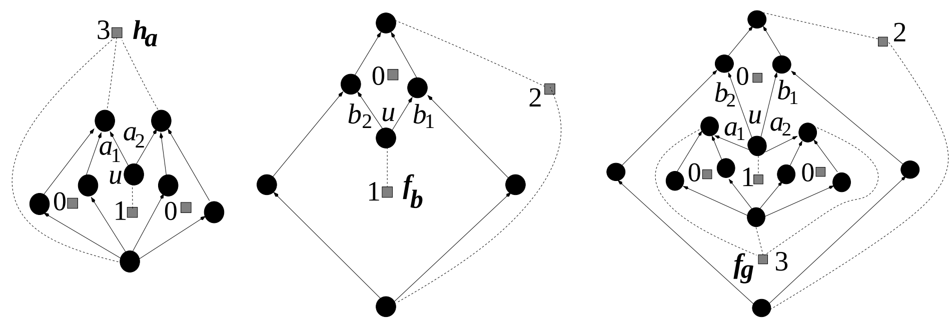


Figure 7: A_α and the consistent assignment $\mathcal{M}_a$ (i); B_β and the consistent assignment $\mathcal{M}_b$ (ii); G_φ and the consistent assignment $\mathcal{M}$ (iii). The small squares are used to represent faces and a dashed line from a vertex x to a face f indicates that vertex x is assigned to face f.

An example of each of the assignments $\mathcal{M}_a$, $\mathcal{M}_b$, and $\mathcal{M}$ is shown in Fig. 7. We now show that the assignment $\mathcal{M}$ is consistent by showing that

$$|\mathcal{M}^{-1}(f)| = \begin{cases} S_{angs}(f) + 1 & \text{if } f \text{ is the external face of } G_\varphi, \\ S_{angs}(f) - 1 & \text{if } f \text{ is an internal face of } G_\varphi. \end{cases}$$

We begin by showing that assignment $\mathcal{M}$ is consistent for each face $f \in faces(G_\varphi) \setminus f_g$. It follows from **P1** that $\left|\mathcal{M}^{-1}(f)\right| = \left|\mathcal{M}_a^{-1}\left(\rho^{-1}(f)\right)\right|$, for each face $f \in G_\varphi \setminus f_g$ such that $\rho^{-1}(f) \in faces(A_\alpha)$. Also, since $\mathcal{M}_b(u) = f_b$, it follows from **P2** that $\left|\mathcal{M}^{-1}(f)\right| = \left|\mathcal{M}_b^{-1}\left(\rho^{-1}(f)\right)\right|$, for each face $f \in G_\varphi \setminus f_g$ such that $\rho^{-1}(f) \in faces(B_\beta)$. But the number of S-angles in each face $f \in faces(G_\varphi) \setminus f_g$ is equal to the number of S-angles in $\rho^{-1}(f)$ (because the facial boundary of f is identical to the facial boundary of $\rho^{-1}(f)$). It follows from the consistency of $\mathcal{M}_a$ and $\mathcal{M}_b$ that the assignment $\mathcal{M}$ is consistent for each face $f \in faces(G_\varphi) \setminus f_g$.

We now show that assignment $\mathcal{M}$ is consistent for f_g. It follows from Property 1 that f_g contains all the S-angles contained by h_a except $\langle a_1, u, a_2\rangle$, all the S-angles contained by f_b except $\langle b_1, u, b_2\rangle$, plus the two "new" S-angles $\langle b_1, u, a_2\rangle$ and $\langle a_1, u, b_2\rangle$. Therefore,

$$\begin{aligned} S_{angs}(f_g) &= S_{angs}(h_a) - 1 + S_{angs}(f_b) - 1 + 2 \\ &= S_{angs}(h_a) + S_{angs}(f_b). \end{aligned}$$

It follows from **P1** and the consistency of $\mathcal{M}_a$ that the number of nodes in $V(A) \cap V(G)$ that are assigned to f_g by $\mathcal{M}$ equals $S_{angs}(h_a) + 1$. The number of nodes in $V(B) \cap V(G) \setminus u$ that are assigned to f_g by $\mathcal{M}$ depends on whether f_b is an internal or external face of B_β. We consider these two cases separately.

Case 1. Assume that f_b is an internal face of B_β. It follows from Definition 1 that f_g is an internal face of G_φ. It follows from the consistency of $\mathcal{M}_b$ and **P2** that the number of nodes in $V(B) \cap V(G) \setminus u$ that are assigned to f_g by $\mathcal{M}$ equals $S_{angs}(f_b) - 2$. Therefore

$$\begin{aligned} |\mathcal{M}^{-1}(f_g)| &= S_{angs}(h_a) + 1 + S_{angs}(f_b) - 2 \\ &= S_{angs}(h_a) + S_{angs}(f_b) - 1 \end{aligned}$$

As $S_{angs}(f_g) = S_{angs}(h_a) + S_{angs}(f_b)$ it follows that if f_b is an internal face of B_β then assignment $\mathcal{M}$ is consistent for f_g.

Case 2. Assume that f_b is the external face of B_β. It follows from Definition 1 that f_g is the external face of G_φ. It follows from the consistency of $\mathcal{M}_b$ and **P2** that the number of nodes in $V(B) \cap V(G) \setminus u$ that are assigned to f_g by $\mathcal{M}$ equals $S_{angs}(f_b)$. Therefore

$$|\mathcal{M}^{-1}(f_g)| = S_{angs}(h_a) + S_{angs}(f_b) + 1$$

As $S_{angs}(f_g) = S_{angs}(h_a) + S_{angs}(f_b)$ it follows that if f_b is the external face of B_β that assignment $\mathcal{M}$ is consistent for f_g.

Therefore $\mathcal{M}$ is a consistent assignment of nodes to the faces of G_φ.

We have shown that G_φ is acyclic, bimodal, and has a consistent assignment of sinks and sources to its faces. It follows from Theorem 2 that G_φ and G are upward planar. We are now finished with the general part of the proof and focus on the proof of the individual statements.

Statement 2.1 True, since we have shown G_φ to be upward planar.

Statement 2.2 Suppose that $\mathcal{B}$ is an UPSL drawing whose external face contains u. Thus the external face of B_β contains u. It then follows directly from Definition 1 that the external face of G_φ also contains u. As we have shown G_φ to be upward planar Statement 2.2 is true.

Statement 2.3 Suppose that $\mathcal{A}$ and $\mathcal{B}$ are both UPSL drawings whose external face contains a large angle centred at u. Therefore f_b is the external face of B_β. Thus f_g is the external face of G_φ (from Definition 1). But it follows from **P1** that $\mathcal{M}(u) = f_g$. Therefore G_φ has consistent assignment $\mathcal{M}$ of sinks and sources to its faces in which u is assigned to its external face f_g. Therefore G_φ and G have an UPSL drawing whose external face contains the large angle centred at u.

□

Lemma 3 is concerned with the case when u is an internal node in both A and B and is illustrated by Fig. 8.

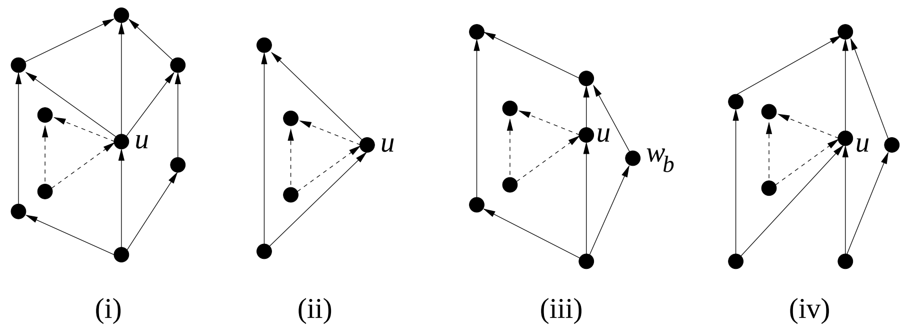


Figure 8: An illustration of Lemma 3. Dashed (Solid) edges are used to represent A (B): Statement 3.1 (i); Statement 3.2 (ii); Statement 3.3 (iii); Statement 3.5 (iv).

Lemma 3 *If u is an internal node in both A and B and A has an UPSL drawing whose external face contains an I-angle centred at u then the following five statements hold.*

3.1 G is upward planar.

3.2 G has an UPSL drawing whose external face contains an I-angle centred at u if B has an UPSL drawing whose external face contains an I-angle centred at u.

3.3 G has an UPSL drawing whose external face contains w_b if B has an UPSL drawing whose external face contains w_b.

3.4 G has an UPSL drawing whose external face contains an edge that leaves u if B has an UPSL drawing whose external face contains an edge that leaves u.

3.5 G has an UPSL drawing whose external face contains an edge that enters u if B has an UPSL drawing whose external face contains an edge that enters u.

Lemma 4 is concerned with the case when u is a source in A and a sink in B, or vice versa, and is illustrated by Fig. 9(i).

Lemma 4 *If u is a source (sink) in A, a sink (source) in B, and A and B both admit an UPSL drawing whose external face contains a large angle with centre u then G admits an UPSL drawing whose external face contains an I-angle with centre u.*

Lemma 5, Lemma 6 and Lemma 7 are concerned with the case when u is an internal node in exactly one of A or B and a source or sink in the other. Lemma 5 is illustrated by Fig. 9(ii) and Fig. 9(iii). Lemma 6 and Lemma 7 are illustrated by Fig. 10.

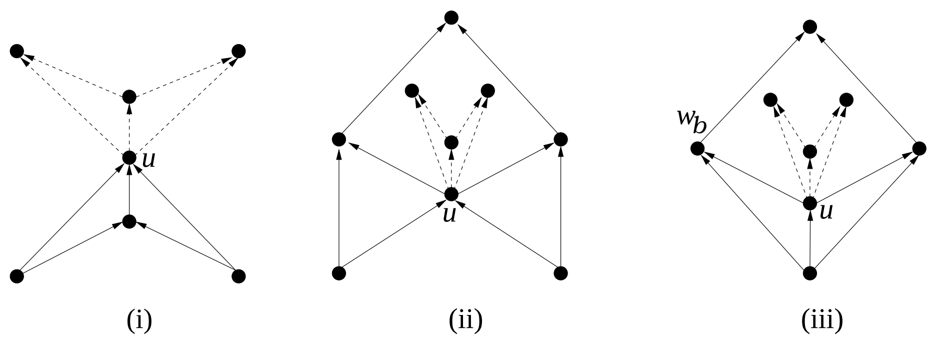


Figure 9: An illustration of Lemma 4, in the case when u is a source (sink) in A (B) (i); an illustration of Statement 1 of Lemma 5 (ii); an illustration of Statement 2 of Lemma 5 (iii). Dashed (Solid) edges belong to A (B).

Lemma 5 *If u is a source in A, an internal node in B, and A admits an UPSL drawing whose external face contains the large angle with centre u then the following statements hold.*

5.1 G admits an UPSL drawing whose external face contains an edge that enters u if B admits an UPSL drawing whose external face contains an edge that enters u.

5.2 G admits an UPSL drawing whose external face contains an `X` *angle with centre w_b if B admits an UPSL drawing whose external face contains an* `X` *angle with centre w_b.*

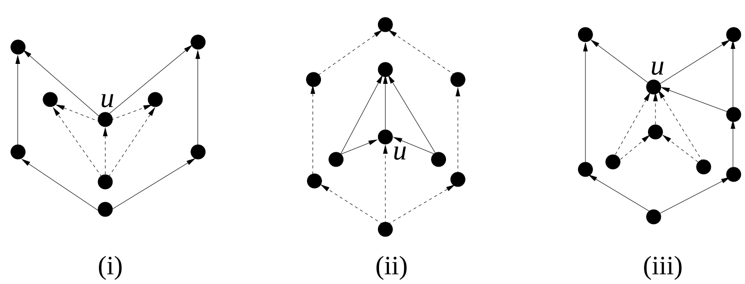


Figure 10: An illustration of Lemma 6 (i); an illustration of Statement 1 of Lemma 7 (ii); an illustration of Statement 2 of Lemma 7 (iii). Dashed (Solid) edges are used to represent A (B).

Lemma 6 *If u is an internal node in A, a source in B, A admits an UPSL drawing whose external face contains an edge that leaves u and B admits an*

UPSL drawing whose external face contains u then G admits an UPSL drawing whose external face contains an edge that leaves u.

Lemma 7 *If u is a sink in A and an internal node in B then the following statements hold.*

7.1 G is upward planar if B admits an UPSL drawing whose external face contains an edge which enters u.

7.2 G admits an UPSL drawing whose external face contains an edge that leaves u if A admits an UPSL drawing whose external face contains a large angle centred at u and B admits an UPSL drawing whose external face contains an edge that leaves u.

4.2 Many Components - Sufficient Conditions

In this subsection we define sufficient conditions for when the union of an arbitrary number of components have an UPSL drawing with certain properties. Lemmas 8, 9 and 10 follow easily by induction from results in Subsection 4.1 and their proofs are similar. Thus we only present the proof of Lemma 8. Fig. 11(i) illustrates Lemma 8 in the case when u is a source in G.

Lemma 8 *Let G be a digraph with a source (sink) u. G has an UPSL drawing whose external face contains u if every component of G wrt u has an UPSL drawing whose external face contains u.*

Proof: Let $G_i, i = 1, \ldots, c$ be the components of G wrt u and let $G^*_x = \bigcup_{i=1}^{x} G_i$. Assume that every component of G wrt u has an UPSL drawing whose external face contains u. Let k be an integer such that $1 \leq k < c$. Assume that G^*_k has an UPSL drawing whose external face contains u. It follows from Statement 2 of Lemma 2 that $G^*_{k+1} = G^*_k \cup G_{k+1}$ has an UPSL drawing whose external face contains u if G_{k+1} has an UPSL drawing whose external face contains u. But $G^*_1 = G_1$ has an UPSL drawing whose external face contains u. Therefore $G^*_1, G^*_2, \ldots G^*_c$ have UPSL drawings whose external face contains u. But $G^*_c = G$. Thus G has an UPSL drawing whose external face contains u if every component of G wrt u has an UPSL drawing whose external face contains u. □

Lemma 9 and Lemma 10 can be proved using Statement 3 of Lemma 2 and Statement 2 of Lemma 3, respectively, along with an inductive argument analogous to that used to prove Lemma 8. Fig. 11(ii) illustrates Lemma 9 in the case when u is a source in G and Fig. 12(i) illustrates Lemma 10.

Lemma 9 *Let G be a digraph with a source (sink) u. G has an UPSL drawing whose external face contains the large angle centred at u if every component of G wrt u has an UPSL drawing whose external face contains the large angle centred at u.*

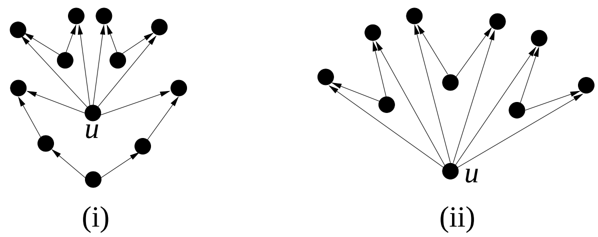


Figure 11: An illustration of Lemma 8 in the case when u is a source in G (i); an illustration of Lemma 9 in the case when u is a source in G (ii).

Lemma 10 *Let G be a digraph with a node u such that all components of G wrt u are internal-components. G has an UPSL drawing whose external face contains an I-angle centred at u if every component of G wrt u has an UPSL drawing whose external face contains an I-angle centred at u.*

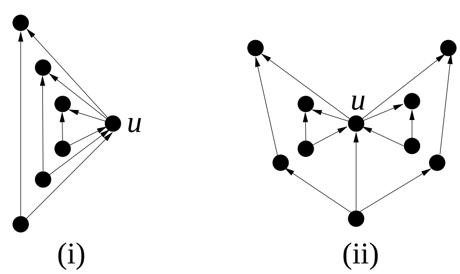


Figure 12: An illustration of Lemma 10 (i); an illustration of Lemma 11 in the case when an edge leaves u (ii).

Lemma 11 *Let G be a digraph with a node u such that all components of G wrt u are internal-components. G has an UPSL drawing whose external face contains an edge that leaves (enters) u if the following two statements hold:*

(i) All components of G wrt u admit an UPSL drawing whose external face contains an edge that leaves (enters) u.

(ii) At most one component of G wrt u does not have an UPSL drawing whose external face contains an I-angle centred at u.

Proof: Case 1. Suppose that all components of G wrt u have an UPSL drawing whose external face contains an I-angle centred at u. Then G admits an UPSL drawing Γ whose external face contains an I-angle centred at u (Lemma 10). But an I-angle consists of an incoming edge and an outgoing edge. Thus the external face of Γ contains an outgoing edge incident on u.

Case 2. Suppose some component G_y of G wrt u does not admit an UPSL drawing whose external face contains an I-angle centred at u, but does admit an UPSL drawing whose external face contains an edge that leaves (enters) u. If all the components of $G \setminus G_y$ wrt u have an UPSL drawing whose external face contains an I-angle centred at u then $G \setminus G_y$ admits an UPSL drawing whose external face contains an I-angle centred at u (Lemma 10). It follows from Statement 4 (Statement 5) of Lemma 3 that $G = G_y \cup (G \setminus G_y)$ admits an UPSL drawing whose external face contains an edge that leaves (enters) u. □

Fig. 12(ii) illustrates Lemma 11 in the case when an edge leaves u.

4.3 Two Components - Necessary Conditions

In this subsection we define necessary conditions for when two components can be merged into an upward planar digraph. Let G be an upward planar digraph with a node u such that there are exactly two components of G wrt u, which we refer to as G_1 and G_2. Let $\mathcal{G}$ be an UPSL drawing of G and let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the drawings induced on G_1 and G_2 by $\mathcal{G}$, respectively.

Property 3 *All of $\mathcal{G}_1$ (resp., $\mathcal{G}_2$) lies in a single face of $\mathcal{G}_2$ (resp., $\mathcal{G}_1$) and at least one of $\mathcal{G}_1$ or $\mathcal{G}_2$ lies in the external face of the other.*

Due to Property 3, in the remainder of this subsection we let f_1 be the face of $\mathcal{G}_1$ that contains $\mathcal{G}_2$ and we let f_2 be the face of $\mathcal{G}_2$ that contains $\mathcal{G}_1$.

Lemma 12 *If G_1 and G_2 are both internal-components then the faces f_1 and f_2 both contain an I-angle centred at u.*

Proof: Assume that G_1 and G_2 are both internal-components wrt u. The edges incident on u in G_1 (resp., G_2) appear contiguously in the clockwise ordering of the edges incident on u in $\mathcal{G}$ (because $\mathcal{G}_1$ lies entirely inside the face f_2 of $\mathcal{G}_2$ and $\mathcal{G}_2$ lies entirely inside the face f_1 of $\mathcal{G}_1$). Therefore exactly one edge, a_1 (resp., a_2) of G_1 (resp., G_2) directly precedes an edge b_2 (resp., b_1) of G_2 (resp., G_1) in the clockwise ordering of edges incident on u in $\mathcal{G}$. Also f_1 (resp., f_2) contains an angle formed from the edges a_1 and b_1 (resp., a_2 and b_2) and u (see Fig. 13). G_1 and G_2 both contain an edge which enters u and an edge which leaves u (because G_1 and G_2 are both internal-components wrt u). If a_1 and b_1 both enter (resp., leave) u then u is not bimodal in $\mathcal{G}$ because edges which leave (resp., enter) u would occur in the clockwise sequence of edges from a_1 to b_1 as well as the clockwise sequence of edges from b_1 to a_1. A similar argument can be used to show that if a_2 and b_2 both enter (resp., leave) u, then u is not bimodal in $\mathcal{G}$. But $\mathcal{G}$ is an UPSL drawing only if it is bimodal planar (Theorem 2). It follows that the set of edges $\{a_1, b_1\}$ (resp., $\{a_2, b_2\}$)

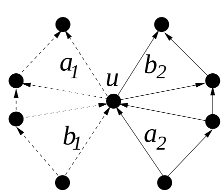


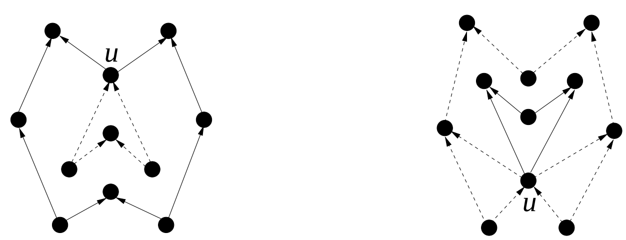
Figure 13: Angles centred at u.

consists of an edge which enters u and an edge which leaves u. Therefore f_1 and f_2 both contain an I-angle centred at u. □

Lemma 13 *If either G_1 or G_2 is a source-component and the other is a sink-component then the faces f_1 and f_2 both contain a large angle centred at u.*

Proof: Assume that G_1 is a source-component and that G_2 is a sink-component. It follows from Lemma 1 that u is the centre of exactly one large angle in $\mathcal{G}_1$ and exactly one large angle in $\mathcal{G}_2$. But u is an internal node in G and therefore is not centred at any large angles in $\mathcal{G}$ (Lemma 1). Thus both large angles are "canceled out" in $\mathcal{G}$. Therefore $\mathcal{G}_1$ (resp., $\mathcal{G}_2$) is drawn within the face of $\mathcal{G}_2$ (resp., $\mathcal{G}_1$) that contains the large angle centred at u. Therefore f_1 and f_2 each contain a large angle centred at u. An analogous argument can be made if G_2 is a source-component and that G_1 is a sink-component. □

Fig. 14(i) illustrates Lemma 13 and Fig. 14(ii) illustrates Lemma 14 in the case when G_1 is a source-component.



(i) (ii)

Figure 14: An illustration of Lemma 13 (i); an illustration of Lemma 14 in the case when G_1 is a source-component (ii). Solid (Dashed) edges are used to represent G_1 (G_2).

Lemma 14 *If G_1 is a source-component (resp., sink-component) and G_2 is an internal-component then the face f_1 contains a large angle centred at u and the face f_2 contains an outgoing (resp., incoming) edge incident on u.*

Proof: Assume that G_1 is a source-component and that G_2 is an internal-component. As u is a source in $\mathcal{G}_1$ but an internal node in $\mathcal{G}$, u is centred at one large angle in $\mathcal{G}_1$ but on no large angle in $\mathcal{G}$ (Lemma 1). Thus the large angle centred at u in $\mathcal{G}_1$ is "canceled out" in $\mathcal{G}$. Therefore $\mathcal{G}_2$ lies in the face of $\mathcal{G}_1$ that contains the large angle centred at u. Therefore f_1 contains a large angle centred at u. The edges incident on u in G_1 (resp., G_2) appear contiguously in the clockwise ordering of the edges incident on u in $\mathcal{G}$ (because $\mathcal{G}_1$ lies entirely inside the face f_2 of $\mathcal{G}_2$ and $\mathcal{G}_2$ lies entirely inside the face f_1 of $\mathcal{G}_1$). Thus f_1 (and therefore G_1) contains edges a_1 and b_1 and f_2 (and therefore G_2) contains edges a_2 and b_2 such that the edge a_1 (resp., b_2) directly precedes the edge a_2

(resp., b_1) in the clockwise ordering of edges incident on u in $\mathcal{G}$. As all edge(s) incident on u in G_1 leave u, but G_2 contains edge(s) which leave u and edge(s) which enter u, it follows that u is bimodal in $\mathcal{G}$ only if b_1 and/or b_2 leave u. The upward planarity of $\mathcal{G}$ implies that u is bimodal in $\mathcal{G}$ (Theorem 2). Therefore f_2 contains an outgoing edge incident on u. An analogous argument can be made if G_2 is a source-component. □

4.4 Many Components - Necessary Conditions

Lemma 15 *Let $G_i, i = 1, \ldots, c$ be the components of a digraph G wrt a node u. Let $\mathcal{G}$ be a drawing of G and let $\mathcal{G}_i$ be the sub-drawing induced on $G_i, i = 1, \ldots, c$. $\mathcal{G}$ is an UPSL drawing only if at least $c-1$ of the sub-drawings $\mathcal{G}_i, i = 1, \ldots, c$, are UPSL drawings whose external face contains u.*

Proof: Clearly the lemma holds if $c = 1$. Henceforth we assume $c > 1$. Let j and k be any two distinct integers such that $1 \leq j, k \leq c$. $\mathcal{G}_j \cup \mathcal{G}_k$ is an UPSL drawing only if both $\mathcal{G}_j$ and $\mathcal{G}_k$ are UPSL drawings and $\mathcal{G}_j \cup \mathcal{G}_k$ is planar. Suppose that $\mathcal{G}_j$ and $\mathcal{G}_k$ are both UPSL drawings whose external face is not incident on u. Let E_j (resp., E_k) be the facial boundary of the external face of $\mathcal{G}_j$ (resp., $\mathcal{G}_k$). As u does not lie on E_j (resp., E_k), u must be a point in the interior of some closed curve E'_j (resp., E'_k) which is a sub-drawing of E_j (resp., E_k). As E'_j and E'_k are both closed curves $\mathcal{G}_j \cup \mathcal{G}_k$ is planar only if all of E'_j lies in the interior of E'_k or vice versa. As $\mathcal{G}_j$ (resp., $\mathcal{G}_k$) is connected a path p_j (resp., p_k) exists from u to a node contained by E'_j (resp., E'_k). But if E'_j (resp., E'_k) is drawn inside E'_k (resp., E'_j) then p_k (resp., p_j) is a curve from a point in the exterior of E'_j (resp., E'_k) to a point in the interior of E'_j (resp., E'_k) and therefore intersects E'_j (resp., E'_k) (see Fig. 15). Thus $\mathcal{G}_j \cup \mathcal{G}_k$ is an UPSL drawing only if at least one of $\mathcal{G}_j$ or $\mathcal{G}_k$ is an UPSL drawing whose external face contains u. □

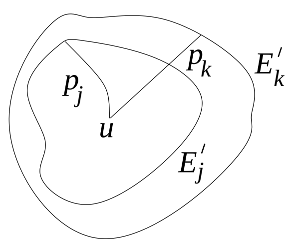


Figure 15: u internal to $\mathcal{G}_j$ and $\mathcal{G}_k$.

Lemma 16 *Let $G_i, i = 1, \ldots, c$ be the components of a digraph G wrt a node u. Let $\mathcal{G}$ be a drawing of G and let $\mathcal{G}_i$ be the sub-drawing induced on $G_i, i = 1, \ldots, c$. $\mathcal{G}$ is an UPSL drawing whose external face contains u only if each $\mathcal{G}_i, i = 1, \ldots, c$, is an UPSL drawing whose external face contains u.*

Proof: Let x be any integer such that $1 \leq x \leq c$. Suppose that $\mathcal{G}_x$ is an UPSL drawing whose outer face $\mathcal{E}_x$ does not contain u. Therefore u is a point in the interior of some closed curve $\mathcal{E'}_x$ that is a sub-drawing of $\mathcal{E}_x$. But $\mathcal{E'}_x$ is a sub-drawing of $\mathcal{G}$. Therefore u is not contained by the external face of $\mathcal{G}$. Therefore $\mathcal{G}$ is an UPSL drawing whose external face contains u only if each $\mathcal{G}_i$, $i = 1, \ldots, c$, is an UPSL drawing whose external face contains u. □

In the following X denotes an arbitrary element of the set {LS, LT, SS, ST, II}. Lemma 17 is illustrated by Fig. 16.

Lemma 17 *Let G be a digraph with a cutvertex u, let C be a component of G wrt u, and let $w \neq u$ be a vertex of C. Let Γ be a drawing of G and let Γ_C (resp., $\Gamma_{G \setminus C}$) be the sub-drawing induced on C (resp., $G \setminus C$) by Γ. If Γ is an UPSL drawing whose external face contains an angle of type* X *centred at w then the following three statements are true:*

1. *Γ_C is an UPSL drawing whose external face contains an angle of type* X *centred at w.*
2. *Γ_C is drawn entirely within the external face of $\Gamma_{G \setminus C}$.*
3. *$\Gamma_{G \setminus C}$ is an UPSL drawing whose external face contains u.*

Proof: Statement 1. As Γ_C can be obtained from Γ by deleting edges (none of which are incident on w) and nodes (none of which are adjacent to w) it follows that Γ_C is an UPSL drawing and that the external face of Γ_C contains an angle of type X centred at w.

Statement 2. Using an argument analogous to that used in Property 3 it can be shown that Γ_C lies entirely within a face f of $\Gamma_{G \setminus C}$. As w is contained by the external face of Γ, but $w \notin \Gamma_{G \setminus C}$, it follows that f must be the external face of $\Gamma_{G \setminus C}$.

Statement 3. There is a path from w to u containing only nodes and edges of C. Thus if u is not contained by the external face of $\Gamma_{G \setminus C}$ this path will cross the boundary of the external face of $\Gamma''_{G \setminus C}$, violating the planarity of Γ. □

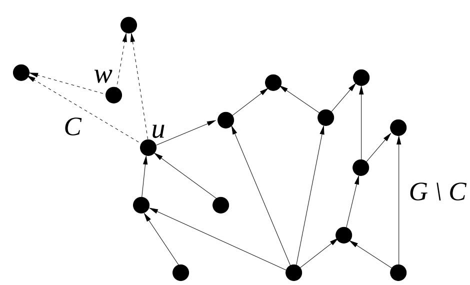


Figure 16: An illustration of Lemma 17. Dashed (Solid) edges belong to C ($G \setminus C$).

4.5 Main Results

We are now ready to present the main results of this section.

Theorem 4 *Let G be a digraph and let u be any vertex in G. G is upward planar if and only if the following four statements are true.*

1. *All components of G wrt u are upward planar.*
2. *At most one component of G wrt u does not admit an UPSL drawing whose external face contains u.*
3. *At most one internal-component of G wrt u does not admit an UPSL drawing whose external face contains an I-angle centred at u.*
4. *At least one of the following three statements is true:*
   (a) *All source-components and sink-components of G wrt u have an UPSL drawing whose external face contains a large angle centred at u.*
   (b) *All source-components of G wrt u admit an UPSL drawing whose external face contains a large angle with centre u and all internal-components of G wrt u admit an UPSL drawing whose external face contains an edge which enters u.*
   (c) *All sink-components of G wrt u admit an UPSL drawing whose external face contains a large angle centred at u and all internal-components of G wrt u admit an UPSL drawing whose external face contains an edge which leaves u.*

Proof: Sufficiency. Assume that G contains at least one source-component, at least one sink-component, and at least one internal-component wrt u. If the conditions are sufficient for upward planarity when all three types of components are present then they must also be sufficient for upward planarity if any of these components are not present. Assume Statements 1, 2, and 3 are true.
Case 1. Assume Statement 4(a) is true. If Statement 4(a) is true then both $S(u)$ and $T(u)$ admit UPSL drawings whose external face contains a large angle centred at u (from Lemma 9). Therefore $S(u) \cup T(u)$ admits an UPSL drawing whose external face contains an I-angle centred at u (from Lemma 4).

We now consider $I(u)$. It follows from the truth of Statement 3 that there exists an internal-component C_i of G wrt u such that all internal-components of $G \setminus C_i$ wrt u admit an UPSL drawing whose external face contains an I-angle centred at u. Therefore $I(u) \setminus C_i$ admits an UPSL drawing whose external face contains an I-angle centred at u (from Lemma 10). Also it follows from the truth of Statement 1 that C_i is upward planar. Therefore $I(u) = (I(u) \setminus C_i) \cup C_i$ is upward planar (Lemma 3.1). As $S(u) \cup T(u)$ admits an UPSL drawing whose external face contains an I-angle centred at u and $I(u)$ is upward planar it follows from Lemma 3.1 that G is upward planar.
Case 2. Assume Statement 4(b) is true. If Statement 4(b) is true then $S(u)$ admits an UPSL drawing whose external face contains a large angle with centre u (from Lemma 9). Also if Statement 3 and Statement 4.2 are both true then $I(u)$ admits an UPSL drawing whose external face contains an edge that enters u (from Lemma 11). It then follows from Lemma 5.1 that $I(u) \cup S(u)$ admits an UPSL drawing whose external face contains an edge that enters u.

Also it follows from the truth of Statement 2 that there exists a sink-component C_t of G wrt u such that all sink-components of $G \setminus C_t$ wrt u admit an UPSL drawing whose external face contains u. Therefore $T(u)\setminus C_t$ admits an UPSL drawing whose external face contains u (from Lemma 8). But as Statement 1 is true C_t must be upward planar. Therefore $T(u) = (T(u) \setminus C_t) \cup C_t$ is upward planar (Lemma 2.1). As $S(u) \cup I(u)$ admits an UPSL drawing whose external face contains an edge that enters u and $T(u)$ is upward planar it follows from Lemma 7.1 that G is upward planar.

Case 3. Assume Statement 4(c) is true. Follows by symmetry from Case 2.

Necessity.

1. A component of G wrt u is a subgraph of G. Thus G is upward planar only if all components of G wrt u are upward planar. It follows that Statement 1 is necessary.

2. The necessity of Statement 2 follows directly from Lemma 15.

3. The necessity of Statement 3 follows directly from Lemma 12.

4. We now show the necessity of Statement 4. We will use the following four statements.

   $\mathcal{A}$ All components of $S(u)$ wrt u admit an UPSL drawing whose external face contains a large angle with centre u.

   $\mathcal{B}$ All components of $T(u)$ wrt u admit an UPSL drawing whose external face contains a large angle with centre u.

   $\mathcal{C}$ All components of $I(u)$ wrt u admit an UPSL drawing whose external face contains an edge that leaves u.

   $\mathcal{D}$ All components of $I(u)$ wrt u admit an UPSL drawing whose external face contains an edge that enters u.

   Suppose Statement $\mathcal{A}$ and Statement $\mathcal{B}$ are both false. Thus there exists a component G_s of $S(u)$ wrt u and a component G_t of $T(u)$ wrt u such that neither G_s nor G_t admits an UPSL drawing whose external face contains the large angle centred at u. It follows from Lemma 13 that $G_s \cup G_t$ is not upward planar. Therefore G is upward planar only if Statement $\mathcal{A}$ and/or Statement $\mathcal{B}$ are/is true.

   Suppose Statement $\mathcal{A}$ and Statement $\mathcal{C}$ are both false. Thus there exists a component G'_s of $S(u)$ wrt u such that G'_s does not admit an UPSL drawing whose external face contains the large angle with centre u and a component G_i of $I(u)$ wrt u such that G_i does not admit an UPSL drawing whose external face contains an edge that leaves u. It follows from Lemma 14 that $G'_s \cup G_i$ is not upward planar. Therefore G is upward planar only if Statement $\mathcal{A}$ and/or Statement $\mathcal{C}$ are/is true.

   An analogous argument can be used to proof that G is upward planar only if Statement $\mathcal{B}$ and/or Statement $\mathcal{D}$ are/is true.

   Therefore G is upward planar only if at least one of the following statements is true.

(a) Statement $\mathcal{A}$ and Statement $\mathcal{B}$ are both true.

(b) Statement $\mathcal{A}$ and Statement $\mathcal{D}$ are both true.

(c) Statement $\mathcal{B}$ and Statement $\mathcal{C}$ are both true.

But these are equivalent to Statements $4(a), 4(b)$, and $4(c)$ respectively. Thus Statement 4 is also necessary. Thus we have proved the necessity.

□

In the following three theorems we use `X` to denote an arbitrary element of the set {`LS`, `LT`, `SS`, `ST`, `II`}.

Theorem 5 *Let G be a digraph with a vertex u such that there is a source-component C of G wrt u which contains a vertex $w \neq u$. G has an UPSL drawing whose external face contains an* `X` *angle with centre w if and only if the following five statements are true.*

A1 *C admits an UPSL drawing whose external face contains an* `X` *angle with centre w.*

A2 *All source-components of $G \setminus C$ wrt u have an UPSL drawing whose external face contains u.*

A3 *All sink-components of G wrt u admit an UPSL drawing whose external face contains a large angle centred at u.*

A4 *All internal-components of G wrt u have an UPSL drawing whose external face contains an edge which leaves u.*

A5 *At most one internal-component of G wrt u does not have an UPSL drawing whose external face contains an I-angle centred at u.*

Proof: We begin by proving the sufficiency for the case when there is at least one source-component, at least one sink-component, and at least one internal-component of $G \setminus C$ wrt u. The sufficiency of all other cases follows from the sufficiency of this case. Assume that Statements A1 - A5 are all true.

It follows from Lemma 9 and the truth of Statement A3 that $T(u)$ admits an UPSL drawing whose external face contains a large angle with centre u. It follows from Lemma 11 and the truth of Statements A4 and A5 that $I(u)$ admits an UPSL drawing whose external face contains an edge that leaves u. Therefore $I(u) \cup T(u)$ admits an UPSL drawing whose external face contains an edge that leaves u (Lemma 7.2).

Let $S'(u)$ denote the union of all source-components of $G \setminus C$ wrt u. It follows from Lemma 8 and the truth of Statement A2 that $S'(u)$ admits an UPSL drawing whose external face contains u. Therefore $(I(u) \cup T(u)) \cup S'(u)$ admits an UPSL drawing whose external face contains an edge that leaves u (Lemma 6).

Thus $G \setminus C = (I(u) \cup T(u)) \cup S'(u)$ admits an UPSL drawing whose external face contains an edge that leaves u. It then follows from the truth of Statement

A1 and Lemma 5.2 that G admits an UPSL drawing whose external face contains an X angle centred at w.

We shall now prove the necessity.

A1 The necessity of Statement A1 follows from Lemma 17.

A2 The necessity of Statement A2 follows from Lemmas 17 and 16.

A3 Suppose that G_T is a sink-component of G wrt u that does not admit an UPSL drawing whose external face contains a large angle centred at u. Let Γ be a drawing of $G_T \cup C$ and let Γ_{G_T} (Γ_C) be the sub-drawing induced on G_T (C) by Γ. It follows from Lemma 13 that if Γ is an UPSL drawing then Γ_C lies entirely within an internal face of Γ_{G_T} and thus the external face of Γ does not contain an X angle with centre w. But $G_T \cup C$ is a subgraph of G. Therefore G does not admit an UPSL drawing whose external face contains an X angle centred at w if Statement A3 is false.

A4 The necessity of Statement A4 can be shown with Lemma 14 and an argument analogous to that used to prove the necessity of Statement A3.

A5 The necessity of Statement A5 follows directly from Lemma 12.

□

Theorem 6 follows by symmetry from Theorem 5.

Theorem 6 *Let G be a digraph with a vertex u such that there is a sink-component C of G wrt u which contains a vertex $w \neq u$. G has an UPSL drawing whose external face contains an* X *angle with centre w if and only if the following five statements are true.*

B1 *C admits an UPSL drawing whose external face contains an* X *angle with centre w.*

B2 *All sink-components of $G \setminus C$ wrt u admit an UPSL drawing whose external face contains u.*

B3 *All source-components of G wrt u have an UPSL drawing whose external face contains a large angle centred at u.*

B4 *All internal-components of G wrt u have an UPSL drawing whose external face contains an edge which enters u.*

B5 *At most one internal-component of G wrt u does not have an UPSL drawing whose external face contains an I-angle centred at u.*

Theorem 7 *Let G be a digraph with a vertex u such that there is an internal-component C of G wrt u which contains a vertex $w \neq u$. G has an UPSL drawing whose external face contains an* X *angle with centre w if and only if the following three statements are true.*

C1 *C admits an UPSL drawing whose external face contains an* X *angle centred at w.*

C2 *All internal-components of $G \setminus C$ wrt u admit an UPSL drawing whose external face contains an I-angle centred at u.*

C3 *All source-components and sink-components of G wrt u admit an UPSL drawing whose external face contains a large angle with centre u.*

Proof: We will prove the sufficiency for the case when there is at least one source-component, at least one sink-component, and at least one internal-component of $G \setminus C$ wrt u. The sufficiency of all other cases follows from this case. Let $I'(u)$ denote the union of all internal-components of $G \setminus C$ wrt u. Assume Statements C1, C2 and C3 are all true. It follows from Lemma 9 and the truth of Statement C3 that both $S(u)$ and $T(u)$ admit an UPSL drawing whose external face contains a large angle centred at u. Therefore $S(u) \cup T(u)$ admits an UPSL drawing whose external face contains an I-angle centred at u (Lemma 4). It follows from Lemma 10 and the truth of Statement C2 that $I'(u)$ admits an UPSL drawing whose external face contains an I-angle centred at u. Therefore $G \setminus C = (S(u) \cup T(u)) \cup I'(u)$ admits an UPSL drawing whose external face contains an I-angle centred at u (Lemma 3.2). It follows from Lemma 3.3 and the truth of Statement C1 that G admits an UPSL drawing whose external face contains an X angle centred at w.

We now prove the necessity of each statement.

C1 The necessity of Statement C1 follows from Lemma 17.

C2 The necessity of Statement C2 follows from Lemma 12 and an argument analogous to that used to prove the necessity of Statement A3 in Theorem 5.

C3 The necessity of Statement C3 can be shown with Lemma 14 and an argument analogous to that used to prove the necessity of Statement A3 in Theorem 5.

□

5 The Algorithm

In this section we give a detailed description of an upward planarity testing algorithm yielded by Theorems 4 – 7 in combination with Bertolazzi et al.'s algorithm for testing the upward planarity of an embedded digraph [2]. In Section 5.1 we introduce the concept of the *rank* of a digraph wrt to one of its vertices and restate Theorems 4 – 7 using this concept. In Section 5.2 we show how to compute the rank of a block wrt one of its vertices by tailoring Bertolazzi et al.'s aforementioned algorithm. Following this, in Section 5.3 we introduce a type of vertex that will play a central role in our testing algorithm. Finally in Section 5.4 we present a description of the whole upward planarity testing algorithm.

5.1 The Rank of a Digraph

In this subsection we show that the classes of UPSL drawings that occur in Theorems 4 – 7 can be ordered according to how "easily" digraphs admitting such UPSL drawings can be merged with other upward planar digraphs in order to form an upward planar digraph. For this purpose we introduce the concept of the *rank* of a digraph wrt one of its vertices. We also restate Theorems 4 – 7 using the concept of rank.

Let A and B represent two distinct components of a digraph G wrt a vertex u of G. Let H be the union of all remaining components of G wrt u (i.e. $H = G \setminus (A \cup B)$). Suppose that A and B are both source-components or both sink-components of G wrt u. It follows from Theorems 4 – 7 that we need to be able to test which, if any, of the following classes of drawings A and B admit: an UPSL drawing, an UPSL drawing whose external face contains u, an UPSL drawing whose external face contains a large angle centred at u. Conveniently, the "flexibility" of source-components and sink-components of G wrt u can be totally ordered according to which of these classes of drawings each allows. Stated more precisely, we can assign an integer to A called the rank of A wrt u and denoted by $rank(A, u)$ such that:

(i) if $rank(A, u) < rank(B, u)$ then $H \cup A$ is upward planar if $H \cup B$ is upward planar;

(ii) if $rank(A, u) = rank(B, u)$ then $H \cup A$ is upward planar if and only if $H \cup B$ is upward planar.

The rank of A wrt u is defined as follows.

$rank(A, u) = 1$ if A admits an UPSL drawing whose external face contains a large angle centred at u.

$rank(A, u) = 2$ if the previous case does not apply and A admits an UPSL drawing whose external face contains u.

$rank(A, u) = 10$ if none of the previous cases apply and A admits an UPSL drawing.

$rank(A, u) = 99$ if there is no UPSL drawing of A.

Suppose that A and B are both internal-components of G wrt to u. It follows from Theorems 4 – 7 that we need to be able to test which, if any, of the following classes of drawings each admits: an UPSL drawing, an UPSL drawing whose external face contains an S-angle centred at u, an UPSL drawing whose external face contains a T-angle centred at u, and an UPSL drawing whose external face contains an I-angle centred at u. The "flexibility" of internal-components of u can be partially ordered according to which of these classes of drawings each allows. Stated more precisely, we can assign an integer to A called the rank of A wrt u and denoted by $rank(A, u)$ such that if:

(i) if $abs\,(rank\,(A, u)) < abs(rank(B, u))$ then $H \cup A$ is upward planar if $H \cup B$ is upward planar;

(ii) if $rank(A, u) = rank(B, u)$ then $H \cup A$ is upward planar if and only if $H \cup B$ is upward planar.

Note the use of the absolute value. The rank of A wrt u is defined as follows.

1. $rank(A, u) = 1$ if A admits an UPSL drawing whose external face contains an I-angle centred at u.
2. $rank(A, u) = 2$ if the previous case does not apply and A admits an UPSL drawing whose external face contains an S-angle centred at u as well as an UPSL drawing whose external face contains a T-angle centred at u.
3. $rank(A, u) = 3$ if none of the previous cases applies and A admits an UPSL drawing whose external face contains an S-angle centred at u.
4. $rank(A, u) = -3$ if none of the previous cases applies and A admits an UPSL drawing whose external face contains a T-angle centred at u
5. $rank(A, u) = 10$ if none of the previous cases applies and A admits an UPSL drawing.
6. $rank(A, u) = 99$ if no UPSL drawing of A exists.

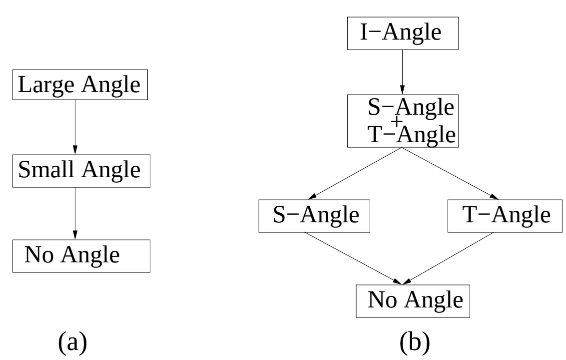


Figure 17: An illustration of the total ordering of the classes of drawings of source-components or sink-components (a); and the partial ordering of the classes of drawings of internal-components (b).

We now restate Theorems 4 – 7 using the concept of rank. This restatement is straight-forward because we have associated a rank with each class of drawing that appears in these theorems. The correctness of our ordering of the classes of drawings that occur in Theorems 4 – 7 follows from these restatements.

Theorem 8 *Let G be a digraph and let u be any vertex in G. G is upward planar if and only if the following four statements are true.*

1. *No component of G wrt u has a rank of* 99.
2. *At most one component of G wrt u has a rank of* 10.
3. *At most one internal-component of G wrt u does not have a rank of* 1.
4. *At least one of the following three statements is true:*
   (a) *All source-components and sink-components of G wrt u have a rank of* 1.
   (b) *All source-components of G wrt u have a rank of* 1 *and all internal-components of G wrt u have a rank r such that $r = -3$ or $abs(r) \leq 2$.*
   (c) *All sink-components of G wrt u have a rank of* 1 *and all internal-components of G wrt u have a rank r such that $r = 3$ or $abs(r) \leq 2$.*

Theorem 9 *Let G be a digraph with a vertex u such that there is a source-component C of G wrt u that contains a vertex $w \neq u$. G has a rank of x wrt w if and only if the following five statements are true.*

A1 *C has a rank of x wrt w.*

A2 *All source-components of $G \setminus C$ wrt u have a rank ≤ 2 wrt u.*

A3 *All sink-components of G wrt u have a rank of* 1 *wrt u.*

A4 *All internal-components of G wrt u have a rank r wrt u such that $r = 3$ or $abs(r) \leq 2$.*

A5 *At most one internal-component of G wrt u has a rank wrt u that is $\neq 1$.*

Theorem 10 *Let G be a digraph with a vertex u such that there is a sink-component C of G wrt u that contains a vertex $w \neq u$. G has a rank of x wrt w if and only if the following five statements are true.*

B1 *C has a rank of x wrt w.*

B2 *All sink-components of $G \setminus C$ wrt u have a rank ≤ 2 wrt u.*

B3 *All source-components of G wrt u have a rank of* 1 *wrt u.*

B4 *All internal-components of G wrt u have a rank r wrt u such that $r = -3$ or $abs(r) \leq 2$.*

B5 *At most one internal-component of G wrt u has a rank wrt u that is $\neq 1$.*

Theorem 11 *Let G be a digraph with a vertex u such that there is an internal-component C of G wrt u that contains a vertex $w \neq u$. G has a rank of x wrt w if and only if the following two statements are true.*

C1 *C has a rank of x wrt w.*

C2 *All components of $G \setminus C$ wrt u have a rank of* 1 *wrt u.*

5.2 Determining Rank

Let B be a block of a digraph G. We now show how to determine the rank of B wrt one of its vertices u, i.e. $rank(B, u)$. B is either a biconnected digraph or its underlying graph is K_2. In the latter case u is either a sink or source in B and B admits an UPSL drawing whose external face contains a large angle centred at u (i.e. $rank(B, u) = 1$). Henceforth we assume that B is a biconnected digraph. We present two algorithms for determining the rank of an embedded digraph wrt one of its vertices: Algorithm 1 computes the rank of an embedded digraph wrt an internal vertex and Algorithm 2 computes the rank of an embedded digraph wrt a source or sink vertex. Using these algorithms as subroutines Algorithm 3 determines the rank of a block wrt one of its cutvertices.

Algorithm 1 *Rank-Internal(G_ϕ, v)*

Input: an acyclic embedded digraph G_ϕ with j nodes that are either sources or sinks and an internal node v of G_ϕ

Output: the rank of G_ϕ wrt v

1: **if** ϕ is not bimodal **then** return 99.

2: Construct a bipartite flow network $\mathcal{N}$ which is defined as follows:

- the vertices of $\mathcal{N}$ are the sources, sinks and faces of G_ϕ; each source or sink $u \in V(G)$ has a supply of one unit and each face $f \in faces(G_\phi)$ has a demand of $S_{angs}(f) - 1$ units;
- $\mathcal{N}$ contains an edge $\langle u, f \rangle$ with a capacity of one unit if and only if u is a source or sink in $V(G)$, $f \in faces(G_\phi)$ and u is contained by the boundary of f.

3: **if** $\mathcal{N}$ does not admit a flow of value $j - 2$ **then** return 99.

4: Let $F = \emptyset$.

for each face f of G_ϕ **do**

Let $\mathcal{N}^f$ be the bipartite flow network formed from $\mathcal{N}$ by increasing the demand of f by two. **if** $\mathcal{N}^f$ admits a flow of value j **then** $F = F \cup \{f\}$.

end for

5: **if** a face in F contains an I-angle with centre v **then** return 1.

else if a face in F contains an S-angle with centre v and a face in F contains a T-angle with centre v **then** return 2.

else if a face in F contains an S-angle with centre v **then** return 3.

else if a face in F contains a T-angle with centre v **then** return -3.

else if $F \neq \emptyset$ **then** return 10.

else return 99.

Suppose that an acyclic embedded digraph G_ϕ with n nodes and an internal node v of G_ϕ are inputted to Algorithm 1. With the exception of Step 5 which inspects the boundary of the successful external faces for angles centred at v and the fact that we have assumed G to be acyclic, Algorithm 1 is identical to Bertolazzi et al.'s [2] algorithm. The inspection performed in Step 5 takes $\mathcal{O}(n)$

time and Bertolazzi et al.'s [2] algorithm takes $\mathcal{O}(n^2)$ time. Thus Algorithm 1 runs in $\mathcal{O}(n^2)$ time.

Algorithm 2 *Rank-Source-Sink*(G_ϕ, v)

Input: an acyclic embedded digraph G_ϕ with j nodes that are either sources or sinks and a source or sink $v \in G$

Output: the rank of G_ϕ wrt v

1: **if** ϕ is not bimodal **then** return 99.

2: Construct a bipartite flow network $\mathcal{N}_v$ which is defined as follows:

- the vertices of $\mathcal{N}_v$ are the sources, sinks and faces of G_ϕ; each source or sink $u \neq v$ has a supply of one, v has a supply of zero and each face f has a demand of $S_{angs}(f) - 1$;
- $\mathcal{N}_v$ contains an edge $\langle u, f \rangle$ with a capacity of one if and only if u is a source or sink in $V(G)$, $f \in faces(G_\phi)$ and u is contained by the boundary of f.

3: **if** $\mathcal{N}_v$ does not admit a flow fl of value $j-3$ **then** return 99.

4: **for** each face f of G_ϕ that contains v **do**
   Let $\mathcal{N}_v^f$ be the bipartite flow network formed from $\mathcal{N}_v$ by increasing the demand of f by one. **if** $\mathcal{N}_v^f$ admits a flow of value $j-1$ **then** return 1.
   end for

5: Let $\mathcal{N}$ be the bipartite flow network formed from $\mathcal{N}_v$ by increasing the supply of v by one.

6: **if** $\mathcal{N}$ does not admit a flow fl' of value $j-2$ **then** return 99.

7: Let $F = \emptyset$.
   for each face f of G_ϕ **do**
   Let $\mathcal{N}^f$ be the bipartite flow network formed from $\mathcal{N}$ by increasing the demand of f by two. **if** $\mathcal{N}^f$ admits a flow of value j **then** $F = F \cup \{f\}$
   end for

8: **if** there is a face in F that contains v **then** return 2.
   else if $F \neq \emptyset$ **then** return 10.
   else return 99.

Algorithm 2 describes how to compute the rank of an embedded digraph G_ϕ which has j source and sink nodes wrt a source or sink v of G_ϕ. The initial bipartite flow network $\mathcal{N}_v$ constructed by Algorithm 2 differs from the initial bipartite flow network constructed by Bertolazzi et al.'s algorithm [2] only in the supply of v. This minor modification allows us to test if G_ϕ admits an UPSL drawing whose external face contains a large angle with centre v. It follows from Bertolazzi et al.'s algorithm [2] and the fact that v has a supply of zero that G_ϕ is upward planar only if $\mathcal{N}_v$ admits a flow of value $j-3$, which is checked in Step 3. In Step 4 we test if G_ϕ admits an UPSL drawing whose external face contains a large angle centred at v. In Step 5 we change the supply of v in $\mathcal{N}_v$ to one and call the resulting bipartite flow network $\mathcal{N}$. Bipartite flow network $\mathcal{N}$

is identical to the initial bipartite flow network that would be constructed if G_ϕ is inputted to Bertolazzi et al.'s algorithm [2]. Finally Step 6, Step 7, and Step 8 are analogous to Step 3, Step 4, and Step 5 of Algorithm 1. We now consider the running time of Algorithm 2. Let n denote the number of nodes in G. The first three steps of Algorithm 2 can be carried out in $\mathcal{O}(n \cdot j)$ time using the same techniques used by Bertolazzi et al.'s algorithm [2]. Step 4 of Algorithm 2 runs in $\mathcal{O}(n^2)$ time as up to two flow augmentations are performed per face of G_ϕ at a cost of $\mathcal{O}(n)$ time each. Step 5 of Algorithm 2 can be performed in constant time. As Step 6, Step 7, and Step 8 of Algorithm 2 are analogous to Step 3, Step 4, and Step 5 of Algorithm 1 it follows that they also run in $\mathcal{O}(n^2)$ time. Thus the total running time of Algorithm 2 is $\mathcal{O}(n^2)$.

We now present Algorithm 3 which follows easily from the previous two algorithms. Algorithm 3 runs in $\mathcal{O}(\#(B) \cdot |V(B)|^2)$ time, where $\#(B)$ represents the number of embeddings of the inputted biconnected digraph B.

Algorithm 3 *Rank-Block*(B, v)

Input: a biconnected digraph B and a vertex v of B.
Output: an integer representing $rank(B, v)$.
1: Test if B is acyclic and planar. If not return 99.
2: Build the SPQR-tree T of B.
3: Use T to generate the embeddings of B.
4: Depending on whether v is a source, sink or internal node use either Algorithm 1 or Algorithm 2 to compute the rank of each embedding of B.
5: Return the integer with lowest absolute value that is calculated in Step 3.

5.3 Star-vertices

In this subsection we introduce a type of cutvertex called a *star-vertex* and prove that a digraph contains a star-vertex if (and only if) it contains a cutvertex (Lemma 18). Star-vertices will play an important role in the decomposition strategy presented in Section 5.4. We begin by introducing some terminology. A *leaf block* of a digraph G is a block of G that contains exactly one cutvertex. A *non-leaf block* of G is a block of G that contains two or more cutvertices. A *star-vertex* is a cutvertex that is contained by at most one non-leaf block. The *block-cutvertex tree* of G, denoted $\mathcal{T}$, is the tree with a vertex for every block and cutvertex of G and an edge between a cutvertex and each block that contains that cutvertex [20]. The vertices in $\mathcal{T}$ that correspond to blocks (cutvertices) in G are referred to as the *block-vertices* (*cutvertices*) of $\mathcal{T}$. The block-cutvertex tree of the digraph in Fig. 18(i) is drawn in Fig 18(ii) with square nodes used to represent the block-vertices of the tree.

Lemma 18 *A digraph G has a star-vertex if and only if it has a cutvertex.*

Proof: The necessity follows from the definition of a star-vertex as a cutvertex with certain properties. We now consider the sufficiency. Assume that G

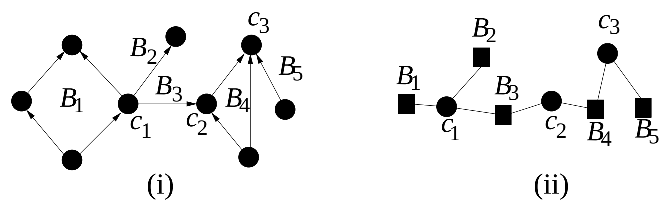


Figure 18: A digraph G (i); the block-cutvertex tree $\mathcal{T}$ of G (ii).

contains a cutvertex and let $\mathcal{T}$ denote the block-cutvertex tree of G. Let $\mathcal{T}^2$ denote the tree that results from deleting all nodes of degree one from $\mathcal{T}$. There is a one-to-one correspondence between the leaf blocks of G and the nodes of degree one in $\mathcal{T}$. Thus all nodes of degree one or zero in $\mathcal{T}^2$ are star-vertices of G. As all cutvertices in $\mathcal{T}$ have a degree of two or more it follows that $\mathcal{T}^2$ contains at least one node of degree one or zero. Thus G contains a star-vertex if G contains a cutvertex. □

5.4 Testing for Upward Planarity

We now describe Algorithm 4, the main algorithm of this paper, in full.

Algorithm 4 *UP-Test(G)*

Input: A connected planar acyclic digraph G.
Output: TRUE if G is upward planar. FALSE otherwise.

Let c be a star-vertex of G;
if all components of G wrt c are blocks **then**
  for each component B of G wrt c **do** $\mathit{Rank\text{-}Block}(B, c)$; **end for**
  if G satisfies Theorem 8 **then** return TRUE; **else** return FALSE; **end if**
else
  Let G' be the multi-block component of G wrt c;
  for each component B of $G \setminus G'$ wrt c **do** $\mathit{Rank\text{-}Block}(B, c)$; **end for**
  if Theorem 8 implies that G is not upward planar **then** return FALSE.
  else if Theorem 8 implies that G is upward planar if and only if G' is upward planar **then** return *UP-Test(G′)*.
  else Theorem 8 implies that G is upward planar if and only if G' admits a drawing of rank r wrt c such that $r = r'$ or $abs(r) < abs(r')$ so return *UP-Test-Rank(G′, c, r′)*; **end if**
end if

Let G be a planar acyclic digraph that we wish to test for upward planarity. Algorithm 4 processes G by splitting it into its components wrt a star-vertex c of G (later in this subsection we will describe how to efficiently locate star-vertices). There are two cases to consider: either all the components of G wrt

c are blocks or there is one multi-block component of G wrt c which we denote by G'. If all the components of G wrt c are blocks then the upward planarity of G can be tested by using Algorithm 3 to calculate their ranks wrt c and then checking if these ranks satisfy the conditions defined by Theorem 8. If there is a multi-block component G' of G wrt c then again using Algorithm 3 calculate the rank of each block component of G wrt c as before. This rank information is then used in conjunction with Theorem 8 to determine a rank r such that G is upward planar if and only if G' admits a drawing of rank r' wrt c such that $r' = r$ or $abs(r') < abs(r)$. The algorithm is then recursively applied to G'. If r is less than 10 then Subroutine 1 (i.e. *UP-Test-Rank(G′, c, r)*) is called instead of Algorithm 4 (i.e. *UP-Test(G′)*). *UP-Test-Rank(G′, c, r)* checks if G' admits a drawing of rank r' wrt c such that $r' = r$ or $abs(r') < abs(r)$.

Subroutine 1 *UP-Test-Rank(G, v, r)*

Input: A connected planar acyclic digraph G, a vertex v of G, and a rank r.
Output: TRUE if G admits a drawing of rank r' wrt v such that $r' = r$ or $abs(r') < abs(r)$. Otherwise returns FALSE.

```
Let c be a star-vertex of G;
Let G_v be the component of G wrt c that contains v;
if all components of G wrt c are blocks then
   if (Rank-Block(G_v, v) ≠ r and abs(Rank-Block(G_v, v)) ≮ abs(r)) then
   return FALSE; end if
   for each component B of G \ G_v wrt c do Rank-Block(B, c); end for
   if G \ G_v satisfies Theorems 9–11 then return TRUE;
   else return FALSE; end if
else
   Let G′ be the multi-block component of G wrt c;
   if G′ = G_v then
      if G \ G_v satisfies Theorems 9–11 then return UP-Test-Rank(G_v, v, r);
      else return FALSE; end if
   else
      if (Rank-Block(G_v, v) ≠ r and abs(Rank-Block(G_v, v)) ≮ abs(r)) then
      return FALSE; end if
      for each component B of G \ (G_v ∪ G′) wrt c do Rank-Block(B, c);
      endfor
      if G \ (G_v ∪ G′) does not satisfy Theorems 9–11 then
         return FALSE;
      else
         Theorems 9–11 and the ranks of the components of G \ (G_v ∪ G′) wrt
         c imply that UP-Test-Rank(G, v, r) is TRUE if and only if G′ admits
         a drawing of rank r† wrt c such that r† = r* or abs(r†) < abs(r*) so
         return UP-Test-Rank(G′, c, r*);
      end if
   end if
end if
```

Subroutine 1 or *UP-Test-Rank(G, v, r)* tests if G admits a drawing of rank r' wrt v such that $r' = r$ or $abs(r') < abs(r)$. Like Algorithm 4, Subroutine 1 processes G by splitting it into its components wrt a star-vertex c of G. Let G_v be the component of G wrt c that contains v. Subroutine 1 considers three cases: all the components of G wrt c are blocks, G_v is the multi-block component of G wrt c and the multi-block component of G wrt c is not G_v. If all the components of G wrt c are blocks then Subroutine 1 uses Algorithm 3 to calculate the rank of G_v wrt v and the ranks of the components of $G \setminus G_v$ wrt c, and checks if these ranks satisfy the conditions defined by Theorems 9–11. If G_v is the multi-block component of G wrt c then again using Algorithm 3 we check if the components of $G \setminus G_v$ wrt c satisfy the conditions defined by Theorems 9–11. If they do then Subroutine 1 is recursively applied to G_v, v and r (i.e. *UP-Test-Rank(G_v, v, r)* is called). If they don't then Subroutine 1 returns FALSE. Recall the third case of Subroutine 1 is when there is a multi-block component G' of G wrt c such that $G' \neq G_v$. In this case we also use Algorithm 3 to calculate the rank of the components of $G \setminus (G_v \cup G')$ wrt c, and the rank of G_v wrt v. If these ranks do not satisfy Theorems 9–11 then Subroutine 1 returns FALSE. If these ranks do satisfy Theorems 9–11 then, there exists a rank r^* such that *UP-Test-Rank(G, v, r)* is TRUE if and only if G' admits a drawing of rank $r^\dagger$ wrt c such that $r^\dagger = r^*$ or $abs(r^\dagger) < abs(r^*)$ (i.e. *UP-Test-Rank(G', c, r^*)* is called).

We now analyse the running time of Algorithm 4. Suppose that G contains n vertices and b blocks which are labeled $B_1, \ldots, B_b$. Let k_i represent the number of embeddings of B_i, for $i = 1, \ldots, b$. Three main factors contribute to the running time of Algorithm 4, the most significant of which is detecting the ranks of the blocks of G (wrt their vertices). Algorithm 4 determines the rank of B_i wrt at most one vertex of B_i, for $i = 1, \ldots, b$. In total this takes $\mathcal{O}\left(\sum_{i=1}^{b} k_i \cdot |V(B_i)|^2\right)$ time.

The second factor that contributes to the running time of Algorithm 4 is the identification of star-vertices. The star-vertices of G can be identified by running Algorithm 5 as a preprocessing step to Algorithm 4. A node in G is either initially a star-vertex or becomes a star-vertex at a latter stage of the decomposition of G by Algorithm 4 (and Subroutine 1) if and only if it is a cutvertex in G. Thus Algorithm 5 orders the cutvertices of G so that if the decomposition of G by Algorithm 4 (and Subroutine 1) is consistent with this order then each cutvertex is a star-vertex of the subgraph of G that remains at that point. We now analyse the running time of Algorithm 5. The block-cutvertex tree $\mathcal{T}$ of G can be built in $\mathcal{O}(n)$ time using depth first search [22]. Also observe that once the degree of all nodes in $\mathcal{T}$ has been calculated (which takes linear time) each node in $\mathcal{T}$ can be processed in $\mathcal{O}(1)$ time by the while loop of Algorithm 5. As $|V(\mathcal{T})| = \mathcal{O}(n)$ the while loop contributes $\mathcal{O}(n)$ time in total to the running time of Algorithm 5. Thus Algorithm 5 runs in $\mathcal{O}(n)$ time.

A third factor that contributes to the running time of Algorithm 4 is checking if the ranks of the single block components of G wrt c satisfy the conditions imposed by Theorem 8 (or Theorems $9 - 11$), for each star-vertex c at which G is decomposed. In order to do this Algorithm 4 counts the number of single block

Algorithm 5 *Star-Vertices*(G)

Input: A connected digraph G with $c \geq 1$ cutvertices.

Output: An ordered list $P_G = \langle v_1, \ldots, v_c \rangle$ of the cutvertices of G such that, v_j is a star-vertex in G_j, for $j = 1, \ldots, c$ (where $G_1 = G$ and G_i is the digraph formed by deleting all leaf-blocks of v_{i-1} in G_{i-1}, for $i = 2, \ldots, c$).

Let P_G be the empty list and let $\mathcal{T}$ be the block-cutvertex tree of G;
while $\mathcal{T}$ contains at least one block-vertex **do**
  Let $\mathcal{T}'$ be the digraph formed by deleting all nodes of degree one from $\mathcal{T}$;
  if $\mathcal{T}'$ consists of a single node **then**
    Add this cutvertex to the end of P_G and let $\mathcal{T} = \mathcal{T}'$;
  else
    Add all nodes of degree one in $\mathcal{T}'$ to the end of P_G (in any order);
    $\mathcal{T}$ = the digraph formed by deleting all nodes of degree one in $\mathcal{T}'$;
  end if
end while
return P_G;

components of G wrt c of each type (i.e. source-component, sink-component, or internal-component) and rank and checks if their quantities satisfy the conditions imposed by Theorem 8 (or Theorems 9 – 11). As G contains $\mathcal{O}(n)$ blocks, in total it takes $\mathcal{O}(n)$ time to count the single block components for all decomposition points of G. Therefore the total running time of Algorithm 4 is $\mathcal{O}\left(n + \sum_{i=1}^{b} k_i \cdot |V(B_i)|^2\right)$ time

6 A Couple of Worked Examples

In this section we present two examples that illustrate the workings of Algorithm 4 (and Subroutine 1). In the first example, Example 1, we describe how Algorithm 4 would process an upward planar digraph G_1 while in the second example, Example 2, we describe how Algorithm 4 would process a digraph, G_2, which is not upward planar. In both examples we assume that Algorithm 5, which looks after the identification of star-vertices, has been run as a preprocessing step.

6.1 Example 1 - An Upward Planar Digraph

In this example we consider how Algorithm 4 would process the upward planar digraph G_1 which is shown in Fig. 19(i). Let $\langle c_3, c_2 \rangle$ denote the ordering of the cutvertices returned by Algorithm 5 when inputted G_1. Thus c_3 is a star-vertex of G_1. There are two components of G_1 wrt c_3: the single block component B_2 and the multi-block component $B_1 \cup B_3 \cup B_4 \cup B_5$ which we denote by G_1' (see Fig. 19(ii)). Using Algorithm 3 we can calculate the rank of B_2 wrt c_3. B_2 is a sink-component of G_1 wrt c_3 and, as shown in Fig. 20(ii), B_2 admits an UPSL drawing whose external face contains a large angle with centre c_3. Thus B_2 has a rank of 1 wrt c_3. It follows from Theorem 8 that there are no restrictions on the number of components of rank 1 wrt a cutvertex and so G_1 is upward planar if and only if G'_1 is upward planar. Thus the next step is to apply Algorithm 4 to G'_1.

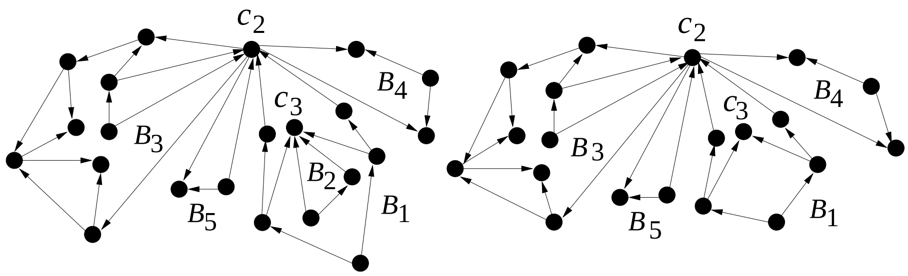


Figure 19: An upward planar digraph G_1 (i); the multi-block component of G_1 wrt c_3 (ii).

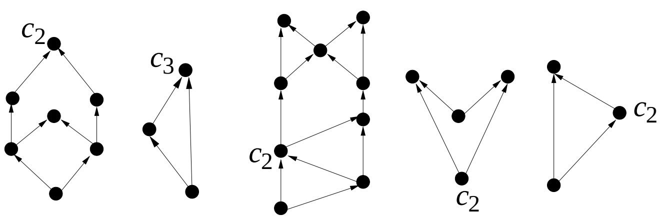


Figure 20: UPSL drawings of B_1 (i), B_2 (ii), B_3 (iii), B_4 (iv) and B_5 (v).

6.1.1 Testing G'_1 for Upward Planarity

The second vertex in the ordering of the cutvertices of G_1 returned by Algorithm 5, $\langle c_3, c_2 \rangle$, is c_2. Thus c_2 is a star-vertex of G'_1. There are four components of G'_1 wrt c_2. These are B_1, B_3, B_4 and B_5. As all are blocks we have reached the base case. Using Algorithm 3 we calculate the ranks of each of these four blocks wrt c_2. It can be seen from Fig. 20(i), (iii), (iv) and (v) that all four blocks have a rank of 1 wrt c_2. As there is no restriction on the number of components of rank 1 in Theorem 8 it follows that G'_1, and therefore G_1, is upward planar.

6.2 Example 2 - A Digraph which is not Upward Planar

In this example we describe how Algorithm 4 would process the non-upward planar digraph G_2 shown in Fig. 21. Let $\langle c_3, c_2 \rangle$ denote the ordering of the cutvertices returned by Algorithm 5 when inputted G_2. Thus c_3 is a star-vertex of G_2.

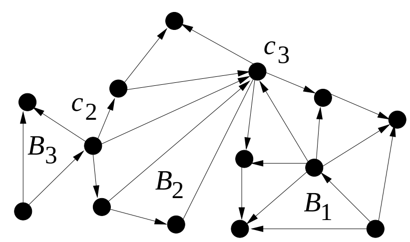


Figure 21: A digraph G_2 which is not upward planar.

There are two components of G_2 wrt c_3. These are the single block component B_1 and the multi-block component $B_2 \cup B_3$ which we denote by G'_2. Using Algorithm 3 we can calculate the rank of B_1 wrt c_3. B_1 is an internal-component of G_1 wrt c_3 and has a rank of 3 wrt c_3. Thus B_1 admits an UPSL drawing whose external face contains an S-angle with centre c_3 (see Fig. 22(i)). It is clear from Theorem 8 that only one internal-component with a rank not equal to 1 is allowed. As G'_2 is also an internal-component of G_2 wrt c_3 it follows that G_2 is upward planar if and only if G'_2 has a rank of 1 wrt c_3. Thus we recursively apply Subroutine 1 to test if G'_2 has a rank of 1 wrt c_3.

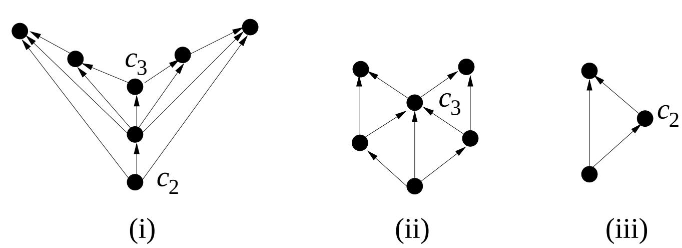


Figure 22: UPSL drawings of B_1 (i), B_2 (ii) and B_3 (iii).

6.2.1 Testing the rank of G'_2.

The second vertex in the ordering of the cutvertices of G_2 returned by Algorithm 5, $\langle c_3, c_2 \rangle$, is c_2. Thus c_2 is a star-vertex of G'_2. G'_2 is shown in Fig. 23. There are two components, B_2 and B_3, of G'_2 wrt c_2. As both components are blocks we have reached the base case. Using Algorithm 3 we calculate the rank of B_2 wrt c_3 and B_3 wrt c_2. B_2 has a rank of 3 wrt to c_3 (see Fig. 22(ii)). It

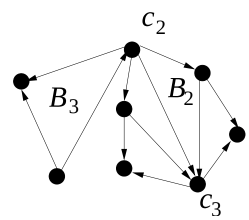


Figure 23: G_2'.

follows from Theorems 9–11 that G_2' does not admit an UPSL drawing of rank 1 wrt c_3. Therefore G_2 is not upward planar, even though all its blocks are upward planar (see Fig. 22).

7 Conclusions

Let G be a planar digraph with b blocks $B_1, \ldots, B_b$ and n vertices. Let k_i represent the number of embeddings of B_i, for $i = 1, \ldots, b$. In this paper we have presented an upward planarity testing algorithm that can process G in $\mathcal{O}\left(n + \sum_{i=1}^{b} k_i \cdot |V(B_i)|^2\right)$ time. Although this running time is acceptable for digraphs whose blocks have few embeddings (e.g. digraphs with a high similarity to trees or digraphs whose blocks are triconnected digraphs) in general it is too high to be of practical use. Thus we see the main contribution of this paper as reducing the problem of testing if a digraph is upward planar to that of testing if its biconnected components admit UPSL drawings with certain properties. We believe that this is an important step towards the development of a practically useful upward planarity testing algorithm.

Currently, all suggested algorithms for testing the upward planarity of general digraphs use one of two high level strategies. The first strategy consists of algorithms which, using Bertolazzi, Di Battista, Liotta and Mannino's algorithm [2] for testing the upward planarity of embedded digraphs as a subroutine, try to examine as few embeddings as possible in order to discern whether or not a given digraph is upward planar [5, 15]. The second strategy for testing the upward planarity of a digraph involves testing if it is a spanning subgraph of a planar *st*-digraph [9, 18]. Both strategies can be tailored to test biconnected digraphs for the properties stated in our characterisation (we showed how to tailor the first strategy to test for these properties in Section 5.2). Thus the running time of algorithms based on both strategies can be improved by dividing a digraph into its blocks and testing each block for the relevant properties. This reduction in running time is not only due to the fact that we are testing subproblems with a smaller input size. Because we are now testing digraphs that are biconnected there is a significant reduction on the number of embeddings that they allow. This can be seen from the following formula that represents the number of embeddings of G [4].

Theorem 12 (Cai [4]) *Let C be the set of cut vertices of G. For each cutvertex $c \in C$ let B^c be the set of biconnected components sharing c and let d_c be the degree of c. For each $c \in C$ and for each $x \in B^c$ let $m_{c,x}$ represent the number of edges in x that are incident on c. Then the total number of embeddings of G is*

$$\prod_{i=1}^{b} k_i \cdot \prod_{c \in C} \Big(\prod_{x \in B^c} m_{c,x} \prod_{j=1}^{|B^c|-2} (d_c - j) \Big).$$

By treating the biconnected components of G separately at most $\sum_{i=1}^{b} k_i$ embeddings need be considered. Clearly this is a significant reduction.

The effectiveness of the decomposition strategy presented in this paper was displayed by the authors when it was invoked in a parameterised algorithm that can test the upward planarity of G in $\mathcal{O}(2^t \cdot t! \cdot n^2)$ time [15], where the parameter t represents the number of triconnected components in G. This improves on the previous best algorithm for this parameter which had a running time of $\mathcal{O}(t! \cdot 8^t \cdot n^3 + 2^{3 \cdot 2^k} \cdot t^{3 \cdot 2^k} \cdot t! \cdot 8^t \cdot n)$ [5], where k is the number of cutvertices in G.

Bertolazzi, Di Battista and Didimo [1] have developed a branch and bound algorithm for testing the upward planarity of biconnected digraphs. Their algorithm uses the first of the high level strategies mentioned previously (i.e. it examines only a subset of the set of embeddings of a biconnected digraph) and thus it should be possible to tailor their algorithm to test for the properties stated in our characterisation. We suggest as an open problem that of devising an upward planarity testing algorithm for biconnected digraphs that works by testing if its triconnected components admit UPSL drawings with certain properties.

Acknowledgments

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